AIAA 2000–3124

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36th AIAA/ASME/SAE/ASEE
Joint Propulsion Conference & Exhibit
16-19 July 2000 / Huntsville, AL

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Active control of longitudinal pressure oscillations in a combustion chamber is studied theoretically by means of a low order model obtained by systematic reduction from a complete representation. The formulation is based on the derivation of a generalized wave equation that accommodates the effects of mean flow, combustion, noise and control action. By using spatial averaging, the equations describing the dynamics of the chamber are reduced to a set of coupled ordinary differential equations, representing the motions of a system of coupled oscillators. The form of the resulting equations is particularly convenient for model reduction and for introducing feedback control terms, while retaining all physical processes.

The oscillator equations are then rewritten in state-space form. Simulations are carried out to investigate in a unified fashion various aspects of the problem. These include the influences of noise, parameter uncertainties, unmodeled modes and a single time-delay.

A criterion is derived that guarantees stability of the controlled closed-loop system in the presence of those quantities. The particular controller used here is based on a standard LQR design, but any design technique can be used as long as the stability criterion is fulfilled.

Introduction

THE need to extend the operating range of combustion systems due to more stringent performance demands (lower emission levels, reduced vibration tolerances,...) has led to increased research activity on control of combustion instabilities. Traditionally these pressure oscillations have been dealt with through passive control techniques. However since these techniques are inadequate under varying operating conditions, active control methods have become more widespread. McManus *et al*[1] gave a review of the different approaches taken nearly a decade ago. Considerable progress has been made since that time, particularly with investigations of the suitability of known control strategies applied to combustor dynamics.

Combustion systems are highly complex and are naturally represented by infinite-dimensional models. So as to reduce the order of the controllers several researchers have made use of an approximate finite-dimensional representation of the flowfield[2] as the basis for their controller design.[3–7] Since any model is prone to uncertainties in the parameters, the controller needs to be robust to changes in those parameters. Among the various approaches taken to achieve this robustness are: a Lyapunov function framework,[5] adaptive control[6] or an H_∞ technique.[7]

The typical characteristics of combustors make designing a controller a challenging task. The model used in the design is normally a considerably reduced representation of the real system; this causes large uncertainties in the parameters. In addition to that, there are very intense internal noise sources due to gasdynamics and combustion and significant time delays that reduce the stability margin. Scaling is also an issue: most of the experiments are conducted in laboratory-scale combustors. Nonlinear processes, especially in the flame models, are not completely understood, and the models normally include a limited representation of those processes. In the case of laboratory combustors, it is possible to ignore the nonlinearities and design a controller on the system linearized around the 'unstable' equilibrium point that the control aims to reach. In the case of classical controllers, they scale linearly with dimensions, but, at this point, it is not clear how the nonlinear processes in the combustor scale: they could become a major issue in large combustors. Moreover, a clearer understanding of nonlinearities is required for the design of nonlinear controllers, that have the potential of being much more efficient in terms of the ratio of the energy going into the control effort versus the total energy released into the system.

*This work was sponsored partly by the California Institute of Technology; partly by the Department of Energy, AGTSR program; partly by the Air Force Office of Scientific Research; and partly by ENEL

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Instabilities in combustion chambers typically manifest themselves as pressure oscillations growing to limit cycles. Experiments[8] show that in some combustors there is a subcritical bifurcation leading to the instability: in this case nonlinear control might be more appropriate, and results from linear simulations could be misleading, especially in the vicinity of the bifurcation point. For the case of heat release depending on velocity fluctuation, Wang,[9] through linear analysis, finds conditions on the heat release function that determine if the bifurcation is subcritical or supercritical. In particular, he has shown that, if the heat release function presents a saturation level, the amplitude of the resulting limit cycle, away from the bifurcation point, is independent of the actual shape of the heat release function. This result plays an important role in simplifying the combustion models for control simulations. If the conditions for supercriticality are satisfied, using the appropriate value for the saturation of the heat release function is enough to capture the correct dynamical behavior of the combustor.

While several of the studies cited include sensor noise and/or parameter variations as part of the system, none of them makes a clear distinction between those uncertainties, the intrinsic noise sources of the system (additive and multiplicative), and the unmodeled dynamics. In this paper we describe how each one of these effects can be included in the control design process. Time-delay, often overlooked in the consideration of combustion systems, can be introduced in the control design as a further uncertainty by using the same framework. A different method, based on an external compensation network, is also discussed.

The controller is designed and tested on the model of a small cylindrical combustor with an instability in the first longitudinal mode. This particular model has been widely used in literature as a test case for the analysis;[5,10,11] even if this might not be realistic in the specific value of the parameters, the model presents a dynamical behavior representative of a combustion chamber, and serves well as a test case. The methods described are obviously not limited to this specific case, and can be applied in general.

In this work the noise sources arise from nonlinear fluid mechanics and as such form an integral part of the system; previous studies have only taken into account external noise sources (such as noisy sensors/actuators). Recently we have shown how these intrinsic noise sources affect the system response and how they can be used to identify the linear parameters of a stable system.[12]

Combustion Instability Model

The fluid dynamical equations (continuity, momentum and energy) governing the flow in the combustion chamber can be combined to yield a nonlinear wave equation for the pressure which in turn, after applying 'spatial averaging' leads to a system of coupled oscillator equations. This system can be truncated to a finite number of modes to get a low order representation of the combustion chamber. This procedure has been described elsewhere in full detail;[10,13] thus only a brief summary is given here.

