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Symmetry Breaking in General Relativity†

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Abstract

Bifurcation theory is used to analyze the space of solutions of Einstein's equations near a spacetime with symmetries. The methods developed here allow one to describe precisely how the symmetry is broken as one branches from a highly symmetric spacetime to nearby spacetimes with fewer symmetries, and finally to a generic solution with no symmetries. This phenomenon of symmetry breaking is associated with the fact that near symmetric solutions the space of solutions of Einstein's equations does not form a smooth manifold but rather has a conical structure. The geometric picture associated with this conical structure enables one to understand the breaking of symmetries. Although the results are described for pure gravity, they may be extended to classes of fields coupled to gravity, such as gauge theories. Since most of the known solutions of Einstein's equations have Killing symmetries, the study of how these symmetries are broken by small perturbations takes on considerable theoretical significance.

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Bifurcation theory deals with the branching of solutions of nonlinear equations. Here we describe a new application of this theory to the determination of how the solution set of Einstein's equations branches near a spacetime with a one-parameter family of symmetries. The directions of these branches are not determined by the linearized theory of gravity alone, but are completely characterized by the second-order terms. Thus the linearized theory of gravity near a spacetime with symmetry is not sufficient to capture the dominant effects of the nonlinear theory.

These conclusions are in accord with the earlier work of Fischer and Marsden [1-3], Moncrief [4-6], and Arms and Marsden [7], which showed that for spacetimes with compact Cauchy surfaces, a solution of Einstein's equations is linearization stable if and only if it has no Killing fields. Our current work extends these results and describes precisely the geometry of the space of solutions of Einstein's equations in a neighborhood of a solution that has a single Killing vector field. In particular, in a neighborhood of such a spacetime, the solutions *cannot* be parameterized in a smooth way by elements of a linear space, such as the space of four functions of three variables. (See Barrow and Tipler [8] for an application of this result.)

We shall begin by describing a theorem from bifurcation theory and give a simple example that is a prototype for what is happening in relativity.

Theorem 1 *Let $\phi:\mathbb{R}^n \to \mathbb{R}^k$ be a smooth mapping satisfying $\phi(0) = 0$ and $D\phi(0) = 0$ (i.e., the matrix $\partial\phi^i/\partial x^j$ of partial derivatives of ϕ vanishes at the origin). Let*

$$Q(v) = D^2\phi(0)\cdot(v, v) = \frac{\partial^2\phi^i}{\partial x^j \partial x^l}(0)v^j v^l$$

and suppose that whenever $Q(v) = 0$ and $v \neq 0$, the matrix

$$a^i_j = \frac{\partial^2\phi^i}{\partial x^j \partial x^l}(0)v^l$$

has rank k, i.e., the map $w \mapsto D^2\phi(0)(v, w)$ is onto. Then the set of solutions of

$$\phi(x) = 0 \tag{1}$$

for x near 0 is homeomorphic to the cone of solutions of

$$Q(v) = 0. \tag{2}$$

Notice that Eq. (2) is a set of k simultaneous homogeneous quadratic equations in the n variables $v^1, \ldots, v^n$. This theorem states that to determine the nature of the solutions of (1), one may restrict one's attention to the second-order terms in the Taylor expansion of ϕ about 0; i.e., one may disregard terms higher than the second order. Thus, roughly, we may say that

singularities of this type are "conical" since they are determined by the second-order terms of their Taylor series.

The above result is proved by a method called "blowing up a singularity," wherein one considers the scaled equation

$$\tilde{\phi}(x, r) = r^{-2}\cdot\phi(rx) = 0 \qquad \text{for a real variable } r \text{ and a unit vector } x$$

in $\mathbb{R}^n$. This change of variables expands the origin to the unit sphere. In these new variables, the implicit function theorem may be applied to deduce the theorem above (see Buchner *et al.* [9] and Marsden [10]).

Some generalizations of Theorem 1 are important for general relativity. For example, if ϕ is invariant under the action of a group G, one may be able to construct a manifold N of solutions of (1) by exploiting this symmetry property. The above theorem may then be applicable if ϕ is restricted to variables transverse to N.

The following well-known theorem illustrates a variant of Theorem 1 for $k = 1$ but in the presence of symmetry groups (see Bott [11]). The theorem is a consequence of the Morse lemma.

Theorem 2 *Let $\phi:\mathbb{R}^n \to \mathbb{R}$ be smooth and satisfy*

$$\phi(0) = 0, \qquad D\phi(0) = 0.$$

Suppose ϕ has a nondegenerate critical manifold $N \subset \mathbb{R}^n$ through 0; i.e., N is a submanifold on which $D\phi = 0$ and $D^2\phi(0)$ restricted to a space transverse to $T_0 N$ (its tangent space at the origin) is nonsingular. Then the conclusions of the previous theorem apply; in particular, the solutions of $\phi(x) = 0$ near N have the structure of a product of a cone with N.

As a simple example, consider the structure of the set of solutions (x, y, z) to the equation

$$\phi(x, y, z) = x^2 + y^2 - 2(x^2 + y^2)^{1/2} - z^2 + 1 = 0$$

near the solution $(0, 1, 0)$. Since ϕ is rotationally invariant about the z axis, the circle of radius 1 in the xy plane, labeled N in Fig. 1, is also a curve of solutions. If ϕ is restricted to variables transverse to N at $(0, 1, 0)$, i.e., if ϕ is restricted to the yz plane, then the above theorem applies, showing that the solution set in these transverse variables is a cone in the yz plane with vertex $(0, 1, 0)$. Due to the rotational invariance of ϕ, the full solution set in $\mathbb{R}^3$ is then given by the circle N from each point of which a cone C of solutions branches, as shown in Fig. 1.

For relativity, the previous theorems need to be generalized to allow ϕ to be defined on an infinite dimensional space. To study spacetimes near one with one Killing field, it suffices to consider a real-valued function on an infinite-dimensional space, as in Theorem 2. For spacetimes with k-Killing

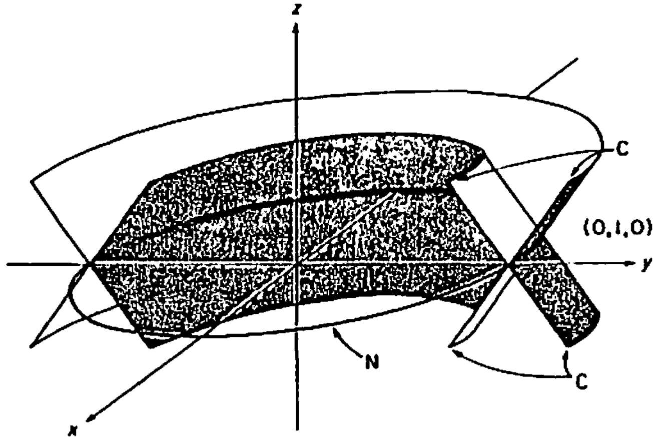

Fig. 1 A cone C of solutions of $\phi(x, y, z) = 0$ branching off a manifold N of solutions.

fields, an $\mathbb{R}^k$-valued function on an infinite-dimensional space is involved. The precise statement is rather technical and we refer to Buchner *et al.* [9] for the statement. The general methods, however, are reflected in the above theorems.

