q-Distributions on boxed plane partitions

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Abstract

We introduce elliptic weights of boxed plane partitions and prove that they give rise to a generalization of MacMahon's product formula for the number of plane partitions in a box. We then focus on the most general positive degenerations of these weights that are related to orthogonal polynomials; they form three two-dimensional families. For distributions from these families we prove two types of results.

First, we construct explicit Markov chains that preserve these distributions. In particular, this leads to a relatively simple exact sampling algorithm.

Second, we consider a limit when all dimensions of the box grow and plane partitions become large, and prove that the local correlations converge to those of ergodic translation invariant Gibbs measures. For fixed proportions of the box, the slopes of the limiting Gibbs measures (that can also be viewed as slopes of tangent planes to the hypothetical limit shape) are encoded by a single quadratic polynomial.

Contents

1 Introduction

The uniform distribution on boxed plane partitions (equivalently, lozenge tilings of a hexagon) is one of the most studied models of random surfaces. There are four principal types of results regarding this model that have been proved.

(1) Law of large numbers: Under the global scaling (the bounding box/hexagon is fixed and the mesh is going to zero), the measure concentrates on surfaces that are close to a certain deterministic *limit shape*. The limit shape can be obtained as the unique solution to a suitable variational problem. The solution is encoded by a second degree polynomial in two variables, see [CKP], [CLP], [DMB], [Des], [KO].

(2) Locally near any point of the limit shape, the measure on tilings converges to a (uniquely defined, see [Sh]) translation-invariant ergodic Gibbs measure on lozenge tiling of the plane of a given slope, and the slope coincides with the slope of the tangent plane to the limit shape at the chosen point, see [Gor] and also [Ke1], [Ke2], [KO], [KOS].

(3) The correlation kernel of the random point process of lozenges of one of the types is explicitly expressed in terms of classical Hahn orthogonal polynomials, see [Gor], [J1], [J2], [JN].

(4) A few algorithms, both asymptotic and exact, have been proposed to generate the random tilings in question, see [BG], [Kr], [P1], [P2], [Wi1], [Wi2].

These are complemented by the classical MacMahon product formula for the total number of plane partitions in a given box, see, e.g., Section 7.21 in [St].

In this paper we study measures on boxed plane partitions that generalize the uniform distribution. The weight of a tiling is defined as the product of certain simple factors over all lozenges of a fixed type, see Section 2.2 for definitions. One special case is the weight q^{volume}, where *volume* is the volume of the corresponding plane partition, and q is an arbitrary positive number.

In the most general case we consider, the weight of a lozenge is *elliptic*. Our initial motivation came from the fact that these weights lead to a nice generalization of the MacMahon formula mentioned above, see Theorem 10.5 in the Appendix.

For the asymptotic analysis, we look at the top degeneration of the elliptic weight that is related to orthogonal polynomials.

Our asymptotic results amount to proving analogs of (2), (3), and (4) above. As for (1), we derive the corresponding variational problem (which differs from the one for the uniform case by the presence of an external potential), and show that the hypothetical limit shape (obtained from (2)) solves the corresponding Euler-Lagrange equation. However, we do not prove the concentration phenomenon rigorously.

One interesting feature of the limit shapes that arise is that the curve that bounds the frozen regions may have one or two nodes in vertices of the hexagon, see Section 9 for illustrations.

In terms of orthogonal polynomials, our models go all the way up to the top of the q-Askey scheme — the most general models we analyze asymptotically are related to the q-Racah classical orthogonal polynomials. We also show that the elliptic weights lead to the biorthogonal functions constructed in [SZ]. We hope to return to the asymptotic analysis of this case in a later publication.

Our proof of (2) follows the same steps as the argument for the uniform case in [Gor]. It is based on the general method of computing limits of correlation kernels suggested in [BO] and [O]. The crucial property we need is the second order difference equation satisfied by the q-Racah orthogonal polynomials.

Our perfect sampling algorithm is obtained from a more general construction of relatively simple Markov chains that change the size of the box (one side increases by 1 and another side decreases by 1), and that map the measures from the class we consider to similar ones. The construction follows the approach of [BF]; the key facts that make that approach possible reduce to certain recurrence relations for the q-Racah polynomials.

A computer simulation of the above-mentioned Markov chains can be found at `http://www.math.caltech.edu/papers/Borodin-Gorin-Rains.exe`.

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2 Model and results

2.1 Combinatorial interpretations

For any integers $a, b, c \geq 1$ consider a hexagon with sides a, b, c, a, b, c drawn on the regular triangular lattice. Denote by $\Omega_{a\times b\times c}$ the set of all tilings of this hexagon by rhombi obtained by gluing two of the neighboring elementary triangles together (such rhombi are called *lozenges*). An element of $\Omega_{3\times 3\times 3}$ is shown in Figure 1.

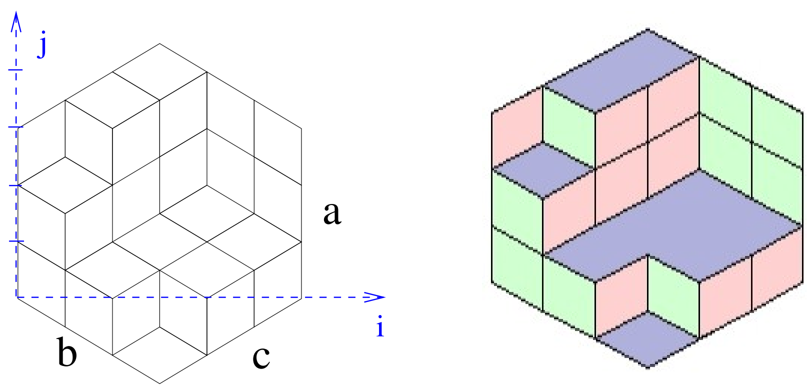


Figure 1. Tiling of a $3 \times 3 \times 3$ hexagon.

Lozenge tilings of a hexagon can be identified with 3-D Young diagrams (equivalently, boxed plane partitions) or with stepped surfaces. The bijection is best described pictorially. We show a 3-D shape corresponding to a tiling in Figure 1.

It is convenient for us to slightly modify both hexagon and lozenges by means of a simple affine transform of the plane.

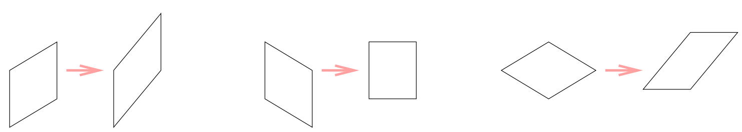


Figure 2. Affine modification of lozenges

We thus obtain a tiling of a slightly different hexagon.

