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Optimal Worst-Case QoS Routing in Constrained AWGN Channel Network

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Abstract— **In this paper, we extend the optimal worst-case QoS routing algorithm and metric definition given in [1]. We prove that in addition to the q-ary symmetric and q-ary erasure channel model, the necessary and sufficient conditions defined in [2] for the Generalized Dijkstra's Algorithm (GDA) can be used with a constrained non-negative-mean AWGN channel.**

The generalization allowed the computation of the worst-case QoS metric value for a given edge weight density. The worst-case value can then be used as the routing metric in networks where some nodes have error correcting capabilities.

The result is an optimal worst-case QoS routing algorithm that uses the Generalized Dijkstra's Algorithm as a subroutine with a polynomial time complexity of $O(V^3)$.

I. Introduction

Mission critical communication networks have very high requirements for performance and reliability. These networks are designed to minimize the possibility of losing even one symbol, packet, or file. Optimizing the Quality of Service (QoS) of such networks requires evaluating network paths (and edges) and selecting those paths and edges optimality that minimizes the worst possible number of observed undesired event (which could be loss, error, etc.). In many applications, such as ad-hoc wireless (sensor) networks, this type of QoS optimization is both critical, and yet highly untrivial.

A novel QoS metric called the worst-case error (or erasure) metric (WCE) was introduced in [1], where it was used to measure, compare, and select the network path whose WCE length is minimum. The WCE metric is derived from the metric more commonly used in specifying communication channel qualities: the Bit Error (or Erasure) Ratio (BER).

In the original introduction, only the cases of q-ary Symmetric Channel and q-ary Erasure Channel were analyzed. It was shown that for these channels, the edge WCE length is a non-decreasing function of its BER length. Thus the optimal BER path is also the optimal WCE path, and vice versa.

In this paper, we extend the optimal worst-case QoS analysis to cover a special case of AWGN channel that has non-negative mean and a constraint on its mean to variance ratio. This channel can be easily used to model delay channel, or in some cases, the Gaussian approximation of Poisson-distributed channels. To calculate a path's BER length from the edges' BER lengths, we use the same algebra as in [1]. The path BER was then applied to the Generalized Dijkstra's Algorithm.

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We show that if some network nodes are capable of correcting erasures, it is often possible to find a path (or paths) with zero WCE length. This partial error correction capability is reasonable. In many wireless and ad-hoc networks, energy (and ultimately compute power) is limited.

Consequently, available resource for error correction that involves complex mathematical operations such as finite fields arithmetic and iterative algorithms is limited as well. In addition, error correction incurs delay and consumes bandwidth. Hence, it is desirable to limit the number of error-correcting nodes. Our algorithm optimizes the worst-case QoS by routing information through the error-correcting nodes.

Forward Error Correction (FEC) is also gaining acceptance in modern networks. Historically, error correction over wired networks has predominantly used the Automatic Repeat reQuest (ARQ) methods over the FEC methods. However, in wireless multimedia applications, FEC outshines ARQ in its ability to significantly improve network QoS [3] [4] [5] [6] without a heavy premium on performance. Unlike TCP and ARQ, FEC does not use return requests and thus consumes less bandwidth [7], especially in large multicast networks typical of wireless multimedia applications [8] [9].

Even in peer-to-peer networks, FEC deployed at strategically positioned error-correcting nodes is superior compared to replication [10]. Multicast algorithms such as *Digital Fountain* [11] and *Bullet* [12] employ FEC-based codes. Adaptive [5] [6] and hybrid (ARQ-FEC) QoS-driven algorithms that dynamically adjust FEC level to network conditions have been proposed [13] [14] [15] [16] to reduce the bandwidth and computation overhead of FEC methods.

Our analysis proves that the worst-case QoS routing metric and algorithm can be applied to a wide class of continuous AWGN channels, which are very versatile and flexible. Many other (QoS) network optimization problems can be transformed into this problem. The algorithm is particularly useful for minimizing the worst-case occurence probability of the "extreme" events that are non-typical, but nevertheless highly catastrophic.

In the next section, we provide a problem formulation. Our notation generalized the previous results which are specific to q-ary channels. In the same section, we define the constrained AWGN channel. In the third section, we prove that the constrained AWGN channel is compatible with the GDA. Finally, we describe the optimal worst-case QoS routing algorithm before concluding with discussions.

