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GENERALIZED POISSON BRACKETS AND NONLINEAR LIAPUNOV STABILITY – APPLICATION TO REDUCED MHD

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Abstract: A method is presented for obtaining Liapunov functionals (LF) and proving nonlinear stability. The method uses the generalized Poisson bracket (GPB) formulation of Hamiltonian dynamics. As an illustration, certain stationary solutions of ideal reduced MHD (RMHD) are shown to be nonlinearly stable. This includes Grad-Shafranov and Alfven solutions.

1. Introduction

To establish stability, the LF method [1-3] relies on the existence of conserved quantities that are used to bound the growth of perturbations from equilibrium. This method has been used to show linearized stability of plasma and fluid equilibria [4-6]. Here we present an algorithm for proving nonlinear stability based on the LF method using the GPB, or noncanonical Hamiltonian formalism. One finds there are often Casimir functionals that Poisson commute with all functionals and these enable one to obtain variational principles for equilibria of various Hamiltonian theories. These equilibria can then be tested for linear Liapunov stability and, in many fluid and plasma examples [2,3,7-9], nonlinear stability (stability to finite perturbations) has been proven. For RMHD, which is a system used for tokamak modeling [10], we find explicit criteria for nonlinear stability of Grad-Shafranov equilibria and equilibria with poloidal flow, including nonlinear Alfven waves. Elsewhere, stability of more realistic tokamak systems is treated [9].

2. GPB Formalism

The GPB formalism uses a conserved functional H (Hamiltonian) together with a Poisson bracket operator on pairs of functionals to represent field equations in the form

$$\frac{\partial\psi^i}{\partial t}(\vec{x},t) = \{\psi^i, H\} \qquad i = 1,\ldots,N \tag{1}$$

where the ψ^i denote the field components, and the GPB, $\{\ ,\ \}$ is (i) bilinear, (ii) antisymmetric, (iii) satisfies the Jacobi identity, and (iv) is a derivation in each argument. The class of field equations that are representable in this form is enlarged by relaxing the requirement that the bracket be in canonical form. In fact, the general formalism can be used to classify Hamiltonian theories. Systems with the same GPB possess common symmetries and have the same Casimirs. If C is a Casimir, then $\{C,F\} = 0$ for all functionals F; hence, C is a constant of motion. Casimirs are an important ingredient of the LF stability method, to which we now turn.

3. Stability Algorithm

The algorithm has four basic steps (A-D, below) that culminate when the norm, $\|\cdot\|$, in the following definition is produced.

Definition: An equilibrium, $\vec{\psi}_e$, is *Liapunov stable* if for every $\varepsilon > 0$ there is a $\delta > 0$, such that, for each nearby solution $\vec{\psi} = \vec{\psi}_e + \Delta\vec{\psi}$ ($\Delta\vec{\psi}$ a finite perturbation) that has $\|\Delta\vec{\psi}\| < \delta$ initially, then $\|\Delta\vec{\psi}\| < \varepsilon$ for all time (for which the solution exists).

A. Constants of motion. There can be either dynamical constants arising from a geometrical symmetry (e.g. energy or momentum conservation, arising from either time or space translation symmetry), or kinematical constants (Casimirs). This first step can be facilitated by understanding the Hamiltonian structure of the problem. From experience, this is *not* a formidable task, since at present these structures are understood for a plethora of systems including the major nondissipative plasma fields (for review and original references, see [11-13]). Understanding the GPB yields the requisite constants. The stability analysis presented here uses the energy and the Casimirs, but we note that additional constants such as momentum may also be utilized when additional symmetries are present.

B. Equilibria are obtained from a variational principle that employs the Hamiltonian and the Casimirs. Evidently from (1) equilibria occur for ψ^i such that $\{\psi^i, H\} = 0$. If we let C denote a linear combination of the Casimirs, then $\{\psi^i, I\} = 0$, where $I \equiv H + C$. Equilibria occur when the first variation of I vanishes; i.e.,

$$DI[\vec{\psi}] \cdot \delta\vec{\psi} = \int \frac{\delta I}{\delta \psi^i} \delta\psi^i \, d\tau = 0 \quad . \tag{2}$$

Usually Casimirs involve free functions; so a whole class of equilibria is often obtained by this step.