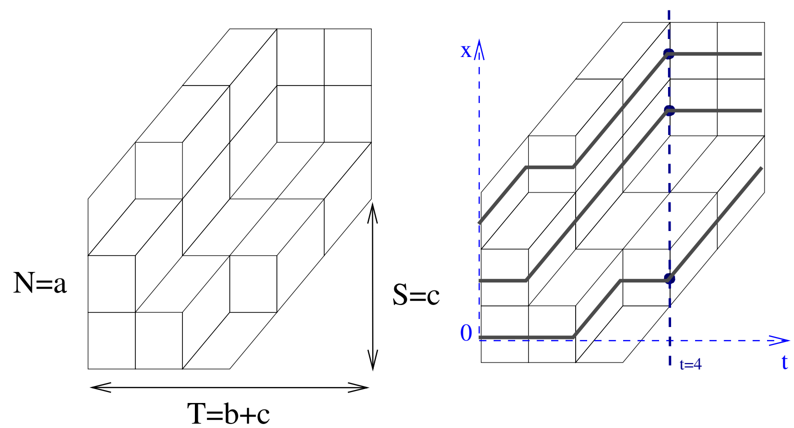


Figure 3. Modified tiling of a $3 \times 3 \times 3$ hexagon and the corresponding family of non-intersecting paths.

In what follows we use different parameters instead of a, b, c. Set $N = a$, $T = b + c$, $S = c$. We will also denote the set $\Omega_{a\times b\times c}$ by $\Omega(N, T, S)$.

Each tiling corresponds to a family of nonintersecting paths as shown in Figure 3.

Consider a section of our family of paths by a vertical line $t = t_0$. Clearly, this gives an N-tuple of points in $\mathbb{Z}$. Thus, our tiling can be viewed as an N-point configuration varying in time $t = 0, 1, \dots, T$. Note that when $t = 0$ the configuration consists of points $\{0, 1, \dots, N-1\}$, while for $t = T$ the configuration consists of points $\{S, \dots, S+N-1\}$.

2.2 Probability models

Let us introduce the probability measures on $\Omega(N, T, S)$ that are studied in this paper. For any $\mathcal{T} \in \Omega(N, T, S)$, we define its weight $w(\mathcal{T})$ and consider the probability distribution on $\Omega(N, T, S)$ given by the formula

$$\mathrm{Prob}\{\mathcal{T}\} = \frac{w(\mathcal{T})}{\sum\limits_{\mathcal{T}' \in \Omega(N,T,S)} w(\mathcal{T}')}$$

The weights we consider are such that the probability of a tiling is proportional to the product of certain weights corresponding to the horizontal lozenges in it, i.e.,

$$w(\mathcal{T}) = \prod_{\lozenge \in \mathcal{T}} w(\lozenge)$$

Note that the number of horizontal lozenges in a tiling of a given hexagon is fixed (i.e., it does not depend on the tiling). Hence, multiplying $w(\lozenge)$ by a nonzero constant does not change the probability distribution.

In the most general case considered in Section 10, $w(\lozenge)$ is an elliptic weight given by

$$w(\lozenge) = \frac{(u_1u_2)^{1/2} q^{j-1/2} \theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)}, \tag{1}$$

where the coordinates of the topmost point of $\lozenge$ are (i, j) (the i and j axes are pictured in Figure 1), u_1, u_2, p, q are (generally speaking, complex) parameters,

$$\theta_p(x) = \prod_{i=0}^{\infty} (1 - p^i x)(1 - p^{i+1}/x)$$

and $\theta_p(a, b, c \dots) = \theta_p(a)\theta_p(b)\theta_p(c) \dots$

Mostly we will not be concerned with this most general case, nor with the most general trigonometric case obtained by taking $p \to 0$ (note $\theta_0(x) = 1 - x$), as in these cases the kernels involve biorthogonal functions. The most general orthogonal polynomial case is the limit

$$p \to 0, \quad u_1 = O(\sqrt{p}), \quad u_2 = O(\sqrt{p}), \quad u_1u_2 = p\kappa^2 q^{-S}, \tag{2}$$

in which case the weight function is

$$w(\Diamond) = \kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}}; \tag{3}$$

this is also the most general case in which the weight is independent of i.

Clearly, the factor $q^{-(S+1)/2}$ can be removed if we replace κ by $\kappa' = \kappa \cdot q^{-(S+1)/2}$. However, this choice is more convenient for our further considerations.

We need to make sure that the probabilities of tilings are positive. This leads to certain restrictions on the parameters. There are three possible cases:

(i). *imaginary q-Racah case:* q is a positive real number, κ is an arbitrary pure imaginary complex number;

(ii). *real q-Racah case:* q is a positive real number, κ is a real number with additional restrictions depending on the size of a hexagon; κ cannot lie inside the interval $[q^{-N+1/2}, q^{(T-1)/2}]$ or $[q^{(T-1)/2}, q^{-N+1/2}]$, depending on whether $q > 1$ or $q < 1$;

(iii). *trigonometric q-Racah case:* q and κ are complex numbers of modulus 1, $q = e^{i\alpha}$, $\kappa = e^{i\beta}$, plus additional restrictions on κ depending on the size of a hexagon: both $-\alpha(T-1)/2 + \beta$ and $\alpha(N-1/2) + \beta$ must lie in the same interval of the form $[\pi k, \pi(k+1)]$, $k \in \mathbb{Z}$. In this case

$$\kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}} = 2\sqrt{-1}\sin\left(\alpha(j-(S+1)/2) + \beta\right),$$

and the factor $2\sqrt{-1}$ here can be omitted.

The names of the cases are related to those of the classical orthogonal polynomials that appear in the analysis.

Denote the resulting measure on $\Omega(N,T,S)$ by $\mu(N,T,S,q,\kappa)$.

There are further limit transitions.

If we send $\kappa \to 0$ then we get the *q-Hahn case*

$$w(\Diamond) = q^{-j}.$$

Thus, the probability of the plane partition of volume (=number of $1 \times 1 \times 1$ boxes) V is proportional to q^{-V}. On the other hand, if we send $\kappa \to \infty$ we will get the weights q^V. Therefore, the case of general κ can be viewed as an interpolation between the measures q^{volume} and $q^{-volume}$.

Another possibility is to set $\kappa = q^K$ and then send $q \to 1$. The weight of a horizontal lozenge tends to

$$w(\Diamond) = K + j - (S+1)/2.$$

We call this case the *Racah case*. One has to impose restrictions to ensure positivity: K cannot lie inside the interval $[-N+1/2, (T-1)/2]$.

