System Identification With Sparse Coprime Sensing

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***Abstract*—Given a continuous time LTI system with impulse response $h_c(t)$, it is shown that the uniformly spaced samples $h_c(nT)$ can be identified for any chosen spacing T by using an impulse train input with an arbitrarily small rate $1/NT$ and sampling the system output with an arbitrarily small rate $1/MT$, provided M and N are coprime. This idea, referred to here as the sparse coprime sensing method for system identification, is closely related to well known results in multirate signal processing. It is shown that the problem can be related to the identification of a decimation filter from input-output measurements. It is also shown that the problem is equivalent to the identification of a discrete time $N \times M$ LTI system from a knowledge of the full rate input and output vector sequences.**

***Index Terms*—Coprime sampling, sparse sampling, system identification.**

I. Introduction

CONSIDER Fig. 1 where a sequence $x(n)$ with sample spacing T is transmitted through a continuous time LTI system with impulse response $h_c(t)$. The system output is given by[1]

$$y_c(t) = \sum_m x(m) h_c(t - mT). \tag{1}$$

Imagine this output is sampled with spacing T to obtain

$$y(n) = \sum_m x(m) h_d(n - m), \tag{2}$$

where $h_d(n) = h_c(nT)$. Thus the discrete-time equivalent of the system in Fig. 1 is an LTI system with impulse response $h_d(n)$. Since $H_d(z) = Y(z)/X(z)$, it is clear that $h_c(nT)$ can be identified from a knowledge of appropriately designed $x(n)$ (e.g., $\delta(n)$) and $y(n)$.

In this letter, we show that the sampled impulse response $h_c(nT)$, with sample spacing T, can be identified by transmitting an impulse train

$$x_c(t) = \sum_m x(m) \delta(t - mNT)$$

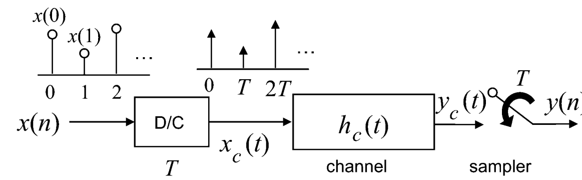

Fig. 1. Pertaining to the system identification problem.

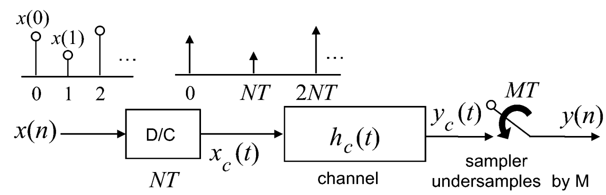

Fig. 2. Input stream transmitted at a lower rate $1/NT$ for system identification. The receiver also performs undersampling by a factor of M.

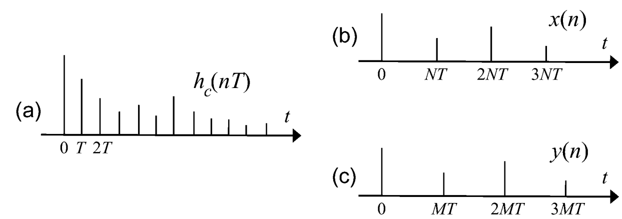

Fig. 3. Locations of (a) the channel samples at the higher rate, and (b), (c) input symbol stream and received output samples at a much lower rate.

at an *arbitrarily small rate* $1/NT$ and taking samples of the received signal at another *arbitrarily small rate* $1/MT$. (The adjective "arbitrarily small" is used here because M and N can be arbitrarily large.) This is schematically shown in Fig. 2. We will see that such identification is possible if and only if the integers M and N are coprime. We shall call this the *coprime sensing method* for system identification. Fig. 3 shows the time scales involved for the input pulse train, the output samples, and the desired impulse response samples $h_c(nT)$.

The proof of the main result, presented in Section II, is based on a simple connection to *fractional sampling rate alteration systems* in multirate signal processing theory [5]. The connection to polyphase representations is described in Section III, which shows in particular that the problem can be reduced to that of identifying decimation filters from input/output measurements. Finally, in Section IV we show that the problem is equivalent to the identification of a discrete time MIMO transfer matrix of size $N \times M$ from a knowledge of the full rate input and output vector sequences.

It should be noticed here that coprime sampling has in the past been used in an entirely different context, namely, for identifying sinusoids in noise (see references in [6]). Coprime pulsing has also been employed for the resolution of range ambiguities in radar [4].

