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Controllability and Stabilizability of Distributed Bilinear Systems: Recent Results and Open Problems

This paper describes recent results for controlling and stabilizing control systems of the form $\dot{u}(t) = Au(t) + p(t)\,B(u(t))$ where A is the infinitesimal generator of a C^0 semigroup on a Banach space X, B' map from X into X, and p(t) is a real valued control. Application to a vibrating beam problem is given for illustration of the theory.

1. INTRODUCTION

The purpose of this paper is to provide a description of some problems and recent results in the theory of distributed bilinear control systems. The paper is in no way comprehensive and should be viewed only as an introduction to selected problems in control theory. Details may be found in Ball, Marsden, and Slemrod [1] and Ball and Slemrod [2].

As a motivation for our later analysis consider the following problem arising in the control of the transverse displacement $w(x,t)$ of a vibrating beam of length ℓ where we use axial force $p(t)$ as a control. A simple model of this situation is provided by the equation

$$w_{tt} + w_{xxxx} + p(t)\, w_{xx} = 0. \qquad (1)$$

Assume the beam has clamped ends so

$$w = w_x = 0 \text{ at } x = 0,\ell. \qquad (2)$$

If p is a constant two possibilities exist: either p is subcritical and the beam has undamped oscillatory motion or p is supercritical and the equilibrium state $\{w,w_t\} = \{0,0\}$ is unstable. Hence if it is our goal to stabilize the motion so the $\{w,w_t\}\to\{0,0\}$ as $t\to\infty$ (in some sense), a choice of constant p is doomed to failure. Similarly if we desire to control the motion to some full neighborhood of a given state, a constant p will allow us to reach only a highly restricted set of states. For these reasons we are led to consider the following two mathematical problems where p(t) is time varying.

<u>Controllability</u>:
Can we find a p(t) such that on a finite time interval [0,T] we can bring $w(x,t)$, $w_t(x,t)$ to a prescribed function values, i.e.

$$w(x,T) = w_1(x),\; w_t(x,T) = w_2(x),\; 0 \le x \le \ell?$$

<u>Stabilizability</u>:
Can we find p(t) such that on the infinite time interval $[0,\infty)$ p(t) can be written as a (feedback) functional of w,w_t in such a way that $\{w,w_t\}\to\{0,0\}$ as $t\to\infty$ (in some sense)?

We note that since $\{w,w_t\} = \{0,0\}$ is an equilibrium point of (1), (2) any reasonable uniqueness theorem will preclude us from being able to control to the state $\{w_1,w_2\} = \{0,0\}$ in finite time.

What methods should be used to attack the controllability and stabilizability problems? A state space theory based on evolution equations in abstract infinite dimensional vector spaces is one useful way to proceed. Such an approach provides both the structure to prove relevant theorems, yet the generality to cover many interesting examples.

2. THE ABSTRACT EVOLUTION EQUATION

Consider the abstract evolution equation

$$\frac{du}{dt}(t) = Au(t) + p(t)\, B(u(t)), \qquad (3)$$

$$u(0) = u_o. \qquad (4)$$

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Here $\mathcal{A}$ generates a C^0 semigroup on a Banach space $X, \mathcal{B}: X \to X$ is a C^1 map and the control p is real valued function of t. To see that (1), (2) is actually in the form (3) we set

$$u = \begin{pmatrix} w \\ w_t \end{pmatrix}, \quad \mathcal{A}u = \begin{pmatrix} 0 & 1 \\ \frac{-d^4}{dx^4} & 0 \end{pmatrix}, \ .$$

$$\text{and } \mathcal{B}u = \begin{pmatrix} 0 & 0 \\ \frac{-d^2}{dx^2} & 0 \end{pmatrix}, \text{ and let}$$

X be the Hilbert space $H_o^2(0,\ell) \times L^2(0,1)$ with inner product

$$<(w_1,w_2),\ (w_1^*,w_2^*)>_X = \int_0^\ell (w_{1_{xx}} w^*_{1_{xx}} + w_2^* w_2^*)\,dx.$$

Here $H_o^2(0,\ell)$ denotes the Sobolev space made up of functions in $L^2(0,\ell)$ whose first and second generalized derivatives lie in $L^2(0,\ell)$ and satisfy (2). (A good reference for Sobolev spaces is the book of Adams [3]; good reference on semigroups are the notes of Pazy [4] and the book of Balakrishnan [5].)