Finally, if we either send $\kappa \to 0$ and set $q = 1$, or send $q \to 1$ and then send $K \to \infty$, then we obtain the *Hahn case*

$$w(\lozenge) = 1$$

In this case our probability distribution on $\Omega(N, T, S)$ is uniform.

Below we mostly work with the imaginary q-Racah case, but all the results can be carried over to all the other cases mentioned above by the appropriate substitutions of parameters and degenerations.

2.3 Representation of the weight

Let us view our tiling as a pile of $1 \times 1 \times 1$ cubes in the box located between the planes $x_1 = 0$, $x_1 = a$, $x_2 = 0$, $x_2 = b$, $x_3 = 0$, $x_3 = c$. Then Figure 1 represents a projection of the border of the 3-D-diagram to the plane $x_1 + x_2 + x_3 = 0$ parallel to the vector $(1, 1, 1)$.

For any $v \in \mathbb{R}^3$, denote by $h(v)$ the Euclidian distance from v to the plane $x_1 + x_2 + x_3 = 0$ divided by $\sqrt{3}$, and denote by $\hat{h}(v)$ the distance from v to the union of coordinate planes $x_1 x_2 x_3 = 0$ computed along the $(1, 1, 1)$-direction and divided by $\sqrt{3}$.

Recall that the weight of a tiling is given by the formula:

$$w(\mathcal{T}) = const \cdot \prod_{\lozenge \in \mathcal{T}} w(j),$$

where j is as in Figure 1.

Grouping all the $1 \times 1 \times 1$ cubes of the plane partitions into columns with fixed coordinates (x_2, x_3), we can rewrite the weight in the form

$$w(\mathcal{T}) = const \cdot \prod_{\text{⬡}} \frac{w(j)}{w(j-1)},$$

where the product is taken over all cubes of the plane partition, and j stands for the j-coordinate (see Figure 1) of the top vertex of the cube.

Collecting factors with the same j, we obtain

$$w(\mathcal{T}) = const \cdot \prod_{v} \left(\frac{w(j)}{w(j-1)} \right)^{h(v) - \hat{h}(v)},$$

where the product is taken over all points v on the border of the plane partition, whose three coordinates (x_1, x_2, x_3) are integers. Equivalently, one can think of the product being taken over all vertices of the triangular lattice inside the hexagon. Note that we may replace $h(v) - \hat{h}(v)$ by $h(v)$ since the remaining product depends only on (N, S, T). This gives

$$w(\mathcal{T}) = const \cdot \prod_{v} \left(\frac{w(j)}{w(j-1)} \right)^{h(v)}.$$

For the q-Racah case we obtain

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)},$$

while for the q-Hahn case

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{q^{-j}}{q^{-j+1}} \right)^{h(v)} = const \cdot \prod_v q^{-h(v)} = const \cdot q^{-volume}.$$

2.4 Results and variational interpretation

Our results are of two kinds.

First, for each of the probability distributions on tilings described above we construct explicit discrete time Markov chains that relate random tilings of hexagons of various sizes. The elementary steps of these chains change the size of the hexagon from $a \times b \times c$ to $a \times (b \mp 1) \times (c \pm 1)$.

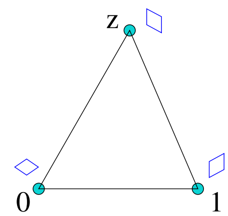
Randomness in each step consists in generating finitely many independent one-dimensional random variables. It takes $O(a(b+c))$ arithmetic operations to produce a tiling of the $a \times (b-1) \times (c+1)$ hexagon using a tiling of the $a \times b \times c$ hexagon.

Together with the trivial observation that there is exactly one tiling of an $a \times (b+c) \times 0$ hexagon, these chains provide, in particular, an efficient perfect sampling algorithm for random tilings distributed according to the real q-Racah, imaginary q-Racah, trigonometric q-Racah, q-Hahn, Racah and Hahn distributions.

A description of the algorithm can be found in Section 6, and in Section 9 we provide some pictures generated using this algorithm.

Second, we evaluate the asymptotics of the local behavior of our measures as all sides of the hexagon tend to infinity comparably.

It is known, see [Ke1], [Ke2], [OR], [Sh], [KOS], [BS], that for any three positive numbers (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = 1$, there exists a unique translation-invariant Gibbs measure on lozenge tilings of the whole plane such that in a large box, the numbers $\{p_j\}_{j=1}^{3}$ provide asymptotic ratios of the number of lozenges of three types. It is convenient to encode the triple (p_1, p_2, p_3) by a complex number z in the upper-half plane so that the triangle $(0, 1, z)$ has angles $(\pi p_1, \pi p_2, \pi p_3)$. The correspondence between angles and lozenge types is indicated in the figure below.



If one of the p_j's tends to 1 (for example, z tends to a point in $\mathbb{R}$ away from $\{0, 1\}$), then the corresponding measure degenerates to the "frozen" (nonrandom) tiling of the plane by lozenges of the corresponding type.

Theorem 2.1. *Introduce a small parameter $\varepsilon \ll 1$, and set*

$$S = \mathsf{S}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \mathsf{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \mathsf{N}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad q = \mathsf{q}^{\varepsilon + o(\varepsilon)}.$$

Then near a given point $(t, x) = (\mathsf{t}\varepsilon^{-1}, \mathsf{x}\varepsilon^{-1})$, the random tiling converges, as $\varepsilon \to 0$, to a certain ergodic translation-invariant Gibbs measure. The parameter z of this measure (encoding the slope (p_1, p_2, p_3) via angles as described) is the unique solution in the upper half-plane to the (quadratic) equation

$$Q(u, v) = 0, \tag{4}$$

where

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \tag{5}$$

and Q is a degree 2 polynomial in (u, v):

$$\begin{aligned} Q(u, v) &= u^2 \\ &+ \Big(\mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + \mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{T}} + \mathsf{q}^{-2\mathsf{S}+\mathsf{T}} + \mathsf{q}^{-\mathsf{S}-\mathsf{N}} - \mathsf{q}^{-\mathsf{S}} - \mathsf{q}^{-\mathsf{S}+\mathsf{T}}) + \kappa^4 \mathsf{q}^{-\mathsf{S}+\mathsf{N}}\Big) v^2 \\ &+ \Big(\mathsf{q}^{\mathsf{T}-\mathsf{S}} + \mathsf{q}^{-\mathsf{N}} + \kappa^2(\mathsf{q}^{\mathsf{N}} + \mathsf{q}^{-\mathsf{S}})\Big) uv - \Big(\mathsf{q}^{\mathsf{T}} + \mathsf{q}^{\mathsf{T}-\mathsf{S}-\mathsf{N}} + \kappa^2(1 + q^{\mathsf{N}-\mathsf{S}+\mathsf{T}})\Big) v \\ &- (1 + \mathsf{q}^{\mathsf{T}})u + \mathsf{q}^{\mathsf{T}}. \end{aligned} \tag{6}$$

If the solutions to this equation in z are real, then the limit measure is frozen.

