ACTA ACADEMIAE SCIENTIARUM TAURINENSIS

Colloque

LA «MÉCANIQUE ANALYTIQUE» DE LAGRANGE ET SON HÉRITAGE

I

Fondation Hugot du Collège de France

Paris, 27-29 Septembre 1988

Supplemento al numero 124 (1990) degli
Atti della Accademia delle Scienze di Torino
Classe di Scienze Fisiche, Matematiche e Naturali

TORINO - 1990

The Energy-Momentum Method

Jerrold E. MARSDEN and Juan C. SIMO

Abstract - *This paper develops the energy momentum method for studying stability and bifurcation of Lagrangian and Hamiltonian systems with symmetry. The method was specifically designed to deal with the stability of rotating structures. The relation with the energy-Casimir method is given and the energy-momentum method is shown to be more general. Stability of rigid body motion is given to illustrate the method. Some discussion of its applicability to general rotating systems and block diagonalization is also given.*

1. Introduction

Lagrange devoted a good deal of attention in Volume 2 of *Mécanique Analytique* to the study of rotational motion of mechanical systems. In fact, in equation A on page 212 he gives the reduced Lie-Poisson equations for $\mathcal{SO}(3)$ for a rather general Lagrangian. His derivation is just how we would do it today-by reduction from material to spatial representation.

In this paper we develop a natural augmentation to the basic work of Lagrange, Riemann, Poincaré and Cartan, which concerns the stability and bifurcation of rotating mechanical systems, be they elastic, coupled rigid bodies, or rotating fluid masses. There have been many developments in stability theory in mechanics, and we will follow the line of Lagrange, Dirichlet et al. by using *energy methods*. These methods have been used extensively in fluid and plasma dynamics under names like the «δW method», «Arnold's method» or the «energy-Casimir method». We shall develop the *energy-momentum method* which includes all of the above. Because some systems such as three-dimensional Euler flow and geometrically exact rod models have a dearth of Casimir functions, which limits the applicability of the energy-Casimir method, and since this is not a limitation in the energy-momentum method, the latter is more general. For an account of the energy-Casimir method and references up to 1985 we refer to Holm et al. [1985]. Some of the ideas in the energy momentum method are already implicit in the work of Holm and Abarbanel (cf. Holm [1986]) in terms of what they call C-frames and of Morrison [1987] in his work on zero modes and stability for the Vlasov-Poisson equation.

The development of the energy-momentum method is motivated by recent work of Simo, Posburgh and Marsden [1988] on non-linear stability of rotating geometrically exact structures and of Lewis and Simo [1989] on rotating pseudo-rigid bodies. That work in fact goes much further by providing a block diagonal structure for the second variation of the augmented Hamiltonian H_ξ defined below. We shall, however, leave these developments for other publications. For an abstract version of those results, see Marsden, Simo, Lewis, and Posbergh [1989].

Acknowledgements. Thanks are due to Bob Grossman, Tim Healey, P.S. Krishnaprasad, Debra Lewis, Jiang-Hua Lu, George Patrick, Tom Posbergh and Tudor Ratiu for their helpful comments and interaction.

2. Symplectic Actions of Lie Groups and Momentum Maps

We develop the abstract energy-momentum method in the context of the reduction theory of Marsden & Weinstein [1974] (see Abraham & Marsden [1978] or Arnold [1978] for expositions). Although the conditions for the stability of the rigid body are classical, it will be worthwhile illustrating the general theory in detail for this case, since many of the ideas and calculations are similar for the context of general rotating structures.

First, we recall a few notions from reduction theory that we shall need in the developments that follow. We refer to Abraham & Marsden [1978] for further details and elaboration of notation not explained here. Let (P, Ω) be a symplectic manifold, possibly infinite dimensional. For us, the case of the cotangent bundle $T^*Q = P$ with the canonical symplectic structure will be used; in fact T^*Q will be connected to TQ via the Legendre transformation in the classical context of mechanics and one can equally well use TQ with the Lagrange symplectic form if desired.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ which acts by canonical transformations on P (i.e., we have a symplectic action). Given any $\xi \in \mathfrak{g}$ we denote by

$$(1) \qquad \xi_P(z) := \frac{d}{dt} \exp(t\xi) \cdot z \big|_{t=0},$$

a vector field on P, the *infinitesimal generator* of the G-action corresponding to ξ. Here $g \cdot z$ denotes the group action. In addition, G acts on g through the *adjoint action*, $Ad: G \times \mathfrak{g} \to \mathfrak{g}$, defined as

$$(2) \qquad Ad_g \xi = T_e(R_{g^{-1}} \circ L_g)\, \xi, \quad \xi \in \mathfrak{g},$$

where L_g and R_g denote left and right translations by $g \in G$, respectively. The infinitesimal generator of the adjoint action, denoted by $\xi_{\mathfrak{g}}$, is the special case of (1) which is given by

$$(3) \qquad \xi_{\mathfrak{g}}(\eta) := \frac{d}{dt} Ad_{\exp(t\xi)}(\eta)\big|_{t=0} = [\xi, v] =: ad_\xi v, \quad v \in \mathfrak{g},$$

where $[\cdot,\cdot]$ denotes the Lie bracket in g. The tangent space to the orbit $O_\xi := \{Ad_g \xi \mid g \in G\}$ at the point $\eta \in O_\xi$ is given by

$$(4) \qquad T_\eta O_\xi = \{\xi_{\mathfrak{g}}(\eta) \mid \xi \in \mathfrak{g}\} \ .$$

The group G also acts on $\mathfrak{g}^*$, the dual of the Lie algebra g, through to *co-adjoint action*, $Ad^*: G \times \mathfrak{g}^* \to \mathfrak{g}^*$, as $\mu \mapsto Ad^*_{g^{-1}}$, for $\mu \in \mathfrak{g}^*$ and $g \in G$. Here Ad^*_g is the transpose of the map Ad_g defined by the relation

$$(5) \qquad \langle Ad^*_g(\mu), \xi \rangle = \langle \mu, Ad_g(\xi) \rangle, \quad \xi \in \mathfrak{g} \ ,$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing on $\mathfrak{g}^* \times \mathfrak{g}$. The corresponding infinitesimal generator, denoted by $\xi_{\mathfrak{g}^*}$, is given at $\mu \in \mathfrak{g}^*$ by

$$(6) \qquad \xi_{\mathfrak{g}^*}(\mu) := \frac{d}{dt} Ad^*_{\exp(-t\xi)}(\mu)\big|_{t=0} = -\langle \mu, [\xi, \cdot] \rangle = -ad^*_\xi \mu .$$

The tangent space to the *co-adjoint orbit* of $\mu \in \mathfrak{g}^*$, which is defined as

$$(7a) \qquad O_\mu := \{Ad^*_{g^{-1}}(\mu) \mid g \in G\},$$

is given by the counterpart of (4); namely,

$$(7b) \qquad T_\nu O_\mu = \{\xi_{\mathfrak{g}^*}(\nu) \mid \xi \in \mathfrak{g}\}, \quad \nu \in O_\mu,$$

where $\xi_{\mathfrak{g}^*}(\nu) = -(\xi_{\mathfrak{g}})^*(\mu)$ is given by

$$\langle \xi_{\mathfrak{g}^*}(\nu), \eta \rangle = \langle \nu, [\eta, \xi] \rangle .$$

With this notation at hand, we let

(8) $\quad \mathbf{J}: P \to \mathfrak{g}^*$

be an *equivariant momentum map* for the action of G on P; that is,

i) $\mathbf{J}$ is *equivariant* relative to the action of G on P and the induced co-adjoint action on $\mathfrak{g}^*$ in the sense that

(9a) $\quad \mathbf{J}(g\cdot z) = Ad^*_{g^{-1}}(\mathbf{J}(z))$,

for all $g \in G$ and $z \in P$.

ii) The infinitesimal generator of the G-action defined by (1) is a Hamiltonian vector field generated by the function $J(\xi): P \to \mathbf{R}$ defined in terms of the momentum map by

(10) $\quad J(\xi)(z) = \langle \mathbf{J}(z), \xi \rangle, \quad$ for all $\xi \in \mathfrak{g}$.