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II. Formulation and Notation

We model the network as a digraph $G = (V, E)$, where the node, edge and path sets of G are denoted by V, E, and $\mathbf{\Pi}$. The nodes s and $d \in V$ are the source and destination nodes, and $\Pi \subset \mathbf{\Pi}$ is the set of all paths from s to d.

A path $\pi \in \mathbf{\Pi}$ whose nodes $V_\pi \subset V$ are connected by $E_\pi \subset E$ is denoted by either $\langle v_0, \ldots, v_J \rangle$, $\langle e_1, \ldots, e_J \rangle$, or $\langle v_0, e_1, \ldots, e_J, v_J \rangle$. The number of nodes (or edges) in π is denoted by $|\pi|_v$ (or $|\pi|_e$). The symbol $\langle v_i, v_{i+1} \rangle$ denotes the edge (path) connecting the two (non-) adjacent nodes v_i and v_{i+1}. A *partial* path π_j of π denotes $\langle v_0, \ldots, v_j \rangle$, with $0 < j \leq J$, and a *truncated* path $\bar{\pi}_j$ is $\langle v_0, e_1, \ldots, v_{j-1}, e_j \rangle$.

Denote the message by $B \in \mathcal{B}$, where $\mathcal{B}$ is the space of all allowable messages in the network. Let B_i denote the value of B as it departs from v_i; and let $\bar{B}_i$ denote the value of B as it leaves e_i. Both v_i and e_i are parts of $\langle v_0, e_1, \ldots, e_J, v_J \rangle$, along which B evolves as follows:

$$B_0 \xrightarrow{e_1} \bar{B}_1 \xrightarrow{v_1} B_1 \xrightarrow{e_2} \bar{B}_2 \xrightarrow{v_2} \cdots \xrightarrow{e_J} \bar{B}_J \xrightarrow{v_J} B_J.$$

Here v_i and e_i can be regarded as operators $\boldsymbol{v}_i, \boldsymbol{e}_i \in \mathcal{E} : \mathcal{B} \to \mathcal{B}$ given by $B_i = \boldsymbol{v}_i(\bar{B}_i)$ and $\bar{B}_{i+1} = \boldsymbol{e}_i(B_i)$. For π, the evolution operator is $\boldsymbol{\pi} = \boldsymbol{v}_J \circ \boldsymbol{e}_J \circ \cdots \circ \boldsymbol{e}_1 \circ \boldsymbol{v}_0$, for π_j, it is $\boldsymbol{\pi}_j = \boldsymbol{v}_j \circ \boldsymbol{e}_j \circ \cdots \circ \boldsymbol{e}_1 \circ \boldsymbol{v}_0$, and for $\bar{\pi}_j$, it is $\bar{\boldsymbol{\pi}}_j = \boldsymbol{e}_j \circ \boldsymbol{v}_{j-1} \cdots \circ \boldsymbol{e}_1 \circ \boldsymbol{v}_0$. Thus, $\boldsymbol{\pi}_j(B_0) = B_j$ and $\bar{\boldsymbol{\pi}}_j(B_0) = \bar{B}_j$.

Define $X : \mathcal{B} \times \mathcal{B} \to \mathcal{M}$ as a function measuring the Hamming distance $x \in \mathcal{M}$ between a message B_0 at v_0 which evolves into B_J at v_J; where v_0 and v_J are connected by $\langle v_0, v_J \rangle$, and $\mathcal{M}$ is the metric space. The distance x from B_i to $B_{i'}$ is denoted by $x = X_{i,i'} = X(B_i, B_{i'})$. Let $X_{i,\bar{i}'}$ denote $X(B_i, \bar{B}_{i'})$. If $e = \langle v_i, v_{i'} \rangle$ then we define the shorthand $X(e) = X(B_i, \bar{B}_{i'})$.

Besides B_0, the messages B_i at v_i are random variables. Therefore, $X_{0,i}$ are also random variables. Consider $x = X_{i,i'}$, the random variables measuring the distance between the message at v_i and its image at $v_{i'}$. Define $P(x, \boldsymbol{\lambda})$ as the probability density of x parameterized by a vector $\boldsymbol{\lambda} \in \Lambda$. For example, for $x \in \mathbb{R}$ and P Gaussian, $\boldsymbol{\lambda}$ is the vector of P's mean and variance (μ, σ^2). The value $\boldsymbol{\lambda} = \infty$ denotes the absence of connection between two nodes. Each edge e_i in the network can be characterized by its parameter $\boldsymbol{\lambda}_i$.