Let us now explain how one could guess these formulas. In Section 8 we present a rigorous proof of Theorem 2.1 which uses an argument of a different kind. The remainder of this section is purely empirical; we hope to address the same issues rigorously in a later publication.

Although Theorem 2.1 describes the microscopic behavior of our model, the parameters (p_1, p_2, p_3) of the limit measure are closely connected with macroscopic properties.

If we view tilings as stepped surfaces in a box and scale them in such a way that the bounding box remains fixed, then it is plausible that in the limit we will observe a deterministic limit shape. The normal vector to this limit shape at any point has to coincide with the vector (p_1, p_2, p_3) of the local limit measure at this point.

The concentration of the measure near the limit shape is known in the q-Hahn case, see [CKP], [KO]. The limit shape is the unique solution of a certain variational problem. It is not hard to pose such a variational problem in the q-Racah case as well.

Recall that in Section 2.3 we found the following representation for the weight of a tiling:

$$w(\mathcal{T}) = const \cdot \prod_v \left(\frac{\kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}}{\kappa q^{j-(S+3)/2} - \dfrac{1}{\kappa q^{j-(S+3)/2}}} \right)^{h(v)} .$$

Taking the logarithm of $w(\mathcal{T})$ and removing the constant we get

$$\sum_v h(v) \left[\ln \left(\kappa q^{j-(S+1)/2} - \frac{1}{\kappa q^{j-(S+1)/2}} \right) - \ln \left(\kappa q^{j-(S+3)/2} - \frac{1}{\kappa q^{j-(S+3)/2}} \right) \right].$$

This is a Riemannian sum for an integral, and as $\varepsilon \to 0$ this yields, up to second order terms,

$$\frac{1}{\varepsilon^2} \int_{\text{hexagon}} \mathsf{h}(\mathsf{x}, \mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}},$$

where x, t are normalized coordinates inside the hexagon, h is the normalized height function, and $\mathsf{j} = \mathsf{x} - \mathsf{t}/2$.

Following [CKP] and [KOS] we know that the number of stepped surfaces in an ε-neighborhood of a given height function is asymptotically

$$\exp \left(\frac{1}{\varepsilon^2} \int \sigma(\nabla \mathsf{h}) \right),$$

where σ is the surface tension that can be expressed through the Lobachevski function. (Note that the signs in [CKP] and [KOS] are different, we follow [CKP].)

Consequently, the probability of the stepped surfaces in the ε-neighborhood of a given height function is asymptotically proportional to

$$\exp \left[\frac{1}{\varepsilon^2} \left(\int \sigma(\nabla \mathsf{h}) + \int \mathsf{h}(\mathsf{x}, \mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}} \right) \right],$$

and to find the limit shape one has to maximize this expression. We conclude that the limit shape can be found as a solution of the variational problem:

$$\int \sigma(\nabla \mathsf{h}) + \int \mathsf{h}(\mathsf{x}, \mathsf{t}) \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}} \to \max$$

Let us write down the Euler-Lagrange equation for this variational problem. It is convenient to use our parameter z instead of partial derivatives of the limit height function h. We know (see [KO]) that the Euler-Lagrange equation for the first term is the complex Burgers equation

$$\frac{z_t}{z} - \frac{z_x}{1-z} = 0.$$

Adding it to the trivial Euler-Lagrange equation for the second term we obtain

$$\frac{z_t}{z} - \frac{z_x}{1-z} = \frac{\partial \ln(\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}} - \frac{1}{\kappa \mathsf{q}^{-\mathsf{S}/2+\mathsf{j}}})}{\partial \mathsf{j}},$$

or

$$\frac{z_t}{z} - \frac{z_x}{1-z} = \ln(\mathsf{q}) \frac{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t}+1}{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t}-1}.$$

This is a quasilinear equation. The equations for characteristics have the form

$$\frac{dt}{ds} = \frac{1}{z}, \quad \frac{dx}{ds} = \frac{1}{z-1}, \quad \frac{dz}{ds} = \ln(\mathsf{q}) \frac{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t}+1}{\kappa^2 \mathsf{q}^{-\mathsf{S}+2x-t}-1}.$$

We find two first integrals

$$u = \frac{z\mathsf{q}^{\mathsf{t}} - \kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}, \qquad v = \frac{(1-z)\mathsf{q}^{\mathsf{x}}}{1 - z\kappa^2 \mathsf{q}^{-\mathsf{S}+2\mathsf{x}-\mathsf{t}}}.$$

Any solution of the partial derivative equation has the form (cf. the proof of Corollary 1 in [KO])

$$Q(u, v) = 0,$$

where Q is a suitable analytic function. This equation defines z as a function of t and x.

In the q-Hahn case, it is known that $Q(u, v)$ is a second degree polynomial. It is natural to assume that the same is true for the q-Racah case, and then the requirement that the (t, x)-curve where z degenerates to $\mathbb{R}$ is tangent to the six sides of the hexagon leads to the formula of Theorem 2.1. It remains a challenge to check if Q is still algebraic for more general polygonal domains, as was shown in [KO] for the q-Hahn case.

3 Weight sums

A horizontal lozenge in the tiling interpretation corresponds to a hole (=absence of a particle) in the nonintersecting paths or N-point configuration interpretation. The coordinates (i, j) of the horizontal lozenge in the formulas (1) and (3) correspond to the coordinates (t, x) of a hole in the following way:

$$\begin{cases} i = t, \\ j = x - t/2 + 1. \end{cases}$$

Now consider any vertical section $t = t_0$ of the family of nonintersecting paths (see Figure 3) and fix the corresponding N-point configuration $x_1 < x_2 < \cdots < x_N$.

Denote by $C(t; x_1, \ldots, x_N)$ the product of weights corresponding to the holes on this vertical line.

Denote by $L(t; x_1, \ldots, x_N)$ the sum of the products of weights corresponding to holes situated to the left of the vertical line. The sum is taken over all families of paths connecting the points $\{(0, 0), (0, 1), \ldots, (0, N-1)\}$ to the points $\{(t_0, x_1), (t_0, x_2), \ldots, (t_0, x_N)\}$.