Now we apply these ideas to the empty space Einstein equations

$$\mathrm{Ein}({}^{(4)}g) = 0, \tag{3}$$

where $\mathrm{Ein}({}^{(4)}g) = \mathrm{Ric}({}^{(4)}g) - \frac{1}{2}{}^{(4)}gR({}^{(4)}g)$. We shall assume that our spacetimes $(V_4, {}^{(4)}g)$ are globally hyperbolic with compact Cauchy hypersurfaces Σ. Let (g, π) be the Cauchy data induced on Σ by the spacetime $(V_4, {}^{(4)}g)$, so that g is a Riemannian metric on Σ with canonically conjugate momentum density $\pi = \pi' \, d\mu_g$ (π' is the "tensor part" of π), and $d\mu_g$ is the volume element associated with g.

Let $(V_4, {}^{(4)}g)$ be a globally hyperbolic Einstein flat spacetime, $\mathrm{Ein}({}^{(4)}g) = 0$. Then $(V_4, {}^{(4)}g)$ is a *maximal spacetime* if it is the maximal development of some triple (Σ, g, π), where Σ is a Cauchy surface with Cauchy data (g, π) (see Hawking and Ellis [12, p. 249], for the definition of a maximal development). Note that if a spacetime is maximal with respect to a Cauchy surface Σ, it is then maximal with respect to any other Cauchy surface. For M compact, and $V_4 = \mathbb{R} \times M$, we let

$$\begin{aligned}\mathscr{E}_{\max} &= \mathscr{E}_{\max}(V_4)\\ &= \{{}^{(4)}g \mid \mathrm{Ein}({}^{(4)}g) = 0, \text{ and } (V_4, {}^{(4)}g) \text{ is a maximal spacetime}\}.\end{aligned}$$

By the initial value theorems of general relativity, solutions to the Einstein

equations are determined up to coordinate transformations by solutions of the constraint equations

$$\Phi(g, \pi) = (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi)) = 0 \tag{4}$$

on Σ, where

$$\begin{aligned}\mathscr{H}(g, \pi) &= (\pi' \cdot \pi' - \tfrac{1}{2}(\mathrm{tr}\, \pi')^2 - R(g))d\mu_g,\\ \mathscr{J}(g, \pi) &= 2\delta_g \pi.\end{aligned}$$

Here $R(g)$ is the scalar curvature of g, and $\delta_g \pi$ is the divergence of π (see Arnowitt *et al.* [13]).

The Eqs. (4) are a powerful "representation" of the Einstein equations (3). This is so because the Fréchet derivative $D\Phi(g, \pi)$ has an elliptic adjoint $D\Phi(g, \pi)^*$, whereas the Einstein equations are hyperbolic in nature. Thus, through the constraint equations (4), methods, techniques, and theorems concerning elliptic operators may be brought to bear on the study of the Einstein equations. (See Berger and Ebin [14] for a discussion of some of these elliptic methods.)

Symmetries of Eq. (3) are reflected in (4) as follows. If ${}^{(4)}X$ is a vector field on V_4 with normal and tangential components $({}^{(4)}X_\perp, {}^{(4)}X_\parallel)$ to Σ, then ${}^{(4)}X$ is a Killing vector field for ${}^{(4)}g$ only if $({}^{(4)}X_\perp, {}^{(4)}X_\parallel)$ lies in ker $D\phi(g, \pi)^*$.

Using these ideas Fischer and Marsden [1-3], Moncrief [4-6], and Arms and Marsden [7] have previously proved the following.

Theorem 3 The space of solutions $\mathscr{E}_{\max}$ is a smooth infinite-dimensional manifold in a neighborhood of a solution ${}^{(4)}g_0 \in \mathscr{E}_{\max}$ with tangent space the space of solutions of the linearized equations if and only if ${}^{(4)}g_0$ has no Killing vector fields.

According to this theorem, near a spacetime ${}^{(4)}g_0 \in \mathscr{E}_{\max}$ with symmetries, $\mathscr{E}_{\max}$ is not a manifold with its natural tangent space. Our current study shows that such singular regions in the space of solutions contain conical structures analogous to those in Fig. 1. In other words, $\mathscr{E}_{\max}$ branches, or bifurcates, in a neighborhood of spacetimes with Killing vector fields.

This branching of solutions is closely related to the phenomenon of linearization instability, but provides much more information about the structure of the space of solutions to Einstein's equations. We say that Eqs. (3) are *linearization stable* at ${}^{(4)}g_0 \in \mathscr{E}_{\max}$ if for any solution ${}^{(4)}h$ of the linearized Einstein equations

$$D\,\mathrm{Ein}({}^{(4)}g_0) \cdot {}^{(4)}h = 0 \tag{5}$$

and compact set $D \subset V_4$, there is a curve ${}^{(4)}g(\rho)$ of exact solutions of (3) on D such that on D, ${}^{(4)}g(0) = {}^{(4)}g_0$ and $d{}^{(4)}g(\rho)/d\rho|_{\rho=0} = {}^{(4)}h$; i.e., ${}^{(4)}h$ is *integrable*.

If there exists some nonintegrable ${}^{(4)}h$ that is a solution of (5), then we say that the Einstein equations (2) are *linearization unstable* at ${}^{(4)}g_0$.

If ${}^{(4)}g_0$ has a Killing vector field ${}^{(4)}X$, we can find a necessary quadratic condition in order that a solution ${}^{(4)}h$ of (5) be integrable. The development of this second-order condition is based on the discovery of a second-order conserved quantity due to Taub [15,16]. We first establish two lemmas, which we shall refer to as the Taub lemmas.

Lemma 4 *If* $\mathrm{Ein}({}^{(4)}g) = 0$, *and* ${}^{(4)}h$ *is any symmetric two tensor, then*

$$\delta_{{}^{(4)}g}(D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h) = 0,$$

where $\delta_{{}^{(4)}g}$ *denotes the covariant divergence with respect to* ${}^{(4)}g$.