Suppose the nodes $U \subseteq V$ can reset $x(\pi_j)$ of any path π_j that passes through $v_j \in U$, as long as $x(\pi_j) < x_{max}$.

$$\begin{aligned} X_{0,j} &= 0 && , v_j \in U \text{ and } X_{0,\bar{j}} \leq x_{max} \\ &\geq X_{0,\bar{j}} && , v_j \in U \text{ and } X_{0,\bar{j}} > x_{max} \\ &= X_{0,\bar{j}} && , v_j \in V \backslash U \end{aligned} \tag{1}$$

We also assume that $d \in U$. For a path π, if $V_\pi \cap U \neq \varnothing$ then $X_{0,j}$ is a non-decreasing function of j.

Finally, we define the *worst-case function* $\bar{x}(\boldsymbol{\lambda}_i, \epsilon)$ that computes the worst-case "possible" value of x, where "possible" values y are defined as those y with $P(y, \boldsymbol{\lambda}_i) > \epsilon$.

$$\bar{x}(\boldsymbol{\lambda}_i, \epsilon) = \max_x \{ x \mid P(x, \boldsymbol{\lambda}_i) \geq \epsilon \, , \, x \in \mathcal{M} \} \tag{2}$$

In general, $\bar{x}(\cdot)$ is a functional accepting $P(x)$. However, here we focus on probability densities with well-defined parameters.

Consider a path $\pi = \langle e_1, \ldots, e_J \rangle \in \mathbf{\Pi}$ and its partial path $\pi_j = \langle e_1, \ldots, e_j \rangle$ with $1 \leq j \leq J$. For convenience, we also define the function $\beta : \mathbf{\Pi} \to \Lambda$ that maps a path π (or an edge e_i) into a density parameter $\boldsymbol{\lambda}_\pi$ (or $\boldsymbol{\lambda}_i$) and the function $\omega : \mathbf{\Pi} \to \mathcal{M}$ that maps a path or an edge (given ϵ) into its worst-case value $x \in \mathcal{M}$.

For e_i, the β and ω are related to $\bar{x}(\boldsymbol{\lambda}_i, \epsilon)$ through: $\omega(e_i) = \bar{x}(\beta(e_i), \epsilon)$. For π, assuming $\boldsymbol{\lambda}_\pi$ is defined, similarly we have $\omega(\pi) = \bar{x}(\beta(\pi), \epsilon)$. The next question is, how does $\boldsymbol{\lambda}_\pi$ depend on $\boldsymbol{\lambda}_i$'s, and how does $\bar{x}(\boldsymbol{\lambda}_\pi, \epsilon)$ depend on $\bar{x}_i = \bar{x}(\boldsymbol{\lambda}_i, \epsilon)$?

The quantities x, $\boldsymbol{\lambda}$, or $\bar{x}$ all have the potential to be used as the routing metric. However, in general, $x_\pi \neq \sum x_i$, $\boldsymbol{\lambda}_\pi \neq \sum \boldsymbol{\lambda}_i$, and $\bar{x}_\pi \neq \sum \bar{x}_i$, where $\sum$ is the standard scalar or vector summation. Let us assume that the addition operation is defined in Λ and $\mathcal{M}$ and is denoted by $\oplus$. If $x_1 = X(e_1)$, $x_2 = X(e_2)$, $\boldsymbol{\lambda}_1 = \beta(e_1)$, $\boldsymbol{\lambda}_2 = \beta(e_2)$, $\bar{x}_1 = \omega(e_1)$, $\bar{x}_2 = \omega(e_2)$, and $\pi = \langle e_1, e_2 \rangle$, then we say $x_\pi = x_1 \oplus x_2$, $\boldsymbol{\lambda}_\pi = \boldsymbol{\lambda}_1 \oplus \boldsymbol{\lambda}_2$, or $\bar{x}_\pi = \bar{x}_1 \oplus \bar{x}_2$. To be useful in a shortest-path algorithm, $\oplus$, Λ and $\mathcal{M}$ have to obey certain algebraic properties.

In the next section, we outline these algebraic properties and how these properties can be used to test whether a certain routing metric is "compatible" with the GDA, in a sense that the GDA will not produce a loop.