Denote by $R(t; x_1, \ldots, x_N)$ the sum of the products of weights corresponding to holes situated to the right of the vertical line. The sum is taken over all families of paths connecting the points $\{(t_0, x_1), (t_0, x_2), \ldots, (t_0, x_N)\}$ to the points $\{(T, S), (T, S + 1), \ldots, (T, S + N - 1)\}$.

The following three propositions correspond to the case of the q-Racah weight (3).

Set

$$\mu_{t,S}(x) = q^{-x} + \kappa^2 q^{x-S-t+1}$$

Here and below we use the q-Pochhammer symbol:

$$(a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1}).$$

Proposition 3.1. *We have*

$$L_t(x_1, \ldots, x_N) = const_t \cdot \prod_{1 \le i < j \le N} (\mu_{t,S}(x_i) - \mu_{t,S}(x_j)) \\ \times \prod_{i=1}^{N} \frac{q^{x_i(t+N-1)}(1 - \kappa^2 q^{2x_i - S - t + 1})}{(q^{-1}; q^{-1})_{t+N-1-x_i} \cdot (q; q)_{x_i} \cdot (\kappa^2 q^{x_i - t - S + 1}; q)_{t+N}}.$$

Proof. Here and in the proof of the next lemma we should consider four cases depending on the value of t (see formulas (8)-(11)). The proofs are similar in all four cases and we consider only the one that corresponds to the pictures in Lemma 10.2 and Lemma 10.3 (i.e., $S < t < T - S$)

Let us use Lemma 10.2 that expresses the required weight sum in terms of the point configuration complementary to $\{x_i\}$, i.e., in terms of the positions of the horizontal lozenges that we call holes. Denote the positions of holes by $\{y_i\}$. Performing the limit transition (2) we get.

$$L_t(x_1, \ldots, x_N) = const \cdot \prod_{1 \le i < j \le S} (\mu_{t,S}(y_i) - \mu_{t,S}(y_j)) \\ \times \prod_{1 \le i \le S} \frac{q^{(S-t)(S+N-1-y_i)}(q^{t-S+1}; q)_{S+N-1-y_i}(q^{-2N-S+t+1}/\kappa^2; q)_{S+N-1-y_i}}{(q; q)_{S+N-1-y_i}(q^{-2N+1}/\kappa^2; q)_{S+N-1-y_i}}.$$

To finish the proof we rewrite the last formula in terms of particles $\{x_i\}$ instead of holes $\{y_i\}$. For the second factor this procedure is simple, while for the first one we use the following observation:

$$\prod_{1\le i<j\le N}(\mu_{t,S}(y_i)-\mu_{t,S}(y_j)) = \prod_{1\le i<j\le N}(\mu_{t,S}(x_i)-\mu_{t,S}(x_j)) \times \prod_{0\le u<v\le S+N-1}(\mu_{t,S}(u)-\mu_{t,S}(v)) \times \prod_{1\le i\le N} \frac{1}{\prod\limits_{0\le u<x_i}(\mu_{t,S}(x_i)-\mu_{t,S}(u)) \prod\limits_{x_i<u\le S+N-1}(\mu_{t,S}(u)-\mu_{t,S}(x_i))}.$$

The product over $u<v$ depends only on t, while the last two products over u are easily expressible in terms of q-Pochhammer symbols.

□

Proposition 3.2. *We have*

$$R_t(x_1,\dots,x_N) = const_t \cdot \prod_{1\le i<j\le N}(\mu_{t,S}(x_i)-\mu_{t,S}(x_j)) \times \prod_{i=1}^{N} \frac{(1-\kappa^2 q^{2x_i-S-t+1})q^{x_i(T-t+N-1)}}{(q^{-1};q^{-1})_{S+N-1-x_i}\cdot(q;q)_{x_i+T-t-S}\cdot(\kappa^2 q^{x_i-T+1};q)_{N+T-t}}.$$

Proof. Performing the limit transition (2) in Lemma 10.3 we get

$$R_t(x_1,\dots,x_N) = const \cdot \prod_{1\le i<j\le S}(\mu_{t,S}(y_i)-\mu_{t,S}(y_j)) \times \prod_{1\le i\le S} \frac{q^{(S-T+t)(S+N-1-y_i)}(q^{-N-S+1};q)_{S+N-1-y_i}(q^{T-S-N+1}/\kappa^2;q)_{S+N-1-y_i}}{(q^{-N+1-T+t};q)_{S+N-1-y_i}(q^{t-N+1}/\kappa^2;q)_{S+N-1-y_i}}$$

Again expressing all the factors in terms of $\{x_i\}$ we obtain the desired result.

□

Proposition 3.3. *We have*

$$C_t(x_1,\dots,x_N) = const_t \cdot \prod_{i=1}^{N} \frac{q^{x_i}}{1-\kappa^2 q^{2x_i-S-t+1}}.$$

Proof. Clearly,

$$C_t(x_1,\dots,x_N) = \prod_{i=1}^{S}\left(\kappa q^{y_i-S/2-t/2+1/2} - \frac{1}{\kappa q^{y_i-S/2-t/2+1/2}}\right),$$

where $\{y_i\}$ is the point configuration complementary to $\{x_i\}$. Expressing the product in terms of $\{x_i\}$ we get the result. □

4 Distributions and transition probabilities

We consider our probability measure on the set of tilings of a given hexagon as a Markov chain in the space of N-tuples of integers. The Markov property can be easily seen in the following form: The past and the future are independent given the present. In this way the Markov property reduces to the fact that a lozenge tiling of the hexagon is a union of a tiling to the left of a vertical line $t = const$ and a tiling to the right of this vertical line. Denote this Markov chain by $X(t)$, $t = 0, 1, \dots T$.

Set

$$\mathfrak{X}_{N,T}^{S,t} = \{x \in \mathbb{Z} : \max(0, t+S-T) \le x \le \min(t+N-1, S+N-1)\}$$

and

$$\mathcal{X}_{N,T}^{S,t} = \{(x_1, x_2, \dots, x_N) \in (\mathfrak{X}_{N,T}^{S,t})^N : x_1 < x_2 < \dots < x_N\}.$$

$\mathfrak{X}_{N,T}^{S,t}$ is the section of our hexagon by the vertical line with coordinate t, and $\mathcal{X}_{N,T}^{S,t}$ is the set of all N-tuples in this section.

Clearly, $X(t)$ takes values in $\mathcal{X}_{N,T}^{S,t}$.