Proof The contracted Bianchi identities assert that

$$\delta_{{}^{(4)}g}\,\mathrm{Ein}({}^{(4)}g) = 0.$$

Differentiation then gives the identity

$$(D\delta({}^{(4)}g)\cdot{}^{(4)}h)\cdot\mathrm{Ein}({}^{(4)}g) + \delta_{{}^{(4)}g}(D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h) = 0,$$

where $\delta({}^{(4)}g)$ indicates the functional dependence of $\delta_{{}^{(4)}g}$ on ${}^{(4)}g$, and $D\delta({}^{(4)}g)\cdot{}^{(4)}h$ is the derivative of this function. The lemma follows since $\mathrm{Ein}({}^{(4)}g) = 0$. ■

Lemma 5 *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$ *and* $D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = 0$. *Then*

$$\delta_{{}^{(4)}g}(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)) = 0.$$

Proof Differentiate the contracted Bianchi identity $\delta_{{}^{(4)}g}\,\mathrm{Ein}({}^{(4)}g) = 0$ twice with respect to ${}^{(4)}g$ and argue as in the preceding lemma. ■

Putting together Taub's lemmas gives us *Taub's Theorem.*

Theorem 6 *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$, $D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = 0$, *and* ${}^{(4)}X$ *is a Killing vector field for* ${}^{(4)}g$. *Then*

$${}^{(4)}T = {}^{(4)}X\cdot(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h))$$

has zero divergence (i.e., ${}^{(4)}T$ *is a conserved quantity; here "·" denotes contraction).*

Proof

$$\begin{aligned}\delta_{{}^{(4)}g}&({}^{(4)}X\cdot(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)))\\ &= \tfrac{1}{2}(L_{{}^{(4)}X}\,{}^{(4)}g)\cdot(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h))\\ &\quad + {}^{(4)}X\cdot\delta_{{}^{(4)}g}(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h))\end{aligned}$$

so the result follows from the hypothesis $L_{{}^{(4)}X}\,{}^{(4)}g = 0$ and Lemma 5. ■

In particular, for two compact spacelike hypersurfaces Σ_1 and Σ_2,

$$\int_{\Sigma_1}{}^{(4)}T\cdot{}^{(4)}Z_{\Sigma_1}\,d\mu_{\Sigma_1} = \int_{\Sigma_2}{}^{(4)}T\cdot{}^{(4)}Z_{\Sigma_2}\,d\mu_{\Sigma_2},$$

where ${}^{(4)}Z_{\Sigma_i}$ is the unit forward pointing timelike normal to Σ_i, and $d\mu_{\Sigma_i}$ is the Riemannian volume element induced on Σ_i by ${}^{(4)}g$. Thus Taub's conserved quantity

$$B(\Sigma, {}^{(4)}h, {}^{(4)}X) = \int_\Sigma {}^{(4)}T\cdot{}^{(4)}Z_\Sigma\,d\mu_\Sigma$$

is independent of the hypersurface on which it is evaluated.

We shall also need the following related result:

Lemma 7 *Suppose* $\mathrm{Ein}({}^{(4)}g) = 0$, ${}^{(4)}X$ *is a Killing field of* ${}^{(4)}g$, *and* ${}^{(4)}h$ *is a symmetric two-tensor field, and* Σ *is a compact spacelike hypersurface. Then*

$$I(\Sigma, {}^{(4)}h) = \int_\Sigma \langle {}^{(4)}X\cdot(D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h), {}^{(4)}Z\rangle\,d\mu_\Sigma = 0.$$

Proof By Lemma 4, $D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h$ has zero divergence, and so the vector field ${}^{(4)}X\cdot(D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h)$ also has zero divergence. Thus for any two compact spacelike hypersurfaces Σ_1 and Σ_2,

$$I(\Sigma_1, {}^{(4)}h) = I(\Sigma_2, {}^{(4)}h).$$

Choose Σ_1 and Σ_2 disjoint and replace ${}^{(4)}h$ by a symmetric two-tensor ${}^{(4)}k$ that equals ${}^{(4)}h$ on a tubular neighborhood of Σ_1 and vanishes on a tubular neighborhood of Σ_2. Then

$$I(\Sigma_1, {}^{(4)}h) = I(\Sigma_1, {}^{(4)}k) = I(\Sigma_2, {}^{(4)}k) = 0. \quad ■$$

These ideas are connected to linearization stability by the following result (see Fischer and Marsden [2,3] and Moncrief [6]):

Theorem 8 *Suppose* $\mathrm{Ein}({}^{(4)}g_0) = 0$, ${}^{(4)}X$ *is a Killing vector field of* ${}^{(4)}g_0$, *and* ${}^{(4)}h$ *is an integrable solution to the linearized equations. Then the conserved quantity of Taub vanishes identically when integrated over any compact spacelike hypersurface* Σ,

$$B(\Sigma, {}^{(4)}X) = \int_\Sigma \langle {}^{(4)}X\cdot(D^2\,\mathrm{Ein}({}^{(4)}g_0)\cdot({}^{(4)}h, {}^{(4)}h)), {}^{(4)}Z_\Sigma\rangle\,d\mu_\Sigma = 0. \qquad (6)$$

Proof Let ${}^{(4)}g(\lambda)$ be a curve of exact solutions through ${}^{(4)}g_0$ and tangent to ${}^{(4)}h$ at ${}^{(4)}g_0$. Differentiating $\mathrm{Ein}({}^{(4)}g(\lambda)) = 0$ twice with respect to λ and evaluating at $\lambda = 0$ gives the identity

$$D^2\,\mathrm{Ein}({}^{(4)}g_0)\cdot({}^{(4)}h, {}^{(4)}h) + D\,\mathrm{Ein}({}^{(4)}g_0)\cdot{}^{(4)}k = 0,$$

where

$${}^{(4)}k = \frac{d^2}{d\lambda^2}\,g(\lambda)|_{\lambda=0}$$

is the "acceleration" of ${}^{(4)}g(\lambda)$ at $\lambda = 0$. Contracting with ${}^{(4)}X$ and integrating over Σ gives

$$\int_\Sigma \langle {}^{(4)}X\cdot(D^2\,\mathrm{Ein}({}^{(4)}g_0)\cdot({}^{(4)}h, {}^{(4)}h)), {}^{(4)}Z_\Sigma\rangle d\mu_\Sigma + \int_\Sigma \langle {}^{(4)}X\cdot(D\,\mathrm{Ein}({}^{(4)}g_0)\cdot{}^{(4)}k), {}^{(4)}Z_\Sigma\rangle d\mu_\Sigma = 0$$

The last integral vanishes by Lemma 7. ∎

This theorem is important when a spacetime has Killing vector fields, since it provides a necessary second-order condition in order that a first-order deformation ${}^{(4)}h$ be integrable. Thus unless the conserved quantity of Taub vanishes, $B(\Sigma, {}^{(4)}h, {}^{(4)}X) = 0$, the first-order solution ${}^{(4)}h$ cannot be tangent to any curve of exact solutions. Theorem 9 below states that the vanishing of Taub's conserved quantity is also sufficient for integrability of a first-order deformation ${}^{(4)}h$. Thus for spacetimes that have Killing vector fields, and hence are not linearization stable, the numerical value of Taub's conserved quantity plays the central role in testing whether or not perturbations ${}^{(4)}h$ are actually tangent to a curve of exact solutions.

