Supporting Information

"Parametric amplification and noise squeezing by a qubit-coupled nanomechanical resonator"

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Fitting Δf vs. V_G in Figure 1(c)

The periodicity of the qubit energy is periodic in n_g, thus Δf also shows a periodic response vs n_g, or $V_G (= 2q_e n_g / C_G)$. The period ΔV_G is 19mV and we get $C_G = 2q_e / \Delta V_G = 17\text{aF}$ which gives us the scaling between n_g and V_G. Now we approximate Equation 2 as,

$$\omega_\pm \cong \omega_0 \pm \frac{2\lambda^2}{\hbar} \frac{E_J^2}{\Delta E^3} = \omega_0 \pm \frac{2\lambda^2 / E_J}{\hbar[(4E_C(1-2n_g)/E_J)^2 + 1]^{3/2}}$$

since $\Delta E >> \hbar\omega_0$. We fit the data $\Delta f (= (\omega_- - \omega_0)/2\pi)$ vs. n_g to this equation, with $E_J = E_{J0}\cos(\Phi/\Phi_0) = 13\text{GHz} \times \cos(0.28\pi) = 8.3\text{GHz}$ taken from the previous spectroscopic measurement[1]. Resulting $E_C / h = 12.5\text{GHz}$ agrees with the spectroscopy data (13GHz)[1]. Also from the fit, we find $\lambda / h =$ 3.2MHz, which gives

$\partial n_g / \partial x = \lambda /(4E_C x_{zp}) = 3.3\times10^9\,\mathrm{m}^{-1}$. This is consistent with our estimate based upon finite element simulations of capacitance between the CPB and nanoresonator, where we have $\partial \mathrm{C_G} / \partial x = 50\mathrm{pF/m}$, which gives $\partial \mathrm{n_g} / \partial x = 2.5\times10^9\,\mathrm{m}^{-1}$.

Fitting the nonlinear dissipation model and estimate of the coefficient

We simplify the equation (4) to $y = Ax^2 + B/x + C$ and fit it with x =(mechanical displacement X before normalization) and $y = V_{2\omega}$. With the calculated fit parameters, a cubic equation $AX^3 + (C - V_{2\omega})X + B = 0$ is solved for each $V_{2\omega}$ and this produces the solid curves in Fig.2(b). From the fit, we get the nonlinear dissipation coefficient $\eta \cong 8\times10^9 (kg/m^2 s)$.

An estimate based on the measured quality factor of the nanoresonator gives a value agreeing within an order of magnitude. The PLL circuit we use in the resonance shift measurement also monitors the magnitude simultaneously, giving us information about the nanoresonator’s quality factor with respect to the gate charge. Figure S1 shows the width of the nanomechanical resonance ($\gamma = \omega_0 / Q$) converted from the measured quality factor. It is clear that there is additional dissipation due to coupling to the CPB, which is maximized at the degeneracy point. The source of this additional dissipation is not clear yet, but a possible reason is the resonance frequency fluctuation due to the finite rate of quasiparticle poisoning. An analogous line broadening effect of atomic transition from random telegraph-type resonance fluctuation was analyzed elsewhere[2]. With the CPB biased at $n_g \cong 0.063$, a small fraction of gate charge is modulated by the nanomechanical motion according to $\delta n_g = ((\partial C_G / \partial x) V_N / 2e) x$, which results in modulation of of γ and ω_0. Assuming that the equation of motion takes the form,

$$m\ddot{x} + m(\gamma + \frac{\partial \gamma}{\partial x} x + \frac{1}{2}\frac{\partial^2 \gamma}{\partial x^2} x^2)\dot{x} + m(\omega_0^2 + 2\omega_0 \frac{\partial \omega_0}{\partial x} x + \omega_0 \frac{\partial^2 \omega_0}{\partial x^2} x^2)x = f$$

up to the second order of x, we can identify higher order terms such as $m(\partial\gamma/\partial x)x\dot{x}$, $m(\partial^2\gamma/\partial x^2)x^2\dot{x}/2$, $2m\omega_0(\partial\omega_0/\partial x)x^2$ and collect them following Lifshitz *et.al.*[3] The nonlinear dissipation coefficient is then given by,

$$\eta = \frac{m}{2}\frac{\partial^2\gamma}{\partial x^2} - \frac{2m}{\omega_0}\left(\frac{\partial\gamma}{\partial x}\right)\left(\frac{\partial\omega_0}{\partial x}\right) = \left(\frac{k_{eff}}{2\omega_0^2}\frac{\partial^2\gamma}{\partial n_g^{\ 2}} - \frac{2k_{eff}}{\omega_0^3}\frac{\partial\gamma}{\partial n_g}\frac{\partial\omega_0}{\partial n_g}\right)\left(\frac{\partial n_g}{\partial x}\right)^2$$

From the measured resonance linewidth vs. gate charge in Fig.S1, we get $\partial^2\gamma/\partial n_g^{\ 2} \approx 5\times10^5 s^{-1}, \partial\gamma/\partial n_g \approx -6\times10^4 s^{-1}$. And from the resonance shift data, we have $\partial\omega_0/\partial n_g \approx 1\times10^4 s^{-1}$, $\partial n_g/\partial x \approx 3.3\times10^9 m^{-1}$, yielding $\eta \cong 1\times10^9 (kg/m^2 s)$.