With $\oplus$, we can now define x_π, $\boldsymbol{\lambda}_\pi$ and $\bar{x}_\pi$ in terms of x_i, $\boldsymbol{\lambda}_i$ and $\bar{x}_i$ using a generalized summation: $x_\pi = \bigoplus x_i$, $\boldsymbol{\lambda}_\pi = \bigoplus \boldsymbol{\lambda}_i$ and $\bar{x}_\pi = \bigoplus \bar{x}_i$. The pairings of Λ and $\mathcal{M}$ with $\oplus$ form algebraic structures which we call the **X**, **B**, and **W** algebras, from the X, β, and ω functions, respectively.

Between two nodes, the optimal path π^* is the path with the "shortest" path length from s to d when measured in the **X**, **B** or **W** algebra (or metric). However, having $\oplus$, X, β, and ω is not enough to calculate π^*. We need to compare path lengths. Therefore we need a total order $\preceq$ on Λ and $\mathcal{M}$ to evaluate expressions like $x_\pi \preceq x_{\pi'}$, $\boldsymbol{\lambda}_\pi \preceq \boldsymbol{\lambda}_{\pi'}$, or $\bar{x}_\pi \preceq \bar{x}_{\pi'}$. Once $\preceq$ is defined, then we can define these optimal values for G:

$$\begin{aligned} x^* &= \min_\pi \{ x_\pi \mid \pi \in \Pi \} \\ \boldsymbol{\lambda}^* &= \min_\pi \{ \boldsymbol{\lambda}_\pi \mid \pi \in \Pi \} \\ \bar{x}^* &= \min_\pi \{ \bar{x}_\pi = \bar{x}(\boldsymbol{\lambda}_\pi, \epsilon) \mid \pi \in \Pi \} \end{aligned} \tag{3}$$

Preferrably, we would like to have the minima $\boldsymbol{\lambda}^*$ and $\bar{x}^*$ related by the expression: $\bar{x}^* = \bar{x}(\boldsymbol{\lambda}^*, \epsilon)$. Indeed, this is true not only for the q-ary symmetric channel and q-ary erasure channel [1], but also for the constrained AWGN channel we present here.

Later in this paper, we prove that if $V_\pi \cap U = \varnothing$, then the lengths x_{π_j}, $\boldsymbol{\lambda}_{\pi_j}$ and $\bar{x}_{\pi_j}$ are non-decreasing functions of j. This means that on any path, the metric $X_{0,j}$ increases as j increases. In this case, reliable communication (in the worst-case sense) becomes very difficult to achieve except when the values $\boldsymbol{\lambda}_i$ are *simultaneously* favorable (which could represent a very small delay or error probability).

Often, this stringent level of QoS could be prohibitively difficult to achieve, forcing the engineers to settle for an unacceptably high catastrophic event probability. However, with U, we later show that with our algorithm, even a *zero* ϵ-worst-case path is well within reach.

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Non-negative-mean AWGN

In this example, the message is a scalar $B \in \mathbb{R}^+$ with non-negative-mean AWGN on each link. This message could represent the amount of non-negative degradation a packet has experienced so far. Therefore, the source always transmits $B_0 = 0$. At each node $v_i \in V_\pi$, two decisions are made: if $x_i = B_i - B_0$ exceeds x_{max}, then the message (or the packet corresponding to the message) is discarded, otherwise it is retransmitted. Let us also assume that the nodes $u_i \in U \subseteq V$ can reset B (and x) back to 0.

The AWGN is characterized by $\boldsymbol{\lambda} = (\mu \geq 0, \sigma^2)$, where μ is the mean, and σ^2 the variance of the Gaussian density and the transition probability density is given by:

$$\begin{aligned} P(\, B_{i+1} \,|\, B_i \,) &= P(x; \mu, \sigma^2) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \end{aligned} \tag{4}$$

where $x = B_{i+1} - B_i$. The $\mathbf{B}$ algebra is defined as follows. Two *independent*, adjacent edges e_1 and e_2 with parameters $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ can be treated as a single edge with parameter $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 \oplus \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$, where the + sign is the standard vector summation operator. Since x is now continuous, we can solve for $\bar{x}$ analytically by solving

$$\bar{x} = P^{-1}(\epsilon; \mu, \sigma^2) = \mu + \sqrt{-\sigma^2 \ln(2\pi\epsilon^2\sigma^2)}, \tag{5}$$

which has one solution if $2\pi\sigma^2 = \epsilon^{-2}$, two solutions $\bar{x}^-$ and $\bar{x}^+$ (with $\bar{x}^- < \bar{x}^+$) if $2\pi\sigma^2 < \epsilon^{-2}$, and no solution if $2\pi\sigma^2 > \epsilon^{-2}$. Since $\bar{x}$ represents the worst-case value for x, we always assume $\bar{x} = \bar{x}^+$. From this result, we can also compute:

$$\bar{x}_1 \oplus \bar{x}_2 = P^{-1}(\epsilon; \mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). \tag{6}$$

It is useful to think of $\boldsymbol{\lambda}$'s as 2D vectors in a half strip $\Lambda = \mathbb{R}^+ \times \{\sigma^2 \leq \epsilon^{-2}/(2\pi)\}$, and the function $\bar{x}$ as an element in $\bar{\mathcal{X}}_\epsilon : \Lambda \to \mathbb{R}$, that defines an isocontours for each value of $\bar{x}$.

