DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

AGGREGATE MATCHINGS

Federico Echenique

SangMok Lee

Matthew Shum

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Abstract

This paper characterizes the testable implications of stability for aggregate matchings. We consider data on matchings where individuals are aggregated, based on their observable characteristics, into types, and we know how many agents of each type match. We derive stability conditions for an aggregate matching, and, based on these, provide a simple necessary and sufficient condition for an observed aggregate matching to be rationalizable (i.e. such that preferences can be found so that the observed aggregate matching is stable). Subsequently, we derive moment inequalities based on the stability conditions, and provide an empirical illustration using the cross-sectional marriage distributions across the US states.

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1 Introduction

The literature on stable matching has grown rapidly, but as a positive empirical theory, stable matchings are still not well understood. There are many advancements and refinements in the theoretical literature, and many normative applications of the theory to real-world matching markets. Positive empirical studies of matching, however, have lagged behind. This is due to three difficulties in deriving observable implications of the theory. The first is a pure dimensionality constraint; many real-world matching markets (such as marriage or housing markets) are huge, featuring hundreds of thousands or millions of individuals on each side of the markets. Most of the theoretical matching models, which are formulated at the individual-level, quickly become intractable at these large dimensions. The second is an indeterminacy in the direction of revealed preference: if Alice matches with Bob and not with Bruce, we cannot know if Alice prefers Bob over Bruce, or if Bruce is unavailable to Alice because he prefers his partner to matching with Alice. The third difficulty is that the theory imposes no restriction on a single matching: one would need to observe different matchings involving the same agents. Multiple observations of different matchings among the same agents are not realistic.

We present one solution to these difficulties. One can aggregate the observed matchings into cells, where the individuals on each side of the market are summed up into cells on the basis of their observed characteristics. Indeed, often data on matching markets is simply not available at the individual level, for privacy concerns, among other reasons, and most empirical studies of marriage matching use aggregate level data. What restrictions on these *aggregate matchings* are implied by the individual-level matching models? This is the motivating question of this paper.

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We find that the theory has very strong implications for aggregate matchings. This may explain why most of the existing empirical literature works with aggregate matchings. The empirical literature also often assumes that agents can make monetary transfers (transferable utility; hereafter TU). Our results are the first deriving the complete observable implications of stability for aggregate matchings: We characterize the observable implications of the models with transfers, and without. It turns out that both models imply strong empirical restrictions, but the theory is strictly more restrictive when transfers are possible. Our results explain, in a sense, why aggregate matchings and transfers have been useful for empirical studies.

We develop an econometric approach for estimating preferences from observed aggregate matchings. Our approach is based on moment inequalities obtained from imposing stability on the data. We develop our estimation technique under the assumption that agents cannot make transfers. In light of the second difficulty mentioned above, this is the most challenging case, and we believe it is realistic for many matching markets. Once transfers between individuals are ruled out, however, multiple stable matchings become a generic feature, which raises important difficulties for the econometric estimation of preferences from observed matching data. We show how the econometric technique can be used by presenting an empirical application to aggregate marriage data.

1.1 General motivation. Discrete choice theory is based on the idea that revealed preferences are unambiguous: if an agent chooses A when B is available then the utility of A is higher than the utility of B. In contrast, in two-sided choice problems, revealed preferences are ambiguous. An agent may choose A over B even when she regards B as the better choice; the reason is that B has a say in the matter, and B may prefer some other choice over matching with the agent. Thus, in a two-sided model, preferences and allocations determine "budgets" endogenously: an agent can only choose among options that are willing to match with the agent, given who their partners are.

For empirical work, the two-sided nature of choices presents a unique challenge. One cannot take choices as given and infer preferences. There is a fundamental simultaneity that must be dealt with, where preferences determine the sets of willing partners ("budgets"), and these sets in turn determine the direction of revealed preferences. Most of the literature deals with the problem by assuming transferable utility, so that matchings maximize total surplus. We tackle the problem directly, in an non-transferable utility model. Our econometric technique is based on deriving a moment inequality from the stability constraints, this technique is quite different from the methods based on discrete choice.

1.2 Related literature. Our paper is close in focus to several other recent papers exploring the empirics of matching markets. These papers can roughly be divided into those in which NTU is assumed, and those in which TU is assumed.

A matching model under TU is equivalent to the Shapley and Shubik (1971) assignment game. A stable matching is one which maximizes the sum of the joint surplus of all matched couples. This is the setup considered in Choo and Siow (2006) and Galichon and Salanie (2009), who consider identification and estimation of TU matching models. Specifically, Choo and Siow derive and estimate an aggregate matching model using marriage cross-sections from the UC Census. Assuming independent logit preferences shocks at the individual level, and a continuum of men and women, they derive a "marriage matching function" of the form:

$$\Pi_{ij} = x_{ij}/\sqrt{x_{i0}x_{j0}}$$

where Π_{ij} is (half of the) total surplus from a match between a type-i man to a type-j woman. x_{ij} are the number such marriages observed, and x_{i0} and x_{j0} are, respectively, the number of single type-i men and type-j women. For the empirical work, they recover the surplus Π_{ij} given the observations of the x's. They do this for two aggregate matchings: one from the early 1970's, and another from the early 1980's. Subsequently, they use their results to explore the dramatic fall in marriage rates between the 1970's and the 1980's and, strikingly, they find that the legalization of abortion can explain about 20% of this drop.

Fox (2007) considers individual-level TU matching models, and develops a maximum-score estimator for these models based on a "pairwise stability" requirement, which implies that, if an observed matching is stable, then no two pairs of agents should profitably be able to swap their partners. (This condition may not hold for stable matchings in NTU settings.) When the matching markets are big, such a comparison of *all* the pairwise stability conditions becomes infeasible. Fox shows that only a subset of the inequalities need to be used in the estimation, so long as a "rank-order" property holds. Subsequently, Bajari and Fox (2008) apply this estimator to analyze the efficiency of allocations in wireless spectrum auctions run by the FCC.

In the NTU setting, Dagsvik (2000) considers the question of inferring preferences from aggregate matching data. Like Choo-Siow, he assumes independent logit-distributed preference shocks at the individual level. Assuming large number of agents (his results are asymptotic in the number of men and women of each type), Dagsvik derives an equilibrium equation, where the supply of matches of a certain type must equal the demand. His aggregate matching function is

$$c_{ij} = \frac{x_{ij}}{x_{i0}x_{j0}}; \quad c_{ij} \equiv a_{ij}b_{ji}$$

where a_{ij} and b_{ji} are, respectively, the deterministic portion of the utility a type-i man gets from a type-j woman, and vice versa, which is very similar to the Choo-Siow matching function, obtained from a TU setting.

Echenique (2008) studies the sets of matchings that can be rationalized as being stable. The focus is on repeated observations of stable individual matchings, not on aggregate matchings.

Hitsch, Hortaçsu, and Ariely (2006) also work in the NTU model, but they employ a dataset from an online dating service to estimate preferences separately from the process of matching. Then they use the estimated preferences to simulate the men- and women-optimal matchings, and compare these optimal matchings to the actual matches observed from the dataset.

2 The Model

2.1 Preliminary definitions. An (undirected) ***graph*** is a pair $G = (V, E)$, where V is a set and E is a subset of $V \times V$. A ***path*** in G is a sequence $p = \langle x_0, \ldots x_N \rangle$ such that for $n \in \{0, \ldots N-1\}$, $(x_n, x_{n+1}) \in E$. We write $x \in p$ to denote that x is a vertex in p. A path $\langle x_0, \ldots x_N \rangle$ ***connects*** the vertices x_0 and x_N. A path $\langle x_0, \ldots x_N \rangle$ is ***minimal*** if there is no proper subsequence of $\langle x_0, \ldots x_N \rangle$ that is also a path connecting the vertices x_0 and x_N.

A ***cycle*** in G is a path $c = \langle x_0, \ldots x_N \rangle$ with $x_0 = x_N$. A cycle is ***minimal*** if for any two vertices x_n and $x_{n'}$ in c, the paths in c from x_n to $x_{n'}$, and from $x_{n'}$ to x_n, are minimal. Say that x and y are ***adjacent in*** c if there is n such that $x_n = x$ and $x_{n+1} = y$ or $x_n = y$ and $x_{n+1} = x$.

If c and c' are two cycles, and there is a path from a vertex of c to a vertex of c', then we say that c and c' are ***connected***.

An ***aggregate matching market*** is described by a triple $\langle M, W, > \rangle$, where:

1. M and W are disjoint, finite sets. We call the elements of M ***types of men*** and the elements of W ***types of women***.

2. $>= ((>_m)_{m \in M}, (>_w)_{w \in W})$ is a profile of strict preferences: for each m and w, $>_m$ is a linear order over $W \cup \{m\}$ and $>_w$ is a linear order over $M \cup \{w\}$.

We call agents on one side men, and on the other side women, as is traditional in the matching literature. Many applications are, of course, to environments different from the marriage matching market.

Consider an aggregate matching market $\langle M, W, > \rangle$, with $M = \{m_1, \ldots, m_K\}$ and $W = \{w_1, \ldots, w_L\}$. An ***aggregate matching*** is a $K \times L$ matrix $X = (X_{ij})$ with non-negative integer entries. The interpretation of X is that X_{ij} is the number of type-i men and type-j women matched to each other. An aggregate matching X is ***canonical*** if $X_{ij} \in \{0, 1\}$. A canonical matching X is a ***simple matching*** if for each i there is at most one j with $X_{ij} = 1$, and for each j there is at most one i with $X_{ij} = 1$. The standard theory of stable matchings studies simple matchings (Roth and Sotomayor, 1990).

An aggregate matching X is ***individually rational*** if $X_{ij} > 0$ implies that $w_j >_{m_i} m_i$ and $m_i >_{w_j} w_j$. A pair of types (m_i, w_j) is a ***blocking pair*** for X if there are $w_l \in W$ with $X_{il} > 0$, and $m_k \in M$ with $X_{kj} > 0$, such that $w_j >_{m_i} w_l$ and $m_i >_{w_j} m_k$. An aggregate matching X is ***stable*** if it is individually rational and there are no blocking pairs for X.

For any aggregate matching X, we can construct a canonical aggregate matching X^c by setting $X^c_{ij} = 0$ when $X_{ij} = 0$ and $X^c_{ij} = 1$ when $X_{ij} > 0$. The following is obvious:

Proposition 1. *An aggregate matching X is stable if and only if X^c is stable.*

Based on this observation, our theoretical results focus on canonical aggregate stable matching.

2.2 Stability conditions. Given a matching market $\langle M, W, > \rangle$, we can construct a graph (V, E) by letting V be the set of pairs (i, j), $i = 1, \ldots, N$ and $j = 1, \ldots, K$. Define

E by $((i,j),(k,l)) \in E$ if either $w_l >_{m_i} w_j$ and $m_i >_{w_l} m_k$ or $w_j >_{m_k} w_l$ and $m_k >_{w_j} m_i$. Then X is stable if and only if

$$((i,j),(k,l)) \in E \Rightarrow X_{ij}X_{kl} = 0. \tag{1}$$

In what follows, we will also make use of the contrapositive to the above statement. Given a canonical matching X, we define an **antiedge** as a pair of couples $(i,j),(k,l)$ with $i \neq k \in M;\ j \neq l \in W$ such that $X_{ij} = X_{kl} = 1$. Then, (1) is equivalent to:

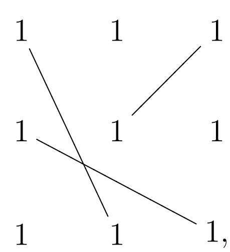
$$(ij),(kl) \text{ is anti-edge } \Rightarrow \begin{cases} \mathbf{1}(w_l >_{m_i} w_j) \cdot \mathbf{1}(m_i >_{w_l} m_k) = 0 \\ \mathbf{1}(w_j >_{m_k} w_l) \cdot \mathbf{1}(m_k >_{w_j} m_i) = 0 \end{cases} \tag{2}$$

In our econometric approach below (Section 4), the contrapositive statement (2) of the stability conditions forms the basis for the moment inequalities.