C. Linear stability; i.e. stability to infinitesimal perturbations $\delta\vec{\psi}$ about $\vec{\psi}_e$, can be shown by taking the second variation of (2). This yields a quadratic form in $\delta\psi^i$. Definiteness of this form implies stability of the linearized equations, but does *not* guarantee stability to finite perturbations for dynamics governed by partial differential equations. A further condition, sometimes called strong positivity, is required. This amounts to a convexity estimate, which we treat in the next step.

D. Showing nonlinear stability requires constructing a norm for the solution space of the system. This will be accomplished if one can find quadratic forms Q_1 and Q_2 that satisfy the following for all finite $\Delta\vec{\psi}$:

$$Q_1[\Delta\vec{\psi}] \leqslant H[\vec{\psi}_e + \Delta\vec{\psi}] - H[\vec{\psi}_e] - DH[\vec{\psi}_e] \cdot \Delta\vec{\psi} \tag{3a}$$

$$Q_2[\Delta\vec{\psi}] \leqslant C[\vec{\psi}_e + \Delta\vec{\psi}] - C[\vec{\psi}_e] - DC[\vec{\psi}_e] \cdot \Delta\vec{\psi} \tag{3b}$$

and

$$\|\Delta\psi\|^2 := Q_1[\Delta\vec{\psi}] + Q_2[\Delta\vec{\psi}] > 0 \text{ for } \Delta\vec{\psi} \neq 0 \quad . \tag{3c}$$

Finding the conditions for the equilibria to satisfy (3a) and (3b) is typically not difficult, but the positivity condition

(3c) can require some ingenuity. To see why this construction gives stability, note that I in step C is a constant of motion; so

$$\begin{aligned} \|\Delta\vec{\psi}\|^2 &\leqslant I[\vec{\psi}(t)] - I[\vec{\psi}_e] - DI[\vec{\psi}_e]\cdot\Delta\vec{\psi}(t) \\ &= I[\vec{\psi}(t=0)] - I[\vec{\psi}_e] \\ &:= \hat{I}[\Delta\vec{\psi}_0] \quad , \end{aligned} \tag{4}$$

where $\Delta\vec{\psi}(t=0) \equiv \Delta\vec{\psi}_0$. Thus, the norm of the perturbation, $\|\Delta\vec{\psi}\|^2$, is bounded by a constant for all time. Suppose this constant is small when $\|\Delta\psi\|$ is small. (This is proven easily by putting quadratic upper bounds on the quantity $I[\Delta\vec{\psi}]$ in (4).) Then, the equilibrium $\vec{\psi}_e$ is nonlinearly Liapunov stable as defined above.

4. RMHD

Assuming helical symmetry, the equations of RMHD are

$$\frac{\partial\psi}{\partial t} = [\psi,\phi] \quad , \qquad \frac{\partial U}{\partial t} = [\psi,J] - [\phi,U] \quad , \tag{5}$$

where $\psi(r,\theta,t)$ is the helical flux, $U(r,\theta,t)$ is the scalar vorticity, $[f,g] = r^{-1}(f_r g_\theta - f_\theta g_r)$, $J = \nabla^2\psi$ and $U = \nabla^2\phi$. This system conserves energy, $H = \frac{1}{2}\int(|\underline{\nabla}\phi|^2 + |\underline{\nabla}\psi|^2)d\tau$, where $d\tau = rdrd\theta$. The GPB for (5) is the Lie-Poisson bracket associated to the semidirect-product Lie group of canonical transformations acting on functions on $\mathbb{R}^2$ [12,13]; hence its Casimirs are known to be $C_1 = \int F(\psi)d\tau$ and $C_2 = \int UG(\psi)d\tau$ where F and G are arbitrary smooth functions of ψ. Varying the functional $I = H + C_1 + C_2$ yields

$$DI\cdot(\delta\phi,\delta\psi) = \int [\delta\phi(-\nabla^2\phi + \nabla^2 G) + \delta\psi(-\nabla^2\psi + UG_\psi + F_\psi)]d\tau \quad , \tag{6}$$

from which we obtain the equilibrium conditions $\phi = G(\psi)$ and $\nabla^2\psi - (\nabla^2 G)G_\psi - F_\psi = 0$. Two special cases are of interest: (i) $G \equiv 0$, which yields the RMHD Grad-Shafranov equation $\nabla^2\psi = F_\psi$, and (ii) $G(\psi) = \psi$, which implies $F = \text{constant}$ and $\phi = \psi$. In the latter case, the specific form of ϕ is not further constrained. The case $\phi = \psi$ corresponds to flow at the poloidal Alfven speed and can be interpreted as nonlinear Alfven waves in the wave frame. We shall investigate nonlinear stability of a class that includes both (i) and (ii).