We begin with the inhomogeneous wave equation and its boundary condition for the pressure fluctuation:

$$\begin{aligned} \nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} &= h \\ \hat{n}\cdot\nabla p' &= -f \end{aligned} \qquad (1)$$

The functions h and f contain *all* relevant physical processes including motion of the boundary, but whereas the contributions from gasdynamics are known explicitly every other process included requires modeling, a separate matter. Explicit forms for h and f are given in the literature[10] and need not be repeated here.

Next 'spatial averaging' is applied, the idea being that any unsteady disturbance in a combustion chamber can be synthesized of an infinite set of basis functions, chosen normally to be acoustic modes. For a rocket motor having a choked exhaust nozzle, the modes are those for a volume enclosed by a rigid boundary having the same shape as the internal surface of the motor, but with no combustion and flow.

Hence we write the familiar series representation of the pressure field,

$$p'(\vec{x};t) = \bar{p}\sum_{j=1}^{\infty}\eta_j(t)\psi_j(\vec{x}) \qquad (2)$$

where $\eta_j(t)$ is the time-dependent amplitude of the j^{th} mode. The spatial distribution or mode shape $\psi_j(\vec{x})$ is calculated as the solution to

$$\begin{aligned} \nabla^2\psi_j + k_j^2\psi_j &= 0 \\ \hat{n}\cdot\nabla\psi_j &= 0 \end{aligned} \qquad (3)$$

and $k_j = \omega_j/\bar{a}$ is the wavenumber, $\bar{a}$ is the mean speed of sound and ω_j is the natural frequency of the mode.

Substitution of (2) in the left-hand side of (1a), multiplication of the equation by ψ_n and integration over the volume (i.e. spatial averaging with the weighting function ψ_n) leads eventually to the system of oscillator equations,

$$\frac{d^2\eta_n}{dt^2} + \omega_n^2\eta_n = F_n \qquad (4)$$

where $$F_n = -\frac{\bar{a}^2}{\bar{p}E_n^2}\left\{\int\psi_n h dV + \oint\psi_n f dS\right\} \qquad (5)$$

and $$E_n^2 = \int\psi_n^2 dV \qquad (6)$$

The central idea motivating the structure of the analysis is that combustion instabilities are dominated by acoustic waves. Hence the pressure field has been represented by the expansion (2) with p' identified as the acoustic pressure. However, following a principle discussed by Chu and Kovaznay,[14] small disturbances are in general made up of three kinds of waves: acoustic, entropy and vorticity waves. Unlike previous analyses in which only the organized oscillatory acoustic field was accounted for all three wave kinds were retained in the analysis given by Burnley.[15,16] The extra waves give rise to stochastic terms in the equation which we choose to retain here.

In lowest approximation (small amplitude, uniform mean flow) the three types of waves propagate independently and hence we can write:

$$\begin{aligned} p' &= p'_a \\ \vec{\Omega}' &= \vec{\Omega}'_\Omega \\ s' &= s'_s \\ \vec{u}' &= \vec{u}'_a + \vec{u}'_\Omega + \vec{u}'_s \end{aligned} \quad (7)$$

The 'forcing function' F_n in equation (6) is a nonlinear function of p' and $\vec{u}'$. In previous applications of spatial averaging[10,17] the velocity fluctuation $\vec{u}'$ is related to p' through the classical linear acoustics equation:

$$\vec{u}'_a = \sum_{j=1}^{\infty} \frac{\dot{\eta}_j(t)}{\bar{\gamma} k_j^2} \nabla \psi_j(\vec{r}) \quad (8)$$

With entropy and vorticity waves present, the terms $\vec{u}'_\Omega$ and $\vec{u}'_s$ make new contributions to F_n. Those contributions can be written explicitly[16] as functions of $\vec{u}'_\Omega$ and $\vec{u}'_s$ but since no model for $\vec{u}'_\Omega$ or $\vec{u}'_s$ exists, we lump them together as stochastic sources. The result of this reasoning is the following system of acoustic equations that are the system used in previous work[10,17,18] with the additional source terms representing stochastic or noise sources:

$$\begin{aligned} \ddot{\eta}_n + \omega_n^2 \eta_n &= -\sum_{l=1}^{\infty} [D_{nl}\dot{\eta}_l + E_{nl}\eta_l] \\ &\quad + \sum_{i=1}^{\infty} [\xi^{\nu}_{ni}(t)\dot{\eta}_i + \xi_{ni}(t)\eta_i] + \Xi_n(t) \end{aligned} \quad (9)$$

Here we have retained only the linear terms in the acoustic amplitudes. Since we are going to design a controller to eliminate the pressure oscillations (i.e. drive all acoustic amplitudes to zero) this is equivalent to linearizing the nonlinear system around the unstable equilibrium point. Note that the linear terms include all linear processes, including linear combustion dynamics. The linear combustion part has in this formulation been lumped together with the linear gasdynamics into the coefficients E_{nl} and D_{nl}; in fact it is the linear combustion that makes the system unstable to begin with. As noted by Wang,[9] this is a valid approximation as long as the system presents a supercritical bifurcation at the point of instability. Note also that while the higher order acoustic terms have been neglected the nonlinear dynamics due to the vorticity and entropy waves are included in the noise terms ξ^{ν}, ξ, and Ξ. The combustor is linearly stable if and only if all modes are linearly stable.

For control applications it is advantageous to reformulate this set of equations in state-space form with state $x = [\eta_1 \ldots \eta_N \dot{\eta}_1 \ldots \dot{\eta}_N]^T$, control input u, and output $y = p'/\bar{p}$. The following definitions are needed:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\Omega^2 - E_{nl} & -D_{nl} \end{bmatrix} \\ B &= \frac{\bar{a}^2}{\bar{p}} \begin{bmatrix} 0 \\ \psi_n(x_{actuator})/E_n^2 \end{bmatrix} \\ C &= [\psi_1(x_{sensor}) \ldots \psi_N(x_{sensor})] \end{aligned} \quad (10)$$

With this notation, equation (9) – without the noise terms – becomes equivalent to

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (11)$$

Chiefly because of the direct connection between the spatially averaged equations and the form (11) favored by the controls community, the procedure leading to equations (4) and (9) is used almost universally for current work in feedback control of combustor dynamics.