Therefore,

(11a) $\quad \mathbf{d}J(\xi)(z)\cdot \delta z = \Omega(\xi_p(z), \delta z)$

for all $\delta z \in T_z P$. Equivalently, condition (11a) is expressed as

(11b) $\quad X_{J(\xi)} = \xi_P$,

where X_f denotes the Hamiltonian vector field associated with the function $f: P \to \mathbf{R}$.

We recall that equivariance in (9a) implies the *classical commutation relations* for the Poisson bracket:

(9b) $\quad \{J(\xi), J(\eta)\} = J([\xi, \eta])$.

If $P = T^*Q$ and G acts on Q, then there is an induced G-action on P as follows: let $\Psi_g: Q \to Q$ denote the action on Q. Define the action Γ on P by letting $\Phi_g: T^*_q Q \to T^*_{g\cdot q} Q$ be defined by

$$\langle \Phi_g(\alpha_q), v_{g\cdot q}\rangle = \langle \alpha_q, T\Psi_g^{-1}\cdot v_{g\cdot q}\rangle \ .$$

This action Φ induced on P is called the *cotangent lift*. Cotangent lifts are the usual way one gets actions induced on T^*Q and they have explicit momentum maps according to the following:

2.1 Proposition. *Let G be a lie group, Q a configuration manifold and $\Psi: G\times Q\to Q$ an action of G on Q. Then, the lifted action on the phase space $P=T^*Q$ is symplectic had has an (Ad^*-equivariant) momentum mapping given by*

$$(12)\qquad \mathbf{J}: P\to \mathfrak{g}^*;\quad J(\xi)(\alpha_q)=\langle \alpha_q, \xi_Q(q)\rangle ,$$

where $\xi_Q(q):=\frac{d}{dt}\Big|_{t=0}\Psi(\exp(t\xi),q)$ *is the infinitesimal generator of the action Ψ on Q, and $J(\xi): P\to\mathbf{R}$ is related to $\mathbf{J}$ by $J(\xi)(\alpha_q)=\langle \mathbf{J}(\alpha_q), \xi\rangle$, as above.*

Proof - See Abrahm and Marsden [1978, p. 283], Corollary 4.2.11. ■.

Let $z\in P$, $\mu=\mathbf{J}(z)\in\mathfrak{g}^*$, and denote by

$$(13)\qquad G_\mu:=\{g\in G\,|\,Ad^*_{g^{-1}}(\mu)=\mu\}\subset G$$

the *isotropy group* of G under the co-adjoint action. The *reduced phase space* (or symplectic quotient) is given by the quotient manifold

$$(14)\qquad P_\mu:=\mathbf{J}^{-1}(\mu)/G_\mu .$$

P_μ is indeed a smooth manifold provided that $\mu\in\mathfrak{g}^*$ is a (clean or) regular value of $\mathbf{J}$ and G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$.

The following result of Marsden & Weinstein [1974] (see also Abraham & Marsden [1978, p. 299] or Arnold [1978, p. 376]) plays a central role in the development of the energy-momentum method.

2.2 Proposition. *Let $z\in\mathbf{J}^{-1}(\mu)$. Further, let $G\cdot z$ and $G_\mu\cdot z$ denote the orbits of z under the actions of G and G_μ, respectively; i.e.,*

$$(15a)\qquad G\cdot z:=\{g\cdot z\,|\,g\in G\},\quad \text{and}\quad G_\mu\cdot z:=\{g\cdot z\,|\,g\in G_\mu\} .$$

Then, the following relation between tangent spaces holds:

$$(15b)\qquad T_z(G_\mu\cdot z)=T_z(G\cdot z)\cap T_z\mathbf{J}^{-1}(\mu).$$

Moreover, $T_z(\mathbf{J}^{-1}(\mu))$ is the Ω-orthogonal complement of $T_z(G\cdot z)$ in T_zP; that is,

$$(15c)\qquad \Omega(\xi_P(z),\delta z)=0,\ \ \textit{for all } \xi\in\mathfrak{g}\ \textit{iff}\ \delta z\in T_z(\mathbf{J}^{-1}(\mu)).$$

Proof - See Abraham & Marsden [1978, p. 299]. ∎

The tangent space, $T_{[z]}P$, to the reduced phase space P_μ is isomorphic to the quotient space:

$$(16) \qquad T_{[z]}P_\mu \cong T_z\mathbf{J}^{-1}(\mu)/T_z(G_\mu\cdot z) \ ,$$

where $[z]=\pi_\mu(z)$ and $\pi_\mu:\mathbf{J}^{-1}(\mu)\to\mathbf{J}^{-1}(\mu)/G_\mu$ is the natural projection. Condition (15c) follows from the definition (11a) of momentum map. Let $i_\mu:\mathbf{J}^{-1}(\mu)\to P$ denote the inclusion.

2.3 Reduction Theorem. *There is a unique symplectique structure Ω_μ on P_μ such that*

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega \ .$$

Proof - See Abraham & Marsden [1978, p. 300]. ∎

Consider the dynamics of a Hamiltonian system with a given G-*invariant* Hamiltonian function $H:P\to\mathbf{R}$. The momentum map $\mathbf{J}:P\to\mathfrak{g}^*$ is conserved for the dynamics of X_H; i.e., the flow F_t of X_H leaves the set $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the action of G_μ on $\mathbf{J}^{-1}(\mu)$. As a results of the G-invariance property of H, it follows that the flow F_t of X_H induces canonically a Hamiltonian flow on the reduced phase space $P_\mu=\mathbf{J}^{-1}(\mu)/G_\mu$, with associated Hamiltonian function $H_\mu:P_\mu\to\mathbf{R}$ defined through the equation $H_\mu\circ\pi_\mu=H\circ i_\mu$ and referrred to as the *reduced Hamiltonian.*

3. Relative Equilibria and the Energy-Momentum Method

Following Poincaré's terminology, a point $z_e\in P$ is called a *relative equilibrium* if the trajectory for Hamilton's equations $\dot{z}=X_H(z)$ through z_e is given by

$$(1) \qquad z(t)=\exp(t\xi)\cdot z_e, \quad \text{for some } \xi\in\mathfrak{g}$$

i.e., *a dynamic orbit equals a group orbit.* Letting $\mu_e=\mathbf{J}(z_e)$, we see that (1) implies $\xi\in\mathfrak{g}_{\mu_e}$ by conservation of $\mathbf{J}$ and Proposition 2.2. For example, if $G=\mathbb{SO}(3)$, the special orthogonal group, the *a relative equili-*

brium is a uniformly rotating solution of Hamilton's equations. Of course there are many classical examples of such solutions such as uniformly rotating rigid bodies, Lagrange's triangular solutions in the three body problem, etc.