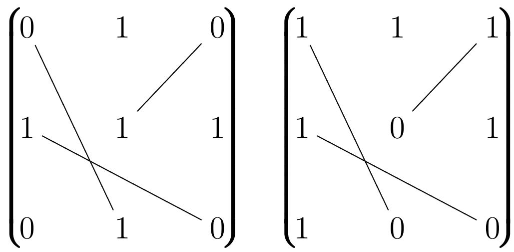
In this section, we use the graph (V,E) to understand stable matchings for given preferences. In the proof of Theorem 7 of Section 3, we use it to infer preferences such that a given matching is stable. For an example, consider the matching market with three types of men and women, and preferences described as follows.

$>_{m_1}$	$>_{m_2}$	$>_{m_3}$	$>_{w_1}$	$>_{w_2}$	$>_{w_3}$
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
w_3	w_1	w_2	m_1	m_2	m_3

The resulting graph can be represented as follows.



where each vertex is indicated with a number 1. The requirement of stability translates into sets of vertexes that must be 0. For example, applying (1) we find that the following two matrices are stable matchings:



2.3 Remarks.

2.3.1 Aggregate matchings are not simple. We show that the testable implications of aggregate stable matchings differs from those of simple stable matchings. In particular, it is tempting to view an aggregate matching as a combination, or the coexistence, of a collection of underlying stable single matchings. This view would be incorrect, as there are additional restrictions imposed when one aggregates.

Consider the following example.

Example 2. Let $\langle M, W, > \rangle$ be an aggregate matching market with $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and where preferences are defined as follows:

m_1	m_2	m_3
w_1	w_2	w_3
w_2	w_3	w_1
w_3	w_1	w_2

w_1	w_2	w_3
m_2	m_3	m_1
m_3	m_1	m_2
m_1	m_2	m_3

Meaning that m_2 ranks w_2 first, followed by w_3, and so on.

The following simple matchings are stable:

$$X^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad X^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Consider the sum of X^1 and X^2:

$$\hat{X} = X^1 + X^2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

One might want to conclude that $\hat{X}$ is stable because it corresponds to the simultaneous matching of agents through X^1 and X^2. Note, however, that $\hat{X}$ is not a stable aggregate matching. The pair (m_1, w_2) is a blocking pair: we have that $w_2 >_{m_1} w_3$ and $m_1 >_{w_2} m_2$ while $\hat{X}_{13} > 0$ and $\hat{X}_{22} > 0$. One cannot view aggregate stable matchings by their decomposition into simple stable matchings.[1]

This example also shows that an aggregate matching cannot be interpreted as a "fractional" solution to the stability constraints in the linear programming formulation of stable matchings (Vande Vate, 1989; Teo and Sethuraman, 1998). Here $\frac{1}{2}\hat{X}$ is a fractional stable matching; but does not correspond to an aggregate stable matching. A similar phenomenon arises with lotteries over matchings and ex-ante stability, see Kesten and Ünver (2009).

Put differently, the testable implications of stability for aggregate matchings cannot be reduced to stability for a collection of simple matchings. There are "cross restrictions" that need to be dealt with; in the example these take the form of instances of m_1 and w_2 who block in a way that is not present in any of the stable simple matchings.

In Section 4.3 we show further how simple disaggregate matchings do not generate empirical implications with traction at the aggregate level.

2.3.2 Single Agents. We have assumed that there are no single agents, but assumption just serves to simplify our notations. We can imagine that, for example, there is $n_i > \sum_j X_{i,j}$ men of type i, and that $n_i - \sum_j X_{i,j}$ of them are single. Our model and results are easily adaptable to this case: One would work with a matrix that has an additional row and column, say i^* and j^*. Then X_{i,j^*} would represent the men of type i who are single; simple adaptations of the results in Section 3 go through.

2.4 The structure of aggregate stable matchings. Let X and X' be aggregate matchings. Say that X ***dominates*** X' if, for any i and j, $X_{ij} = 0$ implies that $X'_{ij} = 0$. The following result is immediate from the definition of a stable aggregate matching.

Proposition 3. *Let X be a stable aggregate matching. If X' is an aggregate matching, and X dominates X', then X' is stable.*

Thus, given an aggregate matching market $\langle M, W, > \rangle$, there is a family of ***maximal***

[1] The conclusion is reinforced by the results of Section 2.4, where we show that the structure of aggregate stable matchings differs from the lattice structure of simple stable matchings.

stable matchings $\mathcal{X}$: this family describes all the stable matchings, as a matching is stable if and only if it is dominated by a member of $\mathcal{X}$.

The members of $\mathcal{X}$ may have different cardinalities (see Example 12 in Section A), so $\mathcal{X}$ does not necessarily have a matroid structure. It is nevertheless straightforward to give an algorithm that, given a matching market $\langle M, W, > \rangle$, computes the set $\mathcal{X}$, and thus finds all the aggregate stable matchings. We proceed to describe the algorithm.

Consider the graph (V, E) associated to $\langle M, W, > \rangle$. Enumerate the vertices, $V = \{1, 2, \ldots N\}$. Start with the matching X^0 that is identically zero. For $v \in V$, given the matching X^{v-1}, define X^v to be identical to X^{v-1} except possibly at entry v. Let entry v be 1 if that does not violate condition (1); let entry v be 0 otherwise. Let $X = X^N$.

The algorithm constructs an aggregate stable matching, as each X^v is an aggregate stable matching. To see that it is maximal, let $\hat{X} \neq X$ be an aggregate matching that dominates X. Let v be a vertex in V such that the entry corresponding to v in X is 0 and the entry in $\hat{X}$ is 1. By definition of X^v, there must be some entry v' such that $(v, v') \in E$ and entry v' in X^v is 1. The entry v' must be 1 in $\hat{X}$, as $\hat{X}$ dominates X and X dominates X^v. Then $\hat{X}$ is not stable because it violates condition (1).

We end this section with a partial result on the structure of $\mathcal{X}$. One may wonder when $\mathcal{X}$ coincides with the simple stable matching for market $\langle M, W, > \rangle$. We show that, typically, $\mathcal{X}$ contains non-simple matchings.

Proposition 4. *Let X be an individual stable matching. $K = |M|$ ($L = |W|$) is the number of types of men (women).*

1. *If $K = L = 3$ then X is not a maximal stable matching.*
2. *If $K > 3$, $L > 3$ and X is a maximal stable matching, then one of the following two possibilities must hold:*
   - *(a) For all (i, j), the submatching $X^{-(i,j)}$ is a maximal stable matching in the $-(i, j)$ submarket.*
   - *(b) There is (h, l) with $X_{hl} = 1$, and a maximal stable matching $\tilde{x}$, for which $\tilde{x}_{h,j} = \tilde{x}_{i,l} = 0$ for all i and j.*

Note that (2) together with (1) is meant to suggest a recursive idea. When $K = L = 4$, (2a) cannot be true so we must have a matched pair in X that is nevertheless "totally single" in another maximal stable matching.

3 Rationalizing Aggregate Matchings.

We suppose that we observe an aggregate matching, and ask when there are preferences that can rationalize it as a stable matching. The property is related to how many entries in the matching matrix are non-zero. Specifically, we consider the graph formed by connecting any two non-zero elements of the matrix, as long as they lie on the same row or column. It turns out that rationalizable of an aggregate matching depends on the number and connectedness of minimal cycles on this graph. We consider the NTU and TU cases in turn.

3.1 Without transfers. Let $M = \{m_1, \ldots, m_K\}$ and $W = \{w_1, \ldots, w_L\}$ be sets of types of men and women. We write i and j for typical types of men and women, and i_l and j_k for specific types of men and women.

We suppose that we are given an aggregate matching X, and we want to understand when there are preferences for the different types of men and women, such that X is a stable aggregate matching. Say that a canonical matching X is **rationalizable** if there exists a preference profile $>= ((>_m)_{m\in M}, (>_w)_{w\in W})$ such that X is a stable aggregate matching in $\langle M, W, >\rangle$.

We present first a simple result, showing that a rationalizable matrix must be relatively *sparse*: it cannot have too many non-zero elements. Proposition 5 is subsumed in Theorem 7, but it has a simple and intuitive proof so we choose to present it here.

Proposition 5. *If X has a 3×2 or a 2×3 submatrix that is identically 1, then X is not a stable aggregate matching for any preference profile.*

Proof. We may assume that X is the submatrix in question. Suppose X is stable. By individual rationality, for all men any woman is preferable to being single. Similarly for the women. We must find a pair (i, j) such that w_j is not last in m_i's preference, and m_i is not last in w_j's preferences. Finding this pair suffices because then there is k and l with $X_{ik} = 1$ and $X_{lj} = 1$ and $w_j >_{m_i} w_k$, $m_i >_{w_j} m_l$. Say that m_1 ranks w_1 last. If either w_2 or w_3 rank m_1 as not-last, then we are done. If both w_2 and w_3 rank m_1 last then consider m_2: m_2 must rank one of w_2 and w_3 as not-last. Since they rank m_1 last then we are done. □

Remark 6. If $K = L = 2$ then the matching X that is identically 1 is stable for the preferences

$>_{m_1}$	$>_{m_2}$	$>_{w_1}$	$>_{w_2}$
w_1	w_2	m_2	m_1
w_2	w_1	m_1	m_2

Fix a matching X. We use the graph defined by the 1-entries in X, where there is an edge between two entries in the same row, and an edge between two entries in the same column. Formally, consider the graph (V, L) for which the set of vertices is $V := \{(i, j) | i \in M, j \in W \text{ such that } X_{ij} = 1\}$, and there is an edge $((i, j), (k, l)) \in L$ if $i = k$ or $j = l$.

The main result of the paper is Theorem 7, a characterization of the rationalizable aggregate matchings. The proof of the sufficiency direction is constructive; it works by using an algorithm to construct a rationalizing preference profile. The construction is not universal, in the sense that some rationalizing preference profiles cannot be constructed using the algorithm (see Example 13).

To simplify the statement and proof of the theorem, we assume that there are no single men or women. Similar arguments apply to the case when some agents may be single.[2]

Theorem 7. *An aggregate matching X is rationalizable if and only if the associated graph (V, L) does not contain two connected distinct minimal cycles.*

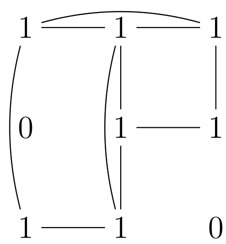
The following example illustrates the condition in the theorem.

Example 8 (minimal cycle). Let X be

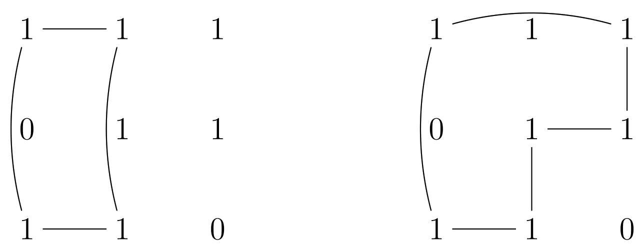
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

[2]Add a column j_s and a row i_s to X. Let X_{i,j_s} be the number of type i men who are single and $X_{i_s,j}$ the number of type j women who are single. A result similar to Theorem 7 holds for this augmented matrix.

The graph (V, L) can be represented as



The following is an example of two minimal cycles that are connected.



3.2 With transfers. So far, we have focused on the NTU model. In this section, we explore the empirical implications of transfers vis-a-vis the result in Theorem 7. We show that if agents can make transfers, then stability has *strictly more* empirical bite than when transfers are not present: any aggregate matching that is rationalizable with transfers is also rationalizable without transfers.[3]

The model of matching with transfers was first introduced by Shapley and Shubik (1971), and applied to the problem of marriage by Becker (1973). A pair of men and women (m, w) generate a surplus $\alpha_{m,w} \in \mathbf{R}$ if they match. The stable matchings are the ones that maximize the total sum of match surplus.

For an aggregate matching X, we suppose that a type i man who matches with a type j woman can generate a surplus $\alpha_{i,j} \in \mathbf{R}$. So the surplus generated by the matchings of types i and j in X is $X_{i,j}\alpha_{i,j}$. The information on surpluses is given by a matrix

$$\alpha = (\alpha_{i,j})_{|M|\times|W|}.$$

Now, in familiar "revealed preference" fashion we ask when, given X, there is a matrix

[3]This contrasts sharply with the results on rationalizing a collection of simple matchings. Chambers and Echenique (2009) show that there are sets of matchings that are rationalizable with transfers but not without transfers, and vice versa.

α such that X is stable for the surpluses in α.