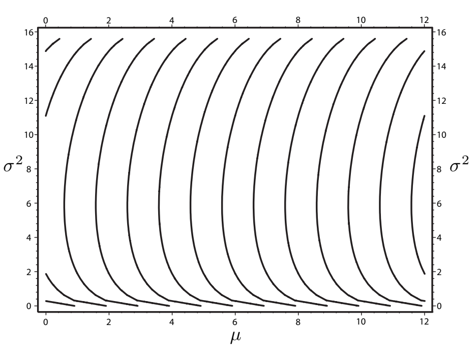

Fig. 1: Isocontour profile of $\bar{x}$ on σ^2 versus μ.

For example, in Figure 1, the contour plot for $\epsilon = 0.1$ is shown. The contour increases from left to right, and at $\sigma^2 = 0$, the value $\bar{x}$ reaches its limit of μ. If we denote G's maximum path variance by σ^2_{max}, then we must have $\sigma^2 \leq \sigma^2_{max} = \epsilon^{-2}_{max}/(2\pi)$ so that for all $\boldsymbol{\lambda} \in \Lambda$, the value $\bar{x}$ exists. For now, we claim that σ^2_{max} is computed with the *longest* path algorithm on G with edge metrics σ_i^2 and usual addition. Later, we show that finding σ^2_{max} is crucial to guarantee closure on Λ.

III. Generalized Dijkstra's Algorithm

Having defined the channel model for each communication link, we now introduce the Generalized Dijkstra's Algorithm (GDA) below [2]. The algorithm can operate on any metric of choice as long as the metric obeys a set of algebraic properties. Specifically, in this paper we would like to show that the $\boldsymbol{\lambda}$ and $\bar{x}$ metrics of the AWGN channel defined in the previous section is compatible with the GDA.

1: **procedure** GDA $(G, \boldsymbol{m}, s)$
2:   **for all** $v \in V$ **do**
3:     $l[v] \leftarrow \infty$
4:     $\pi[v] \leftarrow$ NIL
5:   $Q \leftarrow V$
6:   $l[s] \leftarrow 0$
7:   **while** $Q \neq \varnothing$ **do**
8:     $u \leftarrow$ MIN(Q); $Q \leftarrow Q \backslash u$
9:     **for all** node $v \in N(u)$ **do**
10:       **if** $l[v] \succ l\,[u] \oplus \boldsymbol{m}\,(u,v)$ **then**
11:         $l[v] \leftarrow l\,[u] \oplus \boldsymbol{m}\,(u,v)$
12:         $\pi[v] \leftarrow u$

Upon examination, the GDA is practically identical to the Dijkstra's Algorithm (DA) except for the relaxation step, where the $\oplus$ and $\preceq$ operators act on a generalized metric space $\mathcal{M}$ (instead of the equivalent step in DA, where $+$ and $\leq$ operators act on $\mathbb{R}$). We shall shortly define $\preceq$ for our channel.

On line 9, $N(u)$ denotes the set of all nodes adjacent to u. The argument $\boldsymbol{m}$ is the lengths of the edges in G each of which is an element in $\mathcal{M}$, and $\boldsymbol{m}\,(u,v)$ is the length of $\langle u, v \rangle$. Lines 10–12 perform the relaxation step of the GDA. This step depends on the definitions of $\mathcal{M}$, $\oplus$, and $\preceq$. If the GDA (in)correctly returns the path in G with minimum length measured in $\mathcal{M}$, then $(\mathcal{M}, \oplus)$ and $\preceq$ are said to be *(in)compatible* with the GDA. The following is the required properties for compatibility:

Proposition 1: An algebra $\mathbf{A} = (\mathcal{M}, \oplus)$ and a total order $\preceq$ is compatible with the GDA if and only if it satisfies *all* the properties in the set denoted by $\mathbf{P}$ below:

P1 is a commutative *monoid*, that is, for $a, b, c \in \mathcal{M}$:

- $\mathcal{M}$ is *closed* under $\oplus$: $a \oplus b \in \mathcal{M}$;
- $\oplus$ is *associative* : $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
- 0 is the *identity* : $a \oplus 0 = 0 \oplus a = a$;
- $\oplus$ is *commutative* : $a \oplus b = b \oplus a$.