In addition to (1), two equivalent characterizations of a relative equilibrium are possible:

i) First, by differentiating (1) with respect to t, evaluating at $t=0$, and using Hamilton's equations, one finds

$$(2) \qquad \dot{z}(t)\Big|_{t=0} = X_H(z_e) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi)\cdot z_e \ .$$

Making use of definition (1) in § 2 we find that *$z_e \in P$ is a relative equilibrium if and only if there is a Lie algebra element $\xi \in \mathfrak{g}$ such that*

$$(3) \qquad X_H(z_e) = \xi_P(z_e) \ .$$

ii) Alternatively, *a point $z_e \in P$ is a relative equilibrium if and only if it is a critical point of $H|_{\mathbf{J}^{-1}(\mu_e)}$*; i.e.,

$$(4) \qquad \mathbf{d}H(z_e)\cdot\delta z = 0 \quad \textit{for } \delta z \in \ker[T_{z_e}\mathbf{J}(z_e)].$$

This is equivalent to $\pi_\mu(z_e)$ being a critical point of H_μ by G-invariance of H.

Instead of characterizing relative equilibria as critical points of $H(z)$ subject to the constraint $z \in \mathbf{J}^{-1}(\mu_e)$, it proves more convenient to remove the restriction that $\delta z \in T_{z_e}P$ lie in the tangent space to the constraint set by introducing Lagrange multipliers. In this context, the following result is basic for our subsequent developments.

3.1 Relative Equilibrium Theorem. *A point $z_e \in P$ is a relative equilibrium if and only if there exists a $\xi \in \mathfrak{g}$ such that z_e is a critical point of*

$$(5) \qquad \boxed{H_\xi := H - [J(\xi) - \langle \xi, \mu_e \rangle] = H - \langle \mathbf{J} - \mu_e, \xi \rangle}$$

In (5), $\xi \in \mathfrak{g}$ plays the role of a Lagrange multiplier. The optimality conditions associated with (5) provide a variational characterization of the relative equilibria $z_e \in P_\mu$ and the corresponding multiplier $\xi \in \mathfrak{g}$ as a critical points of H_ξ. For convenience of the reader, we include the proof (See Abraham and Marsden [1978] and Marsden, Simo, Lewis, and Posburgh [1989] for additional conditions).

Proof of the Relative Equilibrium Theorem. First assume that z_e is a relative equilibrium. Then (3) and the definition of the momentum map gives

$$X_H(z_e) - X_{J(\xi)}(z_e) = 0 \tag{6}$$

which, since P is symplectic, is equivalent to z_e being a critical point of $H - J(\xi)$, which is the same as being a critical point of H_ξ. (If P were a Poisson manifold, one would have to add a Casimir to $H - J(\xi)$ at this point and one would be dealing with the *energy momentum Casimir method*).

Conversely, assume z_e is a critical point of H_ξ; i.e., z_e is a stationary point of the dynamical system with Hamiltonian $H - J(\xi)$. Thus z_e is a stationary point of the dynamical system $X_{H-J(\xi)}$. Since H and $J(\xi)$ commute, so do the flows of their Hamiltonian vector fields and so the flow of $X_{H-J(\xi)}$ is $\Phi_{\exp(-t\xi)} \circ F_t$ where F_t is the flow of X_H. Thus

$$\Phi_{\exp(-t\xi)} \circ F_t(z_e) = z_e \quad \text{which gives} \quad z(t) = F_t(z_e) = \exp(t\xi) \cdot z_e$$

which means z_e is a relative equilibrium. ■

4. The Energy-Momentum Method

Theorem **3.1** characterizes the relative equilibria as the critical points of a constrained variational principle, namely, as *the extremals of the Hamiltonian subject to the constraint of constant momentum map.* In this context, the energy-momentum functional $H_\xi := H - \langle \mathbf{J} - \mu_e, \xi \rangle$ is to be optimized and $\xi \in \mathfrak{g}$ is the Lagrange multiplier. The standard criteria for *formal stability* would require that $z_e \in P$ be a *constrained* local minima of the *reduced Hamiltonian*. Note, however, that this condition would place additional unnecessary restrictions on the standard test for positive definiteness of the second variation $\delta^2(H_\xi(z_e)$ on the tangent space, $\ker[T_{z_e}\mathbf{J}(z_e)]$, to the level set $\mathbf{J}^{-1}(\mu_e)$ of the constraint at z_e. In fact there are neutral directions due to the symmetry that must be taken into account. We also caution the reader that we shall be assuming for simplicity that μ is a regular value of **J**, that G_μ acts freely and properly on P and that μ is a generic element of g*; i.e. that μ is on a regular coadjoint orbit. (These points are discussed in Weinstein [1984]; we thank P.S. Krishnaprasad and T. Ratiu for pointing out that without these conditions one can run into trouble with the equivalence of the reduced and unreduces definitions of stability — these singular cases require further work).

The following elementary *gauge invariance condition* will be helpful.

4.1 Proposition. *Let $z_e \in P$ be a relative equilibrium, and let $G\cdot z_e = \{g\cdot z_e | g\in G\}$ be the orbit through z_e with tangent space*

(1) $$T_{z_e}(G\cdot z_e) = \{\eta_P(z_e) | \eta\in \mathfrak{g}\} \ .$$

The, for any $\delta z \in T_{z_e}[\mathbf{J}^{-1}(\mu_e)]$, we have

(2) $$\delta^2 H_\xi(z_e)\cdot(\eta_P(z_e), \delta z) = 0 \quad \textit{for all } \eta\in\mathfrak{g} \ .$$

Proof - Since $H: P\to \mathbf{R}$ is G-invariant, the Ad^*-equivariance condition (9) of § 2 yields

(3) $$\begin{aligned} H_\xi(g\cdot z) &= H(g\cdot z) - \langle \mathbf{J}(g\cdot z), \xi\rangle + \langle \mu_e, \xi\rangle \\ &= H(z) - \langle Ad^*_{g^{-1}}(\mathbf{J}(z)), \xi\rangle + \langle \mu_e, \xi\rangle \\ &= H(z) - \langle \mathbf{J}(z), Ad_{g^{-1}}(\xi)\rangle + \langle \mu_e, \xi\rangle \ , \end{aligned}$$

for any $g\in G$ and $z\in P$. Choosing $g = \exp(t\eta)$ with $\eta\in\mathfrak{g}$, differentiating with respect to t and using (1) and (3) of § 2 we obtain

(4) $$\mathbf{d}H_\xi(z)\cdot\eta_P(z) = -\left\langle \mathbf{J}(z), \frac{d}{dt}\bigg|_{t=0} Ad_{\exp(-t\eta)}(\xi)\right\rangle = \langle \mathbf{J}(z), [\eta, \xi]\rangle.$$

Taking variations relative to $z\in P$ in (4), evaluating at z_e and using the fact that $\mathbf{d}H_\xi(z_e) = 0$, one gets the expression

(5) $$\delta^2 H_\xi(z_e)(\eta_P(z_e), \delta z) = \langle T_{z_e}\mathbf{J}(z_e)\cdot\delta z, [\eta, \xi]\rangle \ ,$$

which vanishes if $T_{z_e}\mathbf{J}(z_e)\cdot\delta z = 0$, i.e., if $\delta z\in \ker[T_{z_e}\mathbf{J}(z_e)] = T_{z_e}\mathbf{J}^{-1}(\mu_e)$. ∎