The following theorem gives the one-dimensional distributions of our chain at the time t, which are probability distributions on N-tuples of integers. Denote by $\rho_{S,t}$ the one-dimensional distribution of the process $X(t)$ (below we explain why we keep S in the notation but omit all the other parameters).

Theorem 4.1.

$$\mathrm{Prob}\{X(t) = (x_1, x_2, \dots, x_N)\} = const \cdot \prod_{i<j} (\mu_{t,S}(x_i) - \mu_{t,S}(x_j))^2 \prod_{i=1}^{N} w_{t,S}(x_i),$$

where

$$\mu_{t,S}(x) = q^{-x} + \kappa^2 q^{x-S-t+1}$$

and

$$w_{t,S}(x) = \frac{(-1)^{t+S} q^{x(2N+T-1)} (1 - \kappa^2 q^{2x-t-S+1})}{(q;q)_x (q;q)_{T-S-t+x} (q^{-1};q^{-1})_{t+N-x-1} (q^{-1};q^{-1})_{S+N-x-1}} \times \frac{1}{(\kappa^2 q^{x-T+1};q)_{T+N-t} (\kappa^2 q^{x-t-S+1};q)_{N+t}}.$$

Proof. Clearly,

$$\mathrm{Prob}\{X(t) = (x_1, x_2, \dots, x_N)\} \propto L(t; x_1, \dots, x_N) C(t; x_1, \dots, x_N) R(t; x_1, \dots, x_N)$$

Propositions 3.1, 3.2, 3.3 imply the result. □

Observe that $w_{t,S}(x)$ is (up to the factor not depending on x) the weight function of the q-Racah orthogonal polynomials, see e. g. [KS, Section 3.2]. These polynomials are given by the formula

$$R_n(\mu(x); \alpha, \beta, \gamma, \delta \mid q) = {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix} \middle| q; q \right), \qquad (7)$$

where

$$\mu(x) = q^{-x} + \gamma\delta q^{x+1},$$

and $\alpha q = q^{-M}$ or $\beta\delta q = q^{-M}$ or $\gamma q = q^{-M}$ for a nonnegative integer M. They are orthogonal on $\{0, 1, \ldots, M\}$ with respect to the weight function

$$w(x) = \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_x}{(q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_x} \frac{1 - \gamma\delta q^{2x+1}}{(\alpha\beta q)^x (1 - \gamma\delta q)}.$$

The correspondence between the parameters of polynomials and parameters of our model is established in the following way.

(i). $t < S, \quad T - t - S > 0, \quad 0 \leq x \leq t + N - 1,$

$$\begin{array}{lcl} q_{(qRacah)} & = & q \\ \alpha_{(qRacah)} & = & q^{-S-N} \\ \beta_{(qRacah)} & = & q^{S-T-N} \\ \gamma_{(qRacah)} & = & q^{-t-N} \\ \delta_{(qRacah)} & = & \kappa^2 q^{-S+N} \end{array} \tag{8}$$

(ii). $S - 1 < t < T - S + 1, \quad 0 \leq x \leq S + N - 1,$

$$\begin{array}{lcl} q_{(qRacah)} & = & q \\ \alpha_{(qRacah)} & = & q^{-t-N} \\ \beta_{(qRacah)} & = & q^{t-T-N} \\ \gamma_{(qRacah)} & = & q^{-S-N} \\ \delta_{(qRacah)} & = & \kappa^2 q^{-t+N} \end{array} \tag{9}$$

(iii). $T - S - 1 < t < S, \quad 0 \leq x - (t + S - T) \leq T - S + N - 1,$

$$\begin{array}{lcl} q_{(qRacah)} & = & q \\ \alpha_{(qRacah)} & = & q^{-T+t-N} \\ \beta_{(qRacah)} & = & q^{-t-N} \\ \gamma_{(qRacah)} & = & q^{-T-N+S} \\ \delta_{(qRacah)} & = & \kappa^2 q^{-T+t+N} \\ x_{(qRacah)} & = & T - t - S + x \end{array} \tag{10}$$

(iv). $t > T - S - 1, \quad t > S - 1, \quad 0 \leq x - (t + S - T) \leq T - t + N - 1,$

$$\begin{array}{lcl} q_{(qRacah)} & = & q \\ \alpha_{(qRacah)} & = & q^{-T-N+S} \\ \beta_{(qRacah)} & = & q^{-S-N} \\ \gamma_{(qRacah)} & = & q^{-T+t-N} \\ \delta_{(qRacah)} & = & \kappa^2 q^{-T+N+S} \\ x_{(qRacah)} & = & T - t - S + x \end{array} \tag{11}$$

Let us also describe what happens if one performs the limit transitions described in Section 2.2.

If we send $\kappa \to 0$ (the weight of a plane partition becomes proportional to $q^{-volume}$), then we obtain the weight function

$$w(x) = \frac{q^{x(2N+T-1)}}{(q;q)_x (q;q)_{T-S-t+x} (q^{-1};q^{-1})_{t+N-x-1} (q^{-1};q^{-1})_{S+N-x-1}}$$

This is exactly (up to the factor not depending on x) the weight function of the q-Hahn polynomials.

If we set $\kappa = q^K$ and send $q \to 1$, then the weight function becomes

$$w(x) = \frac{1}{x!(T-S-t+x)!(t+N-x-1)!(S+N-x-1)!} \\ \times \frac{K+2x-t-S+1}{(K+x-T+1)_{T+N-t}(K+x-t-S+1)_{N+t}}.$$

This is the weight function of the Racah orthogonal polynomials.

Finally, if we send $\kappa \to 0$ and set $q = 1$ (this case corresponds to the uniform measure on the set of lozenge tilings of the hexagon) then we get the weight function

$$w(x) = \frac{1}{x!(T-S-t+x)!(t+N-x-1)!(S+N-x-1)!}.$$

This is the weight function of the Hahn polynomials. This case was previously studied in [J1], [J2], [Gor], see also references therein.

We also need the transition probabilities of the Markov chain $X(t)$.

Proposition 4.2.

$$\mathrm{Prob}\{X(t+1) = Y | X(t) = X\} \\ = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i) - \mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \prod_{y_i = x_i + 1} w_1(x_i) \prod_{y_i = x_i} w_0(x_i),$$

where

$$w_0(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}}$$

and

$$w_1(x) = q^{T+N-1-t}(1 - q^{x-S-N+1}) \frac{1 - \kappa^2 q^{x-T+1}}{1 - \kappa^2 q^{2x-t-S+1}}.$$

Proof. We use

$$\mathrm{Prob}\{X(t+1) = Y | X(t) = X\} = \frac{L_t(X) C_t(X) C_{t+1}(Y) R_{t+1}(Y)}{L_t(X) C_t(X) R_t(X)} \\ = \frac{R_{t+1}(Y) C_{t+1}(Y)}{R_t(X)}$$

and Propositions 3.3, 3.2. □

Next let us compute the *cotransition* probabilities ($t \to t-1$).