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[1]Here the notation D/C stands for the discrete to continuous converter which converts $x(n)$ into an impulse train $x_c(t) = \sum_m x(m)\delta(t - mT)$.

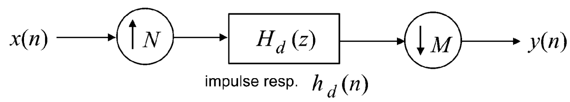

Fig. 4. Discrete time representation of the system sensing scheme of Fig. 2.

II. Identifying the LTI System

The signal $y_c(t)$ in Fig. 2 can be expressed as

$$y_c(t) = \sum_m x(m)h_c(t - mNT) \tag{3}$$

so that the M-fold undersampled version $y(n)$ takes the form

$$y(n) = y_c(nMT) = \sum_m x(m)h_c(nMT - mNT). \tag{4}$$

Defining the desired-rate samples of the system

$$h_d(n) = h_c(nT) \tag{5}$$

as before, the discrete time model for the system of Fig. 2 is given by

$$y(n) = \sum_m x(m)h_d(Mn - Nm). \tag{6}$$

This equation can be interpreted using standard multirate signal processing notations as shown in Fig. 4. Here $\downarrow M$ and $\uparrow N$ represent the M-fold decimator and N-fold expander respectively, as defined in [5]. Thus the problem of identifying the samples $h_c(nT)$ is equivalent to identifying the impulse response $h_d(n)$ of the digital filter $H_d(z)$ in Fig. 4 from a knowledge of the output $y(n)$, in response to an appropriately designed input $x(n)$.

Need for Coprimality: We claim that the identification of $H_d(z)$ described above cannot be done when M and N are not coprime. To see this, assume that M and N have the greatest common factor (GCD) K. Then from (6) we have

$$y(n) = \sum_m x(m)h_d\left(K(M_1 n - N_1 m)\right) \tag{7}$$

for appropriate integers M_1, N_1. Those samples of $h_d(i)$ for which i is not a multiple of K are not present in this equation, and cannot therefore be identified from a knowledge of $x(n)$ and $y(n)$. Thus, unless M and N are coprime (i.e., $K = 1$), there will always be a subset of samples $h_d(i)$ that are not "observable" at the output, and cannot therefore be identified.

Assume therefore that M and N are coprime. The argument of $h_d(.)$ in (6) has the form $i = Mn - Nm$. By Euclid's theorem [1], [2], coprimality implies that every integer i can be expressed in the above form, for an appropriate integer pair (m, n), say, (m_i, n_i):

$$i = Mn_i - Nm_i. \tag{8}$$

Thus if $x(m_i)$ were the only nonzero input sample, then $y(n_i) = h_d(i)x(m_i)$, from which $h_d(i)$ can be identified. This is the intuitive reason why $h_d(i)$ is identifiable from $x(m)$ and $y(n)$ when M and N are coprime. To develop a formal proof we will assume that $H_d(z)$ is FIR:

$$H_d(z) = \sum_{i=0}^{L} h_d(i)z^{-i}. \tag{9}$$

In this case, we can design $x(m)$ with its nonzero samples sufficiently spaced apart, so that any output sample is affected by at most one input sample. This is the idea behind the proof of the main result given below:

Lemma 1. LTI System Identification: Consider the scheme of Fig. 2 where an input stream $x(n)$ is transmitted with uniform spacing NT and the LTI system output is uniformly sampled with spacing MT to obtain $y(n)$. Assume that the sampled impulse response $h_d(n) = h_c(nT)$ is FIR. Then $h_d(n)$ can be identified from the received signal $y(n)$ (for an appropriately designed finite duration input $x(n)$) if and only if M and N are coprime. $\diamondsuit$

Stated equivalently, the FIR system $H_d(z)$ in Fig. 4 can be identified from a finite-duration observation of $y(n)$ for appropriately designed input $x(n)$, if and only if M and N are coprime.