In particular, from the above result and Proposition **2.1** we have

4.2 Corollary. *$\delta^2 H_\xi(z_e)$ vanishes identically on* $\ker[T_{z_e}\mathbf{J}(z_e)]$ *along the directions tangent to the orbit $G_{\mu_e}\cdot z_e$; that is*

(6) $$\delta^2 H_\xi(z_e)\cdot(\eta_P(z_e), \zeta_P(z_e))=0 \quad \textit{for any } \eta, \zeta\in \mathfrak{g}_{\mu_e}.$$

Proof - By proposition **2.1**, $T_{z_e}(G_{\mu_e}\cdot z_e)=T_{z_e}(G\cdot z_e)\cap\ker[T_{z_e}\mathbf{J}(z_e)]$. Since $T_{z_e}(G_{\mu_e}\cdot z_e)\subset T_{z_e}(G\cdot z_e)$ the result follows from (2) by taking $\delta z=\xi_P(z_e)$ with $\xi\in\mathfrak{g}_{\mu_e}$. ■

From this corollary we conclude taht *formal stability* of a relative equilibrium requires *positives definiteness* of the second variation $\delta^2 H_\xi(z_e)$ on $T_{z_e}\mathbf{J}^{-1}(\mu)$ modulo the gauge directions $T_{z_e}(G_{\mu_e}\cdot z_e)=\{\eta_P(z_e)\,|\,\eta\in\mathfrak{g}_{\mu_e}\}$ which by (16) of § 2, coincides with the tangent space to the reduced phase space. To summarize

Formal stability of $z_e\in P$ is equivalent to $\delta^2 H_\xi(z_e)\cdot(v, v)>0$ for $v\in T_{z_e}\mathbf{J}^{-1}(\mu_e)/T_{z_e}(G_{\mu_e}\cdot z_e)$.

Here the quotient space is identified with some subspace

(7) $$\mathcal{S}\equiv T_{z_e}\mathbf{J}^{-1}(\mu_e)/T_{z_e}(G_{\mu_e}\cdot z_e)=T_{[z_e]}P_{\mu_e},$$

transverse to the orbit $G_{\mu_e}\cdot z_e$ in $T_{z_e}\mathbf{J}^{-1}(\mu_e)=\ker[T_{z_e}\mathbf{J}]$. The definition of $\mathcal{S}$ requires the enforcement of two restrictions on variations $\delta z\in T_{z_e}P$:

i) $\delta z\in\mathcal{S}$ is such that $T_{z_e}\mathbf{J}\cdot\delta z=0$, and

ii) Elements δz in $\mathcal{S}$ are taken modulo the gauge directions: $T_{z_e}(G_{\mu_e}\cdot z_e):=\{\eta_P(z_e)\,|\,\eta\in\mathfrak{g}_{\mu_e}\}$, where $\mu_e=\mathbf{J}(z_e)$, G_μ denotes the isotropy subgroup of $\mu\in\mathfrak{g}^*$ (relative to the co-adjoint action) and $\mathfrak{g}_\mu$ is its Lie algebra.

The fact that the definiteness of the second variation is to be examined restricted to the quotient space $\mathcal{S}$ is an important aspect of the energy-momentum method which is justified by the standard test for constrained optimization problems along with Corollary **4.2**. For convenience, a step-by-step procedure outlining the energy-momentum method is contained in the table below. We emphasize that the type of stability one gets in P_μ is *Liapunov stability*, while in P it is *orbital stability* of the relative equilibrium orbit $\exp(t\xi)\cdot z_e$.

We conclude this section with a few general remarks. First, as noted above, since the original Hamiltonian H is G-invariant, it induces a Hamiltonian H_μ on each reduced phase space P_μ. The reduction theorem shows that the dynamics of X_H projects to that of X_{H_μ}. In addition, the point $[z_e]=\pi_{\mu_e}(z_e)$ in P_{μ_e} which is the orbit of z_e is indeed a fixed point of H_{μ_e}.

Conditions i and ii also show that the second variation of H_ξ at z_e induces on the quotient space (14) of § 2 the second variation $\delta^2 H_{\mu_e}(z_e)$ of the reduced Hamiltonian H_{μ_e}.

It can be much easier to calculate $\delta^2 H_\xi$ than $\delta^2 H_{\mu_e}$ since computations are carried out with unconstrained variations. This is an essential advantage of the energy-momentum method. This is also one reason the energy-Casimir method is useful (see the remarks below). The formal reason that the energy-momentum method produces a stability criterion is simply the fact that conditon 3 and 4 in the table below insure stability on the reduced space, which corresponds to stability modulo the group action on the original space. The other basic advantage of the energy momentum method is the block diagonalization work of Marsden, Simo, Posberg, and Lewis, already noted.

We also note that in many examples (like the nonlinear stability of vortex patches, as in Wan & Pulvirente [1984]), one needs to be careful about what type of stability is concluded. For the applications to geometrically exact rod models with quadratic constitutive relations, these delicate functional analytic difficulties do not cause problems.

The Energy-Momentum Method

- Typical *set-up* in Mechanics

$Q,\ P=T^*Q$	Configuration manifold and phase space
$H:P\to\mathbf{R}$	Hamiltonian
$G,\ \mathfrak{g}$	Symmetry group and Lie algebra
$\Psi:G\times Q\to Q$	Symplectic action of G on Q
$\xi_Q(q):=\left.\dfrac{d}{dt}\right\vert_{t=0}\Psi(\exp(t\xi),q)$	Infinitesimal generator of Ψ

- *Computation of relative equilibria* $z_e \in P$, and test for *formal stability* involves the following steps:

  1 Momentum map - Compute $J(\xi): P \to \mathbf{R}$ associated with $\xi \in \mathfrak{g}$. Typically, on T^*Q, use

  $$J(\xi)(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle, \quad \alpha_q \in P.$$

  2 First variation - Construct $H_\xi = H - [J(\xi) - \langle \mu_e, \xi \rangle]$ and find $z_e \in P$ and $\xi \in \mathfrak{g}$ such that

  $$\mathbf{d} H_\xi(z_e) \cdot \delta z = 0, \quad \text{and} \quad \mathbf{J}(z_e) - \mu_e = 0 \ ,$$

  for all $\delta z \in T_{z_e} P$ (no restrictions placed on δz at this stage).

  3 Admissible variations for second variant test - Choose a linear subspace $\mathcal{S} \subset T_{z_e} P$ such that

  i) $T_{z_e} \mathbf{J} \cdot \delta z = 0$ for all $\delta z \in \mathcal{S}$.

  ii) $\mathcal{S}$ complements $T_{z_e}(G_{\mu_e} \cdot z_e)$ in $T_{z_e} \mathbf{J}^{-1}(\mu_e) = \ker [T_{z_e} \mathbf{J}]$; i.e., every variation $\delta z \in T_{z_e} P$ satisfying **i** is uniquely written as

  $$\delta z = \nu + \chi_P(z_e), \quad (\chi_P(z_e) \text{ is tangent to the orbit})$$

  for some $\nu \in \mathcal{S}$ and $\chi \in \mathfrak{g}_{\mu_e}$ (so that $\chi_P(z_e) \in T_{z_e}(G_{\mu_e} \cdot z_e)$).

  4 Test the second variation $\delta^2 H_\xi$ for definiteness on $\mathcal{S}$; i.e.,

  $$\delta^2 H_\xi(z_e) \cdot (\nu, \nu) > 0 \ ,$$

  for all $\nu \in \mathcal{S}$. Definiteness means ***formal stability*** of $z_e \in P$.