Proposition 4.3.

$$\mathrm{Prob}\{X(t-1)=Y|X(t)=X\} = \\ const \cdot \prod_{i<j} \frac{\mu_{t-1,S}(y_i)-\mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)} \tilde{w}_1(x_i) \prod_{y_i=x_i} \tilde{w}_0(x_i),$$

where

$$\tilde{w}_0(x) = -(1-q^{x-t-N+1}) \frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}$$

and

$$\tilde{w}_1(x) = q^{-(t+N-1)}(1-q^x) \frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}$$

Proof. We use

$$\mathrm{Prob}\{X(t-1)=Y|X(t)=X\} = \frac{L_{t-1}(Y)C_{t-1}(Y)C_t(X)R_t(X)}{L_t(X)C_t(X)R_t(X)} = \frac{L_{t-1}(Y)C_{t-1}(Y)}{L_t(X)}$$

and Propositions 3.1, 3.3. ☐

5 Families of stochastic matrices

This section and the next one are similar to [BG], where the Hahn case was treated, and we have tried to keep the notations and statements of theorems unchanged where possible.

5.1 Definition of matrices

We want to introduce four families of stochastic matrices $P^{S,t}_{t+}$, $P^{S,t}_{t-}$, $P^{S,t}_{S+}$, $P^{S,t}_{S-}$.

$P^{S,t}_{t+}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S,t+1}|$ matrix, $X=(x_1<\cdots<x_N) \in \mathcal{X}^{S,t}$, $Y=(y_1<\cdots<y_N) \in \mathcal{X}^{S,t+1}$;

if $y_i - x_i \in \{0,1\}$ for every i, then

$$P^{S,t}_{t+}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i)-\mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)} \prod_{y_i=x_i+1} w_1(x_i) \prod_{y_i=x_i} w_0(x_i),$$

where

$$w_0(x) = -(1-q^{x+T-t-S}) \frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}},$$

$$w_1(x) = q^{T+N-1-t}(1-q^{x-S-N+1}) \frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}},$$

and $P^{S,t}_{t+}(X,Y)=0$ otherwise.

$P^{S,t}_{S+}(X,Y)$ is an $|\mathcal{X}^{S,t}|\times|\mathcal{X}^{S+1,t}|$ matrix, $X=(x_1<\dots<x_N)\in\mathcal{X}^{S,t}$, $Y=(y_1<\dots<y_n)\in\mathcal{X}^{S+1,t}$;

If $y_i-x_i\in\{0,1\}$ for every i, then

$$P^{S,t}_{S+}(X,Y)=const\cdot\prod_{i<j}\frac{\mu_{t,S+1}(y_i)-\mu_{t,S+1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\prod_{y_i=x_i+1}w_1(x_i)\prod_{y_i=x_i}w_0(x_i),$$

where

$$w_0(x)=-(1-q^{x+T-t-S})\frac{1-\kappa^2q^{x+N-S}}{1-\kappa^2q^{2x-t-S+1}},$$

$$w_1(x)=q^{T+N-1-S}(1-q^{x-t-N+1})\frac{1-\kappa^2q^{x-T+1}}{1-\kappa^2q^{2x-t-S+1}},$$

and $P^{S,t}_{S+}(X,Y)=0$ otherwise.

$P^{S,t}_{t-}(X,Y)$ is an $|\mathcal{X}^{S,t}|\times|\mathcal{X}^{S,t-1}|$ matrix, $X=(x_1<\dots<x_N)\in\mathcal{X}^{S,t}$, $Y=(y_1<\dots<y_n)\in\mathcal{X}^{S,t-1}$;

If $y_i-x_i\in\{-1,0\}$ for every i, then

$$P^{S,t}_{t-}(X,Y)=const\cdot\prod_{i<j}\frac{\mu_{t-1,S}(y_i)-\mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\prod_{y_i=x_i-1}\tilde{w}_1(x_i)\prod_{y_i=x_i}\tilde{w}_0(x_i),$$

where

$$\tilde{w}_0(x)=-(1-q^{x-t-N+1})\frac{1-\kappa^2q^{x-S-t+1}}{1-\kappa^2q^{2x-t-S+1}},$$

$$\tilde{w}_1(x)=q^{-(t+N-1)}(1-q^{x})\frac{1-\kappa^2q^{x+N-S}}{1-\kappa^2q^{2x-t-S+1}},$$

and $P^{S,t}_{t-}(X,Y)=0$ otherwise.

$P^{S,t}_{S-}(X,Y)$ is an $|\mathcal{X}^{S,t}|\times|\mathcal{X}^{S-1,t}|$ matrix, $X=(x_1<\dots<x_N)\in\mathcal{X}^{S,t}$, $Y=(y_1<\dots<y_n)\in\mathcal{X}^{S-1,t}$;

If $y_i-x_i\in\{-1,0\}$ for every i, then

$$P^{S,t}_{S-}(X,Y)=const\cdot\prod_{i<j}\frac{\mu_{t,S-1}(y_i)-\mu_{t,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\prod_{y_i=x_i-1}\tilde{w}_1(x_i)\prod_{y_i=x_i}\tilde{w}_0(x_i),$$

where

$$\tilde{w}_0(x)=-(1-q^{x-S-N+1})\frac{1-\kappa^2q^{x-S-t+1}}{1-\kappa^2q^{2x-t-S+1}},$$

$$\tilde{w}_1(x)=q^{-(S+N-1)}(1-q^{x})\frac{1-\kappa^2q^{x+N-t}}{1-\kappa^2q^{2x-t-S+1}},$$

and $P^{S,t}_{S-}(X,Y)=0$ otherwise.

Looking at the sets that parameterize rows and columns of these matrices one can say that $P^{S,t}_{t+}$ increases t, $P^{S,t}_{t-}$ decreases t, while $P^{S,t}_{S+}$ increases S and $P^{S,t}_{S-}$ decreases S. This explains our notation.