In terms of the constraint equations, the second-order condition becomes

$$\int_\Sigma \langle({}^{(4)}X_\perp, {}^{(4)}X_\parallel), D^2\Phi(g,\pi)\cdot((h,\omega),(h,\omega))\rangle d\mu_g = 0 \tag{7}$$

where (h, ω) is the perturbation of (g, π) induced by ${}^{(4)}h$. These quadratic conditions (6) or (7) are analogous to Equations (2) in the bifurcation analysis sketched earlier. Indeed, the main result of our current work is that this bifurcation analysis applies to the Einstein equations (3) via the constraint equations (4). In terms of linearization stability, our main result may be stated as a converse to Theorem 8.

Theorem 9 *Let $(V_4, {}^{(4)}g)$ be a maximal spacetime such that $\mathrm{Ein}({}^{(4)}g) = 0$. Assume that $(V_4, {}^{(4)}g)$ has a compact Cauchy hypersurface with $\mathrm{tr}\,\pi' =$ constant, and that ${}^{(4)}g$ has a one-dimensional space of Killing vector fields.*

Let ${}^{(4)}h$ satisfy the linearized equations

$$D\,\mathrm{Ein}({}^{(4)}g)\cdot{}^{(4)}h = 0.$$

Then ${}^{(4)}h$ is integrable if and only if ${}^{(4)}h$ satisfies the second-order condition

$$\int_\Sigma \langle {}^{(4)}X\cdot(D^2\,\mathrm{Ein}({}^{(4)}g)\cdot({}^{(4)}h, {}^{(4)}h)), {}^{(4)}Z_\Sigma\rangle d^3\Sigma = 0$$

on any Cauchy hypersurface Σ.

Remarks

(1) A similar result holds when the solution ${}^{(4)}g$ has more than one Killing vector field. The precise details of this more general case are now under investigation.

(2) The above results should hold for a variety of systems coupled to gravity, such as the Einstein–Maxwell system, or the Einstein–Yang–Mills system. (See Arms [18,19] for linearization stability results regarding these systems.)

(3) It is believed that the requirement of a hypersurface with constant mean curvature is not very restrictive (see Choquet-Bruhat *et al.* [20]).

Theorem 9 states that in the presence of a Killing vector field the second-order condition is not only necessary but is also sufficient for integrability of first-order deformations. It is the sufficiency of this second-order quadratic condition that tells us that the singular regions in the solution space $\mathscr{E}_{\max}$ are at worst singularities of a conical type, and that the second-order condition above defines the tangent direction to the conical singularity. As has been pointed out by Moncrief [21], it is important to take these singularities into account in the quantum theory of gravity.

In order to understand the geometry behind the bifurcations that are taking place, it is necessary to isolate the directions of degeneracy analogous to the manifold N in the finite dimensional example shown in Fig. 1. For general relativity these degeneracy directions occur not only because of the coordinate covariance of the Einstein equations, but also because there will be nonisometric spacetimes nearby with one Killing vector field.

Since we are dealing with spacetimes with only one Killing vector field and which satisfy the $\mathrm{tr}\,\pi' =$ constant condition, there are two cases to analyze in the proof of Theorem 9; the case when ${}^{(4)}X$ is spacelike, and the case when ${}^{(4)}X$ is timelike.

We shall comment on the case when ${}^{(4)}X$ is timelike later. First, suppose that ${}^{(4)}X$ is spacelike. In this case, the conical singularity that occurs in $\mathscr{E}_{\max}$ in a neighborhood of a solution ${}^{(4)}g$ is closely related to the purely geometric conical singularity that occurs in the divergence equation

$$\mathscr{J}(g,\pi) = 2\delta_g\pi = 0 \tag{8}$$

near a solution (g, π) that has precisely one Killing vector field X; i.e., a vector field X such that

$$L_X g = 0 \qquad \text{and} \qquad L_X \pi = 0.$$

An important ingredient in the analysis of the singularity that occurs in (8) about such a (g, π) is the Ebin–Palais slice theorem (see Ebin [22]) extended to a contangent bundle action. We now describe this analysis.

Let $\mathcal{M} = \mathrm{Riem}(M)$ denote the space of Riemannian metrics on a compact manifold M, and let $\mathcal{D} = \mathrm{Diff}(M)$ denote the group of diffeomorphisms of M. Then $\mathcal{D}$ acts on $\mathcal{M}$ by pull-back, and $\mathcal{D}$ lifts naturally to a symplectic action on the L_2-cotangent bundle $T^*\mathcal{M}$;

$$\mathcal{D} \times T^*\mathcal{M} \to T^*\mathcal{M},$$
$$(f, (g, \pi)) \mapsto (f^*g, f^*\pi),$$

where $f^*\pi = (f^{-1})_*\pi$ is the pull-back of contravariant tensor densities.

For $(g, \pi) \in T^*\mathcal{M}$, let

$$\psi_{(g,\pi)}: \mathcal{D} \to T^*\mathcal{M}\,;\; f \mapsto (f^*g, f^*\pi)$$

denote the orbit map through (g, π), and let $\mathcal{O}_{(g,\pi)} = \mathrm{Image}\ \psi_{(g,\pi)} = \{(g', \pi') \in T^*\mathcal{M} \mid g' = f^*g,\ \pi' = f^*\pi \text{ for some } f \in \mathcal{D}\}$ denote the orbit through (g, π).

Let $\mathcal{J}: T^*\mathcal{M} \to \Lambda^1_d\,;\, (g, \pi) \mapsto \mathcal{J}(g, \pi) = 2\delta_g \pi$ denote the divergence map, and let

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}: S^2_d \times S_2 \to S_2 \times S^2_d\,;$$

$$\begin{pmatrix} \omega \\ h \end{pmatrix} \mapsto J\begin{pmatrix} \omega \\ h \end{pmatrix} = \begin{pmatrix} h \\ -\omega \end{pmatrix}$$

be the symplectic matrix on $T^*\mathcal{M}$. Here Λ^1_d is the space of one-form densities on M; S_2 is the space of symmetric two-covariant tensor fields on M; S^2_d is the space of symmetric two-contravariant tensor densities on M; and $\mathcal{X}$ is the space of vector fields on M.