Proof of Lemma 1: We have already shown above that it is necessary for M and N to be coprime. So assume coprimality. Then any integer i can be expressed as in (8), which can be rewritten as

$$i = M(n_i + Nk_i) - N(m_i + Mk_i)$$

for any integer k_i. Thus, for fixed i and k_i the output $y(n_i+Nk_i)$ has the term $h_d(i)x(m_i + Mk_i)$. For each i suppose we have identified one initial (m_i, n_i) pair such that (8) holds. Suppose we modify (m_i, n_i) to

$$m_i' = m_i + Mk_i, \quad n_i' = n_i + Nk_i$$

for some set of integers k_i, and construct an input $x(m)$ which is nonzero only at the points $m_i + Mk_i, 0 \le i \le L$. Then the output at $n_i' = n_i + Nk_i$ is given by

$$y\left(n_i'\right) = \sum_{p=0}^{L} x\left(m_p'\right) h_d\left(Mn_i' - Nm_p'\right), \quad 0 \le i \le L. \tag{10}$$

Note that Mn_i' in the right hand side is independent of p. Since the initial set $\{m_i\}$ is fixed and the above equation holds for any choice of the integers $k_0, k_1, \ldots, k_L$, we can always choose them such that

$$Nm_p' > Nm_{p-1}' + L, \quad 1 \le p \le L.$$

where L is the order of $H_d(z)$. Then $h_d(Mn_i' - Nm_p')$ cannot be nonzero for more than one value of p in (10). Hence

$$y\left(n_i'\right) = x\left(m_i'\right) h_d\left(Mn_i' - Nm_i'\right) = x\left(m_i'\right) h_d(i), \quad 0 \le i \le L.$$

Since $x(m_i')$ and $y(n_i')$ are known, we can identify $h_d(i)$ from this, for each i in $0 \le i \le L$. $\bigtriangledown\bigtriangledown\bigtriangledown$

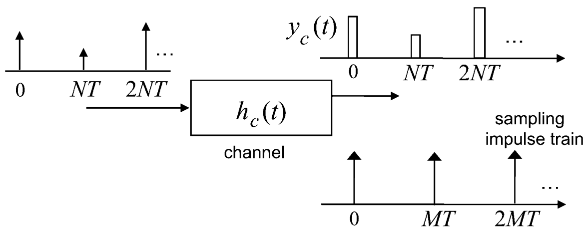

Fig. 5. Example where the system $h_c(t)$ is a narrow pulse.

The following remarks help to develop further intuition:

1) Bandlimited Channel: When $h_c(t)$ is a bandlimited channel with Nyquist rate $1/T$, the preceding lemma shows that the input pulse train, and the output sampling rate, can both be at rates that are arbitrarily smaller than the Nyquist rate.

2) Case of Conventional Sampling: If the input signal is $x(n) = \delta(n)$, then since $y_c(t) = h_c(t)$, the only way to obtain the samples $h_c(nT)$ is to choose the obvious sampling rate $1/T$ (i.e., set $M = 1$). In this case the "rate" of the input $x(n)$ is zero (since there is only one sample). So this is an extreme case where $N = \infty$ and $M = 1$.

3) Case Where $h_c(t)$ is a Narrow Pulse: Next consider the example where $h_c(t) = 1$ for $|t| < \epsilon < T/2$, and zero otherwise. So $h_c(t)$ is a very short pulse, and $h_c(nT) = \delta(n)$. So the system $h_c(nT)$ to be identified has only one nonzero sample. The system output $y_c(t)$ in Fig. 2 is therefore a train of narrow pulses separated by NT. See Fig. 5. If this is sampled at spacing MT as demonstrated in the figure, then the sampler's impulse train overlaps with the output pulse train $y_c(t)$ exactly at one point, namely $t = 0$ (since M and N are coprime). So $y_c(nT) = x(0)h_c(nT)$ from which $h_c(nT)$ can be identified trivially.

III. Polyphase View of System Identification Based on Coprime Sampling

We now give a second view of identifiability of $h_d(n)$ in Fig. 4, based on the polyphase approach [5]. When M and N are coprime, the discrete time system in Fig. 4 can be redrawn as in Fig. 6 where $H_k(z)$ are the N polyphase components of $H_d(z)$ (with some delays inserted). This is a well-known result in multirate signal processing theory (see [5, Sec. 4.3.3]). If $H_d(z)$ is FIR, all the components $H_k(z)$ are also FIR. In this equivalent structure, $y(n)$ is the interleaved version of the N signals $y_k(n)$. Thus knowing $y(n)$ for all n is equivalent to knowing $y_k(n)$ for all n and k. Identifying $H_d(z)$ from a knowledge of $y(n)$ and $x(n)$ is therefore equivalent to identifying each component $H_k(z)$ from a knowledge of $y_k(n)$ and $x(n)$. We will now argue that it is possible to design the signal $x(n)$ such that, from a knowledge of $y_k(n)$ we can indeed identify $H_k(z)$, when it is FIR. For this we first make a simple observation about decimation filters.