Control Theory

General Considerations

Even before the development of models including combustor dynamics and feedback control, experimental application of feedback control of combustion instabilities was successfully tested on small systems (mainly using loudspeakers as actuators). Those laboratory demonstrations report examples in which the amplitudes of limit cycles in linearly unstable combustors have been significantly reduced, sometimes even to vanishingly small values.[19,20] In most cases, the 'practical' controller was a simple proportional feedback or a variation of a *PID* controller. One might wonder why that simple approach works or, conversely, ask why we need more sophisticated control methods. From a general viewpoint, experiments show that an unstable combustion chamber is a system exhibiting a linear instability (rapidly) growing to a limit cycle (defined by the non-linearities) that typically shows a marked predominant frequency. In terms of dynamical systems, the combustor is characterized by two unstable complex-conjugate poles and then a series of stable poles with relatively large damping. Provided that the combustor is observable and controllable, for this kind of system, a proportional feedback or a *PID* controller can be successfully tuned to obtain a stable feedback loop.[21] Regarding the issue of controllability (and observability) of the system, for the purpose of this argument, we will say that controllability has been proved in practice by the success of the experiments cited. A detailed analysis of this point would allow optimization of the position of actuators and sensors, but that is out of the scope of the present discussion. The need for more sophisticated control methods derives mainly from two aspects: first one might want to impose performance specifications on the controller, for example on the maximum control action, or on the noise or disturbance rejection. Second, combustion systems show a high degree of uncertainty and variability,[22] and a controller 'tuned' on a particular operating point does not guarantee a reliable performance. Modern control design methods allow for the introduction of this kind of consideration during the synthesis of the controller.

All the considerations above and the design method presented in the following section are based on a linear model of the combustor. On the other hand, the real system is manifestly nonlinear: the main indication of that is the fact that the pressure oscillations in the combustion chamber rapidly reach a limit cycle. A complete understanding of the dynamics of the combustor would allow tracing the source of the nonlinear behavior observed in the experiments (limit cycles, hysteresis[8,22]) to its origin: nonlinear gasdynamics or nonlinear combustion. In that case nonlinearities in the system could be exploited by an 'ad hoc' form of (nonlinear) control to overcome the main limitations of linear control: requirement of a relatively high control effort and actuation frequency at the same frequency of the instability. Since such a complete model is not available, we decided to limit the analysis to the linear case. Note that the linear model of the combustion chamber presented in the previous section is actually a linearization of the full model around the operating point. Since the main purpose here is to keep the system 'stable', i.e. as close as possible to the linearized equilibrium point, the linear model and simulation is a valid and realistic approximation to the real case, provided that the nonlinearities do not give rise to a subcritical bifurcation.[9] Note that nonlinearities have the effect of limiting the amplitude of the oscillations: hence the linear model is in this sense a 'conservative' approach to the problem (for example, in terms of required control action, we will find an upper limit). In short, within the present approach, nonlinearities can actually be neglected, except as a formal vehicle for rigorously introducing noise sources. As a consequence, we will not be able to capture the effects of any instability mechanism different from the linear growth and phase shifting included in the model presented above. On the other hand, the present approach allows for a clear distinction of the effects of uncertainties, intrinsic noise sources, external noise sources, unmodeled dynamics and time-delay.

Robustness

The controller strategy proposed in this section is based on results described in Chou *et al*[23] and Biswas.[24] It allows treating parameter uncertainty (due to uncertainties in the modeling or system identification process), multiplicative noise (in this case intrinsic to the system, arising from vorticity and entropy waves) and residual dynamics (neglected in the control design in order to achieve a low order controller) in a unified way.

Including parameter uncertainty in the controller design

Consider the following system:[23]

$$\begin{aligned}
\dot{x}_c(t) &= [A_c + \Delta A_c(t)]x_c(t) + A_{cr}x_r(t) \\
&\quad +[B_c + \Delta B_c(t)]u(t) \\
\dot{x}_r(t) &= A_{rc}x_c(t) + A_r x_r(t) + B_r u(t) \\
y &= [C_c + \Delta C_c(t)]x_c + C_r x_r \qquad (12)
\end{aligned}$$

In this description the system has been split into two parts: the controlled dynamics (state x_c) which will be used in the design of the controller and the residual dynamics (state x_r) which are neglected in that design. The reasoning behind this splitting is that we want to achieve a controller that is as simple as possible. This desire leads to a need for a low order model of the system; the controlled dynamics describe that low order system (which at the minimum needs to include all unstable modes) whereas the residual modes describe those parts of the original system that the designer chooses to disregard (the higher acoustic modes which are strongly attenuated in the combustion chamber).

The uncertain parameters of the controlled system are assumed to be bounded ($|\ |$ denotes the modulus matrix):

$$\begin{aligned}
|\Delta A_c(t)| &\le Q^{(A)} \\
|\Delta B_c(t)| &\le Q^{(B)} \\
|\Delta C_c(t)| &\le Q^{(C)} \qquad (13)
\end{aligned}$$

$Q^{(A)}$, $Q^{(B)}$ and $Q^{(C)}$ are nonnegative constant matrices that describe the highly structured uncertainty of the parameters.