The derivatives of $\psi_{(g,\pi)}$, $\mathcal{J}$, and their natural L_2 adjoints are related as follows:

Lemma 10

$$T_{id}\psi_{(g,\pi)} = J \circ (D\mathcal{J}(g, \pi))^*: \mathcal{X} \to S_2 \times S^2_d\,;\; X \mapsto (L_X g, L_X \pi),$$

$$(T_{id}\psi_{(g,\pi)})^* = D\mathcal{J}(g, \pi) \circ J^* = -D\mathcal{J}(g, \pi) \circ J: S^2_d \to \Lambda^1_d\,;$$

$$(\omega, h) \mapsto D\mathcal{J}(g, \pi) \cdot \begin{pmatrix} -h \\ \omega \end{pmatrix}.$$

Moreover $S_2 \times S^2_d$ splits L_2 orthogonally as

$$\begin{aligned} S_2 \times S^2_d &= \mathrm{range}\ T_{id}\psi_{(g,\pi)} \oplus (\ker(T_{id}\psi_{(g,\pi)})^*)^* \\ &= \mathrm{range}\ J \circ (D\mathcal{J}(g, \pi))^* \oplus (\ker(D\mathcal{J}(g, \pi) \circ J))^* \end{aligned}$$

where $(\ )^*$ denotes the L_2 adjoint map

$$(\ )^*: S^2_d \times S_2 \to S_2 \times S^2_d\,;\, (\omega, h) \mapsto ((\omega^\flat)', h^\sharp d\mu_g).$$

Proof (Here $\omega^\flat$ is the covariant form of ω and $h^\sharp$ the contravariant form of h.) $D\mathcal{J}(g, \pi)^* \cdot X = (-L_X\pi, L_X g)$. Hence $J \circ D\mathcal{J}(g, \pi)^* \cdot X = (L_X g, L_X \pi) = (T_{id}\psi_{(g,\pi)}) \cdot X$, and thus $(T_{id}\psi_{(g,\pi)})^* = D\mathcal{J}(g, \pi) \circ J^* = -D\mathcal{J}(g, \pi) \circ J$ since $J^* = -J$. The splitting follows from injectivity of the symbol of $D\mathcal{J}(g, \pi)^*$ (see Fischer and Marsden [17]). ∎

We shall also need the following result:

Lemma 11 The orbit $\mathcal{O}_{(g,\pi)}$ through (g, π) is a closed submanifold of $T^*\mathcal{M}$ with tangent space at (g, π) given by

$$\begin{aligned} T_{(g,\pi)}\mathcal{O}_{(g,\pi)} &= \mathrm{range}\ J \circ D\mathcal{J}(g, \pi)^* \\ &= \{(L_X g, L_X \pi) \mid X \text{ is a vector field on } M\}. \end{aligned}$$

Proof Let $\Psi: \mathcal{D} \mapsto T^*\mathcal{M}\,;\, \eta \mapsto (\eta^*g, \eta^*\pi)$. We have

$$T_\eta\Psi(X) = (\eta^* L_X g, \eta^* L_X \pi).$$

Since $X \mapsto L_X g$ is elliptic, $T_\eta\Psi$ has closed range and finite-dimensional kernel. By the arguments of Ebin and Marsden ([23], Appendix B), $\ker T_\eta\Psi$ is a subbundle of $T\mathcal{D}$. It follows from the implicit function theorem that the range of Ψ is an immersed submanifold. From Ebin ([22], Proposition 6.13), it follows that Ψ is an open map onto its range and that the range is closed. The lemma then follows. ∎

The proof of our results depends on a carefully constructed slice for the action of $\mathcal{D}$ on $T^*\mathcal{M}$. For the action of $\mathcal{D}$ on $\mathcal{M}$, the Ebin–Palais slice theorem (see Ebin [22]) asserts the existence of a slice. To avoid unnecessary technicalities, our slice will be an affine one, L_2 orthogonal to the orbit of $\mathcal{D}$. Let

$$I_g = \{f \in \mathcal{D}(M) \mid f^*g = g\},$$
$$I_\pi = \{f \in \mathcal{D}(M) \mid f^*\pi = \pi\},$$

and

$$I_{(g,\pi)} = I_g \cap I_\pi.$$

Since the isometry group I_g is a compact finite-dimensional Lie group, and since I_π is a closed subgroup of $\mathcal{D}(M)$, $I_{(g,\pi)} = I_g \cap I_\pi$ is also a compact finite-dimensional Lie group. $I_{(g,\pi)}$ is the isotropy group at (g, π) for the action of $\mathscr{D}$ on $T^*\mathscr{M}$.

Theorem 12 *The action*

$$\mathscr{D} \times T^*\mathscr{M} \to T^*\mathscr{M};$$
$$(f, (g, \pi)) \mapsto (f^*g, f^*\pi),$$

has a slice $S_{(g,\pi)} \subset T^*\mathscr{M}$ at each $(g, \pi) \in T^*\mathscr{M}$; i.e., $S_{(g,\pi)}$ is a submanifold of $T^*\mathscr{M}$ containing (g, π) such that

(i) *if $f \in I_{(g,\pi)}$, then $f^*(S_{(g,\pi)}) = S_{(g,\pi)}$;*
(ii) *if $f \in \mathscr{D}$ and $f^*(S_{(g,\pi)}) \cap S_{(g,\pi)} \neq \phi$, then $f \in I_{(g,\pi)}$;*

and

(iii) *there is a local cross section $\chi: \mathscr{D}/I_{(g,\pi)} \to \mathscr{D}$.*

defined in a neighborhood U of the identity coset such that the map $(\phi, (g', \pi')) \mapsto (\chi(\phi))^*(g', \pi')$ is a homeomorphism of $U \times S_{(g,\pi)}$ onto a neighborhood of (g, π) in $T^*\mathscr{M}$. In particular the slice $S_{(g,\pi)}$ sweeps out a neighborhood of (g, π) under the group action.

The tangent space to the slice at (g, π) is given by

$$T_{(g,\pi)}S_{(g,\pi)} = (\ker(T_{id}\psi_{(g,\pi)})^*)^* = (\ker(D\mathscr{J}(g, \pi) \circ J))^*.$$

Sketch of the Slice Construction

Fix (g, π) and let G denote the L_2-metric on $T^*\mathscr{M}$ given by

$$G_{(g',\pi')}((h, \omega), (h, \omega)) = \int_M (\langle h, h\rangle_g + \langle \omega', \omega'\rangle_g) d\mu_g$$

for $(h, \omega) \in T_{(g',\pi')}T^*\mathscr{M} = S_2 \times S_d^2$, where $\langle , \rangle$ indicates contraction using g (note that the inner product is independent of (g', π')).