Lemma 2. Identifying a Decimation Filter: Consider the M-fold decimation filter $F(z)$ shown in Fig. 7 and assume $F(z)$ is FIR. Then there exists a finite duration input $x(n)$ such that $F(z)$ can be identified from observation of a finite duration portion of the output $y(n)$. ◇

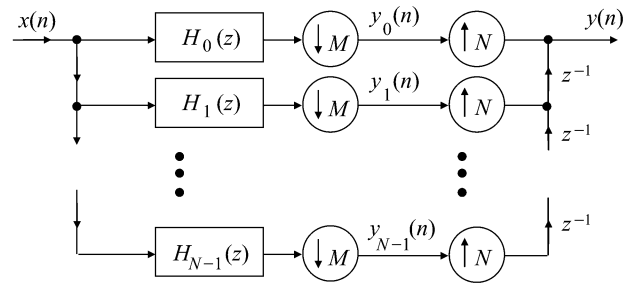

Fig. 6. Polyphase representation of Fig. 4.

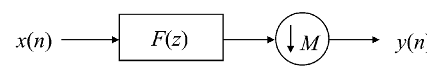

Fig. 7. Decimation filter.

The intuition behind this is as follows: if we apply the input $x(n) = \delta(n)$ in Fig. 7, then $y(n) = f(Mn)$, which allows us to identify a subset of the samples of $f(n)$. If we apply the shifted input $x(n) = \delta(n-1)$ instead, then we can similarly identify $f(Mn-1)$ for all n. But if we apply $\delta(n) + \delta(n-1)$ then the responses $f(Mn)$ and $f(Mn-1)$ overlap in time and cannot be separated. By leaving enough room between nonzero input samples, such overlap can be avoided.

Proof of Lemma 2: Suppose we design $x(n)$ such that it is nonzero for M specific values of n and zero elsewhere. More specifically we choose

$$X(z) = z^{-k_0M} + z^{-1}z^{-k_1M} + \ldots + z^{-(M-1)}z^{-k_{M-1}M}. \quad (11)$$

With $F(z)$ expressed in the polyphase form $F(z) = \sum_{\ell=0}^{M-1} z^{\ell} R_\ell(z^M)$, the decimated output has z-transform

$$Y(z) = [F(z)X(z)]_{\downarrow M} = \quad (12)$$

$$\left[F(z)z^{-k_0M}\right]_{\downarrow M} + \left[z^{-1}F(z)z^{-k_1M}\right]_{\downarrow M} + \left[z^{-2}F(z)z^{-k_2M}\right]_{\downarrow M} \cdots \quad (13)$$

We now observe that

$$\left[z^{-\ell}F(z)z^{-k_\ell M}\right]_{\downarrow M} = z^{-k_\ell}\left[z^{-\ell}F(z)\right]_{\downarrow M} = z^{-k_\ell}R_\ell(z) \quad (14)$$

for $0 \le \ell \le M-1$. The preceding expression for $Y(z)$ therefore simplifies to

$$Y(z) = z^{-k_0}R_0(z) + z^{-k_1}R_1(z) + \ldots + z^{-k_{M-1}}R_{M-1}(z). \quad (15)$$

Equivalently in the time domain:

$$y(n) = r_0(n-k_0) + r_1(n-k_1) + \ldots + r_{M-1}(n-k_{M-1}). \quad (16)$$

If $F(z)$ is FIR then so are the polyphase components $R_k(z)$. So we can always choose the delays k_i such that the impulse responses $r_i(n-k_i)$ on the preceding right hand side do not overlap. Thus, from a measurement of $y(n)$, the components $R_k(z)$ can be identified by inspection, and $F(z)$ can therefore be identified. ▽ ▽ ▽

Returning to Fig. 6 we now see that there exists an input $x(n)$ such that $H_m(z)$ can be identified from $y_m(n)$, for each m.