P2 There exists $\infty \in \mathcal{M} \mid a \oplus \infty = \infty \oplus a = \infty$.

P3 $\preceq$ is a *total order* on $\mathcal{M}$, i.e., $\preceq$ is :

- *reflexive*: $a \preceq a$;
- *anti-symmetric*: if $a \preceq b$ and $b \preceq a$ then $a = b$;
- *transitive*: if $a \preceq b$ and $b \preceq c$ then $a \preceq c$;
- *total*: for every $a, b \in \mathcal{M}$ either $a \preceq b$ or $b \preceq a$.

P4 There exists the least element 0 that satisfies $0 \preceq a$.

P5 $a \oplus c \prec b \oplus c$ if $a \prec b$ and $c \in \mathcal{M} - \{\infty\}$.

Proof: Refer to [2] for a complete proof. □

Next, we prove that the $\mathbf{B}$ algebra we defined for the constrained AWGN channel is compatible with the GDA.

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The proof requires us to define and use $\preceq$ to compare any $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda$. Since $\Lambda \subset \mathbb{R}^{2+}$ and $\mathbb{R}^{2+}$ is not a totally ordered set, we must define $\preceq$ in terms of $\bar{x}(\boldsymbol{\lambda}, \epsilon)$ as follows:

$$\boldsymbol{\lambda} \preceq \boldsymbol{\lambda}' \Leftrightarrow \bar{x}(\boldsymbol{\lambda}, \epsilon) < \bar{x}(\boldsymbol{\lambda}', \epsilon) \text{ or } \bar{x}(\boldsymbol{\lambda}, \epsilon) = \bar{x}(\boldsymbol{\lambda}', \epsilon) \text{ and } \sigma^2 \leq \sigma'^2 \tag{7}$$

Graphically, this means that the $\bar{x}$ isocontours defined on Λ determines whether $\boldsymbol{\lambda} \prec \boldsymbol{\lambda}'$. If they lie on the same contour line, then the variances break the tie. We also define Λ as follows:

$$\Lambda = \left(\mathbb{R}^+ \times [0, \sigma^2_{max}] \cap \left\{ \boldsymbol{\lambda} : \frac{\mu}{\sigma^2} \geq \frac{\partial \mu}{\partial \sigma^2}\bigg|_{\sigma^2_{max}} \right\} \right) \cup \{\infty\}$$

Λ is depicted by the shaded region in figure 2. For a fixed value of σ^2, along any $\bar{x}$ contour, the slope $\mu' = \partial\mu/\partial\sigma^2$ is equal for all μ and is maximized at σ^2_{max}. If $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$ have slopes larger than μ', their sum lies in a higher contour.

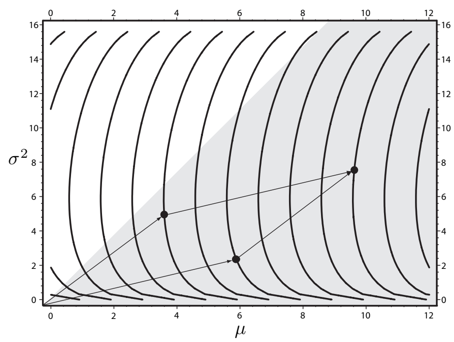

Fig. 2: The metric space Λ (shaded).

Theorem 2: The algebras $\mathbf{B} = (\Lambda, \oplus)$ defined for the AWGN channel and its total order $\preceq$ satisfy all the properties in $\mathbf{P}$, and thus compatible with the GDA.

PROOF: **P1** Except for closure, the other monoid properties are obvious because $\oplus$ is the standard vector addition. Because Λ does not occupy the full non-negative octant, but rather bounded by $\sigma^2 \leq \sigma^2_{max} = \epsilon^{-2}_{max}/(2\pi)$, in general closure is not guaranteed. However, if we can compute:

$$\sigma^2_{max} = \max_{\pi \in \Pi} \left\{ \sum\nolimits_{i:v_i \in V_\pi} \sigma^2_i \right\} \tag{8}$$

then closure is guaranteed. This value of σ^2_{max} can be obtained by running the DA once on G, costing $O(V^2)$. In case one (or both) of the operands is ∞, then as in q-SC and q-EC, by definition, the $\oplus$ sum is also $\infty \in \Lambda$.