---

5. Relationship with the Energy-Casimir Method

The energy-Casimir method is concerned with Poisson reduction, rather than symplectic reduction, which was used above. The Poisson reduced space is simply

$$P_G := P/G \ , \tag{1}$$

assuming it is a manilfod, with its inherited Poisson structure: functions on P_G are identified with G-invariant functions on P. If $\mathcal{O}_\mu$ is the coadjoint orbit through μ, then the reduced symplectic manifolds, written as

$$P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu \cong \mathbf{J}^{-1}(\mathcal{O}_\mu)/G \,, \tag{2}$$

are the symplectic leaves of P_G. We saw above that the definiteness of $\delta^2 H_\xi$ on $\mathcal{S} \subset T_{z_e}P$ corresponds to definiteness of $\delta^2 H_\mu$. The function H on P induces a function h on P/G and the restriction of h to P_μ is H_μ.

Now suppose $\Phi: \mathfrak{g}^* \to \mathbf{R}$ is an Ad^*-invariant function, (so is a Casimir on $\mathfrak{g}^*$ in the Lie-Poisson bracket structure on $\mathfrak{g}^*$). Further, let

$$\Phi \circ \mathbf{J} : P \to \mathbf{R} \tag{3}$$

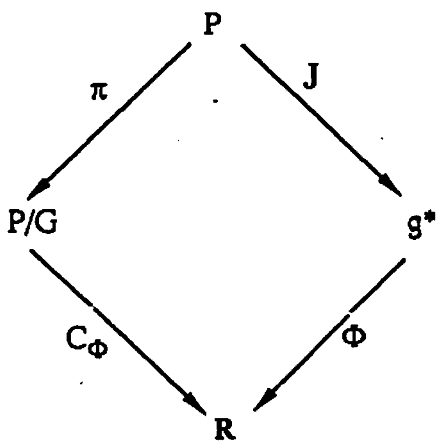
be a «collective» Hamiltonian on P. This function is G-invariant since $\Phi(\mathbf{J}(g \cdot z)) = \Phi(Ad^*_{g^{-1}}(\mathbf{J}(z)) = \Phi(\mathbf{J}(z))$. Consequently, it defines a function

$$C_\Phi: P_G \to \mathbf{R}. \tag{4}$$

One checks that C_Φ is in fact a Casimir on P/G in the sense that

$$\{C_\Phi, F\} = 0 \,, \tag{5}$$

for any $F: P_G \to \mathbf{R}$. This is illustrated in the following commutative diagram:



Assume that $z_e \in P$ is a relative equilibrium, so that there is an associated multiplier $\xi \in g$, as above. Furthermore, assume that there is at least one function $\Phi_e : g^* \to \mathbf{R}$ satysfying

$$(6) \qquad \left. \frac{\delta \Phi_e}{\delta \mu} \right|_e = \xi \, .$$

Then, the functional

$$h + C_{\Phi_e} : P/G \to \mathbf{R}$$

has a critical point at $[z_e] = \pi(z_e)$ where $\pi : P \to P/G$ is the projection. The *energy-Casimir method* is essentially a test for definiteness of the second variation

$$(7) \qquad \delta^2 (h + C_{\Phi_e}) |_e \, ,$$

on the tangent space $T_{[z_e]}P/G$ to orbit $[z_e] \in P/G$; see Holm et al. [1985]. Normally one chooses a Φ_e satisfying (6) to optimize the definiteness of (7).

For any Φ_e satisfying (6), the restriction of the second variation (7) to $T_{[z_e]}P_{\mu_e}$, the tangent space to the symplectic leaves in P/G is equal to the second variation of H_{μ_e} at $[z_e]$. This is simply because $H_{\mu_e} = h | P_\mu$, H_{μ_e} has a critical point at $[z_e]$, and C_{Φ_e} is constant on P_{μ_e}.

Thus, assuming (6) can be satisfied, the second variation (7) restricted to the reduced space, and the quadratic form induced by $\delta^2 H_\xi$ both coincide with $\delta^2 H_{\mu_e}([z_e])$. Therefore, if the energy-Casimir method works, i.e., the form (7) is definite, then so is $\delta^2 H_\xi(z)$ (restricted to $\mathfrak{S}$); i.e., the energy-momentum method works.

On the other hand, there are situations, such as those concerned with geometrically exact rod models, where the energy-momentum method can be applied successfully, but there appears to be no function Φ satisfying (6) and so the energy-Casimir method fails — see Simo, Posburgh, and Marsden [1988]. This also appears to be the essence of the results of Abarbanel & Holm (cf. Holm [1986]) and Morrison [1987].

Of course one can synthesize the energy-momentum and energy-Casimir methods; this is suitable when a group commuting with G is present. This results in the *energy-momentum-Casimir method*. it is implicitly used in Holm et al. [1985] for examples like the symmetric heavy top.

As in the energy-Casimir method, for some problems in the infinite dimensional case, if one wants to deduce dynamical stability, convexity estimates for H_ξ on $\mathcal{S}$ are required. The situation is analogous to that in Holm et al. [1985].

6. Example: The Rigid Body

We illustrate how to use the energy-momentum method by considering the dynamics of a freely spinning rigid body. Of course we will recover the classic results that uniform rotation about longest and shortest principal axes are stable motions. The energy-momentum method is also used in more sophisticated examples of rotating structures, as in Simo, Posbergh and Marsden [1988].

The rotation group $\mathcal{S}O(3)$ consists of all orthogonal linear transformations of Euclidean three space to itself which have determinant one. Its Lie algebra, denoted $so(3)$, consists of all 3×3 skew matrices, which we identify with $\mathbf{R}^3$ *by the isomorphism* $\hat{}:\mathbf{R}\to so(3)$ defined by

$$(1)\qquad \Omega\mapsto\hat{\Omega}=\begin{bmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega^1 \\ -\Omega^2 & \Omega^1 & 0 \end{bmatrix}$$

where $\Omega=(\Omega^1,\Omega^2,\Omega^3)$. One checks that for any vectors $\mathbf{r}$ and Θ,

$$(2a)\qquad \hat{\Omega}\mathbf{r}=\Omega\times\mathbf{r},$$

and

$$(2b)\qquad \hat{\Omega}\hat{\Theta}-\hat{\Theta}\hat{\Omega}=(\Omega\times\Theta)\hat{}\,.$$

Equations (1) and (2a, b) give the usual identification of the Lie algebra $so(3)$ with $\mathbf{R}^3$ and the Lie algebra bracket with the cross product of vectors. Moreover, if $\Lambda\in\mathcal{S}O(3)$ and $\Omega\in so(3)$, the *adjoint action* defined in (2) of § 2 is given by

$$(3)\qquad [\Lambda\Theta]\hat{}=Ad_\Lambda\hat{\Theta}:=\Lambda\hat{\Theta}\Lambda^T\,.$$

Since the adjoint action is a Lie algebra homomorphism, (3) gives the elementary identity

(4) $\Lambda(\mathbf{r}\times\mathbf{s})=\Lambda\mathbf{r}\times\Lambda\mathbf{s}$,

for all $\mathbf{r},\mathbf{s}\in\mathbf{R}^3$.