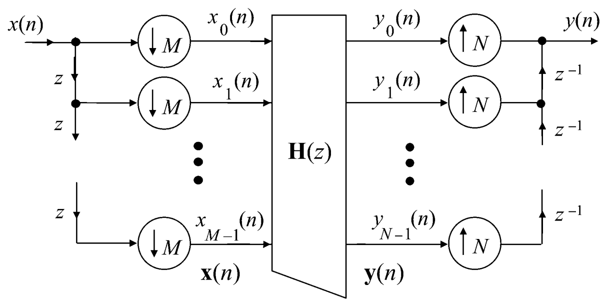

Fig. 8. Further development of the polyphase form in Fig. 6.

From the proof of Lemma 2 we see that the choice of $x(n)$ itself depends on $H_m(z)$ (the lengths of its polyphase components in particular). But it is easy to see that there exists an $x(n)$ which works for all $H_m(z)$. For this, observe that the integers k_m in the proof of Lemma 2 need only satisfy the following:

$$k_1 > k_0 + l_0, \quad k_2 > k_1 + \ell_1, \quad k_3 > k_2 + \ell_2 \ldots \tag{17}$$

where ℓ_i are the lengths of the FIR filters $R_i(z)$. If we redefine ℓ_i in (17) to be

$$\ell_i = \max_m (\text{length of } i\text{th polyphase component of } H_m(z)) \tag{18}$$

then the above set $k_0, k_1, \ldots$, will work for all the decimation filters in Fig. 6. Summarizing, we see that there exists $x(n)$ such that from a measurement of $y(n)$ we can identify all the FIR filters $H_m(z)$ in Fig. 6. This gives a second proof of Lemma 1.

IV. Relation to MIMO System Identification

Suppose each filter $H_m(z)$ in Fig. 6 is expressed in the polyphase form

$$H_m(z) = \sum_{k=0}^{M-1} z^k R_{mk}(z^M), \quad 0 \le m \le N-1. \tag{19}$$

Defining the $N \times M$ matrix $\mathbf{H}(z)$ with elements $[\mathbf{H}(z)]_{mk} = R_{mk}(z)$, and using an appropriate noble identity [5], we can then redraw the system of Fig. 6 as in Fig. 8.

In this figure $y(n)$ is an interleaved version of $y_k(n)$ and similarly $x(n)$ is an interleaved version of $x_k(n)$. Thus, a knowledge of $y(n)$ for all n is equivalent to knowledge of $\mathbf{y}(n)$ for all n, and similar remarks hold for $x(n)$ and $\mathbf{x}(n)$. Since the system in Fig. 8 is nothing but an equivalent redrawing of Fig. 4 when M and N are coprime, we conclude therefore that the identification of the scalar LTI system $H_d(z)$ from a knowledge of $y(n)$ is equivalent to the identification of the MIMO system $\mathbf{H}(z)$ from a knowledge of its vector-output sequence $\mathbf{y}(n)$. Summarizing, we have shown the following.

Lemma 3: Consider the system identification problem in Fig. 2 where we seek to identify the samples $h_c(nT)$ by inputting an impulse train at the low rate $1/NT$ and sampling the system output at the low rate $1/MT$. When M and N are coprime integers, this problem is equivalent to identifying a $N \times M$ discrete-time MIMO LTI system $\mathbf{H}(z)$ from its output $\mathbf{y}(n)$ (in response to an appropriately designed input $\mathbf{x}(n)$.) $\diamondsuit$

Notice from Fig. 8 that $\mathbf{y}(n) = \sum_k \mathbf{h}(k)\mathbf{x}(n-k)$ which represents MIMO convolution. The preceding lemma therefore shows that identification of $h_c(nT)$ from the sparse input and sparse output samples of Fig. 2 is equivalent to a MIMO deconvolution problem.

V. Concluding Remarks

The results presented here are quite basic, and remain valid in any situation where a linear time invariant system has to be identified by "sounding out" the system with an impulse train. Applications of the result include channel identification, in which case additive channel noise should also be taken into account. Another potential application is in the identification of target signature in an active sensing scenario. This problem is more sophisticated because of the presence of signal driven interference such as clutter. It will be interesting to explore these applications in greater detail. Note that the proofs of Lemmas 1 and 2 are constructive in the sense of showing how the input $x(n)$ should be chosen. However these proofs do not indicate what the best choice of the input is. For example, for fixed M and N, what is the shortest $x(n)$ which will identify the system? In the presence of additive noise at the output of $h_c(t)$, what is the best choice of $x(n)$ subject to some constraint such as a total power constraint or peak power constraint? It appears that the deconvolution formulation offered by Lemma 3 is well suited to answer these questions. Further work along these directions will be of interest.

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