Taking the second variation of I and rearranging terms yields

$$D^2 I\cdot(\delta\phi,\delta\psi)^2 = \int\Big(|\underline{\nabla}\delta\phi - \underline{\nabla}(G_\psi\delta\psi)|^2 + |\underline{\nabla}\delta\psi|^2\,(1-G_\psi^2) + (\delta\psi)^2[G_{\psi\psi}\nabla^2 G + F_{\psi\psi} + G_\psi\underline{\nabla}\cdot(G_{\psi\psi}\underline{\nabla}\psi)]\Big)d\tau\ .$$

This quantity is positive definite if $|G_\psi| < 1$ and $G_{\psi\psi}\nabla^2 G + F_{\psi\psi} + G_\psi\underline{\nabla}\cdot(G_{\psi\psi}\underline{\nabla}\psi) > 0$. In case (i) the latter condition becomes $F_{\psi\psi} > 0$, which is a severe restriction (monotonicity) on the toroidal current; while in case (ii) we obtain $D^2 I\cdot(\delta\phi,\delta\psi)^2 = \int|\underline{\nabla}\delta\phi - \underline{\nabla}\delta\psi|^2 d\tau$. In this Alfven wave case, we see that ϕ and ψ can each grow arbitrarily large; however their difference is bounded in time. This is consistent with the kink mode instability that RMHD is known to possess [14].

For nonlinear stability it is necessary to show convexity. Since H is already quadratic, we let $Q_1 = H$. If we let $\hat{C}$ be the right-hand side of (3b) then for RMHD we obtain

$$\hat{C} = \int[U_e(F(\psi_e+\Delta\psi) - F(\psi_e) - F'(\psi_e)\Delta\psi) + \Delta U(F(\psi_e+\Delta\psi) - F(\psi_e)) + G(\psi_e+\Delta\psi) - G(\psi_e) - G'(\psi_e)\Delta\psi]d\tau\ .$$

If we assume the functions F and G satisfy $F_\psi \geqslant q$, $2F_{\psi\psi} \geqslant p$ and $2G_{\psi\psi} \geqslant s$ for constants q,p,s, then $\hat{C} \geqslant Q_2$ where

$$Q_2 \equiv \int[pU_e(\Delta\psi)^2 + q\Delta U\Delta\psi + s(\Delta\psi)^2]d\tau\ .$$

Hence we obtain

$$Q_1 + Q_2 = \tfrac{1}{2}\int[|\underline{\nabla}(\Delta\phi) - q\underline{\nabla}(\Delta\psi)|^2 + (1-q^2)|\underline{\nabla}\Delta\psi|^2 + (pU_e+s)(\Delta\psi)^2]d\tau\ .$$

Thus $Q_1 + Q_2 \geqslant 0$ when (i) $|q| \leqslant 1$ and (ii) $pU_e + s \geqslant 0$. Liapunov stability is established upon regarding $(Q_1+Q_2)^{1/2}$ as a norm and further requiring $F_\psi \leqslant Q$, $2F_{\psi\psi} \leqslant P$ and $2G_{\psi\psi} \leqslant S$

for constants Q,P,S, in order for $\hat{I}[\Delta\psi_e,\Delta\psi_0]$, which equals $\hat{I}[\Delta\vec{\psi}_0]$ in (4), to have a quadratic upper bound.

In case (ii) for Alfven waves, the second variation analysis for linearized stability is equivalent to the convexity analysis for nonlinear stability, since I is quadratic. For Grad-Shafranov equilibria we obtain nonlinear stability provided F_ψ (negative of the toroidal current) is a decreasing function of ψ (with a bound on its slope, so that $\hat{I}[\Delta\phi_0,\Delta\psi_0]$ has a quadratic upper bound).

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