Now consider a controller based on a Kalman filter (used to reconstruct the state from the measured output, i.e. the pressure signal from the sensor) with estimator gain L and feedback gain K:

$$\begin{aligned}
\dot{\hat{x}}_c(t) &= A_c\hat{x}_c(t) + B_c u(t) + L[y(t) - C_c\hat{x}_x(t)] \\
u(t) &= -K\hat{x}_c(t) \qquad (14)
\end{aligned}$$

Then the controlled closed-loop system is described by H_c and its interaction with the residual system is governed by H_{cr} and H_{rc} (note that H_{cr} and H_{rc} have been changed to allow for the presence of A_{cr} and A_{rc} when compared to Chou *et al*[23]).

$$\begin{aligned}
H_c &= \begin{bmatrix} A_c - B_cK & -B_cK \\ 0 & A_c - LC_c \end{bmatrix} \\
H_{cr} &= \begin{bmatrix} A_{cr} \\ LC_r - A_{cr} \end{bmatrix} \\
H_{rc} &= [A_{rc} - B_rK \quad -B_rK] \qquad (15)
\end{aligned}$$

Define the matrices G_c and G_r as follows:

$$\begin{aligned}
G_r &\geq [\,\sup |g_{ik}^{(r)}(j\omega)|\,] \\
G_c &= [\,\sup |g_{ik}^{(c)}(j\omega)|\,] \qquad (16)
\end{aligned}$$

Where $\omega > 0$ and $g_{ik}^{(r)}$ and $g_{ik}^{(c)}$ are the ikth elements of $(j\omega I - A_r)^{-1}$, respectively $(j\omega I - H_c)^{-1}$. Also define the uncertainty matrix U as

$$U = \begin{bmatrix} Q^{(A)} + Q^{(B)}|K| & Q^{(B)}|K| \\ Q^{(A)} + Q^{(A)}|K| + |L|Q^{(C)} & Q^{(B)}|K| \end{bmatrix} \quad (17)$$

According to the results cited[23] (with the trivial extension to include A_{cr} and A_{rc}), the closed-loop system will be stable if the matrices H_c and A_r are stable matrices and the following inequality

$$\rho\,[\,G_cU \;+\; G_c\,|H_{cr}|\,G_r\,|H_{rc}|\,] < 1 \qquad (18)$$

is satisfied (here $\rho[\,]$ denotes the spectral radius).

Including multiplicative noise in the controller design

Now consider the system below:[24]

$$\begin{aligned}
\dot{x}(t) &= [A + \xi^{(A)}(t)]x(t) + [B + \xi^{(B)}(t)]u(t) \\
y &= [C_c + \xi^{(C)}(t)]x \qquad (19)
\end{aligned}$$

Here $\xi^{(A)}$, $\xi^{(B)}$,and $\xi^{(C)}$ are random time functions. Lacking precise information about their nature (as in many practical applications) they are assumed to be described by Gaussian white noise processes with zero mean and can be characterized through the quantities $\Xi^{(AA)}$, $\Xi^{(BB)}$, $\Xi^{(AB)}$, and $\Xi^{(CC)}$ where ($E[\,]$ denotes the expected value)

$$\begin{aligned}
\Xi_{ij}^{(AA)} &= \sum_{k=1}^{N} E[\xi_{ki}^{(A)}(t)\xi_{kj}^{(A)}(t)] \quad 1 \leq i,j \leq n \\
\Xi_{ij}^{(BB)} &= \sum_{k=1}^{N} E[\xi_{ki}^{(B)}(t)\xi_{kj}^{(B)}(t)] \quad 1 \leq i,j \leq N_u \\
\Xi_{ij}^{(AB)} &= \sum_{k=1}^{N} E[\xi_{ki}^{(A)}(t)\xi_{kj}^{(B)}(t)] \quad \begin{matrix} 1 \leq i \leq N \\ 1 \leq j \leq N_u \end{matrix} \\
\Xi_{ij}^{(CC)} &= \sum_{k=1}^{N} E[\xi_{ki}^{(C)}(t)\xi_{kj}^{(C)}(t)] \quad 1 \leq i,j \leq N_y (20)
\end{aligned}$$

As before a controller with estimator gain L and feedback gain K is considered.

Following the approach of Biswas[24] (expanded to include $\Xi^{(CC)}$ and modified by considering the process $V(t) = \frac{1}{2}||x(t)|| + \frac{1}{2}||\hat{x}(t) - x(t)||$) we conclude that the closed-loop system is exponentially mean square stable if there exist K and L such that the expected closed-loop matrix H is negative definite. H is given by

$$\begin{aligned}
H &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \qquad (21) \\
\text{where}\quad H_{11} &= A + BK + \Xi^{(AA)} + \Xi^{(AB)}K \\
&\quad + (\Xi^{(AB)}K)^T + K^T\Xi^{(BB)}K \\
&\quad + L^T\Xi^{(CC)}L \\
H_{12} &= BK + \Xi^{(AB)}K + K^T\Xi^{(BB)}K \\
H_{21} &= (\Xi^{(AB)}K)^T + K^T\Xi^{(BB)}K \\
H_{22} &= A - LC + K^T\Xi^{(BB)}K
\end{aligned}$$

Combining both approaches and designing a controller

We can combine both approaches (i.e account for bounded parameter uncertainty and random Gaussian noise perturbations) by including the Ξ-terms in equation (21) in the uncertainty of the closed-loop matrix H_c in equation (16). Thus by redefining the uncertainty matrix U,

$$\begin{aligned}
U &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \qquad (22) \\
\text{where}\quad U_{11} &= Q^{(A)} + Q^{(B)}|K| + \Xi^{(AA)} + \Xi^{(AB)}|K| \\
&\quad + (\Xi^{(AB)}|K|)^T + |K|^T\Xi^{(BB)}|K| \\
&\quad + |L|^T\Xi^{(CC)}|L| \\
U_{12} &= Q^{(B)}|K| + \Xi^{(AB)}|K| + |K|^T\Xi^{(BB)}|K| \\
U_{21} &= Q^{(A)} + Q^{(A)}|K| + |L|Q^{(C)} \\
&\quad + (\Xi^{(AB)}|K|)^T + |K|^T\Xi^{(BB)}|K| \\
U_{22} &= Q^{(B)}|K| + |K|^T\Xi^{(BB)}|K|
\end{aligned}$$

and fulfilling equation (18), robust stability can still be guaranteed.