Formally, let X be an aggregate matching. Say that X is ***TU-rationalizable*** by a matrix of surplus α if X is the unique solution to the following problem.

$$\begin{array}{l} \max_{\tilde{X}} \sum_{i,j} \alpha_{i,j} \tilde{X}_{i,j} \\ s.t. \begin{cases} \forall j \ \sum_i \tilde{X}_{i,j} = \sum_i X_{i,j} \\ \forall i \ \sum_j \tilde{X}_{i,j} = \sum_j X_{i,j} \end{cases} \end{array} \tag{3}$$

Remark 9. We restrict $\tilde{X}$ in (3) to have the same number of agents of each type as X. The restriction is obviously needed, as one could otherwise generate high surplus by reclassifying agents into high-surplus types. Essentially, we consider situations where the number of agents of each type is given, and we focus on how they match.

Note also that we require X to be the unique maximizer in (3). There is a clear contrast with Section 3, where we did not require that X was the unique stable matching. This difference is inevitable, though. If we instead would require X to be only one of the maximizers of (3), then any matching can be rationalized. In a sense, without transfers multiplicity is unavoidable (only very strong conditions ensure a unique stable matching), while uniqueness in the TU model holds for almost all real matrices α.

Theorem 10. *An aggregate matching X is TU-rationalizable if and only if the associated graph (V, L) contains no minimal cycles.*[4]

Corollary 11. *If an aggregate matching X is TU-rationalizable, then it is rationalizable.*

4 Empirical implementation

Starting in this section, we consider how to estimate agents' preferences from observed aggregate matchings. Throughout, we assume the following parameterized preferences:

$$u_{ij} = Z_{ij}\beta + \varepsilon_{ij}, \tag{4}$$

where u_{ij} denotes the utility received by a type i individual if he/she matches with a type j individual. Z_{ij} is a vector of observed covariates; β is the vector of parameters we want to estimate; and ε_{ij} denotes unobserved components of utility. In the empirical

[4] A graph contains a cycle if and only if it contains a minimal cycle. We stress minimality in the results because they play a crucial role in our proofs.

work, we assume that $\varepsilon_{i,j}$ is ii.d. distributed according to a $N(0,1)$ distribution, across all pairs of types (i,j), and also independent of the observables $Z_{i,j}$. Given the utility specification, then, we define

$$d_{ijk} \equiv \mathbf{1}(u_{ij} \geq u_{ik}).$$

4.1 Estimating equations. The antiedge condition (2) implies that

$$\begin{aligned} Pr((ij),(kl) \text{ antiedge}) &\leq (1 - Pr(d_{ilj} = d_{lik} = 1))(1 - Pr(d_{jki} = d_{kjl} = 1)) \\ &= Pr(d_{ilj}d_{lik} = 0, d_{jki}d_{kjl} = 0). \end{aligned} \tag{5}$$

For given parameter values β, Eq. (5) can be written as a moment inequality:

$$\mathbb{E}\underbrace{\left[\mathbb{1}((ij),(kl) \text{ antiedge}) - Pr(d_{ilj}d_{lik} = 0, d_{jki}d_{kjl} = 0; \beta))\right]}_{g_{ijkl}(X_t;\beta)} \leq 0. \tag{6}$$

The identified set is defined as

$$\mathbb{B}_0 = \left\{\beta : \mathbb{E}g_{ijkl}(X_t;\beta) \leq 0, \ \forall i,j,k,l\right\}.$$

These moment inequalities are quite distinct from other types of inequalities considered in the existing empirical matching literature. A number of papers in this literature (including Choo and Siow (2006), Dagsvik (2000), Fox (2007)) use inequalities similar to those in the multinomial choice literature, that each observed pair (i,j) represents, for both i and j, an "optimal choice" from some "choice set". The restrictions in (2) cannot be expressed in such a way.

Assume that we observe multiple aggregate matchings. Let T be the number of such observations, and X_t denote the t-th aggregate matching that we observe. Then the sample analog of the expectation in (6) is

$$\begin{aligned} &\frac{1}{T}\sum_t \mathbb{1}((ij),(kl) \text{ is antiedge in } X_t) - Pr(d_{ilj}d_{lik} = 0, d_{jki}d_{kjl} = 0; \beta) \\ =&\frac{1}{T}\sum_t g_{ijkl}(X_t;\beta). \end{aligned} \tag{7}$$

If the number of types of men and woman were equal ($M = W$), then there would be $\frac{W^2 * (W-1)^2}{2}$ such inequalities, corresponding to each couple of pairs. Note that the expectation $\mathbb{E}$ above is over both the utility shocks ε's, as well as over the "equilibrium selection" process (which we are agnostic about).

There is by now a large methodological literature on estimating confidence sets for parameters in partially identified moment inequality models that cover the identified set $\mathbb{B}_0$ with some prescribed probability. (An incomplete list includes Chernozhukov, Hong, and Tamer (2007), Andrews, Berry, and Jia (2004), Romano and Shaikh (2009), Pakes, Porter, Ho, and Ishii (2007).) While there are a variety of objective functions one could use, we use here the simple sum of squares objective:

$$\mathbb{B}_n = \text{argmin}_\beta Q_n(\beta) = \sum_{i,j,k,l} \left[\frac{1}{T} \sum_{t=1}^{T} g_{ijkl}(X_t; \beta) \right]_+^2$$

where $[x]_+$ denotes $x * \mathbb{1}(x > 0)$.

4.2 Relaxing the stability constraints. Stability (rationalizability) places very strong demands on the data that can be observed. The condition in Theorem 7 will very often be violated by aggregate matchings with many non-zero elements. We propose a relaxation of the stability constraint that is particularly useful in applied empirical work.

We assume that blocking pairs may not necessarily form. So if preferences are such that the pair (m, w) would block X, the block only actually occurs with probability less than 1. The reason for not blocking could be simply the failure of m and w to meet or communicate (as in the literature on search and matching).

Specifically, we allow for the possibility that an observed edge between pairs (i, j) and (k, l) may imply nothing about the preferences of the affected types i, j, k, l, simply because the couples (i, j) and (k, l) fail to meet. In particular, define

$$\delta_{ijkl} = P(\text{types } (i, j), (k, l) \text{ communicate}).$$

We then modify the stability inequalities (2) as:

$$\begin{pmatrix} (ij), (kl) \text{ is anti-edge} \\ (ij), (kl) \text{ meet} \end{pmatrix} \Rightarrow \begin{cases} d_{ilj} d_{lik} = 0 \\ d_{jki} d_{kjl} = 0 \end{cases} \tag{8}$$

This leads to the modified moment inequality:

$$Pr((ij), (kl) \text{ antiedge}) \leq \frac{Pr(d_{ilj} d_{lik} = 0, d_{jki} d_{kjl} = 0; \beta)}{\delta_{ijkl}} \tag{9}$$

Note that as $\delta_{ijkl} \to 1$, we expect that the identified set $\mathbb{B}_0$ shrinks to the empty set. The reason is that most aggregate matchings violate the condition in Theorem 7; thus they cannot be rationalized without a positive probability that potential blocking pairs do not form. On the other hand, as $\delta_{ijkl} \to 0$, the identified set converges to the whole parameter space: the right-hand side of the moment inequality becomes larger than 1.

Here, we are assuming that the events $((ij),(kl)$ is an edge) and $((ij),(kl)$ meet) are independent events. The first event depends on preferences and process that produces a stable matching in the first place. So we are making the assumption that the probability of communication is independent of preferences and the matching.[5]

In our empirical work, we allow δ_{ijkl} to depend on the relative number of matched (i,j) and (k,l) couples in each observation. Specifically, letting γ denote a scaling parameter, we set

$$\delta^t_{ijkl} = \min\{\ 2 \cdot \gamma \cdot \frac{|X_{T_i^M,T_j^W}|}{|X_t|} \cdot \frac{|X_{T_k^M,T_l^W}|}{|X_t|}\ ,\ 1\ \}$$

where $|X_{T_i^M,T_j^W}|$ denotes the number of type i men (type j women) married to a type j woman (type i man) in observation t, and $|X_t|$ denote the number of observed men (women) in observation t.

To interpret this, consider a given pair of couples $(i,j),(k,l)$. If this couple constitutes an antiedge, and the stability conditions fails, then two potential blocking pairs can be formed: (i,l) and (k,j). The specification for δ^t_{ijkl} represents one story for when a blocking pair which is present in the agents' preferences, actually blocks. With $|X_{T_i^M,T_j^W}|$ (resp. $|X_{T_k^M,T_l^W}|$) being the number of (i,j) (resp. (k,j)) couples, and $|X_t|^2$ being the total number of potential couples in the entire market, then δ^t_{ijkl} is set proportional to the frequency of potential blocking pairs $(j,l),(k,j)$ in the market; it is scaled by γ (and capped from above by 1). We scale by γ to allow the probability that a blocking pair forms to be smaller or larger than this frequency, with a larger γ implying that blocking pairs form more frequently, so that there is less slackness in the stability restrictions.

More broadly, the δ's weight the anti-edges in the sample moment inequalities. Intuitively, an antiedge $((i,j),(l,k))$ should receive a higher weight when it involves many potential blocking pairs than when it only involves a few. Our specification achieves this

[5]We could relax this assumption by making δ dependent on the same covariates that enter into the agents preferences.

idea, as it makes the probability of forming a blocking pair dependent on the number of agents involved.

4.3 Individual-level heterogeneity: remarks. We have assumed that agents who are of the same type have identical preferences. A different starting point is to assume (as most of the empirical literature has assumed) that individuals of the same type have the same preferences up to individual-specific i.i.d. shocks: for example see Choo and Siow (2006) and Galichon and Salanie (2009) in the TU model, and Dagsvik (2000) for the NTU model. In this section, we compare our empirical approach with the approach taken in these papers, especially with respect to the role of individual-level heterogeneity.

Note that the i.i.d. assumption allows two (say) type i men to differ in the utility they would obtain from a matching with a (say) type j woman. However, each of these men still remains indifferent between all type j women.[6] Thus the unobserved heterogeneity is limited, as two agents of the same type are still perceived as identical by the opposite side of the market. Put differently, under the i.i.d. assumption, individual-level preference shocks only perturb agents' preferences among their potential partners' types, but do not affect how they are perceived by their potential partners.

In the papers cited previously, individual-level preference shocks are added so that each agent type ends up with a non-zero probability of being matched with any agent type on the opposite side of the market, in order to better reconcile the theory to the observed data. In this respect, the role of the preference shocks in these papers plays the same role as the "communication probability" δ_{ijkl} in our empirical analysis. Our results on aggregate matchings (especially Theorems 7 and 10) imply that many of the observed marriages in the data would be labeled "blocking pairs" which would make the observed matchings unstable; by introducing the communication probability, we transform the either-or notion of a blocking pair to a probabilistic notion, which allows us to rationalize the observed matchings.

Moreover, it turns out that i.i.d. preference shocks have no aggregate implications on the moment restrictions implied by stability. Specifically, with i.i.d. preference shocks, stability leads to vacuous empirical implications, which are trivially satisfied. To see this, we consider a simplified problem where every woman (man) is acceptable to all men

[6] Galichon and Salanie (2009) also discuss this point (cf. pg. 10).

(women), and work from the stability inequalities: for all pairs (i,j):

$$\sum_{k:k>_i j} x_{i,k} + \sum_{k:k>_j i} x_{k,j} + x_{i,j} \geq 1.$$

Here, $k >_i j$ means that i prefers k over j, and $k >_j i$ means that j prefers k over i. Letting $d_{ikj} = \mathbb{1}_{k>_i j}$, this can be written as:

$$\sum_k x_{i,k} d_{ikj} + \sum_k x_{k,j} d_{jki} + x_{i,j} \geq 1. \tag{10}$$

Here (i,j,k) all denote individual agents, not types. These inequalities cannot be taken directly to the data, because we do not observe the individual-level matching, but rather an aggregate-level matching.

One starting point is to treat both the x's and the d's as random variables, where the randomness derives from both the individual-level preference shocks, as well as from the procedure whereby the observed matching is selected among the set of stable matchings. We partition the men and women into types $t_1^M, \dots t_L^M$ $t_1^W, \dots t_L^W$. Since individual-level preference shocks are i.i.d. we obtain that

$$P(d_{ijk} = 1) = P(d_{i'j'k'} = 1): \quad \forall (i,i') \in t_i^M,\ (j,j') \in t_j^M,\ (k,k') \in t_k^M. \tag{11}$$

That is, the distribution of d_{ijk} is identical for all individuals of the same type. Hence, below we will use the notation $P(d_{ijk} = 1)$ and $P(t_j^W >_{t_i^M} t_k^W)$ interchangeably.