P2 The proof is derived from closure on ∞.

P3 The proof follows from the definition of Λ and $\preceq$.

P4 Both terms in equation (5) are minimized when they are zero, i.e., $\mu = 0$ and either $\sigma^2 = 0$ or $\sigma^2 = \epsilon^{-2}/(2\pi)$. However, only $\boldsymbol{\lambda} = (0, 0) \in \Lambda$, and is thus the 0 element.

P5 Obvious from figure 2 and the definition of Λ. □

From the definition of $\boldsymbol{\lambda} \preceq \boldsymbol{\lambda}'$, it follows that a path's $\bar{x}$ value is a non-decreasing function of its $\boldsymbol{\lambda}$ values, which means that the path with minimum $\boldsymbol{\lambda}$ is also the path with minimum $\bar{x}$.

IV. OPTIMAL WORST-CASE QOS ROUTING

The preceding assertion – that $\bar{x}$ is a non-decreasing function of $\boldsymbol{\lambda}$ – allows us to construct the optimal worst-case QoS routing algorithm. The path algebra for $\boldsymbol{\lambda}$'s is relatively simple. In the case of our constrained AWGN channel, since the edges are independent, the $\oplus$ operation is simply a vector addition. In contrast, the algebra for $\bar{x}$ depends on computing $\boldsymbol{\lambda}$ first. First, we argue for the need of an optimal worst-case QoS routing algorithm that routes information through error correction nodes by proving our earlier claim:

Proposition 3: If $V_{\pi_J} \cap U = \varnothing$, then $\beta(\pi_j)$ is an non-decreasing function of j. The minimum $\beta(\pi_j) = 0$ is only possible if the edge lengths $p_j = 0$ for all $j = 0 \ldots J$.

PROOF: **P5** with $0 = a \prec b = p_j$, and $c = p_{\pi_{j-1}}$ gives us $\beta(\pi_{j-1}) = p_{\pi_{j-1}} \prec p_{\pi_{j-1}} \oplus p_j = \beta(\pi_j)$, proving that $\beta(\pi_j)$ is a non-decreasing function of j. The second part of the proof can be derived directly from **P1**. □

The preceding proposition shows the practical benefit of including error correcting nodes in the network. If the path includes $u \in U$, then for some $j \in [0, J]$, we can have $\beta(\pi_j) = 0$ even with $p_j \neq 0$ for all j. This feature could be added into most existing shortest-path based routing protocols.

Suppose these protocols compute the path based on $\bar{x}$. A path $\phi = \langle v_1, v_2 \rangle$ is feasible iff $\omega(\langle s, v_1 \rangle) \oplus \omega(\phi) \leq x_{max}$ — the value of $\bar{x}$ up to v_1 added to the $\bar{x}$ of ϕ must be less than x_{max}. Denote by $\Phi \subseteq \mathbf{\Pi}$ the feasible paths in $\mathbf{\Pi}$, and by $\Phi(v_1, v_2)$ the feasible paths between v_1, v_2.

Theorem 4: A path π^* is the path with minimum $\omega(\pi)$ iff it solves the Shortest Path Problem (SPP) given by $G' = (V', E')$, where $V' = \{s\} \cup U$.

An edge connecting two nodes $v_1, v_2 \in V'$ is the shortest path in $\Phi(v_1, v_2)$, i.e.,

$$E' = \left\{ \operatorname{argmin}_\phi \{ \omega(\phi) \mid \phi \in \Phi(v_1, v_2) \} \,\middle|\, v_1, v_2 \in V' \right\}$$

PROOF: Suppose π^* contains $n+1$ segments ϕ_i connecting the nodes in $V'' = \{s, U_{\pi^*}, d\}$, where $U_{\pi^*} = U \cap V_{\pi^*}$. In segment notation, π^* is denoted by $s \rightsquigarrow u_1 \rightsquigarrow \cdots \rightsquigarrow u_j \rightsquigarrow d$, with $\{u_i\} = U_{\pi^*}$, and $0 \leq j \leq |U|$.

Then ϕ_i must be the shortest feasible paths between adjacent nodes in V'', and U_{π^*} must be the set that minimizes $\sum \beta(\phi_i)$. Otherwise, a better path ξ^* can be obtained by modifying ϕ_i or U_{π^*}, contradicting the claim that π^* is optimal.