Up to this point we have derived conditions that L and K need to fulfill without specifying how to design them. Note that the controller is to be designed for the nominal, undisturbed plant; we can then check equation (18) to make sure it is also effective on the perturbed system - or we can use that equation to see how much noise or parameter variation our controller can handle.

The design of the controller depends of course on the performance we wish to achieve. Any method can be used: Biswas[24] uses a pole-placement technique, here we follow Chou *et al*[23] in using a standard LQG method.

LQG is advantageous because it allows for the inclusion of additive system noise ($\xi^{(x)}$) as well as sensor noise ($\xi^{(y)}$). Thus the complete system we try to control is given by:

$$\dot{x}_c(t) = [A_c + \Delta A_c(t) + \xi^{(A)}(t)]x_c(t) + A_{cr}x_r(t)$$

$$
\begin{aligned}
&\quad + [B_c + \Delta B_c(t) + \xi^{(B)}(t)]u(t) + \xi^{(x)} \\
\dot{x}_r(t) &= A_{rc}x_c(t) + A_r x_r(t) + B_r u(t) \\
y &= [C_c + \Delta C_c(t) + \xi^{(C)}(t)]x_c + C_r x_r + \xi^{(y)}
\end{aligned}
\quad (23)
$$

In using this method we minimize the performance index J given by

$$
J = \int_0^\infty \left[x_c^T(t) Q x_c(t) + \rho_c u^T(t) R u(t)\right] dt \qquad (24)
$$

In this case ρ_c is the design parameter that will be changed to fulfill condition (18). Since we want to reduce the pressure oscillations at a specific location x_p, it is natural to choose Q such that $x_c^T Q x_c$ represents our best estimate of p' at this point

$$
Q = \bar{p}^2 \cdot \begin{bmatrix} \psi_1^2(x_p) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \psi_N^2(x_p) \end{bmatrix} \quad \text{and} \quad R = 1 \qquad (25)
$$

Time Delay

Time delays often arise in combustion systems: for example, even when no control is present, there is delay between injection of the fuel mixture and fully developed combustion for the case of liquid or gas combustors. When feedback control is present, there are delays intrinsic to the controller due to finite rates of actuators and sensors, time spent for signal acquisition and processing, and clock time in case a digital computer is used. Even for the typical laboratory-scale combustor, when a loudspeaker is used as an actuator, time delays might play an important role: suppose the first unstable acoustic mode has a frequency of $1kHz$, then a typical reaction time for the controller (if we consider linear approach, the bandwidth of the controller should at least match the typical frequencies of the instability in the plant) is of the order of $1ms$. Modern electronic equipment can certainly process the required computation for determining the control input much faster than that; the bottleneck for this case is the time it takes for the pressure input (from the loudspeaker) to influence the chamber acoustic response. This time, for a $50cm$ chamber, is of the order of 1-$2ms$, just the same order of the instability. In the case of industrial scale combustors, or when using secondary fuel injection as control actuation, the necessity of considering time delays becomes even more compelling, since in these cases the time delay can easily be larger than the characteristic timescale of the instability.

Time delays always reduce the stability of a system,[21] hence it is very important to take them into consideration when simulating a realistic combustor and when designing a suitable controller. Regarding the controller design phase, three general approaches are possible.

- *Classical Control.* If we look at the transfer function of the system, and indicate with τ the time delay, the problem with time delay is reduced to a conventional one by expressing the nonrational function $e^{\tau s}$ in terms of a rational function. Note that the function $e^{\tau s}$ is analytic (for finite values of s), so approximation with a rational function is allowed. A typical approach is to use a Padé approximant, based on a McLauren series expansion of the exponential function. The value of the method is limited by two factors. First, the rational approximation of the delay rapidly increases the effective order of the plant, making the control design problematic. Second, large values of time delay will decrease the available phase margin to the point where it is no longer possible to design a stabilizing controller. Also, Wang[9] shows that a low order polynomial approximation of the time delay is not enough to have a satisfactory model in terms of dynamical behavior of the original system. In the present paper we do not take this method into consideration, since we focus our attention on control design methods capable of incorporating robustness requirements.

- *Modern Control.* In this case, time delay can be viewed as an uncertainty in the system and incorporated in the design as a perturbation to the original plant. More details are given in the following section.

- *Delay Compensation.* This category includes all the other methods used to compensate for time delay. An important group includes compensation networks that bring the delay 'out of the loop', and hence allow to design the controller using conventional methods applied to the plant without time delay. A typical example is the Smith Regulator. A caveat here is constituted by the fact that most networks based on linear elements generally do not modify the eigenvalues of the original plant, so they only apply to stable (or marginally stable) plants. On the other hand, by using these methods, arbitrarily large time delays can be accounted for without loss of stability margin.