We note that, from the fluctuation-dissipation theorem, this nonlinear dissipation should have been accompanied by displacement-dependent terms in the nanoresonator's force noise correlation[4]. However, we did not attempt to measure this force noise correlation in the current experiment.

Noise model

The nanoresonator biased at $V_{dc} = V_N - V_{NG}$ at its resonance can be represented as a resistor($R_m = \frac{k_{eff}}{G\omega_0 Q} \Big/ \left(\frac{\partial C_{NG}}{\partial x} V_{dc} \right)^2$)[5]. The impedance matching circuit transforms the impedance as $N^2:1$ where $N = \omega_T L_T / Z_0 \cong 50$ ($\omega_T = 1/\sqrt{L_T C_T}$) . The amplifier is assumed to have noise sources represented by two uncorrelated ones[6] (Fig.1(a)), spectral densities of noise voltage and current (S_V , S_I), with a noiseless input impedance $Z_{in} = Z_0 = 50\Omega$. In our set-up, $R_m \cong 0.8M\Omega$ for G =1 and $R_m / N^2 >> Z_{in}$. The mechanical displacement is proportional to the current through Z_{in}, thus the noise current through Z_{in} gives the displacement noise. The spectral density of noise current through Z_{in} is,

$$S_{I,Zin} = \frac{S_V}{\left(R_m / N^2 + Z_{in}\right)^2} + S_I \left(\frac{R_m / N^2}{R_m / N^2 + Z_{in}} \right)^2 \cong \frac{S_V}{\left(R_m / N^2\right)^2} + S_I$$

We see the second term does not depend on the mechanical resonator and it is simply additive contribution from the amplifier. The first term, by contrast, increases when R_m

decreases, i.e. the coupling to the amplifier increases. Thus we identify the amplifier back-action noise as,

$$S_{x,BA} = S_V /\left(R_m / N^2\right)^2 /\left(N\omega_0 V_{dc}(\partial C_{NG} / \partial x)\right)^2 = S_V (GNQV_{dc}(\partial C_{NG} / \partial x) / k_{eff})^2$$

and the additive noise as,

$$S_{x,ADD} = S_I \big/ \left(N\omega_0 V_{dc}(\partial C_{NG} / \partial x)\right)^2$$

It is evident that only the back-action noise is amplified or squeezed depending on the parametric gain G. And also, since only $S_{x,BA}$ depends on the mechanical Q, the noise floor under the motional peak of the displacement noise spectrum is $S_{x,ADD}$.

For the first step, the parametric pump is set to zero, and the total displacement noise spectrum $S_x = S_{x,ADD} + S_{x,BA}$ is recorded for each of quadratures(X and Y) of lock-in. The two noise spectrum show no difference in peak height and noise floor level with each other as expected, and we choose X quadrature data to plot in Fig.3(a) and to estimate $S_{x,ADD}$. Since the noise floor has a slope due to slight offset of LC matching circuit resonance and mechanical resonance, a quadratic polynomial is fitted to the noise floor as an estimate (dashed line in Fig.S2) and the fitted noise density at the mechanical resonance is chosen as $S_{x,ADD}$. Then we turn on the parametric pump, and the phase of the resonator excitation is swept, while S_x at the mechanical resonance is monitored. $S_{x,BA} = S_x - S_{x,ADD}$ gives us the back-action noise for each phase.

Now the total displacement noise is,

$$S_x = S_{x,BA} + S_{x,ADD} \cong S_V \left(\frac{GNQV_{dc}(\partial C_{NG} / \partial x)}{k_{eff}} \right)^2 + \frac{S_I}{\left(N\omega_0 V_{dc}(\partial C_{NG} / \partial x) \right)^2}$$

where we do not include the thermomechanical noise of the nanoresonator considering the noise temperature of the amplifier (~30K) which is much higher than the sample temperature. We confirm this assumption later in the section. The displacement noise S_x at the nanomechanical resonance $\omega = \omega_0$ gives the noise in the force measurement by, $S_f = (k_{eff} / GQ)^2 S_x$. Thus the force noise due to the amplifier is,

$$\begin{aligned} S_f &= \left(NV_{dc} \frac{\partial C_{NG}}{\partial x} \right)^2 S_V + \left(\frac{k_{eff}}{NGQ\omega_0 V_{dc}(\partial C_{NG} / \partial x)} \right)^2 S_I \\ &= S_{f,BA} + S_{f,ADD} \end{aligned}$$

It is minimized when

$$S_{f,BA} = S_{f,ADD} \quad \text{or} \quad R_n \equiv \sqrt{S_V / S_I} = R_m / N^2$$

which gives the noise matching condition. The minimum force noise is then,

$$\min(S_f) = \frac{2k_{eff}}{G\omega_0 Q} \sqrt{S_V S_I} = \frac{4k_{eff} k_B T_N}{G\omega_0 Q}$$

where k_B is the Boltzmann constant and T_N is the minimum noise temperature of the amplifier.