Given $\Lambda\in\mathcal{SO}(3)$, let $\hat{\mathbf{v}}_\Lambda$ denote an element of the tangent space to $\mathcal{SO}(3)$ at Λ. Since $\mathcal{SO}(3)$ is a submanifold of $\mathcal{GL}(3)$, the general linear group, we can identify $\hat{\mathbf{v}}_\Lambda$ with a 3×3 matrix, which we denote with the same letter. Linearizing the defining (submersive) condition $\Lambda\Lambda^T=1$ gives

(5) $\Lambda\hat{\mathbf{v}}_\Lambda^T+\hat{\mathbf{v}}_\Lambda\Lambda^T=0$,

which defines $T_\Lambda\mathcal{SO}(3)$. We can identify $T_\Lambda\mathcal{SO}(3)$ with so(3) by two isomorphisms:

i) **Left translations** - Given $\hat{\Theta}\in so(3)$ and $\Lambda\in\mathcal{SO}(3)$ we define $(\Lambda,\hat{\Theta})\mapsto\hat{\Theta}_\Lambda\in T_\Lambda\mathcal{SO}(3)$ by setting

(6a) $\hat{\Theta}_\Lambda := T_eL_\Lambda\cdot\hat{\Theta}\cong(\Lambda,\Lambda\hat{\Theta})$.

Thus $\hat{\Theta}_\Lambda$ is the left invariant extension of $\hat{\Theta}$.

ii) **Right translations** - Given $\hat{\theta}\in so(3)$ and $\Lambda\in\mathcal{SO}(3)$ we define $(\Lambda,\hat{\theta})\mapsto\hat{\theta}_\Lambda\in T_\Lambda\mathcal{SO}(3)$ through right translations by setting

(6b) $\hat{\theta}_\Lambda := T_eR_\Lambda\cdot\hat{\theta}\cong(\Lambda,\hat{\theta}\Lambda)$.

Thus $\hat{\theta}_\Lambda$ is the right invariant extension of $\hat{\theta}$.

As in Simo, Marsden & Krishnaprasad [1988], the notation is dictated by continuum mechanics considerations; uppercase letters are used for the body (or convective) variables and lower case for the spatial (or Eulerian) variables. Often, the base point is omitted and with an abuse of notation we write $\Lambda\hat{\Theta}$ and $\hat{\theta}\Lambda$ for $\hat{\Theta}_\Lambda$ and $\hat{\theta}_\Lambda$, respectively.

The dual space to so(3) is identified with $\mathbf{R}^3$ by using the *standard dot product*:

(7) $\Pi\cdot\Theta=\frac{1}{2}tr[\hat{\Pi}^T\hat{\Theta}]$.

This extends to the *left-invariant pairing* on $T_\Lambda \mathcal{S}O(3)$ given by

(8) $$\langle \hat{\Pi}_\Lambda, \hat{\Theta}_\Lambda \rangle = \frac{1}{2} tr[\hat{\Pi}_\Lambda^T \hat{\Theta}_\Lambda] = \frac{1}{2} tr[\hat{\Pi}^T \hat{\Theta}] = \Pi \cdot \Theta \ .$$

We shall, thereby, write elements of $so(3)^*$ as $\hat{\Pi}$, where $\Pi \in \mathbf{R}^3$, (or $\hat{\pi}$ with $\pi \in \mathbf{R}^3$) and elements of $T_\Lambda^* \mathcal{S}O(3)$ as

(9) $$\hat{\Pi}_\Lambda = (\Lambda, \Lambda\hat{\Pi}) \ ,$$

for the body representation, and for the spatial representation

(10) $$\hat{\pi}_\Lambda = (\Lambda, \hat{\pi}\Lambda) \ .$$

Again, explicit indication of the base point will often be omitted and we shall simply write $\Lambda\hat{\Pi}$ and $\hat{\pi}\Lambda$ for $\hat{\Pi}_\Lambda$ and $\hat{\pi}_\Lambda$, respectively. If (9) and (10) represent the same covector, then

(11) $$\hat{\pi} = \Lambda\hat{\Pi}\Lambda^T,$$

which coincides with the co-adjoint action. Equivalently, using the isomorphism (2) we have

(12) $$\pi = \Lambda\Pi \ .$$

The mechanical set-up for rigid body dynamics is as follows: the configuration manifold Q and the phase space P are

(13) $$Q = \mathcal{S}O(3); \quad P = T^*\mathcal{S}O(3)$$ with the canonical symplectic structure

i) The *Hamiltonian H* is the kinetic energy of a free rigid body. One shows in standard fashion (see for instance Marsden, Ratiu & Weinstein [1984]) that

(14) $$H = \frac{1}{2}\pi \cdot \mathbb{I}^{-1}\pi; \quad \mathbb{I} := \Lambda \mathbb{J} \Lambda^T,$$

where $\mathbb{I}$ is the *time dependent inertia tensor* (in spatial coordinates) and $\mathbb{J}$ is the ***constant inertia dyadic*** given by

$$(15)\qquad \mathbb{J} = \int_{\mathcal{B}} \varrho_{ref}(X)\,[\,\|X\|^2\,\mathbf{1} - X \otimes X\,]\,d^3X\ .$$

Here, $\mathcal{B} \subset \mathbf{R}^3$ is the ***reference configuration*** of the rigid body and $\varrho_{ref}:\mathcal{B}\,\mathbf{R}$ the ***reference density***. We regard H in (14) as a function $H: \mathbb{SO}(3) \times \mathrm{so}^*(3) \to \mathbf{R}$ where $\mathrm{so}(3) \cong \mathbf{R}^{3*}$. This is essentially equivalent to regarding H as a function on P because of the isomorphism

$$(16)\qquad (\Lambda,\ \pi) \in \mathbb{SO}(3) = \mathbf{R}^{3*} \mapsto (\Lambda,\ \hat{\pi}\Lambda) \cong \hat{\pi}_\Lambda \in T^*_\Lambda\mathbb{SO}(3).$$

However, the former view, i.e., $H(\Lambda,\ \pi)$, is computationally more convenient.

ii) **Invariance Properties** - Making use of (12), H in (14) can be written in the ***convective representation*** (***body coordinates***) as

$$(17)\qquad H = \frac{1}{2}\Pi \cdot \mathbb{J}^{-1}\Pi$$

which reflects the (manifest) *left invariance* of H under the action of $\mathbb{SO}(3)$. Thus *left reduction by* $\mathbb{SO}(3)$ to *body coordinates* induces a function on the quotient space $T^*\mathbb{SO}(3)/\mathbb{SO}(3) \cong \mathrm{so}^*(3)$. The symplectic leaves are spheres, $|\Pi| = \text{constant}$. The induced function h on these spheres is given by (17) regarded as a function of Π. The dynamics on this sphere is given by the usual picture obtained by intersection of the sphere $|\Pi|^2 = \text{constant}$ and the ellipsoid $H = \text{constant}$.