The splitting of $T_{(g,\pi)}T^*\mathscr{M}$ defined by Lemma 10 is, by construction, an orthogonal splitting with respect to G. We may exponentiate the subspace $(\ker(D(g, \pi) \circ J))^*$ to obtain an "affine" submanifold $A_{(g,\pi)} = \{(g, \pi)\} + [\ker(D\mathscr{J}(g, \pi) \circ J)]^*$ of $T^*\mathscr{M}$, which intersects $\mathscr{O}_{(g,\pi)}$ orthogonally at (g, π). The action of $I_{(g,\pi)}$ on $T^*\mathscr{M}$ leaves $A_{(g,\pi)}$ invariant. This follows from the fact that $I_{(g,\pi)}$ is an isometry group of G. We may now intersect $A_{(g,\pi)}$ with a sufficiently small ball (in a Sobolev norm invariant under $I_{(g,\pi)}$) to obtain a slice $S(g, \pi)$ for the action of $\mathscr{D}$ on $T^*\mathscr{M}$.

The proof that such "affine slices" are indeed slices now follows from the methods of Ebin and Palais.

An important consequence of the existence of a slice for the action of $\mathscr{D}$ on $T^*\mathscr{M}$ is the local decreasing property of the groups $I_{(g,\pi)}$:

Corollary 13 *Let $(g, \pi) \in T^*\mathscr{M}$, and let $S_{(g,\pi)}$ be a slice at (g, π). Then if $(g', \pi') \in S_{(g,\pi)}$,*

$$I_{(g',\pi')} \subseteq I_{(g,\pi)}.$$

Proof The inclusion follows from property (ii) of a slice. ■

Let $B_{(g,\pi)} = \{(g', \pi') \in S_{(g,\pi)} \mid I_{(g',\pi')} = I_{(g,\pi)}\}$. Thus $B_{(g,\pi)}$ is the set of elements of the slice with the same symmetry group $I_{(g,\pi)}$. In the terminology of superspace (Fischer [24]), $B_{(g,\pi)}$ is a local stratum of (g, π)s with the same symmetry type.

Let $I^0_{(g,\pi)}$ denote the connected component of the identity of the Lie group $I_{(g,\pi)}$ and let $\mathscr{I}_{(g,\pi)}$ denote its Lie algebra. Since $\mathscr{O}_{(g,\pi)}$ is closed, the discrete part $I_{(g,\pi)}/I^0_{(g,\pi)}$ of $I_{(g,\pi)}$ never accumulates on $\mathscr{O}_{(g,\pi)}$. Since $B_{(g,\pi)} \subset S_{(g,\pi)}$, $B_{(g,\pi)}$ can also be described using $I^0_{(g,\pi)}$ rather than the full isotropy group $I_{(g,\pi)}$; i.e., if $S_{(g,\pi)}$ is sufficiently small, then

$$\begin{aligned} B_{(g,\pi)} &= \{(g', \pi') \in S_{(g,\pi)} \mid I^0_{(g',\pi')} = I_{(g,\pi)}\} \\ &= \{(g', \pi') \in S_{(g,\pi)} \mid \mathscr{I}_{(g',\pi')} = \mathscr{I}_{(g,\pi)}\}. \end{aligned}$$

Proposition 14 *$B_{(g,\pi)}$ is a submanifold of $S_{(g,\pi)}$ with tangent space at (g, π) given by*

$$\begin{aligned} &T_{(g,\pi)}B_{(g,\pi)} \\ &= \{(h, \omega) \in (\ker(D\mathscr{J}(g, \pi) \circ J))^* \mid f^*h = h, f^*\omega = \omega \quad \text{for all } f \in I^0_{(g,\pi)}\} \\ &= \{(h, \omega) \in (\ker(D\mathscr{J}(g, \pi) \circ J))^* \mid L_X h = 0, L_X\omega = 0 \quad \text{for all } X \in \mathscr{I}_{(g,\pi)}\}. \end{aligned}$$

We shall use the submanifold $B_{(g,\pi)}$ to study the set

$$\mathscr{C}_\delta = \{(g, \pi) \in T^*\mathscr{M} \mid \delta_g\pi = 0\}.$$

As is shown in Fischer and Marsden [3], if $(g, \pi) \in \mathscr{C}_\delta$ satisfies $\mathscr{I}_{(g,\pi)} = 0$, then $\mathscr{C}_\delta$ is a submanifold in a neighborhood of (g, π). In such a neighborhood the slice $S_{(g,\pi)}$ is a cross section for the action of $\mathscr{D}$ on $T^*\mathscr{M}$, and $S_{(g,\pi)} \cap \mathscr{C}_\delta$ is also a submanifold of $\mathscr{C}_\delta$.

To study the structure of $\mathscr{C}_\delta$ and $S_{(g,\pi)} \cap \mathscr{C}_\delta$ when $\mathscr{I}_{(g,\pi)} \neq 0$, we shall utilize the bifurcation analysis sketched out earlier. The result in the case $\dim \mathscr{I}_{(g,\pi)} = 1$ is the following:

Theorem 15 *Let $(g, \pi) \in \mathscr{C}_\delta$, let $S_{(g,\pi)}$ be a slice at (g, π) and let $B_{(g,\pi)} = \{(g', \pi') \in S_{(g,\pi)} \mid I_{(g',\pi')} = I_{(g,\pi)}\}$. Assume that $\dim \mathscr{I}_{(g,\pi)} = 1$; i.e., (g, π) has a single Killing vector field X, a vector field on M such that $L_Xg = 0$ and $L_X\pi = 0$.*

Then

(a) *$N = \mathscr{C}_\delta \cap B_{(g,\pi)}$ is a submanifold of the set $\mathscr{C}_\delta \cap S_{(g,\pi)}$;*
(b) *$\mathscr{C}_\delta \cap S_{(g,\pi)}$ is a product $C \times N$, where C is a cone; and*
(c) *$\mathscr{C}_\delta$ is locally homeomorphic to $C \times N \times (\mathscr{D}/I_{(g,\pi)})$.*

(See Fig. 2.)