Modern Control

Time delay can be incorporated in the design of a controller by considering the time delay as a multiplicative perturbation to the plant. Let $\tilde{P}(s) = P(s)e^{-\tau s}$ be the perturbed plant. The perturbed plant can be included in the set

$$
\{(1 + \Delta_{plant} W_{unc})P : \| \Delta_{plant} \|_\infty \leq 1\} \qquad (26)
$$

with the choice of an appropriate weight function W_{unc}

$$
|e^{-\tau j\omega} - 1| \leq |W_{unc}(j\omega)| \qquad \forall \omega, \tau \qquad (27)
$$

The design of the controller then proceeds in the same way as before. Note that, if the time delay is 'large', condition (27) typically imposes a significant limitation on the controller; in general performance is degraded and, if τ is large enough, it might be impossible to design a stabilizing controller for the delayed system. On the other hand, when a solution exists, stability and performance are guaranteed according to the design. Uncertainty in the numerical value of the time delay, as it is typical in combustion systems, is automatically taken into consideration by the design method. Application of this approach is included in the example presented later.

Delay Compensation

In this section we will examine a method based on *predictive control*: the time delay is compensated by a predictor that acts on the measured or estimated state and feeds the controller with the appropriate signal to perform the feedback action at the compensated time. The control system consists of a predictor and a controller; the closed loop equations can be written as

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) - Bu(t-h) \\ y(t) & = & Cx(t) \end{array} \right. \tag{28}$$

$$p(t) = e^{A\tau}x(t) + \int_{-\tau}^{0} e^{-A\tau}Bu(t+\tilde{t})d\tilde{t} \tag{29}$$

$$u(t) = Kp(t) + i(t) \tag{30}$$

where $i(t)$ is the external input to the system and might not be present. The predictor written as in (29) is simply derived by integrating (28) from the current time t to the time $t+\tau$. A change of variables produces the form (29), which contains information only up to the current time and consequently it can be physically implemented.

It can be proved[25] (based on direct computation of the closed-loop characteristic equation) that if the pair (A, B) is controllable, then the predictor (29) and the controller (30) yield a finite spectrum of the closed-loop system, located at arbitrarily preassigned points in the complex plane.

Note that the predictor (29) contains an integral term up to the current time. It is impossible to integrate up to current time without solving an integral equation, or iterating on the solution, but it can be shown[26] that the limits in the integral term appearing in (29) can be substituted by $-\tau - \epsilon$ and $-\epsilon$ if ϵ is sufficiently small. Let us now consider the following scheme:

$$\dot{x}(t) = Ax(t) - Bu(t-h) \tag{31}$$

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}(t) & = & A\tilde{z}(t) - Bu(t) \\ z(t) & = & \tilde{z}(t) - e^{A\tau}\tilde{z}(t-\tau) \\ p(t) & = & e^{A\tau}x(t) + z(t) \end{array} \right. \tag{32}$$

$$u(t) = Kp(t) \tag{33}$$

A simple substitution shows that computing $z(t)$ from the equations above results in the evaluation of the correct predictor term (29). The scheme (31)-(32) can be physically implemented in Simulink by using the network connection presented in figure 1, where symbols refer to the letters used in (31)-(33) and p is the signal sent to the controller. Since the time delay is compensated in this secondary predictor-loop, application of this scheme to the plant presented above allows including significant time delays in the system without compromising performance in the design of the controller. On the other hand, the use of a second loop reduces the robustness of the system to uncertainties in the value of the parameters.

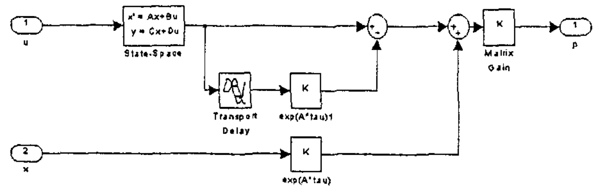

Fig. 1 SIMULINK realization of the predictor block, equations (32)

Example

In the following we will briefly demonstrate the described design procedures on a particular example. The example has been chosen solely because it has been used previously in the literature.[5,10,11] The methods used are obviously much more general and can be applied to any linearized (combustion) system – as mentioned previously, *all* processes, including combustion, have been linearized and are embedded in the model parameters.

The numerical example used is the same one as given in Haddad *et al.*[5] The combustion chamber is assumed to be cylindrical of length L and only longitudinal modes are considered. The chamber is closed on the upstream end and has a nozzle at the downstream end which acts as an acoustically closed boundary. The sensor detecting the instability is a microphone located at x_s whereas the actuator used to control the pressure oscillations is a loudspeaker located at x_a. The internal dynamics of the loudspeaker are modeled as a second order system and included in the state-space formulation, i.e. they form an integral part of A_c in equation (23) through augmentation of the state x_c. In this way the actuator dynamics are accounted for in the design of the controller. In this example the actuator (and sensor) are treated as perfectly known systems; within the framework described here it is straightforward to include uncertainties or noise in those systems too. Note that the model of the actuator as a 'loudspeaker', a second order system with a high gain, is actually more general than it seems. If we wanted to represent an injector, the same

D_{nl}	$n=1$	$n=2$	$n=3$	$n=4$
$l=1$	-.01	.007	-.001	.007
$l=2$	.01	.1	.007	-.001
$l=3$	-.01	.01	.75	.008
$l=4$	.02	-.005	.01	1.5
E_{nl}	$n=1$	$n=2$	$n=3$	$n=4$
$l=1$	-.005	-.005	.0025	.0016
$l=2$	-.0025	-.015	.01	.01
$l=3$	-.005	.0	-.02	.02
$l=4$	.01	.02	.02	-.025

Table 1 Combustion chamber parameters

model would still apply, with a longer time delay, and some difference in the numerical value of the parameters, but substantially the same structure, i.e. second order dynamics and very high gain.[27]

The linearized model of the combustion chamber is characterized through the parameters E_{nl} and D_{nl} which are given (after non-dimensionalization of time by $\frac{\pi L}{\bar{a}}$) in table 1 for the first 4 longitudinal modes – the parameters are originally taken from Yang *et al*[11] and are typical for solid propellant rockets.