For the forward proof, note that $V'' = \{s, d, U_{\pi^*}\} \subseteq V'$. Further, since each ϕ_i is a shortest path between nodes in V', then it has a representation in E', i.e., $\phi_i \in E'$. Therefore π^* is the solution to the SPP given by $G' = (V', E')$.

For the reverse proof, suppose ξ^* is the SPP solution but is not the minimum $\bar{x}$ path π^*. From the forward proof, if π^* minimizes $\bar{x}$, then it also solves the SPP, thus $\omega(\pi^*) \leq \omega(\xi^*)$.

However, if $\omega(\pi^*) < \omega(\xi^*)$, then ξ^* is not the SPP solution – a contradiction. Thus, $\omega(\pi^*) = \omega(\xi^*)$, and if path lengths are unique, $\pi^* = \xi^*$. Hence, the SPP solution π^* is also the minimum $\bar{x}$ path. □

Theorem 4 essentially proves the correctness of the optimal worst-case QoS routing algorithm listed below. Although in this paper $\bar{x}$ is linked to the worst-case value of x for the constrained AWGN channel, other worst-case metrics can be used.

1: **procedure** MIN-$\bar{x}$-PATH $(G\,, \boldsymbol{m}\,, s)$
2: $\quad E \setminus= \{\ e \in E \mid \omega(e) > x_{max}\ \}$
3: $\quad V \setminus= \{\ v \in V \mid \deg(v) = 0\ \}$
4: $\quad$ **for** $v_1 \in V'$ **do**
5: $\quad\quad \mathrm{SP}_1 = \mathrm{GDA}(\ G\,, \boldsymbol{m}\,, v_1\ )$
6: $\quad\quad$ **for** $v_2 \in V' \neq v_1$ **do**
7: $\quad\quad\quad E' = E' \cup \langle v_1, v_2 \rangle$
8: $\quad E' \setminus= \{\ e' = (v_1, v_2) \in E' \mid e' \notin \Phi(v_1, v_2)\ \}$
9: $\quad V' \setminus= \{\ v' \in V' \mid deg(v') = 0\ \}$
10: $\quad \mathrm{SP}_2 = \mathrm{GDA}(\ G'\,, \boldsymbol{m}\,, s\ )$

On lines 2 and 3, the algorithm prunes all infeasible edges. Then, on line 5, it runs the GDA on all $v_1 \in V'$, every time producing a shortest path tree SP_1 rooted at v_1. On line 7, the edges connecting v_1 and $v_2 \in V'$ are added into E' based on SP_1. This finishes the first stage and starts the second stage. On lines 8 and 9, the infeasible edges in E' are pruned, and any isolated nodes in V' are removed. On line 10, π^* is finally calculated using GDA. □

Let us denote $|V'|$ by α. The first stage produces a complete graph with α nodes and $\alpha(\alpha - 1)$ directed edges by executing the GDA α times on line 5, and thus has a time complexity of $O(\alpha V^2)$ (if the GDA is implemented using Fibonacci heap, then its complexity could reach $O(V \log V + E)$ [18]).

Lines 8 and 9 search linearly over them with $O(\alpha^2)$ time complexity. The GDA on line 10 has a time complexity $O(\alpha^2)$. Hence, overall time complexity is $O(\alpha V^2 + \alpha^2)$, or conservatively, $O(V^3)$.

V. Conclusion and Discussion

In this paper, we extended the optimal worst-case QoS routing algorithm and metric definition given in [1]. We proved that in addition to the q-ary symmetric and q-ary erasure channel model, the necessary and sufficient conditions defined in [2] for the Generalized Dijkstra's Algorithm can be used with a constrained non-negative-mean AWGN channel.

The generalization allowed the computation of the worst-case QoS metric value for a given edge weight density. We showed that this worst-case value can then also be used as the routing metric. This also allowed an exact analysis for the case where some network nodes have error correcting capabilities. The result is an optimal worst-case QoS routing algorithm that uses the Generalized Dijkstra's Algorithm as a subroutine with a polynomial time complexity of $O(V^3)$.

Future work includes supplementing the theoretical results with simulation and experimental verification. We believe there are still many important probability densities that are compatible with this routing framework. Specific implementation of this algorithm into existing and future protocols is also of interest. Finally, the parameter ϵ can be used to evaluate the worst-case performance of real life communication networks.

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