Of course many other physical problems can be put in the form (3), (4). A list of such problems may be found in the papers of Ball and Slemrod [2], [6].

3. Controllability:

Having argued that (3), (4) is natural way to view distributed bilinear systems, we now proceed to the question of controllability of (4) with an eye to providing a taste for the methods and problems involved in developing an adequate theory.

For (3), (4) the variation of constants formula gives

$$u(t;p,u_o) = e^{\mathcal{A}t}u_o + \int_0^t e^{\mathcal{A}(t-s)}p(s)\mathcal{B}(u(s;p,u_0))\,ds. \tag{5}$$

Here $e^{\mathcal{A}t}$ denotes the semigroup generated by $\mathcal{A}$ and $u(t;p,u_o)$ means that the state vector u is evaluated at time t with control p and initial datum u_o.

One iteration of (5) yields the equation

$$u(t;p,u_o) = e^{\mathcal{A}t}u_o + \int_0^t e^{\mathcal{A}(t-s)}p(s)\mathcal{B}(e^{\mathcal{A}s}u_o + \int_0^s e^{\mathcal{A}(s-\tau)}p(\tau)\mathcal{B}(u(\tau;p,u_o))d\tau)ds. \tag{6}$$

Continuing in this fashion we see that at least formally we can write

$$u(t;p,u_o) = e^{\mathcal{A}t}u_o + \int_0^t e^{\mathcal{A}(t-s)}p(s)\mathcal{B}(e^{\mathcal{A}s}u_o)\,ds \tag{7}$$

+ higher order terms in p.

(Actually this idea is due to Volterra [7]). Hence for small p we could take

$$u(t;p,u_o) \simeq e^{\mathcal{A}t}u_o + \int_0^t e^{\mathcal{A}(t-s)}p(s)\mathcal{B}(e^{\mathcal{A}s}u_o)\,ds. \tag{8}$$

Thus a linear approximation to our control problem of finding a p such that

$$u(T;p,u_o) = h \tag{9}$$

for $h \in X$ prescribed is the problem of finding a p such that

$$e^{\mathcal{A}T}u_o + \int_0^T e^{\mathcal{A}(T-s)}p(s)\mathcal{B}(e^{\mathcal{A}s}u_o)\,ds = h. \tag{10}$$

If we set

$$Lp \stackrel{\text{def}}{\equiv} \int_0^T e^{\mathcal{A}(T-s)}p(s)\mathcal{B}(e^{\mathcal{A}s}u_o)\,ds$$

(10) can be rewritten as

$$Lp = h - e^{\mathcal{A}T}u_o. \tag{11}$$

Here L is a linear operator acting on the space where the p's lie. If we were able to solve the linear approximating problem (11) for all $h \in X$, then an application of the generalized inverse function theorem [8] would tell us that the original equation (9) could be solved for p as well if $h - e^{\mathcal{A}T}u_o$ is sufficiently small.

All this sounds rather easy. However it is not usually simple to check the surjectivity of L except in rather special cases, e.g. $\mathcal{A},\mathcal{B}$ bounded linear operators and X finite dimensional. The reason for this difficulty is that typically L will map the space of p's into but not onto X. In such situations, X must be modified to a smaller space appropriate to the problem. We investigate this subject in detail in [1].

As mentioned above, the surjectivity of L is relatively easy to check if L has closed range R(L) and the Fredholm alternative holds. For instance, if X is finite dimensional $X = R(L) \oplus N(L^*)$ (and not just $X = \overline{R(L)} \oplus N(L^*)$). Hence if X is finite dimensional, $N(L^*) = \{0\}$ implies $R(L) = X$ and surjectivity holds. But if the space of controls,is, say $p \in L^2[0,T]$, then it is easy to see $L^*q = 0$ if and only if $<q, e^{-\mathcal{A}s}\mathcal{B}e^{\mathcal{A}s}u_o> = 0$ for $0 \le s \le T$. Expansion of

$$e^{-\mathcal{A}s}\mathcal{B}e^{\mathcal{A}s}u_o = \mathcal{B}u_o - s\,[\mathcal{A},\mathcal{B}]\,u_o + \frac{s^2}{2}\,[\mathcal{A},[\mathcal{A},\mathcal{B}]]u_o + \ldots$$

where $[\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ yields the well known controllability result [9]:

<u>Assume $X = \mathbb{R}^n$ and that dim span $\{\mathcal{B}u_o, [\mathcal{A},\mathcal{B}]u_o, [\mathcal{A},[\mathcal{A},\mathcal{B}]]u_o, \ldots\} = n$. There is an $\epsilon > 0$ such that if $\|e^{\mathcal{A}T}u_o - h\| < \epsilon$ then there is a $p \in L^2[0,T]$ such that $u(T;p,u_o) = h$.</u>

4. Stabilizability:

The theory of stabilizability of (3), (4) is much more developed. It has been covered to a large extent in the papers of Ball and Slemrod [2], [6]. We shall not go into very much detail here, only sketching the main ideas and open problems.

Assume that A is dissipative, i.e. $\langle A\psi,\psi\rangle_X \leq 0$ for all $\psi \in D(A)$. For example in (1), (2) $\langle A\psi,\psi\rangle_X = 0$.

Let us now (at least formally) compute:

$$\frac{1}{2}\frac{d}{dt}||u(t)||_X^2 = \langle u, Au\rangle_X + p(t)\langle u, \mathcal{B}(u)\rangle_X.$$

Since A is assumed to be dissipative we have

$$\frac{d}{dt}||u(t)||_X^2 \leq 2p(t) \langle u, \mathcal{B}(u)\rangle_X.$$

Hence if we choose

$$p(t) = -\langle u, \mathcal{B}(u)\rangle_X \qquad (12)$$

then

$$\frac{d}{dt}||u(t)||_X^2 \leq -2 \langle u(t), \mathcal{B}(u(t))\rangle_X^2 \quad .$$

and $||u(t)||_X^2$ is non-increasing. If we knew that $u(t)$, $0 \leq t < \infty$, belonged to a compact subset of X, then Hale's generalization [10] of LaSalle's Invariance Principle [11] would tell us that $t\to\infty$, $u(t)$ approached the set of y such that

$$\langle y(t), \mathcal{B}(y(t))\rangle_X = 0 \text{ for all } t \geq 0 \qquad (13)$$

where $y(t)$ is a solution of (3) with $p(t) = -\langle y(t), \mathcal{B}(y(t))\rangle$. But this means that

$$\langle e^{At}y_o, \mathcal{B}(e^{At}y_o)\rangle_X = 0 \text{ for all } t \geq 0 \qquad (14)$$

where $y(0) = y_o$. Hence we may conclude that <u>if $u(t)$ belonged to a compact subset of X and the only solution to (14) is $y_o = 0$, then $u(t)\to 0$ as $t\to\infty$ in the strong (norm) topology of X</u>. Unfortunately it is not clear in general that $u(t)$ will belong to a compact subset of X. To overcome this difficulty Ball and Slemrod [2], [6] resorted to use of the weak topology on X. They were then able to show under natural assumptions on A and $\mathcal{B}$, that if the only solution to (14) is $y_o = 0$ then $u(t)\to 0$ as $t\to\infty$ <u>weakly</u>. Strong convergence is still an open question.

For example this weak topology approach leads for (1), (2) to the following result

(Theorem 4.3 of [2]):

<u>Set $p(t) = \int_0^{\ell} w_t w_{xx} dx$. Then for all initial data $\{w_o, \dot{w}_o\} \in H_o^2(0,\ell) \times L^2(0,\ell)$, (1), (2) possesses a unique weak solution $\{w, w_t\} \in C([0,\infty); H_o^2(0,\ell) \times L^2(0,\ell))$ and $\{w, w_t\} \to \{0,0\}$ weakly in $H_o^2(0,\ell) \times L^2(0,\ell)$ as $t\to\infty$.</u>

Unfortunately this is not as nice as it looks. Even if we ignore the fact that we have only proven weak decay there is a more important difficulty, namely $p(t)$ as given (12) in general or $\int_0^{\ell} w_t w_{xx} dx$ in our example depends on complete knowledge of $u(t)$, i.e. <u>full state feedback</u>. In practice we can only sense a <u>finite dimensional</u> projection of u. Thus it would be desirable to have a theory in which $p(t)$ could be chosen with less than a full knowledge of u. Based on ideas of Balas [12], [13] it seems unlikely that this can be done if the uncontrolled system has no damping since "spill-over" of energy from controlled to uncontrolled modes will occur. Nevertheless this problem and its understanding may be crucial in real world applications of the abstract theory.

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