iii) **Momentum map** - Consistent with the preceding discussion, we choose $G = \mathbb{SO}(3)$ acting from the left on $Q = \mathbb{SO}(3)$ by left translation, i.e.,

$$(18)\qquad \Psi(Q,\ \Lambda) = L_Q\Lambda = Q\Lambda\ ,$$

for all $\Lambda \in \mathbb{SO}(3)$ and $Q \in G \equiv \mathbb{SO}(3)$. Hence, the action of $G = \mathbb{SO}(3)$ on $P = T^*\mathbb{SO}(3)$ is by *cotangent lift* of *left translations*. Since the infinitesimal generator associated with $\xi \in \mathrm{so}(3)$ is obtained as

$$(19)\qquad \xi_{\mathrm{so}(3)}(\Lambda) = \frac{d}{dt}\exp[t\hat{\xi}]\,\Lambda\Big|_{t=0} = \hat{\xi}\Lambda\ ,$$

by **Proposition 2.1** the momentum map associated with the left $\mathcal{SO}(3)$ action is given by

$$(20)\qquad J(\hat{\xi})(\hat{\pi}_\Lambda) = -\frac{1}{2}\, tr\,[\hat{\pi}_\Lambda^T \xi_{so(3)}] = -\frac{1}{2}\, tr\,[\Lambda^T \hat{\pi}^T \hat{\xi} \Lambda] = -$$

$$-\frac{1}{2}\, tr\,[\hat{\pi}^T \hat{\xi}] = \pi \cdot \xi ,$$

Thus, (20) gives

$$(21)\qquad \mathbf{J}(\hat{\pi}_\Lambda) = \hat{\pi}, \quad \text{or} \quad J(\hat{\xi}) = \pi \cdot \xi .$$

This constitutes our first step in the application of the energy-momentum method in the box. According to the second step of the energy-momentum method, we consider

$$(22)\qquad H_\xi = H - [J(\xi) - \pi_e \cdot \xi] = \frac{1}{2}\pi \cdot \mathrm{II}^{-1}\pi - \xi \cdot (\pi - \pi_e) ,$$

and examine its critical points. To compute the first variation we recall that although $\hat{\pi}_\Lambda \in T^*\mathcal{SO}(3)$, including its base point Λ, are the basic variables, it is more convenient to regard H_ξ as a function of $(\Lambda, \pi) \in \mathcal{SO}(3) \times \mathbf{R}^{3*}$ through the isomorphism (2).

Thus, let $\hat{\pi}_{\Lambda_e} \cong (\Lambda_e, \hat{\pi}_e \Lambda_e) \in T^*\mathcal{SO}(3)$ be a relative equilibrium point. For any $\delta\theta \in \mathbf{R}^3$ we construct the curve

$$(23)\qquad \epsilon \mapsto \Lambda_\epsilon = \exp[\epsilon \delta\hat{\theta}]\, \Lambda_e \in \mathcal{SO}(3) ,$$

which starts at Λ_e since

$$(24)\qquad \Lambda_\epsilon \Big|_{\epsilon=0} = \Lambda_e, \quad \text{and} \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Lambda_\epsilon = \delta\theta\Lambda .$$

Let $\delta\pi \in \mathbf{R}^{3*}$ and consider the curve in $\mathbf{R}^{3*}$ defined as

$$(25)\qquad \epsilon \mapsto \pi_\epsilon = \pi_e + \epsilon\delta\pi \in \mathbf{R}^{3*} ,$$

which starts at π_e. These constructions induce a curve $\epsilon \mapsto \hat{\pi}_{\Lambda_\epsilon} \in T^*SO(3)$ through the isomorphism (3): that is, $\hat{\pi}_{\Lambda_\epsilon} := (\Lambda_\epsilon, \hat{\pi}_\epsilon \Lambda_\epsilon)$. With this notation at hand we proceed to compute the first variation.

i) **First variation** - Using the chain rule, set

(26a) $$\delta H_\xi \Big|_e := \frac{d}{d\epsilon} H_{\xi,\epsilon} \Big|_{\epsilon=0} = 0,$$

where

(26b) $$H_{\xi,\epsilon} := \frac{1}{2} \pi_\epsilon \cdot \mathbb{I}_\epsilon^{-1} \pi_\epsilon - \xi \cdot \pi_\epsilon \quad \text{and} \quad \mathbb{I}_\epsilon^{-1} := \Lambda_\epsilon \mathbb{J}^{-1} \Lambda_\epsilon^T .$$

In addition, at equilibrium we have the additional optimality condition coming from varying the Lagrange multiplier

(27) $$(\pi - \pi_e) \cdot \eta = 0, \quad \text{for all } \eta \in \mathbf{R}^{3*}; \quad \text{i.e., } \pi = \pi_e .$$

To compute (26a) observe that

(28) $$\begin{aligned} \frac{1}{2} \pi_e \cdot \frac{d}{d\epsilon} \mathbb{I}_\epsilon^{-1} \pi_e \Big|_{\epsilon=0} &= \frac{1}{2} \pi_e \cdot [\delta\hat{\theta}\, \mathbb{I}_e^{-1} - \mathbb{I}_e^{-1} \delta\hat{\theta}]\, \pi_e \\ &= \frac{1}{2} [\pi_e \cdot \delta\theta \times \mathbb{I}_e^{-1} \pi_e - \mathbb{I}_e^{-1} \pi_e \cdot \delta\theta \times \pi_e] \\ &= \delta\theta \times (\mathbb{I}_e^{-1} \pi_e \times \pi_e) , \end{aligned}$$

where we have made use of elementary vector product identities. By (28), expression (26) reduces to

(29) $$\delta H_\xi \Big|_e = \delta\pi \cdot [\mathbb{I}_e^{-1} \pi_e - \xi] + \delta\theta \cdot [\mathbb{I}_e^{-1} \pi_e \times \pi_e] = 0.$$

From this relation we obtain the two expected equilibrium conditions:

(30a) $$[\mathbb{I}_e^{-1} \pi_e \times \pi_e] = 0, \quad \text{and} \quad [\mathbb{I}_e^{-1} \pi_e] = \xi ,$$

Equivalently, we have

$$(30b)\qquad \xi\times\pi_e=0, \quad \text{and} \quad \mathbb{I}_e^{-1}\xi=\lambda\xi \ ,$$

where $\lambda>0$ by positive definiteness of $\mathbb{I}_e=\Lambda_e\mathbb{J}\Lambda_e^T$. These conditions constitute the statement that π_e is aligned with a principal axis, and that the rotation must be about this axis. Note that $\pi=\mathbb{I}_e\omega_e$, where $\mathbb{I}_e$ is the spatial inertial dyadic and $\pi_e=\mathbb{I}_e\xi$, so that ξ does correspond to the angular velocity, as it should.

ii) Second variation - From (29) we compute the second variation at equilibrium by again making use of the direction derivative formula and setting

$$(31)\qquad \delta H_\xi\Big|_e := \frac{d}{d\epsilon}[\delta\pi\cdot(\mathbb{I}_\epsilon^{-1}\pi_\epsilon-\xi)\times\delta\theta(\mathbb{I}_\epsilon^{-1}\pi_\epsilon\times\pi_\epsilon)]\Big|_{\epsilon=0} \ .$$

By performing manipulations similar to those leading to (29) and making use of the equilibrium conditions (30) we obtain the following quadratic form at equilibrium

$$(32)\qquad \delta^2H_\xi\Big|_e((\delta\pi,\delta\theta),(\delta\pi,\delta\theta))=$$

$$[\delta\pi^T\delta\theta^T]\begin{bmatrix}\mathbb{I}_e^{-1} & (\mathbb{I}_e^{-1}-\lambda 1)\hat{\pi}_e \\ -\hat{\pi}_e(\mathbb{I}_e^{-1}-\lambda 1) & -\hat{\pi}_e(\mathbb{I}_e^{-1}-\lambda 1)\hat{\pi}_e\end{bmatrix}\begin{bmatrix}\delta\pi \\ \delta\theta\end{bmatrix} \ .$$

Note that the matrix in (32) in 6×6.