Given these assumptions, we can derive an aggregate version of Eq. (10). First, we take expectations:

$$\begin{aligned} &\sum_k E\left[x_{i,k} d_{ikj}\right] + \sum_k E\left[x_{k,j} d_{jki}\right] + E\left[x_{i,j}\right] \geq 1 \\ \Leftrightarrow &\sum_k \bar{x}_{i,k,j} \cdot P(d_{ikj} = 1) + \sum_k \bar{x}_{k,j,i} \cdot P(d_{jki} = 1) + E\left[x_{i,j}\right] \geq 1 \end{aligned}$$

with $\bar{x}_{i,k,j} \equiv E\left[x_{i,k} d_{ikj} | d_{ikj} = 1\right]$. Next, we aggregate up to the type-level:

$$\sum_l \left\{ P\left\{ t_l^W >_{t_i^M} t_j^W \right\} \bar{X}_{t_i^M, t_l^W, t_j^W} \right\} + \sum_l \left\{ P\left\{ t_l^M >_{t_j^W} t_i^M \right\} \bar{X}_{t_l^M, t_j^W, t_i^M} \right\} \geq \left| t_j^W \right| \left| t_i^M \right| (1 - E[X_{i,j}]) \tag{12}$$

Here $\bar{X}_{t_i^M, t_l^W, t_j^W} \equiv \sum_{k \in t_l^W} \sum_{i \in t_i^M} \sum_{j \in t_j^W} \bar{X}_{i,k,j}$ and $\bar{X}_{t_l^M, t_j^W, t_i^M} \equiv \sum_{j \in t_i^M} \sum_{j \in t_j^W} \sum_{i \in t_i^M} \bar{X}_{k,j,i}$. In the above inequality, only the $\left| t_j^W \right|$ and $\left| t_i^M \right|$ are observed, but nothing else. This is of little use empirically.

On the other hand, because $d_{ijk} \geq 0$, for all (i, j, k), we also have

$$\begin{aligned} E(X_{ik}d_{ikj}) =& E(X_{ik}d_{ikj}|d_{ikj} = 1)P(d_{ikj} = 1) \leq E(X_{ik}) \\ &\Rightarrow \sum_{k \in t_l^W} E(X_{ik}d_{ikj}|d_{ikj} = 1)P(d_{ikj} = 1) \leq \sum_{k \in t_l^W} E(X_{ik}) \\ &\Leftrightarrow P(t_l^W >_i j) \sum_{k \in t_l^W} \bar{X}_{ikj} \leq \sum_{k \in t_l^W} E(X_{ik}) \\ &\Rightarrow \sum_{i \in t_i^M} P(t_l^W >_i j) \sum_{k \in t_l^W} \bar{X}_{ikj} \leq \sum_{i \in t_i^M} \sum_{k \in t_l^W} E(X_{ik}) \\ &\Leftrightarrow P(t_l^W >_{t_i^M} j) \sum_{i \in t_i^M} \sum_{k \in t_l^W} \bar{X}_{ikj} \leq X_{t_i^M, t_l^W} \\ &\Rightarrow P(t_l^W >_{t_i^M} t_j^W) \sum_{j \in t_j^W} \sum_{i \in t_i^M} \sum_{k \in t_l^W} \tilde{X}_{ikj} \leq \left|t_j^W\right| X_{t_i^M, t_l^W} \\ &\Leftrightarrow P(t_l^W >_{t_i^M} t_j^W) \bar{X}_{t_i^M, t_l^W, t_j^M} \leq \left|t_j^W\right| X_{t_i^M, t_l^W} \end{aligned} \tag{13}$$

Combining inequalities (12) and (13), we get

$$\sum_l \left|t_j^W\right| X_{t_i^M, t_l^W} + \sum_l \left|t_i^M\right| X_{t_l^M, t_j^W} \geq \left|t_j^W\right| \left|t_i^M\right| (1 - E[X_{i,j}])$$

By the equalities $\sum_l X_{t_i^M, t_l^W} = \left|t_i^M\right|$ and $\sum_l X_{t_l^M, t_j^W} = \left|t_j^W\right|$, the above reduces to

$$2\left|t_i^M\right| \left|t_j^W\right| \geq \left|t_i^M\right| \left|t_j^W\right| (1 - E[X_{ij}) \;\Rightarrow\; 2 \geq (1 - E[X_{ij})$$

which is vacuous.

In summary, then, i.i.d. individual-level preference shocks seem inappropriate in the aggregate NTU setting of our empirical work. Furthermore, the communication probability δ_{ijkl} plays a similar role in our empirical work as do preference shocks in others' work: namely, to better reconcile the theory to the data by enlarging the the sets of marriages which one could observe in a stable matching.

5 Estimation results

5.1 Data and empirical implementation. In the empirical implementation, we use data on new marriages, as recorded by the US Bureau of Vital Statistics. We consider new marriages in the year 1988, and treat data from each state as a separate, independent matching. We aggregate the matchings into age categories, and create canonical matchings. Table 1 has examples of aggregate matchings, and the corresponding canonical matchings, for several states. In these matching matrices, rows denote age categories for the husbands, and the columns denote the age categories for the wives.

Table 1: Aggregate Matchings and the corresponding Canonical Matchings.

	Age:	Aggregate Matchings							Canonical Matchings						
	♂↓, ♀→	12-20	21-25	26-30	31-35	36-40	41-50	51-94	12-20	21-25	26-30	31-35	36-40	41-50	51-94
MI	12-20	231	47	8	0	0	1	0	1	1	1	0	0	1	0
	21-25	329	798	156	32	11	7	0	1	1	1	1	1	1	0
	26-30	71	477	443	136	27	8	0	1	1	1	1	1	1	0
	31-35	11	148	249	196	83	21	0	1	1	1	1	1	1	0
	36-40	2	41	105	144	114	51	1	1	1	1	1	1	1	1
	41-50	0	15	42	118	121	162	25	0	1	1	1	1	1	1
	51-94	0	2	11	11	35	137	158	0	1	1	1	1	1	1
NV	12-20	8	1	0	0	0	0	0	1	1	0	0	0	0	0
	21-25	17	31	4	0	0	0	0	1	1	1	0	0	0	0
	26-30	2	21	22	7	1	0	0	1	1	1	1	1	0	0
	31-35	0	4	10	5	3	0	0	0	1	1	1	1	0	0
	36-40	0	3	8	2	2	2	0	0	1	1	1	1	1	0
	41-50	0	1	1	2	6	3	3	0	1	1	1	1	1	1
	51-94	0	0	0	0	0	5	3	0	0	0	0	0	1	1
PA	12-20	307	83	12	6	0	0	0	1	1	1	1	0	0	0
	21-25	453	1165	214	64	10	6	1	1	1	1	1	1	1	1
	26-30	113	698	703	190	51	17	0	1	1	1	1	1	1	0
	31-35	17	184	393	277	78	26	2	1	1	1	1	1	1	1
	36-40	9	73	152	191	148	84	5	1	1	1	1	1	1	1
	41-50	3	27	83	146	187	273	28	1	1	1	1	1	1	1
	51-94	1	7	12	38	48	182	268	1	1	1	1	1	1	1

These aggregate canonical matchings have many 1's. Indeed it is apparent from simply eye-balling the table that the rationalizability condition in Theorem 7 is violated: the matchings for all three of these states contain more than two connected cycles, implying that they are not rationalizable. For example, consider the following submatrix for Michigan:

♂↓, ♀→	12-20	21-25	26-30
12-20	1	1	1
21-25	1	1	1
26-30	1	1	1

which has two connected cycles. As a consequence of the non-rationalizability of these matchings, we use the approach in Section 4.2 to relax the requirements of stability.

Finally, one feature of the table is relevant for the discussion below. Note that the matchings in Table 1 contain more non-zero entries below the diagonal, which means that in a preponderance of marriages, the husband is older than the wife.

In our empirical exercise, the specification of utility (Eq. (4)) is very simple, and it only involves the ages of the two partners to a match. Suppose that man m of age Age^m is matched to woman w of age Age^w. The following utility functions capture preferences over age differences, and partner's age.

$$Utility^m = \beta_1|Age^m - Age^w|^- + \beta_2|Age^m - Age^w|^+ + \varepsilon^m$$
$$Utility^w = \beta_3|Age^m - Age^w|^- + \beta_4|Age^m - Age^w|^+ + \varepsilon^w,$$

where ε^m and ε^w are assumed to follow a standard normal distributions. In this specification, we assume that utility is a piecewise-linear function of age, with the "kink" occurring when the age-gap between husband and wife is zero.

The sample moment inequality (Eq. (7)), with the modification in Eq. (8), becomes:

$$\frac{1}{T}\sum_t g_{ijkl}(X_t;\beta) = \Big(\frac{1}{T}\sum_t \mathbb{1}((ij),(kl) \text{ is antiedge in } X_t) * \delta^t_{ijkl}\Big)$$
$$- (1 - Pr(d_{ilj} = 1;\beta_{1,2})Pr(d_{lik} = 1;\beta_{3,4}))(1 - Pr(d_{jki} = 1;\beta_{3,4})Pr(d_{kjl} = 1;\beta_{1,2}))$$

for all combinations of pairs, (i, j) and (k, l).

5.2 Component-wise identified sets. Table 2 summarizes the identified set for several levels of γ, and presents the highest and lowest values that each parameter attains in the identified set. The unrestricted interval in which we searched for each parameter was $[-2, 2]$. So we see that, for a value of $\gamma = 27$, the identified set contains the full parameter space, implying that the data impose no restrictions on parameters. At the other extreme, when $\gamma \geq 36$, the identified set becomes empty, implying that the observed matchings can no longer be rationalized. The latter is consistent with our discussion above, where we noted that when the communication probability δ becomes very large (which is the case when γ is large), then the observed matchings will violate the rationalizability conditions in Theorem 7.

For $\gamma = 35$, we see that β_1 and β_3 take negative values, while the values of β_2 and

Table 2: Unconditional Bounds of β.

	β_1		β_2		β_3		β_4	
γ	min	max	min	max	min	max	min	max
27	-2.00	2.00	-2.00	2.00	-2.00	2.00	-2.00	2.00
30	-2.00	2.00	-2.00	2.00	-2.00	2.00	-2.00	2.00
33	-2.00	0.25	-2.00	1.75	-2.00	0.25	-2.00	1.50
35	-2.00	-0.75	-2.00	1.00	-2.00	-0.75	-2.00	0.75

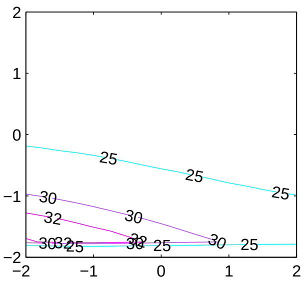
β_4 tend to take negative values but also contain small positive values. This suggests that husbands' utilities are decreasing in the wife's age when the wife is older, but when the wife is younger, his utility is less responsive to the wife's age. A similar picture emerges for wives' utilities, which are increasing in the husband's age when the husband is younger, but when the husband is older, the wife's utility is less responsive to her husband's age. All in all, our findings here support the conclusion that husbands' and wives' utilities are more responsive to the partner's age when the wife is older than the husband.

5.3 Joint identified sets. A richer picture emerges when we consider the joint values of parameters in the identified set. Figure 1 illustrates the contour sets (at different values of γ) for the husband's preference parameters (β_1, β_2), holding the wife's preference parameters (β_3, β_4) fixed. To simplify the interpretation of these findings in light of the stability restrictions, we recall two features of our aggregate matchings (as seen in Table 1): first, there are more anti-edges below the diagonal, where $age^m > age^w$. Second, there are more "downward-sloping" anti-edges than "upward-sloping" ones. That is, there are more anti-edges (i, j), (k, l) with $k > i, l > j$ than with $k < i, l < j$, as illustrated here.

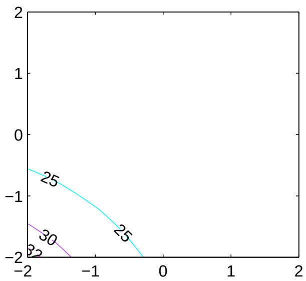
Because of these features, we initially focus on the parameters (β_2, β_4), which describe preferences when the husband is older than the wife.