In this design study we only consider uncertainty in the parameters E_{nl} and D_{nl} and noise in those terms given in equation (9). In other words there is no noise or uncertainty in the sensing or actuation process and the stochastic sources act (as described by the model) as random perturbations of E_{nl} and D_{nl}. Thus we put (keeping the notation of the previous section)

$$Q^{(B)} = 0 \quad Q^{(C)} = 0$$
$$\Xi^{(AB)} = 0 \quad \Xi^{(BB)} = 0 \quad \Xi^{(CC)} = 0 \qquad (34)$$

Furthermore we assume that all stochastic sources (due to the vorticity and entropy modes in the chamber) are uncorrelated and have the same variance σ^2. Finally we assume that the parameter uncertainties can all be described by a single variable ϵ. These assumptions are only made to simplify the expressions as we can now write:

$$\begin{aligned} Q^{(A)} &= \epsilon \cdot \begin{bmatrix} 0 & 0 \\ |E_{nl}| & |D_{nl}| \end{bmatrix} \\ \Xi^{(AA)} &= N \cdot \sigma^2 \cdot 1 \end{aligned} \qquad (35)$$

A controller was designed using an LQG technique by taking only 1 mode (the unstable first mode) into consideration. It is assumed that the complete system is given by 4 modes and thus the remaining 3 modes are considered to give the residual dynamics. Basing the controller on a minimal set of modes is desired as it reduces reduces the order of the controller and thus allows for easier implementation. Figure 2 shows the response of the reduced and complete system to the controller (turned on at $\tilde{t} = 100$). As expected the presence of the extra modes (not considered in the design) in the full system reduces the performance of the controller (slower decay).

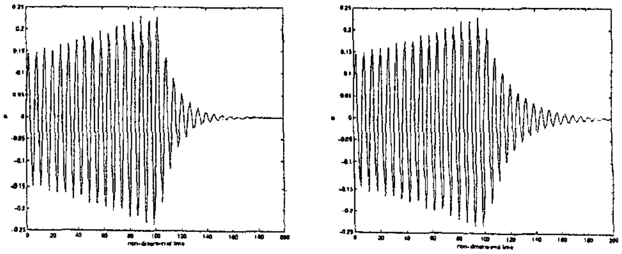

Fig. 2 System response to controller action: 1 mode nominal system (left) and complete 4 modes system (right)

Figure 3 shows the guaranteed stability limits (in terms of ϵ and σ) of the controller. The solid line is the limit predicted by equation (18) for the truncated system where only 1 mode is used in the simulation. The other lines describe the stability region as more modes are added in the simulation, i.e. as the system approaches the 'complete' system. The region shrinks as additional modes are introduced into the simulation while the same controller (based on 1 mode) is retained. In this extreme case (where we considered only one mode to base the controller on) the changes are drastic, but as the neglected modes become more heavily damped their influence grows smaller (as can be seen by the lines moving closer together).

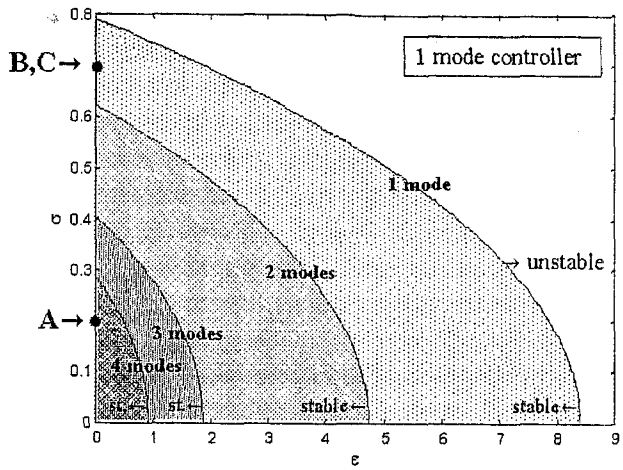

Fig. 3 Guaranteed stability limits for closed-loop system in $\sigma - \epsilon$ plane (noise variance and uncertainty) depending on total number of modes considered

Figure 4 illustrates the effect the (stable) residual system can have in the presence of noise. The same controller (based only on the unstable first mode) is used in all three cases. In case A (low noise) the controller is able to stabilize the full system (all 4 modes included in the simulation); however at a higher noise level (case B) the pressure oscillations do not decay to zero. Note that this noise level is well within the stability limits as predicted with the truncated (1 mode) system and thus underlines the importance of the ne-