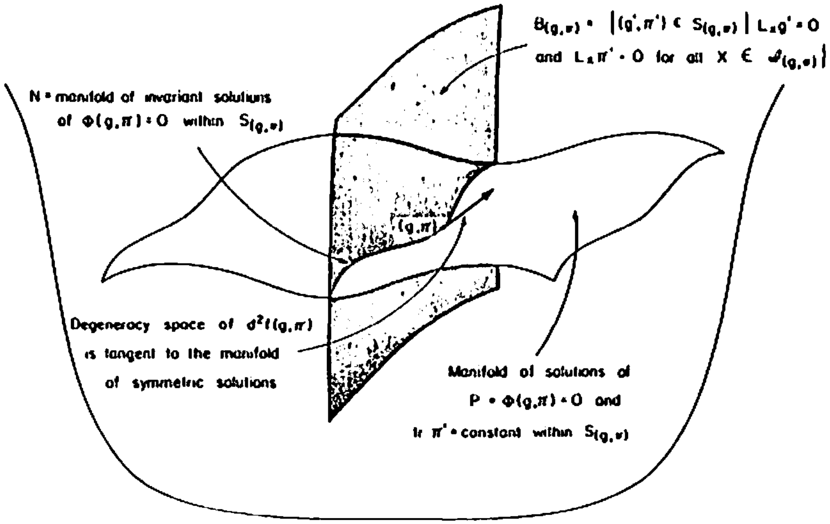

Fig. 2 The structure of the constraint manifold $\mathscr{C}$ in a neighborhood of a (g, π) with a Killing vector field X. The space of the picture is the slice $S_{(g,\pi)}$.

Sketch of Proof

As in the proof of Lemma 10, Λ_δ^1 splits L_2 orthogonally as

$$\text{range } D\mathscr{J}(g, \pi) \oplus \ker(D\mathscr{J}(g, \pi))^*.$$

Let P denote the orthogonal projection onto the first factor. Since $D\mathscr{J}(g, \pi)^* \cdot X = (-L_X \pi, L_X g)$, the second factor is one dimensional, and is spanned by $\{X\}$. Assume X has L_2 length one and identify $\ker D\mathscr{J}(g, \pi)^*$ with the real line $\mathbb{R}$. Thus the projection $I - P$ is given by

$$(I - P) \circ \mathscr{J}(g, \pi) = \int_M X \cdot \mathscr{J}(g, \pi).$$

Let

$$\mathscr{C}_P = \{(g, \pi) \mid P \circ \mathscr{J}(g, \pi) = 0\}.$$

The map $\mathscr{J}$ is clearly transverse to $\ker D(g, \pi)^*$ and its kernel splits, so that in a neighborhood of (g, π); it follows from the implicit function theorem that $\mathscr{C}_P$ is a smooth manifold with tangent space

$$T_{(g,\pi)}\mathscr{C}_P = \ker D\mathscr{J}(g, \pi),$$

the space of solutions to the linearized constraint equation.

Let $f: \mathscr{C}_P \to \mathbb{R}$ be defined by

$$f(g, \pi) = (I - P) \circ \mathscr{J}(g, \pi)$$
$$= \int_M X \cdot \mathscr{J}(g, \pi) = \int_M \pi \cdot L_X g = 2 \int_M X \cdot \delta_g \pi.$$

Clearly the constraint set $\mathscr{C}_\delta = \mathscr{J}^{-1}(0)$ is given by

$$\mathscr{C}_\delta = f^{-1}(0).$$

Thus our problem is to analyze the zero set of f near (g, π). In bifurcation theory, the above construction of P, $\mathscr{C}_P$, and f is called the Liapunov–Schmidt procedure (see Marsden [10]).

For $(h, \omega) \in S_2 \times S_d^2$,

$$df(g, \pi) \cdot (h, \omega) = \int_M \omega \cdot L_X g + \int_M \pi \cdot L_X h = \int_M \omega \cdot L_X g - \int_M L_X \pi \cdot h.$$

Thus (g, π) is a critical point of f if and only if $L_X g = 0$ and $L_X \pi = 0$; i.e., if and only if $X \in \ker D\mathscr{J}(g, \pi)^*$. At a critical point, the Hessian of f may be computed in the ambient space $T^*\mathscr{M}$ and then restricted to $\mathscr{C}_P$. Thus

$$d^2 f(g, \pi) \cdot ((h_1, \omega_1), (h_2, \omega_2)) = \int_M \omega_1 \cdot L_X h_2 + \int_M \omega_2 \cdot L_X h_1.$$

Note that $d^2 f$ is independent of (g, π).

Since $B_{(g,\pi)} = \{(g', \pi') \in S_{(g,\pi)} \mid \mathscr{I}_{(g',\pi')} = \mathscr{I}_{(g,\pi)}\}$, $B_{(g,\pi)}$ is a critical submanifold of f. Moreover, it is easy to see that the degeneracy of the bilinear form $d^2 f(g, \pi)$ is tangent to this submanifold of critical points intersected with the constraint set,

$$N = \mathscr{C}_\delta \cap B_{(g,\pi)}.$$

Thus on a submanifold transverse to N, $d^2 f$ is weakly nondegenerate.

Within this setting a generalization of Theorem 2 to infinite dimensions applies and gives the structure of $\mathscr{C}_\delta \cap S_{(g,\pi)}$ as the product of a (manifold) × (cone), where the cone directions at (g, π) are given by the solutions (h, ω) of the second-order condition

$$d^2 f(g, \pi) \cdot ((h, \omega), (h, \omega)) = 2 \int \omega \cdot L_X h = 0.$$

For the map $\mathscr{J}$ restricted to $S_{(g,\pi)}$, the set $N = \mathscr{C}_\delta \cap B_{(g,\pi)}$ plays the role of N in Theorem 2. ■

Remark

When $k = \dim \mathscr{I}_{(g,\pi)} > 1$, the set $\mathscr{C}_\delta \cap S_{(g,\pi)}$ will have the structure of "cones on cones."

The result of Theorem 15 can be generalized to give the structure of the constraint equations

$$\Phi(g, \pi) = (\mathscr{H}(g, \pi), \mathscr{J}(g, \pi)) = 0$$

in a neighborhood of $(g, \pi) \in \mathscr{C} = \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_\delta$, which satisfies the hypotheses

(i) $\dim \mathscr{I}_{(g,\pi)} = 1$; and
(ii) $\operatorname{tr} \pi' = \text{const.}$

The conclusion is identical to (a)-(c) of Theorem 15 with $\mathscr{C}$ replacing $\mathscr{C}_\delta$; see Fig. 2. Thus the conical singularities in $\mathscr{C}_\delta$ are carried over to $\mathscr{C}$. Using this result on the structure of $\mathscr{C}$, we can now sketch a proof of Theorem 9 (see [25] for details).