From the measured $S_{x,ADD}$ and $S_{x,BA}$, we calculate S_V and S_I and get $S_V^{1/2} = 360\, pV / Hz^{1/2}$ and $S_I^{1/2} = 2.2\, pA / Hz^{1/2}$ which are close to what measured in a separate measurement on the cryogenic amplifier, $S_V^{1/2} = 340\, pV / Hz^{1/2}$ and $S_I^{1/2} = 2.2\, pA / Hz^{1/2}$. Also, we extract the force sensitivity $83 aN / Hz^{1/2}$ with no parametric gain. Comparing this with the expected thermomechanical noise force, $\sqrt{4k_{eff} k_B T / (\omega_0 Q)} = 5.1 aN / Hz^{1/2}$, we see the preamplifier noise is dominant and the parametric amplification indeed improves the force sensitivity and it also confirms the validity of our initial assumption neglecting the thermomechanical noise.

References

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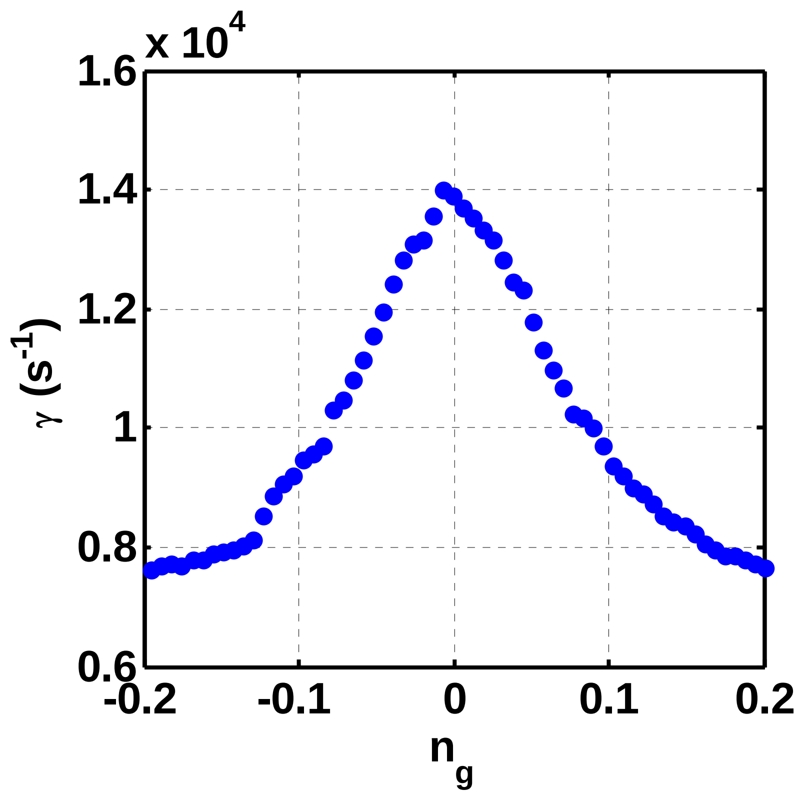

Figure S1. Nanomechanical resonance width vs. gate charge

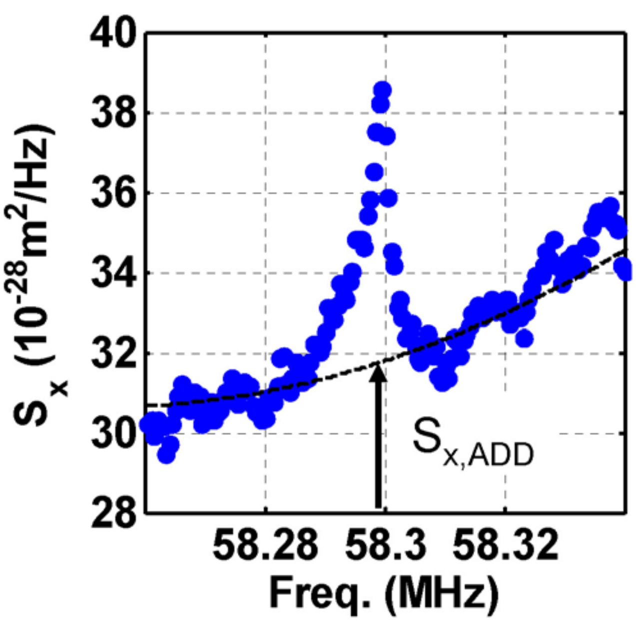

Figure S2. Noise floor estimate