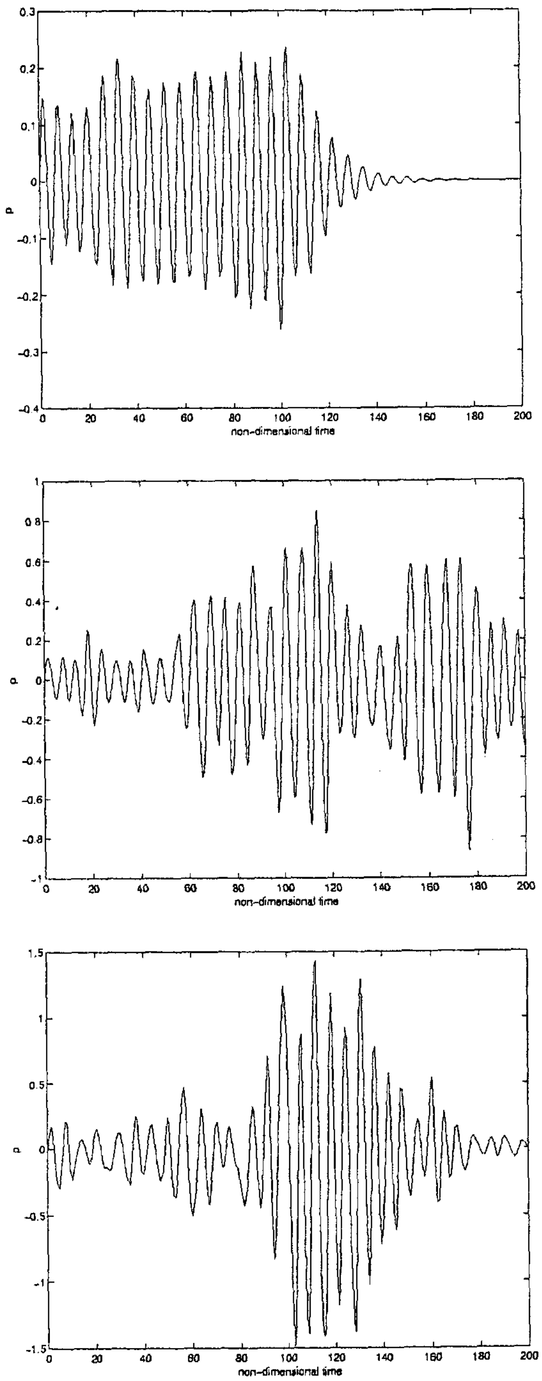

Fig. 4 **Time simulations of cases A (complete system; top),B (complete system; middle) ,and C (reduced system; bottom) as defined in figure 3.**

glected dynamics. If the simulation is performed with the reduced system, the instability does indeed decay as anticipated (case C).

Example with Time Delay Compensation

The delay compensation approach allows separating the control problem from compensation of the time delay. Figure 5 presents the results of the application of the method to our model combustor. The non-dimensional time delay is chosen to be $\tau = 10$, which corresponds to a delay of about $10ms$, i.e. 5 periods of an oscillation at $500Hz$ and constitutes a reasonable upper limit to the delay that can be expected in a real combustor controlled by modulating the injection of a secondary fuel. Note how the predictor works: the controller (control action is plotted in the bottom half of figure 5) starts sending commands immediately when it is activated. The control is computed on a prediction of the future state of the system, i.e. the state of the system when the control signal will effectively reach the plant. The system response, plotted in the top half of 5, shows that the system effectively starts reacting to the control at a non-dimensional time of 40 when the controller is put online at a non-dimensional time of 30.

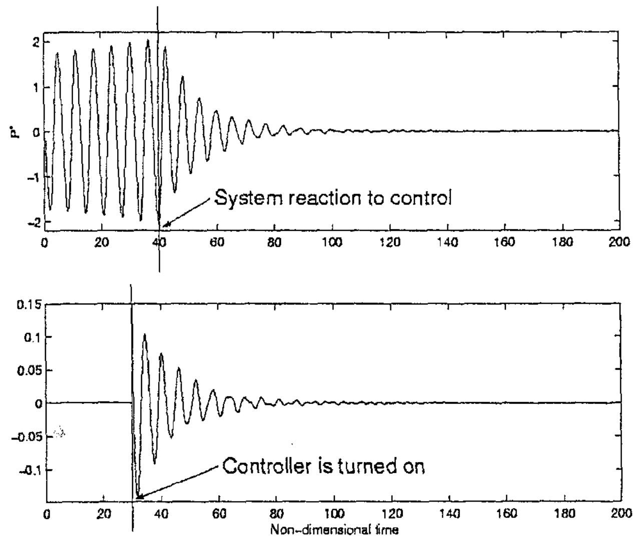

Fig. 5 Control with time-delay, ($\tau = 10$). Top half: system response. Bottom half: control action.

Concluding Remarks

In this paper we have shown how uncertainties, noise, unmodeled dynamics and time-delay can be included in the controller design for combustion instabilities.

A clear distinction has been made between the uncertainty and the noise. This is necessary as the parameter uncertainty can be bounded; e.g. in practical applications we might know that in the operating range of interest the various parameters are located within certain numerical bands. In contrast, true noise sources can in general not be bounded, and thus do not fit in the common control frameworks; they are characterized by their mean values, which we include in the system parameters and by their variation.

Explicit consideration of the neglected modes allows studying their influence on the controller robustness. This is especially important since in most experimental implementations to date the controller has been designed by taking only the unstable mode(s) into account. In the example given here only the first mode is unstable and it is in fact possible to stabilize the system by solely controlling this one mode. Note that the controller is designed to accommodate large uncertainties (or noise) since we anticipate that the residual modes will affect the dominant first mode. This is the way unmodeled dynamics are traditionally handled: by including them in the uncertainties of the system. The framework presented here shows how much of that uncertainty can be attributed to the neglected modes. In the example given, the damping of the ignored modes (notably the second one) is rather small and thus we see that the actual parameter uncertainty ϵ (or noise intensity σ) that the controller can tolerate declines dramatically as additional modes are considered. Therefore we conclude that the residual dynamics dominate the uncertainty unless the neglected modes are highly damped.

Inclusion of a time-delay in the modern-design framework as an uncertainty is adequate when the time-delay is of the same order of the characteristic time of the instability, defined as the inverse of the frequency of the unstable mode. Cases with longer time delays, as it might be the case in full-scale combustors, can be treated by adding a second loop to compensate for the delay: simulation shows very good performance, but issues about robustness to uncertainty and perturbations need to be addressed carefully. An adaptive observer might be needed for application to real systems. Future work in this area should include testing of the concepts on an experimental combustor; and system identification to define better models of real actuators, in particular injectors and fuel flow modulators. More analysis is also needed to characterize (and eventually take advantage of) nonlinearities naturally present in combustion chambers.

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