Sketch of Proof of Theorem 9 We have already remarked on the necessity of the second-order condition for integrability of first-order deformations ${}^{(4)}h$. To prove sufficiency, we first consider the case that ${}^{(4)}X$ is spacelike. From the fact that Σ has constant mean curvature, one can show that ${}^{(4)}X$ is parallel to Σ, so that the perpendicular-parallel decomposition of ${}^{(4)}X$ along Σ is $({}^{(4)}X_\perp, {}^{(4)}X_\parallel) = (0, X)$, where X is now a vector field on the submanifold Σ (identified with M).

Let (g, π) be the Cauchy data induced on Σ by ${}^{(4)}g$. In Moncrief [4], it is proven that

$$\ker(D\Phi(g, \pi)^*) = \{({}^{(4)}X_\perp, {}^{(4)}X_\parallel) | {}^{(4)}X \in \mathscr{I}_{{}^{(4)}g}\},$$

where $\mathscr{I}_{{}^{(4)}g}$ is the space of Killing vector fields for ${}^{(4)}g$. In the case at hand, ${}^{(4)}g$ has only a single spacelike Killing vector field, so $\ker(D\Phi(g, \pi))^*$ is the one-dimensional space spanned by $(0, X)$. Moreover,

$$D\Phi(g, \pi)^* \cdot (0, X) = D\mathscr{J}(g, \pi)^* \cdot X = (-L_X \pi, L_X g),$$

so that if $(0, X) \in \ker D\Phi(g, \pi)^*$, then $L_X g = 0$ and $L_X \pi = 0$.

Let ${}^{(4)}g_0 \in \mathscr{E}_{\max}$ be a maximal development of (Σ, g, π), so that $({}^{(4)}g, V_4) \subseteq ({}^{(4)}g_0, V_4)$. Let $I_{{}^{(4)}g_0}$ denote the isometry group of ${}^{(4)}g_0$ and let

$$B_{{}^{(4)}g_0} = \{{}^{(4)}g' \in \mathscr{E}_{\max} | I_{{}^{(4)}g} = F^{-1} \circ I_{{}^{(4)}g_0} \circ F \quad \text{for some } F \in \operatorname{Diff}(V_4)\},$$

i.e., $B_{{}^{(4)}g_0}$ is the set of solutions to the field equations with the same *symmetry type* as ${}^{(4)}g_0$. Thus $B_{{}^{(4)}g_0}$ includes all solutions ${}^{(4)}g'$ related to ${}^{(4)}g_0$ by a transformation F; ${}^{(4)}g' = F^{*(4)}g_0$, but it also includes other nonisometric solutions as well.

Let P be the projection to range $D\Phi(g, \pi)$, $\mathscr{C}_P = \{(\bar{g}, \bar{\pi}) | P\Phi(\bar{g}, \bar{\pi}) = 0\}$ and define the function

$$f : \mathscr{C}_P \cap S_{(g,\pi)} \to \mathbb{R}$$

by

$$\begin{aligned} f(h, \omega) &= \int_\Sigma ({}^{(4)}X_\perp, {}^{(4)}X_\parallel) \cdot \Phi(g + h, \pi + \omega) \\ &= \int_\Sigma {}^{(4)}X_\perp \cdot (g + h, \pi + \omega) + \int_\Sigma X \cdot \mathscr{J}(g + h, \pi + \omega) \\ &= \int_\Sigma X \cdot \mathscr{J}(g + h, \pi + \omega), \end{aligned}$$

since $X_\perp = 0$. Applying the methods used in Theorem 15, one can show that the degeneracy subspace of the second derivative of this function at (g, π) is described by the Cauchy data on Σ corresponding to elements of the manifold $B_{{}^{(4)}g_0}$, and that conical structures appear in the transverse direction. Thus, in a neighborhood of ${}^{(4)}g_0$, the solution space $\mathscr{E}_{\max}$ looks like a set of cones bifurcating off of the manifold $B_{{}^{(4)}g_0}$ as is depicted in Fig. 1. Thus, locally, in a neighborhood of ${}^{(4)}g_0$, $\mathscr{E}_{\max}$ has the product structure (manifold) × (cone) where the manifold is $B_{{}^{(4)}g_0}$ and the cone C consists of the zeros ${}^{(4)}h$ of the second-order condition (6), representing solutions with less symmetry. Thus the directions of the cone field are delineated precisely by this second-order condition.

A similar argument can be given for the case of one timelike Killing vector field. In this case the solutions with the same symmetry are all stationary and hence all flat. As in the spacelike case, a cone of solutions of lower symmetry branches from each flat spacetime. ■

The conical structure when ${}^{(4)}g_0$ has just one Killing vector field can be visualized as follows. Consider an open book. The spine of the book corresponds to the manifold $B_{{}^{(4)}g_0}$ and the sides of the book correspond to the conical structures bifurcating off $B_{{}^{(4)}g_0}$. Going along the spine of the book corresponds to moving in $\mathscr{E}_{\max}$ in a direction that preserves the symmetry type of ${}^{(4)}g_0$, whereas moving up the sides of the book in the conical direction corresponds to breaking the symmetry and moving to generic empty space solutions of the Einstein equations that do not have any symmetry.

The results of our analysis show that the space of solutions of Einstein's equations has an unexpected complexity near a spacetime with symmetries. The complexity points out the cautions required when doing perturbation analysis and the dangers of function counting arguments. It is hoped that these results shed light on the nature of solutions of Einstein's equations near spacetimes with Killing vector fields, their stability as regards the breaking of these symmetries, and corresponding difficulties that occur with the quantum theory. Since our own Universe apparently is highly symmetrical in the large, the structure of the space of solutions near spacetimes with Killing vector fields is a question of some importance.

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8

Gauge Invariant Perturbation Theory in Spatially Homogeneous Cosmology

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Abstract

Lie group harmonic analysis is applied to the solution of tensor equations on LRS class A spatially homogeneous spacetimes. The techniques developed are used to discuss the Hamiltonian dynamics of the linearized vacuum Einstein equations and Moncrief's orthogonal decomposition of the linearized phase space adapted to the linearized gauge transformations. This leads to what Moncrief has called "gauge invariant perturbation theory" for the class of spacetimes under consideration.

I. Introduction

In relativity, it is especially important to have examples that illustrate the ideas and techniques of the general theory. Spatially homogeneous cosmology, introduced by Taub several decades ago [1], has served a very important role in providing a wealth of examples with which to investigate various aspects of the theoretical perspective that has evolved in relativity over the intervening years. Misner, Ryan and Ellis, among many others, have used this class of spacetimes to study such things as Hamiltonian dynamics, singularities, and global structure. Taub participated in these

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