Finally we restrict the admissible variations $(\delta\pi,\pi\theta)\in\mathbf{R}^{3*}\times\mathbf{R}^3$ by the conditions in step **3** of the energy-momentum method. By (21), $J(\hat{\pi}_\Lambda)=\hat{\pi}$; hence $\mu_e=\hat{\pi}_e$ and $T_{z_e}(G_{\mu_e}\cdot z_e)=$ infinitesimal rotations about the axis π_e; i.e., multiplies of π_e, or equivalently ξ. Variations that are orthogonal to this space and also in the space $\delta\pi=0$ (which is the condition **i**: $\delta J=\delta\pi=0$) are of the form $\delta\theta$ with $\delta\theta\perp\pi_e$. Thus, we choose

$$(33)\qquad \mathcal{S}=\{(\delta\pi,\delta\theta)\,|\,\delta\pi=0,\ \delta\theta\perp\pi_e\} \ ,$$

which completes step **3** of the energy-momentum method. Note that $\delta\theta$

is a variation that infinitesimally rotates π_e on the sphere

(34) $$\mathcal{O}_{\pi_e} := \{\pi \in \mathbf{R}^3 \mid \|\pi\|^2 = \|\pi_e\|^2\},$$

which is the co-adjoint orbit through π_e. The second variation (32) restricted to the subspace V is given by

(35) $$\left. \delta^2 H_{\xi,\mu} \right|_e = \delta\theta \cdot (\hat{\pi}_e^T (\mathbb{I}_e^{-1} - \lambda 1)\, \hat{\pi}_e)\, \delta\theta$$
$$= (\pi_e \times \delta\theta) \cdot (\mathbb{I}_e^{-1} - \lambda 1)\, (\pi_e \times \delta\theta).$$

If λ is the largest or smallest eigenvalue of $\mathbb{I}$, (35) will be definite; note that the null space of $\mathbb{I}^{-1} - \lambda 1$ in (35) consists of vectors parallel to π_e, *which have been excluded. Also note that in thus example,* $\mathcal{S}$ is a 2-dimensional space and (35) in fact represents a 2×2 matrix.

7. Comments on Block Diagonalization

In the energy-momentum method, for mechanical systems with Hamiltonian H of the form kinetic energy (K) plus potential (V), it is possible to choose variables in a way that makes the determination of stability conditions sharper and more computable. These variables are related to what are known as Eckart frames, but they can also be related to horizontality conditions for an associated connection. In this set of variables (with the conservation of momentum constraint and a gauge symmetry constraint imposed on $\mathcal{S}$) the second variation of $\delta^2 H_\xi$ block diagonalizes; schematically

$$\delta^2 H_\xi = \begin{bmatrix} \begin{bmatrix} 2\times 2 \text{ rigid} \\ \text{body block} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \text{Internal vibration} \\ \text{block} \end{bmatrix} \end{bmatrix}$$

Furthermore, the internal vibrational block takes the form

$$\begin{bmatrix} \text{Internal vibration} \\ \text{block} \end{bmatrix} = \begin{bmatrix} \delta^2 V_\mu & 0 \\ 0 & \delta^2 K_\xi \end{bmatrix}$$

where $V_\mu = V + \mu \, \mathbb{I}(q)_\mu^{-1}$ is the amended potential of Smale, $K_\xi(q, p) = = \frac{1}{2} \| p - A_\xi \|^2$ and $A_\xi(q)$ is the metric flat (or the Legendre transform) of $\xi_Q(q)$. Here $\delta^2 K_\xi > 0$ so formal stability is equivalent to $\delta^2 V_\xi > 0$ which separates out the overall rigid body motions from the internal motions of the system under consideration (for a geometrically exact rod, this includes shear and torsion).

The dynamics of the internal vibrations (such as the elastic wave speeds) depend on the rotational angular velocity. That is, the internal vibrational block is ξ-dependent, but in way we can explicitly calculate. On the other hand, these two types of motions do not *dynamically decouple*, since the symplectic form does *not* block diagonalize. However, we can compute the off-diagonal terms explicitly (they turn out to be momentum maps that play a crucial role in how the block diagonalizing variables are constructed in the first place!) which determines the dynamic coupling. See Marsden, Simo, Lewis, and Posbergh [1989] and Simo, Lewis and Marsden [1990] for further information and references.

REFERENCES

ABRAHAM R. and MARSDEN J. [1978], *Foundation of Mechanics*. Second Edition, Addison-Wesley Publishing Co., Reading, Mass.

ARNOL'D V.I. [1978], *Mathematical methods of classical mechanics*. Graduate Texts in Math. **60**, Springer-Verlag.

HOLM D.D. [1986], Liapunov stability of ideal compressible and incompressible fluid equilibria in three dimensions. *Séminaire de Mathématiques supérieurs*. Les Presses de l'Université de Montréal, **100**, 125-208.

HOLM D.D., MARSDEN J.E., RATIU T. and WEINSTEIN A. [1985], Nonlinear stability of fluid and plasma equilibria. *Physics Reports* **123**, 1-116.

LEWIS D. and SIMO J.C. [1989], Nonlinear stability of pseudo-rigid bodies, *Proc. Roy. Soc. Lon.* (to appear).

MARSDEN J.E., SIMO, LEWIS D.R. and POSBERGH T.A. [1989], A block diagonalization theorem in the energy momentum method *Cont. Math. AMS.* **97**, 297-314.

MARSDEN J.E., RATIU T. and WEINSTEIN A. [1984], Semi-direct products and reduction in mechanics, *Trans. Am. Math. Soc.* **281**, 147-177.

MARSDEN J.E. and WEINSTEIN A. [1983], Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. *Physica* **7D**, 305-323.

MORRISON P.J. [1987], Variational principle and stability of nonmonotone Vlasov-Poisson equilibria. *Z. Naturforsch.* **42a** 1115-1123.

MORRISON P.J. and ELIEZER S. [1986], Spontaneous symmetry breaking and neutral stability on the noncanonical Hamiltonian formalism *Phys. Rev. A* **33**, 4205.

SIMO J.C., LEWIS D.R., MARSDEN J.E. [1990], Stability of Relative Equilibria I: The Reduced Energy-Momentum Method (*Arch. Rat. Mech. An.*, to appear).

SIMO J.C., MARSDEN J.E. and KRISHNAPRASAD [1988], The Hamiltonian structure of nonlinear elasticity: The material, spatial, and convective representations of solids, rods, and plates. *Arch. Rat. Mech. An.* **104**, 125-183.

SIMO J.C., POSBERGH T.A. and MARSDEN J.E. [1988], Nonlinear stability of geometrically exact rods by the energy-momentum method SUDAM Report # 88.1. Division of Applied Mechanics, Stanford University (to appear in Physics Reports).

WAN Y.-H. and PULVIRENTE M. [1984], Nonlinear stability of circular vortex patches. *Commun. Math. Phys.* **99**, 435-450.

WEINSTEIN A. [1984], Stability of Poisson-Hamilton equilibria. *Cont. Math. AMS* **28**, 3-14.

J.E. MARSDEN - Department of Mathematics, University of California, Berkeley, CA 94720.
Research partially supported by DOE contract DE-AT03-88ER-12097 and MSI at Cornell University.

J.C. SIMO - Division of Applied Mechanics, Stanford University, Stanford, CA 94305.
Research partially supported by AFOSR contract 2-DJA-544 and 2-DJA-771.