The graphs in the bottom row of Figure 1 correspond to $\beta_4 = -2$, corresponding to the case that the wife prefers a younger husband: with a downward-sloping anti-edge,

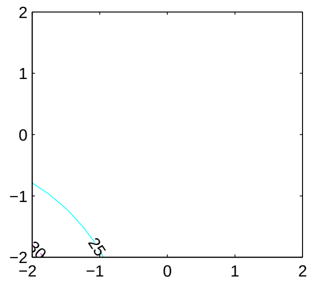
Figure 1: Identified sets of (β_1, β_2) given (β_3, β_4) and γ.



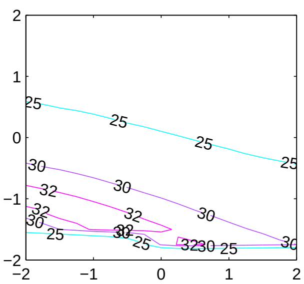
(a) $\beta_3 = -2$ and $\beta_4 = 1$



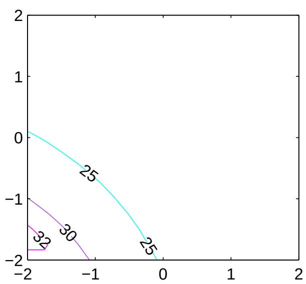
(b) $\beta_3 = 0$ and $\beta_4 = 1$



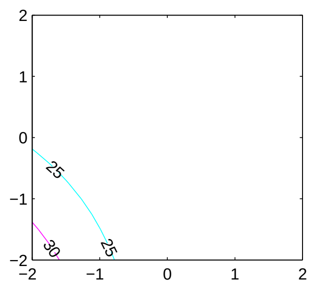
(c) $\beta_3 = 1$ and $\beta_4 = 1$



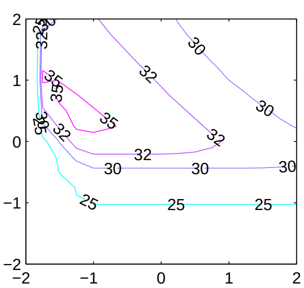
(d) $\beta_3 = -2$ and $\beta_4 = 0$



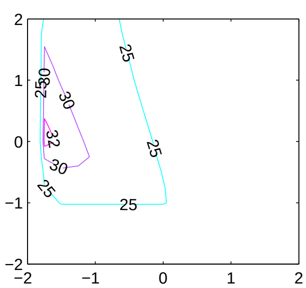
(e) $\beta_3 = 0$ and $\beta_4 = 0$



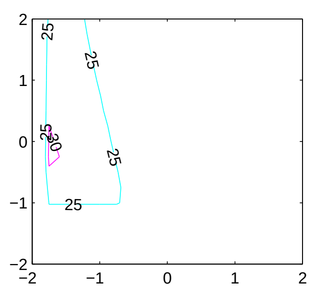
(f) $\beta_3 = 1$ and $\beta_4 = 0$



(g) $\beta_3 = -2$ and $\beta_4 = -2$



(h) $\beta_3 = 0$ and $\beta_4 = -2$



(i) $\beta_3 = 1$ and $\beta_4 = -2$

this implies that it is likely that $d_{jik} = 1$ and $d_{lki} = 0$. In turn, using the stability restrictions (2), this implies that $d_{ilj} = 0$ (that husbands prefer younger wives), but places no restrictions on the sign of d_{kjl}. For this reason, we find that in these graphs, β_2 tends to take positive values at the highest contour levels so that, when husbands are older than their wives, they prefer the age gap to be as large as possible.

By a similar reasoning, β_2 takes negative values when $\beta_4 = 1$. When wives prefer older husbands (which is the case when $\beta_4 = 1$), then with a downward-sloping anti-edge, this implies that $d_{jik} = 0$ and $d_{lki} = 1$. Consequently, stability considerations would restrict the husband's preferences so that $d_{kjl} = 0$ (and husbands prefer older wives), leading to $\beta_2 < 0$.

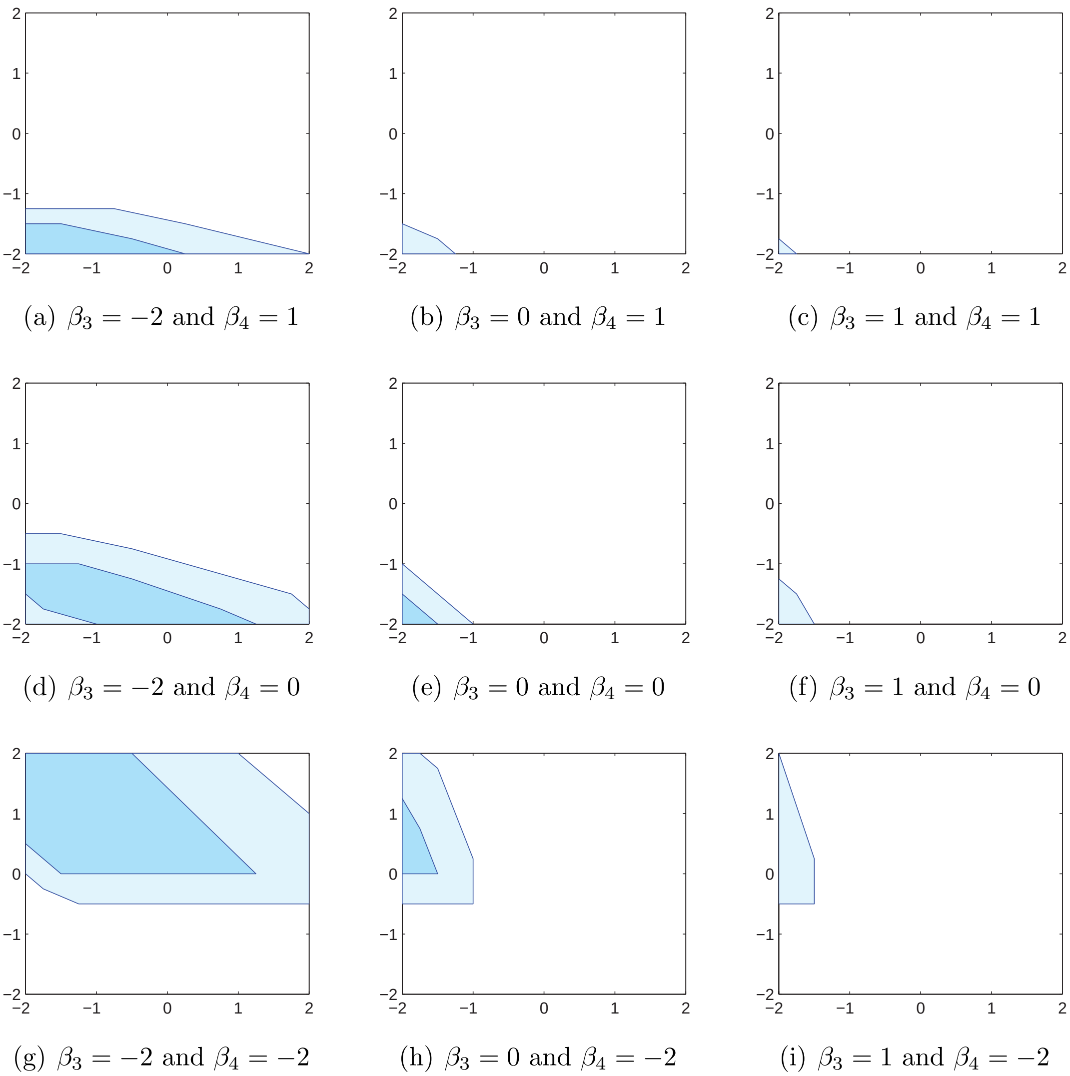
On the other hand, because there are more matchings below the diagonal, when the wife is older than the husband, restriction (2) implies that one of two cases – either the husband prefers a younger wife, or the wife prefers an older husband – must be true. In Figure 1, as β_3 increases from -2 to 1 (from the left to the right column), the wife's utilities becomes more favorable towards a younger husband. As a result, fewer restrictions are imposed to the husbands' utilities, which yields a tighter range for β_1 in the identified sets.

Overall, we see that $\beta_1 < 0$ and $\beta_3 < 0$, implying that as long as the wife is older than the husband, both prefer a smaller age gap. On the other hand, β_2 and β_4 are negatively correlated: as β_4 increases, β_2 decreases. This suggests that, when the husband is older than the wife, one side prefers a smaller gap but the other side is less responsive on the age gap.

5.4 Confidence sets. Figure 2 summarizes the 95% confidence sets with $\gamma = 32$ (in light blue) and 35 (dark blue). In computing these confidence sets, we use the subsampling algorithm proposed by Chernozhukov, Hong, and Tamer (2007). Comparing the confidence sets in Figure 2 to their counterpart identified sets in Figure 1, the confidence sets are apparently larger than the identified sets. This is not surprising, given the modest number of matchings (fifty-one: one for each state) which we used in the empirical exercise.

Nevertheless, the main findings from Figure 1 are still apparent; $\beta_1 < 0$ across a range of values for (β_3, β_4), and $\beta_2 < 0$ (resp. > 0) when $\beta_4 > 0$ (resp. < 0). These somewhat "antipodal" preferences between a husband and wife are a distinctive consequence of the stability conditions of the matching model, in a non-transferable utility setting.

Figure 2: 95% confidence sets of (β_1, β_2) given (β_3, β_4) and $\gamma = 32$ (light blue) and 35 (dark blue).



(a) $\beta_3 = -2$ and $\beta_4 = 1$

(b) $\beta_3 = 0$ and $\beta_4 = 1$

(c) $\beta_3 = 1$ and $\beta_4 = 1$

(d) $\beta_3 = -2$ and $\beta_4 = 0$

(e) $\beta_3 = 0$ and $\beta_4 = 0$

(f) $\beta_3 = 1$ and $\beta_4 = 0$

(g) $\beta_3 = -2$ and $\beta_4 = -2$

(h) $\beta_3 = 0$ and $\beta_4 = -2$

(i) $\beta_3 = 1$ and $\beta_4 = -2$

6 Conclusions

We have characterized the full observable implications of stability for aggregate matchings: with transfers and without them. The implications are easy to check, and strongly restrict the data. We have developed an econometric procedure for estimating preference parameters from aggregate data; our procedure is based on moment inequalities derived from the stability restrictions.

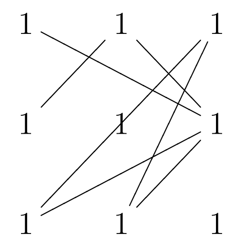
We focused on aggregate matching data because it seems that often data come in an aggregate form, and because many applied researchers have already looked at aggregate matchings. More broadly, though, the idea of stability is akin to the absence of arbitrage, and as such it is a very weak notion of equilibrium for a market; thus, our emphasis on stability represents an attempt to derive results for matching markets which are robust to the exact matching process, which we remain agnostic about.

An alternative approach would have been to specify a detailed structural model of how agents match, and estimate this model by traditional means. This would have some clear advantages. One could empirically back out some of the details involved in how a matching is produced, and understand the source of frictions that may prevent a market from reaching a fully stable matching. On the other hand, it would also require very strong assumptions about how agents act, and on the technology involved in matching, and one worries that the estimation results may be unrobust if these assumptions were wrong. Our focus on stability avoids these problems, and the results here show that it is enough to yield nontrivial empirical implications which can be used for estimating preference parameters.

Appendix A Examples

Example 12. The following example shows that two maximal stable matchings may have a different number of non-zero entries.

$>_{m_1}$	$>_{m_2}$	$>_{m_3}$	$>_{w_1}$	$>_{w_2}$	$>_{w_3}$
w_3	w_2	w_3	m_2	m_3	m_3
w_2	w_1	w_1	m_1	m_2	m_1
w_1	w_3	w_2	m_3	m_1	m_2



Then both X and X' are maximal stable matchings:

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The following example is rationalizable using many different preference profiles. The algorithm used in the proof of Theorem 7 can only construct some of them.

Example 13. Consider the following aggregate matching.

$$X = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

We illustrate the algorithm used in the proof of Theorem 7.

There is a minimal cycle, $\{(i_1, j_1), (i_4, j_1), (i_4, j_3), (i_1, j_3)\}$.

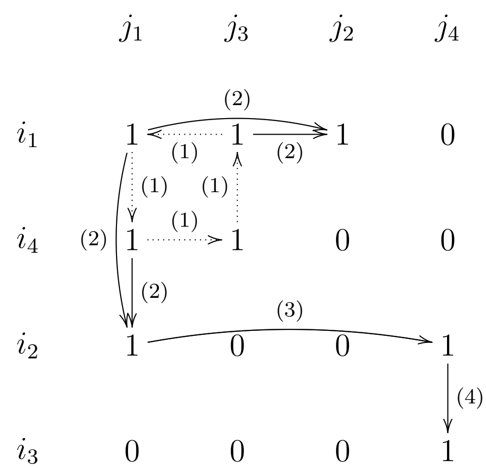
$\bar{I}_1 = \{i_1, i_4\}$, $\bar{J}_1 = \{j_1, j_3\}$

$\bar{I}_2 = \{i_2\}$, $\bar{J}_2 = \{j_2\}$

$\bar{I}_3 = \emptyset$, $\bar{J}_3 = \{j_4\}$

$\bar{I}_4 = \{i_3\}$, $\bar{J}_4 = \emptyset$

All orientations labeled (1) are determined by the minimal cycle. The orientations denoted (2), (3), and (4) are determined as we apply the algorithm.



Appendix B Proofs

B.1 Proof of Theorem 7.. We first record a simple fact about minimal cycles:

Lemma 14. *If $c = \langle x_0, \ldots, x_N \rangle$ is a minimal cycle, then no vertex appears twice in c.*

B.1.1 Proof of necessity. We break up the proof into a collection of simple lemmas.

An ***orientation*** of (V, L) is a mapping $d : L \to \{0, 1\}$. We shall often write $d((i, j), (i, k))$ as $d_{i,j,k}$ and $d((i, j), (l, j))$ as $d_{j,i,l}$. A preference profile $(>_{m_i}, >_{w_j})$ defines an orientation d by setting $d_{j,i,l} = 1$ iff $m_i >_{w_j} m_l$ and $d_{i,j,k} = 1$ iff $w_j >_{m_i} w_k$.

Let d be an orientation defined from a preference profile. Then X is stable if and only if, for all (i_1, j_1) and (i_2, j_2), if $X_{i_1 j_1} = X_{i_2 j_2} = 1$ then

$$d_{i_1 j_2 j_1} d_{j_2 i_1 i_2} = 0 \text{ and } d_{i_2 j_1 j_2} d_{j_1 i_2 i_1} = 0. \tag{14}$$

We say that the pair $((i_1, j_1), (i_2, j_2))$ is an ***antiedge*** if $i_1 \neq i_2$, $j_1 \neq j_2$ and $X_{i_1 j_1} = X_{i_2 j_2} = 1$.

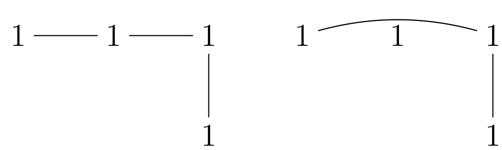
Fix an orientation d of (V, L). A path $\{(i, j)_n : n = 0, \dots, N\}$ is a ***flow*** for d if either $d((i, j)_n, (i, j)_{n+1}) = 1$ for all $n \in \{0, \dots N - 1\}$, or $d((i, j)_n, (i, j)_{n+1}) = 0$ for all $n \in \{0, \dots N - 1\}$. If the second statement is true, we call the path a ***forward flow***.

Our first observation is an obvious consequence of the property of being minimal:

Lemma 15. *Let $\{(i, j)_n : n = 0, \dots, N\}$ be a minimal path with $N \geq 2$, then for any $n \in \{0, \dots N - 2\}$,*

$$(i_n = i_{n+1} \Rightarrow j_{n+1} = j_{n+2}) \text{ and } (j_n = j_{n+1} \Rightarrow i_{n+1} = i_{n+2})$$

That is, any two subsequent edges in a path must be at a right angle:



The path on the left is not minimal; the path on the right is.

Fix an orientation d derived from the preferences rationalizing X.

Lemma 16. *Let $p = \langle (i, j)_n : n = 0, \dots, N \rangle$ be a minimal path. If $d((i, j)_1, (i, j)_0) = 1$ or $d((i, j)_N, (i, j)_{N-1}) = 0$, then p is a flow for d.*

Proof. By Lemma 15, for any $n \in \{1, \dots N - 1\}$ the pair of vertices $(i, j)_{n-1}$ and $(i, j)_{n+1}$ form an antiedge: we have $X_{(i,j)_{n-1}} = X_{(i,j)_{n+1}} = 1$, $i_{n-1} \neq i_{n+1}$ and $j_{n-1} \neq j_{n+1}$. Further, $(i, j)_n$ has one element in common with $(i, j)_{n-1}$ and the other in common with $(i, j)_{n+1}$. Thus by Equation 14, $d((i, j)_n, (i, j)_{n-1}) = 1$ implies that $d((i, j)_n, (i, j)_{n+1}) = 0$, i.e. $d((i, j)_{n+1}, (i, j)_n) = 1$.

The argument in the previous paragraph shows that the existence of some n' with $d((i, j)_{n'}, (i, j)_{n'-1}) = 1$ implies $d((i, j)_n, (i, j)_{n-1}) = 1$ for all $n \geq n'$. So if $d((i, j)_1, (i, j)_0) =$

1 then $d((i,j)_{n+1},(i,j)_n) = 1$ for all $n \in \{1, \dots N-1\}$; and if $d((i,j)_N,(i,j)_{N-1}) = 0$, then $d((i,j)_{n+1},(i,j)_n) = 0$ for all $n \in \{0, \dots N-1\}$. Either way, p is a flow. □

As an immediate consequence of Lemma 16, we obtain the following

Lemma 17. *Let $p = \langle (i,j)_n \rangle$ be a minimal cycle, then p is a flow for d.*

Let $p = \langle (i,j)_n \rangle$ be a path and $(i,j) \notin p$. A path $\bar{p} = \langle (\bar{i},\bar{j})_n : n = 0,\dots,\bar{N} \rangle$ connects p and (i,j) if $(\bar{i},\bar{j})_0 \in p$ and $(\bar{i},\bar{j})_N = (i,j)$.

Lemma 18. *Let $c = \langle (i,j)_n \rangle$ be a minimal cycle, and $p = \langle (\bar{i},\bar{j})_n : n = 0,\dots,\bar{N} \rangle$ be a minimal path connecting c to some $(\bar{i},\bar{j})$. Then $\langle (\bar{i},\bar{j})_n : n = 1,\dots,\bar{N} \rangle$ is a forward flow.*

Proof. Let $c = \langle (i,j)_n : n = 0,\dots,N \rangle$ be the cycle in the hypothesis of the lemma. We write $(i,j)_n$ for $(i,j)_{n \mod (N)}$, so we can index the cycle by any positive integer index. By Lemma 17, c is a flow for d: we can in fact suppose that it is a forward flow, otherwise, if $d((i,j)_1,(i,j)_0) = 0$, then we can re-index by setting $(i,j)_k = (i,j)_{N-k}$.

To prove Lemma 18 we need to deal with two different cases. Let $(i,j)_{n^*} = (\bar{i},\bar{j})_0$. By definition of a cycle, then, $(\bar{i},\bar{j})_0$ shares either i or j with $(i,j)_{n^*-1}$. Suppose, without loss of generality, that they share i, so $\bar{i}_0 = i_{n^*-1}$. The two cases in question are represented below, where the center vertex is $(\bar{i},\bar{j})_0$. Case 1 on the left has $(\bar{i},\bar{j})_1$ also sharing i with $(\bar{i},\bar{j})_0$, while Case 2 has $(\bar{i},\bar{j})_0$ sharing j with $(\bar{i},\bar{j})_1$.

Case 1: Suppose that $\bar{i}_1 = \bar{i}_0 = i_{n^*-1}$. Consider the minimal path

$$p' = \langle (i,j)_{n^*-1}, (\bar{i},\bar{j})_1, \dots, (\bar{i},\bar{j})_{\bar{N}} \rangle.$$

Since $i_{n^*-2} \neq \bar{i}_1$, the path

$$\hat{p} = \langle (i,j)_{n^*-2}, (i,j)_{n^*-1}, (i,j)_1 \rangle$$

is a minimal path from $(i,j)_{n^*-2}$ to $(i,j)_1$. We have that $d((i,j)_{n^*-1},(i,j)_{n^*-2})=1$, as c is a forward flow. It follows by Lemma 16 that $d((\bar{i},\bar{j})_1,(i,j)_{n^*-1})=1$ and thus $\hat{p}$ is also a forward flow. Then, by Lemma 16 again, p' is a forward flow; in particular, $d((\bar{i},\bar{j})_{n+1},(\bar{i},\bar{j})_n)=1$ for $n\in\{1,\dots\bar{N}-1\}$.

Case 2: Suppose that $\bar{i}_1\neq\bar{i}_0=i_{n^*-1}$. Then the path

$$\langle(i,j)_{n^*-1},(\bar{i},\bar{j})_0,(\bar{i},\bar{j})_1\rangle$$

is a minimal path connecting $(i,j)_{n^*-1}$ and $(\bar{i},\bar{j})_1$.

We have that $d((i,j)_{n^*-1},(i,j)_{n^*-2})=1$, as c is a forward flow. By an application of Lemma 16, analogous to the one in Case 1, we obtain that p is a forward flow.

Regardless of whether we are in Case 1 or 2 we establish that $\langle(\bar{i},\bar{j})_n:n=1,\dots,\bar{N}\rangle$ is a forward flow. □

Lemma 19. *There are no two connected distinct minimal cycles.*

Proof. Suppose, by way of contradiction, that there are two minimal cycles c_1 and c_2, and a path $p=\langle(i,j)_n:n=0,\dots,N\rangle$ connecting $(i,j)_0\in c_1$ with $(i,j)_N\in c_2$. We can suppose without loss of generality that p is minimal. We can also suppose that $N\geq 3$ because if $N<3$ we can add $(i',j')\in c_1$ to p with $((i',j'),(i,j)_0)\in L$, and $(i'',j'')\in c_2$ to p with $((i'',j''),(i,j)_N)\in L$; the corresponding path will also be a minimal path connecting c_1 and c_2.

By Lemma 18 applied to c_1 and p,

$$\langle(i,j)_n:n=1,\dots N\rangle$$

is a forward flow. On the other hand, Lemma 18 applied to c_2 and p implies that

$$\langle(i,j)_{N-k}:k=1,\dots N\rangle$$

is a forward flow. The first statement implies that $d((i,j)_2,(i,j)_1)=1$ and the second that $d((i,j)_1,(i,j)_2)=1$, a contradiction. □

B.1.2 Proof of sufficiency. To prove sufficiency, we explicitly construct an orientation d that satisfies Equation 14. We then show that there is a rationalizing preference profile.

We first deal with the case where all vertices in X are connected and there is at most one minimal cycle. By decomposing an arbitrary X into connected components, we shall later generalize the argument. If there is no cycle in X, choose a singleton vertex and treat it as the "cycle" in the sequel.

Let C be the submatrix having the indices in the minimal cycle. If $c = \langle (i,j)_n \rangle$ is the minimal cycle, let $I_1 = \cup_n \{i_n\}$ and $J_1 = \cup_n \{j_n\}$. Then C is the matrix $(x_{i',j'})_{(i',j') \in I_1 \times J_1}$. Thus C contains the minimal cycle.

We re-arrange the indices of X to obtain a matrix of the form:

$$\begin{array}{c|cccc} & (J_1) & (J_2) & (J_3) & \\ \hline (I_1) & C & X_1 & O & \cdots \\ (I_2) & Y_1 & O & X_2 & \cdots \\ (I_3) & O & Y_2 & O & \cdots \\ & \vdots & \vdots & \vdots & \end{array} \tag{15}$$

We define the submatrices X_n and Y_n by induction. For $n \geq 1$, let

$$\begin{aligned} I_{n+1} &= \{i \notin \cup_1^n I_k | \exists j \in \cup_1^n J_k \text{ s.t. } (i,j) \in V\} \\ J_{n+1} &= \{j \notin \cup_1^n J_k | \exists i \in \cup_1^n I_k \text{ s.t. } (i,j) \in V\} \end{aligned}$$

Now, let X_n be the matrix $(x_{i',j'})_{(i',j') \in I_n \times J_{n+1}}$ and Y_n be the matrix $(x_{i',j'})_{(i',j') \in I_{n+1} \times J_n}$. Finally, re-label the indices such that if $i \in I_n$ and $i' \in I_{n'}$ and $n < n'$ then $i < i'$. The numbering of indexes in I_n is otherwise arbitrary. Re-label j's in a similar fashion.

For every $i \in I_n$ there is a $k < n$ and $j \in J_k$ such that $(i,j) \in V$, and similarly, for every $j \in J_n$ there is a $k < n$ and $i \in I_k$ such that $(i,j) \in V$. Thus, for $i \in I_n$ there is a sequence

$$(i, j_{k_0}), (i_{k_1}, j_{k_0}), \ldots (i_{k_N}, j_{k_{N'}}),$$

with $N = N' + 1$ or $N' = N - 1$, which defines a path connecting (i, j_{k_0}) to the cycle c. Similarly, if $j \in J_n$ there is a path connecting (i_{k_0}, j) to c.

The observation in the previous paragraph has two consequences:

Claim 20. *If $i \in I_n$ and $j \in J_n$ $(n > 1)$, then $(i,j) \notin V$.*

Claim 20 is true because otherwise there would be two different paths connecting

(i, j) to c, one having (i, j_{k_0}) and the other (i_{k_0}, j) as second element. Then we would have a distinct second cycle.

Claim 21. *Let $i \in I_n$ $(n > 1)$, and let there be two distinct j and j' $(j' > j)$ such that $(i, j), (i, j') \in V$. Then $(i', j') \in V$ implies that $i' \in I_{n'}$ with $n' > n$.*

Claim 21 is true because otherwise we would again have two different paths connecting (i, j') to c; one path with (i, j) and one with (i', j') as its second element.

Define the orientation d as follows.

1. If $(i, j) \in c$ and $(i, j') \in c$ then define $d_{i,j,j'}$ to be 1 if (i, j) comes immediately after (i, j') in c. That is, $d_{i,j,j'} = 1$ if there is n such that

$$(i, j') = (i, j)_{n \mod (N)} \text{ and } (i, j) = (i, j)_{n+1 \mod (N)}.$$

2. If $(i, j) \in c$ and $(i', j) \in c$ then define $d_{j,i,i'}$ to be 1 if (i, j) comes immediately after (i', j) in c.
3. If $(i, j) \notin c$ and $(i, j') \in c$ then define $d_{i,j,j'}$ to be 1.
4. If $(i, j) \notin c$ and $(i', j) \in c$ then define $d_{j,i,i'}$ to be 1.
5. If $(i, j) \notin c$ and $(i, j') \notin c$ then define $d_{i,j,j'}$ to be 1 iff $j > j'$.
6. If $(i, j) \notin c$ and $(i', j) \notin c$ then define $d_{j,i,i'}$ to be 1 iff $i > i'$.
7. If $(i, j) \in V$ and $(i', j) \notin V$, then define $d_{j,i,i'}$ to be 1.

Let $d_{i,j',j} = 0$ when 1-7 imply that $d_{i,j,j'} = 1$; similarly $d_{j,i',i} = 0$ when 1-7 imply that $d_{j,i,i'} = 1$.

Lemma 22. *If (i, j) is a vertex in c, then there is at most one j' such that $j' \neq j$ and $(i, j') \in c$; in addition, (i, j) and (i, j') are adjacent in c. Similarly, there is at most one $i' \neq i$ such that $(i', j) \in c$; in addition, (i, j) and (i', j) are adjacent in c*

Proof. We let the index of c range over all the integers by denoting $(i, j)_{n \mod (N)}$ by $(i, j)_n$.

Let (i, j) be a vertex in c, and $n > 0$ be such that $(i, j) = (i, j)_n$. Suppose there is j' such that $j' \neq j$ and $(i, j') \in c$. If it does not exist, we are done. Since now $N \geq 2$, (i, j) is in the minimal path connecting $(i, j)_{n-1}$ and $(i, j)_{n+1}$. By Lemma 15, then, either

$i_{n-1} = i$ or $i_{n+1} = i$, and exactly one of these is true. In the first case, we can set $j' = j_{n-1}$ and in the second we can set $j' = j_{n+1}$. Suppose, without loss of generality, that $j' = j_{n+1}$.

We show that there is not a $j'' \neq j, j'$ with $(i, j'') \in c$. Suppose that there is such a j''. Let $(i, j'') = (i, j)_m$. By Lemma 15, we have either $m < n - 1$ or $m > n + 1$. When $m > n+1$, the path $\langle (i,j)_{n-1}, \ldots, (i,j)_m \rangle$ is not minimal because $\langle (i,j)_{n-1}, (i,j)_n, (i,j)_m \rangle$ is a proper subset connecting $(i,j)_{n-1}$ and $(i,j)_m$. When $m < n-1$, the path $\langle (i,j)_m, (i,j)_n, (i,j)_{n+1} \rangle$ is not a minimal because $(i,j)_m$ and $(i,j)_{n+1}$ are directly connected. Thus c is not a minimal cycle, a contradiction. □

Lemma 23. *Let (i, j) be a vertex in c. If $(i, j') \in V$ is not a vertex in c, then, for all $i' \neq i$, $(i', j') \notin c$. Similarly, if $(i', j) \in V$ is not a vertex in c, then, for all $j' \neq j$, $(i', j') \notin c$.*

Proof. Suppose, by way of contradiction, that $(i, j) \in c$, $(i', j') \in c$, with $i \neq i'$, $j \neq j'$, and $(i, j') \notin c$. Since $(i, j), (i', j') \in c$, there is a minimal path $\langle (i, j)_k : k = 0, \ldots, K \rangle$ connecting (i', j') to (i, j). Then, since $(i, j') \notin c$, the minimal cycle

$$\langle (i,j)_0, \ldots, (i,j)_K, (i,j'), (i',j') \rangle$$

is distinct from c and connected to c. □

Lemma 24. *1. If $d_{i,j,j'} = 1$ and $d_{i,j',j''} = 1$ then $d_{i,j,j''} = 1$.*

2. If $d_{j,i,i'} = 1$ and $d_{j,i',i''} = 1$ then $d_{j,i,i''} = 1$.

Proof. We prove only the first statement. The second statement can be proved by similar fashion to the following first three cases.

First, we can rule out that $d_{i,j,j'} = 1$ because $(i, j) \in c$, $(i, j') \in c$, and (i, j) comes immediately after (i, j') in c (case 1). To see this, note that $d_{i,j',j''} = 1$ would imply that either $(i, j'') \in c$, which is not possible by Lemma 22.

Second, suppose that $d_{i,j,j'} = 1$ because $(i, j) \notin c$ and $(i, j') \in c$. Then $d_{i,j',j''} = 1$ implies that $(i, j'') \in c$. Thus $d_{i,j,j''} = 1$ by case 3.

Third, suppose that $d_{i,j,j'} = 1$ because $(i, j) \notin c$ and $(i, j') \notin c$ and $j > j'$. If $d_{i,j',j''} = 1$ because $(i, j'') \notin c$ and $j' > j''$ then $d_{i,j,j''} = 1$ by case 5 by the transitivity of $>$. On the other hand, if $d_{i,j',j''} = 1$ because $(i, j'') \in c$ then $d_{i,j,j''} = 1$ (case 3) as well. Finally, if $d_{i,j,j'} = 1$ because of Case 7 then we obtain $d_{i,j,j''} = 1$ by Case 7 as well. □

Lemma 25. *The orientation d satisfies* (14).

Proof. Let $((i, j), (i', j'))$ be an antiedge: so $(i, j), (i', j') \in V$, $j \neq j'$ and $i \neq i'$. Suppose that $d_{i,j',j} = 1$. We shall prove that $d_{j',i,i'} = 0$.

Suppose first that $d_{i,j',j} = 1$ because of case 1. Then $(i, j') \in c$. So, if $(i', j') \notin c$ we obtain that $d_{j',i,i'} = 0$ by case 3. On the other hand, if $(i', j') \in c$ then the edges $((i, j), (i, j'))$ and $((i, j'), (i', j'))$ are in c. In fact, these edges must be consecutive, or (i, j') will appear twice in c. Then, $d_{i,j',j} = 1$ because of case 1 implies that (i, j') comes immediately after (i, j) in c; the edge $((i, j'), (i', j'))$ comes after $((i, j), (i, j'))$ in c, so we obtain that $d_{j',i,i'} = 0$ by case 1.

Suppose second that $d_{i,j',j} = 1$ because of case 3. So $(i, j) \in c$ and $(i, j') \notin c$. Then $i \in I_1$ because i is an index for a vertex in the minimal cycle c. Now, by Lemma 23, there is no $\tilde{i}$ with $(\tilde{i}, j') \in c$. Since $(i', j') \in V$ we must have $i' \in I_n$ for $n > 1$. By the labeling we adopted, then, $i < i'$. Hence, $d_{j',i',i} = 1$ by case 6.

Thirdly, suppose that $d_{i,j',j} = 1$ because of case 5. If $i \in I_1$, there exists j'' such that $(i, j'') \in c$ and $d_{i,j',j''} = 1$ because of case 3, and $d_{j',i',i} = 1$ by the previous result. If $i \in I_n$ $(n > 1)$, then we have shown in Claim 21 that $(i', j') \in V$ implies that $i' \in I_k$ with $k > n$. Hence $d_{j',i',i} = 1$ because of Case 5.

Finally, note that we cannot have $d_{i,j',j} = 1$ because of Case 7 because $(i, j) \in V$. □

Given the orientation d we have constructed, define two collections of partial orders, $(>_i : i \in I)$ and $(>_j : j \in J)$ where we say that $j >_i j'$ when $d_{i,j,j'} = 1$ and that $i >_j i'$ when $d_{j,i,i'} = 1$. By Lemma 24, these are well-defined strict partial orders.

Now define the preferences of man i to be some complete strict extension of $>_i$ to J, and similarly for the women. By Lemma 25, these preferences rationalize the matching X.

The previous construction assumed that X had one minimal cycle. If X has more than one minimal cycle, these must not be connected in the graph. Therefor, if we partition the graph into connected components, there will be at most one minimal cycle in each.

In particular, we can partition the set of vertices V of X to be $V = V_1 \cup \cdots \cup V_N$ and $V_m \cap V_n = \emptyset$. All vertices in each V_n are connected, but no pair of vertices in different sets are connected. The partition corresponds to the connected components of the graph.

Now re-label the indices of types such that the aggregate canonical matching X is a diagonal block matrix:

$$X = \begin{pmatrix} X_1 & O & \cdots & O \\ O & X_2 & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & X_N \end{pmatrix}$$

All vertices in V_n correspond to X_n.

The previous construction, applied to each X_n separately, yields a rationalizing preference profile of each X_n. Now, extend the preferences of each man i: say that i indexes rows in X_n, then define a partial order $\succ_i$ on J to agree with $>_i$ on the indexes of columns of X_n, and such that any index of a column of X_n is ranked above any other index; then define i's preferences to be any complete extension of $\succ_i$. Women's preferences are defined analogously.

The resulting profile of preferences rationalizes X because if (v, v') is an antiedge with $v, v' \in V_n$, for some n, then (14) is satisfied by the previous construction of preferences, and if v and v' are in different components of the partition of V, then (14) is satisfied because any agent ranks an index in their component over an index in a separate component.

B.2 Proof of Proposition 4.

Proof. We shall first prove Statement 1. Suppose, by way of contradiction, that X is a maximal stable matching for a preference profile $((>_m)_{m \in M}, (>_w)_{w \in W})$. Without loss of generality, suppose that $X_{13} = X_{22} = X_{31} = 1$.

We have $X_{32} = 0$ and X is maximal. Then there is $X_{ij} = 1$ s.t. $((3, 2), (i, j)) \in E$. We must have $3 \neq i$ and $2 \neq j$ so we must have $(i, j) = (1, 3)$. Now, there are two possibilities:

$$(m_1 >_{w_2} m_3) \wedge (w_2 >_{m_1} w_3) \tag{16}$$

$$(m_3 >_{w_3} m_1) \wedge (w_3 >_{m_3} w_2) \tag{17}$$

Suppose first that (16) holds. Since X is maximal and $X_{12} = 0$, $(1,2)$ must be part of an edge. By a similar reasoning to above, we must have that $((1,2),(3,1)) \in E$. By (16) we have that $m_1 >_{w_2} m_3$ so $((1,2),(3,1)) \in E$ implies that $m_1 >_{w_1} m_3$ and $w_1 >_{m_1} w_2$. Then, by (16), we have

$$w_1 >_{m_1} w_2 >_{m_1} w_3.$$

Then $m_1 >_{w_1} m_3$ implies that $((1,3),(3,1)) \in E$ which is impossible as $X_{13} = X_{31} = 1$.

Suppose, second, that (16) does not hold and that (17) holds. Since X is maximal and $X_{33} = 0$, $(3,3)$ must be part of an edge. By a similar reasoning to above, we must have that $((3,3),(2,2)) \in E$. By (17) we have that $w_3 >_{m_3} w_2$ so $((3,3),(2,2)) \in E$ implies that $m_2 >_{w_3} m_3$ and $w_3 >_{m_2} w_2$. Then, by (17), we have

$$m_2 >_{w_3} m_3 >_{w_3} m_1.$$

Then $w_3 >_{m_2} w_2$ implies that $((1,3),(2,2)) \in E$ which is impossible as $X_{13} = X_{22} = 1$.

We prove Statement 2 next. Let X be an individual matching. Suppose there is (h,l) s.t. $X_{hl} = 1$ and the submatrix $X^{-(hl)}$ is not maximally stable. Clearly, since X is stable, so is $X^{-(hl)}$. Since $X^{-(hl)}$ is not maximally stable, there is a stable $(K-1) \times (L-1)$ aggregate matching X' that dominates $X^{-(hl)}$, in fact there is a stable matrix X' which dominates $X^{-(hl)}$ and exactly one (i^*, j^*) has $X'_{i^*j^*} = 1$ and $x^{-(hl)}_{i^*j^*} = 0$.

Consider the $K \times L$ matrix $\hat{x}$ that coincides with X everywhere except that $\hat{X}_{i^*j^*} = 1$. Since X is maximally stable it must be that $((i^*,j^*),(h,l)) \in E$, as the stability of X' ensures that there is no other pair (i,j) with $((i^*,j^*),(i,j)) \in E$ and $\hat{X}_{ij} = 1$.

Note that, for all $j \neq l$, $X_{hj} = 0$ implies that there is some (s,t) with $s \neq h$, $X_{st} = 1$ and $((s,t),(hj)) \in E$. Additionally, since X is an individual matching, $X_{st} = 1$ implies that also $t \neq l$. In a similar fashion, for all $i \neq h$ there is (s,t) with $s \neq h$, $t \neq l$, $X_{st} = 1$ and $((s,t),(h,j)) \in E$.

Now consider the matching $\tilde{X}$ that coincides with X everywhere except that $\tilde{X}_{hl} = 0$ and $\tilde{X}_{i^*j^*} = 1$. Note that $\forall (i,j)(\tilde{X}_{il} = \tilde{X}_{hj} = 0)$. We claim that $\tilde{X}$ is a stable matching: the submatrix $\tilde{X}^{-(hl)}$ coincides with X', so there are no edges among pairs (i,j) with $i \neq h$ and $j \neq l$. As for (i,j) with $i = h$ or $j = l$, we have $\tilde{X}_{ij} = 0$ so they cannot be part of an edge.

Finally, consider a maximal stable matching $\hat{\hat{X}}$ that dominates $\tilde{X}$. Note that we prove that for any $j \neq l$, there is some (s,t) with $s \neq h$ and $t \neq l$ such that $((s,t),(h,j)) \in E$ and $\hat{\hat{X}}_{st} = x_{st} = 1$. Thus the stability of $\hat{\hat{X}}$ requires that $\hat{\hat{X}}_{hj} = 0$. Similarly we get that $\hat{\hat{X}}_{il} = 0$ for any $i \neq h$. We also have that $\hat{\hat{X}}_{i^*j^*} = 1$ because $\hat{\hat{X}}$ dominates $\tilde{X}$. Then $((i^*,j^*),(h,l)) \in E$ and the stability of $\hat{\hat{X}}$ implies $\hat{\hat{X}}_{hl} = 0$. Thus we prove that $\hat{\hat{X}}$ satisfies the property in the statement. □

B.3 Proof of Theorem 10. We prove necessity first. Let X be an aggregate matching that is rationalizable by the matrix α. Suppose, by way of contradiction, that the graph (V,L) associated to X has a minimal cycle $c = \langle y_0, \ldots, y_N \rangle$.

We say that an edge $((i,j),(i',j')) \in L$ is ***vertical*** if $j = j'$ and that it is ***horizontal*** if $i = i'$. Since the cycle c is minimal, a horizontal edge in c must be followed by a vertical edge; and a vertical edge in c must be followed by a horizontal edge (Lemma 15). Thus c has an even number of vertices. Since $y_0 = y_N$, this implies that N is an even number.

Consider the aggregate matching X', which coincides with X on all entries except the ones in c. For the entries that are vertices in c, let

$$\begin{aligned} X'_{y_{2n-1}} &= X_{y_{2n-1}} + 1, \quad n = 1, \ldots, \tfrac{N}{2} \\ X'_{y_{2n}} &= X_{y_{2n}} - 1, \quad n = 0, \ldots, \tfrac{N}{2} - 1 \end{aligned}$$

Fix a row i of X'. For each column j, if $y_n = (i,j)$ for some n, then (modulo N) either y_{n-1} or y_{n+1} share the same j. Without loss of generality, say that y_{n+1} shares the same j. By definition of X', then $X_{y_n} + X_{y_{n+1}} = X'_{y_n} + X'_{y_{n+1}}$. Thus $\sum_j X'_{i,j} = \sum_j X_{i,j}$. A similar argument implies that, for each j, $\sum_i X'_{i,j} = \sum_i X_{i,j}$. Hence X' is a feasible aggregate matching in program (3).

Since α rationalizes X, we have that $\sum_{i,j} \alpha_{i,j} X_{i,j} > \sum_{i,j} \alpha_{i,j} X'_{i,j}$. Thus,

$$\sum_{i,j} \alpha_{i,j}(X'_{i,j} - X_{i,j}) = \sum_{n=1,\ldots,\frac{N}{2}} \alpha_{y_{2n-1}} - \sum_{n=0,\ldots,\frac{N}{2}-1} \alpha_{y_{2n}} < 0 \tag{18}$$

But then we can consider the aggregate matching X'' defined as

$$\begin{aligned} X''_{y_{2n-1}} &= X_{y_{2n-1}} - 1, \quad n = 1, \ldots, \tfrac{N}{2} \\ X''_{y_{2n}} &= X_{y_{2n}} + 1, \quad n = 0, \ldots, \tfrac{N}{2} - 1, \end{aligned}$$

on the vertices of c, and which coincides with X on all entries that are not vertexes of c.

By the same argument we made for X', X'' is feasible in program (3).

Now, Equation (18) implies that

$$\sum_{i,j} \alpha_{i,j}(X''_{i,j} - X_{i,j}) = -\sum_{n=1,\dots,\frac{N}{2}} \alpha_{y_{2n-1}} + \sum_{n=0,\dots,\frac{N}{2}-1} \alpha_{y_{2n}} > 0;$$

a contradiction of X being rationalized by α.

Second, we prove sufficiency. Suppose that X is an aggregate matching such that the associated graph contains no cycles. Let α be the canonical matching derived from X. We shall prove that α rationalizes X.

Clearly, $\sum_{i,j} \alpha_{i,j} X_{i,j} = \sum_{i,j} X_{i,j}$. Suppose that X' is an aggregate matching such that X' is feasible in program (3) for X, and that $\sum_{i,j} \alpha_{i,j} X'_{i,j} \geq \sum_{i,j} X_{i,j}$. We shall prove that $X' = X$.

Give α as surplus matrix, $\sum_{i,j} X_{i,j}$ is the maximal surplus that can be achieved in Program (3). To see this, note that all pairs who are matched generate the same value: 1 if they are a pair that is matched under $X_{i,j}$ and 0 otherwise. The number of different men is $\sum_{i,j} X_{i,j}$ $(= \sum_i \sum_j X_{i,j})$. The number of different women is also $\sum_{i,j} X_{i,j}$ $(= \sum_j \sum_i X_{i,j})$. Thus there are at most $\sum_{i,j} X_{i,j}$ pairs that can be formed. The maximum value in (3) obtains when all of them generate a surplus of 1. Thus we have $\sum_{i,j} \alpha_{i,j} X'_{i,j} = \sum_{i,j} X_{i,j}$.

As a consequence, $X'_{i,j} = 0$ when $X_{i,j} = 0$. Otherwise we would have a pair (i, j) that are generating a surplus of 0 under α, and we cannot have $\sum_{i,j} \alpha_{i,j} X'_{i,j} = \sum_{i,j} X_{i,j}$. Thus $X'_{i,j} = 0$ for all $(i, j) \notin V$.

We shall assume that (V, L) has exactly one connected component. When that assumption fails, we can apply the argument in the sequel to each component separately.

Choose a vertex v_0 in V. Since (V, L) contains no cycle, for each $v \in V$ there is a unique path connecting v_0 to v in (V, L). Let $\eta(v)$ be the length of the path connecting v_0 to v. We shall prove the result by induction on $\eta(v)$. Specifically, we show that for each v with maximal η, either the row or the column of v must be identical in both X and X'. We can then consider the submatrix that omitting that row or column, and repeat our argument.

Specifically, define a partial order $\succ$ on V, such that $v_1 \succ v_2$ if and only if v_1 is on the unique path from v_0 to v_2. Then $(V, \succ)$ defines a set of maximal chains denoted as $\{V_1, \dots V_L\}$. Each maximal chain has a unique vertex with highest value of $\eta(v)$. The following argument can be made for each of these chains.

Let (i, j) be a vertex with a maximal value of $\eta(v)$. Since $\eta(v)$ is maximal, one of the following two cases hold.

1. there is no i' with $((i, j), (i', j)) \in L$
2. there is no j' with $((i, j), (i, j')) \in L$

That is, there are either no horizontal edges, or no vertical edges, incident to (i, j).

Suppose that Case 1 holds, so $X_{h,j} = 0$ for all $h \neq i$. Then, $X'_{h,j} = 0$ for all $h \neq i$, and $\sum_h X_{h,j} = \sum_h X'_{h,j}$, imply that $X_{i,j} = X'_{i,j}$. Thus, column j in both matrices X' and X coincide.

Consider the submatrices $X_{\setminus j}$ and $X'_{\setminus j}$, obtained after eliminating column j. Then $\alpha_{\setminus j}$ is the canonical matching of $X_{\setminus j}$; an entry of $X'_{\setminus j}$ is 0 when the corresponding entry of $X_{\setminus j}$ is 0, and

$$\sum_{(i,h):h \neq j} \alpha_{i,h} X'_{i,h} = \sum_{(i,h):h \neq j} \alpha_{i,h} X_{i,h}.$$

Finally, the resulting graph $(V_{\setminus j}, L_{\setminus j})$ contains no cycle.

Similarly, when Case 2 holds, row i of both matrices must coincide. We can then consider the submatrices obtained after eliminating row i.

By applying the above argument to this sequence of submatrices, we will show that $X'_{i,j} = X_{i,j}$ for all $(i, j) \in V$. We have already shown that $X'_{i,j} = X_{i,j} = 0$ for all $(i, j) \notin V$. Hence $X = X'$.

Appendix C Detailed Data Description

We use Marriage and Divorce Data of the National Vital Statistics System of the National Center for Health Statistics (NCHS).[7]

The data are based on marriage and divorce certificates, and include all records for States with small numbers of events and a sample of records for States with larger

[7]http://www.nber.org/data/marrdivo.html

numbers of events. Since the sample size significantly decreased from year 1989, and NCHS stopped producing data after 1995 due to lack of funds, we use data of year 1988.

In order to produce cross-sectional marriage distributions across the states in US, we restrict our attention to marriage samples (i) of states of the United States or District of Columbia, (ii) in which both groom and bride reside in a same state. In 784,211, total number of observations, 10,204 from Puerto Rico, Virgin Islands, Guam, Canada, Cuba, Mexico, or Remainder of the world are eliminated, and also samples with states not stated are eliminated. In addition, 47,289 observations are deleted since groom and bride are reported to reside in distinct states. In all, total sample size is 726,718.

In categorizing men and women by there types, we only used ages; although Marriage microdata also includes variables such as education or previous marital status, there are significant number missing observations, so we do not use other variables. Marriage age varies from 12 to 94 for groom and from 12 to 92 for bride. Both men and women are categorized as 7 different age groups, and the thresholds are 12-20, 21-25, 26-30, 31-35, 36-40, 41-50, and 51-94.

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