Theorem 5.1. *With appropriate choices of normalizing constants, all four types of matrices defined above are stochastic. They preserve the family of measures $\rho_{S,t}$. In other words*

$$\sum_{Y \in \mathcal{X}^{S,t\pm 1}} P_{t\pm}^{S,t}(X,Y) = 1, \qquad \sum_{Y \in \mathcal{X}^{S,S\pm 1}} P_{t\pm}^{S,t}(X,Y) = 1, \tag{12}$$

$$\rho_{S,t\pm 1}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P_{t\pm}^{S,t}(X,Y) \cdot \rho_{S,t}(X),$$
$$\rho_{S\pm 1,t}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P_{S\pm}^{S,t}(X,Y) \cdot \rho_{S,t}(X).$$

Proof. Propositions 4.2 and 4.3 imply the claim for $P_{t+}^{S,t}(X,Y)$ and $P_{t-}^{S,t}(X,Y)$.

Now observe that the space $\mathcal{X}^{S,t}$ is unaffected when we interchange parameters t and S, i.e.,

$$\mathcal{X}^{S,t} = \mathcal{X}^{t,S}.$$

Moreover, the measures $\rho_{S,t}$ are also invariant under $S \leftrightarrow t$, i.e.,

$$\rho_{S,t} = \rho_{t,S}.$$

(This is a consequence of our special choice of the parameter κ which included additional factor $q^{-S/2}$.)

Finally, note that $P_{t+}^{S,t}(X,Y)$ becomes $P_{S+}^{S,t}(X,Y)$ under $S \leftrightarrow t$ and $P_{t-}^{S,t}(X,Y)$ becomes $P_{S-}^{S,t}(X,Y)$.

Therefore, applying $S \leftrightarrow t$ to the relations for $P_{t\pm}$ we obtain the needed relations for $P_{S\pm}$. □

5.2 Determinantal representation

In this section we write our stochastic matrices in a determinantal form. This representation is very convenient for various computations.

First, we introduce 4 new two-diagonal matrices.

For $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S,t+1}$,

$$U_{t+}^{S,t}(x,y) = \begin{cases} -q^{T+N-1-t}(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x+1, \\ (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S+1,t}$,

$$U_{S+}^{S,t}(x,y) = \begin{cases} -q^{T+N-1-S}(1-q^{x-t-N+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x+1, \\ (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S,t-1}$,

$$U_{t-}^{S,t}(x,y) = \begin{cases} q^{-(t+N-1)}(1-q^x)\frac{1-\kappa^2 q^{x+N-S}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x-1, \\ -(1-q^{x-t-N+1})\frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise;} \end{cases}$$

and for $x \in \mathfrak{X}^{S,t}$, $y \in \mathfrak{X}^{S-1,t}$,

$$U_{S-}^{S,t}(x,y) = \begin{cases} q^{-(S+N-1)}(1-q^x)\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x-1, \\ -(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x-S-t+1}}{1-\kappa^2 q^{2x-t-S+1}}, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

It is possible to express the stochastic matrices $P_{t\pm}^{S,t}$, $P_{S\pm}^{S,t}$ as certain minors of the matrices defined above.

Proposition 5.2. *We have*

$$P_{t+}^{S,t}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t+1,S}(y_i) - \mu_{t+1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U_{t+}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P_{S+}^{S,t}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S+1}(y_i) - \mu_{t,S+1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U_{S+}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P_{t-}^{S,t}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t-1,S}(y_i) - \mu_{t-1,S}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U_{t-}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}$$

$$P_{S-}^{S,t}(X,Y) = const \cdot \prod_{i<j} \frac{\mu_{t,S-1}(y_i) - \mu_{t,S-1}(y_j)}{\mu_{t,S}(x_i) - \mu_{t,S}(x_j)} \det[U_{S-}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}$$

Proof. Straightforward computation using the definitions of the stochastic matrices $P_{t\pm}^{S,t}$, $P_{S\pm}^{S,t}$ and the matrices $U_{t\pm}^{S,t}$, $U_{S\pm}^{S,t}$.

Any submatrix of a two-diagonal matrix, which has a nonzero determinant, is block-diagonal, where each block is either an upper or a lower triangular matrix. Thus, any nonzero minor is a product of suitable matrix elements. □

5.3 Commutativity

Theorem 5.3. *The families of stochastic matrices $P_{t\pm}^{S,t}$ and $P_{S\pm}^{S,t}$ commute, that is*

$$\begin{aligned} P_{t+}^{S,t} \cdot P_{S-}^{S,t+1} &= P_{S-}^{S,t} \cdot P_{t+}^{S-1,t}, \\ P_{t-}^{S,t} \cdot P_{S-}^{S,t-1} &= P_{S-}^{S,t} \cdot P_{t-}^{S-1,t}, \\ P_{t+}^{S,t} \cdot P_{S+}^{S,t+1} &= P_{S+}^{S,t} \cdot P_{t+}^{S+1,t}, \\ P_{t-}^{S,t} \cdot P_{S+}^{S,t-1} &= P_{S+}^{S,t} \cdot P_{t-}^{S+1,t}, \end{aligned}$$

for any meaningful values of S and t.

Proof. Proofs of all four cases are very similar and we consider only the first one.

$$
\begin{aligned}
(P_{t+}^{S,t}\cdot P_{S-}^{S,t+1})(X,Y) &= \sum_{Z\in\mathcal{X}^{S,t+1}} P_{t+}^{S,t}(X,Z)\cdot P_{S-}^{S,t+1}(Z,Y)\\
&= const\cdot\prod_{i<j}\frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\\
&\times\sum_{Z\in\mathcal{X}^{S,t+1}}\det[U_{t+}^{S,t}(x_i,z_j)]_{i,j=1,\dots,N}\det[U_{S-}^{S,t+1}(z_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}
$$

Applying the Cauchy-Binet identity we obtain

$$
\begin{aligned}
&\sum_{Z\in\mathcal{X}^{S,t+1}}\det[U_{t+}^{S,t}(x_i,z_j)]_{i,j=1,\dots,N}\det[U_{S-}^{S,t+1}(z_i,y_j)]_{i,j=1,\dots,N}\\
&\qquad=\det[(U_{t+}^{S,t}\cdot U_{S-}^{S,t+1})(x_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(P_{t+}^{S,t}\cdot P_{S-}^{S,t+1})(X,Y)\\
&= const\cdot\prod_{i<j}\frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\det[(U_{t+}^{S,t}\cdot U_{S-}^{S,t+1})(x_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(P_{S-}^{S,t}\cdot P_{t+}^{S-1,t})(X,Y)\\
&= const\cdot\prod_{i<j}\frac{\mu_{t+1,S-1}(y_i)-\mu_{t+1,S-1}(y_j)}{\mu_{t,S}(x_i)-\mu_{t,S}(x_j)}\det[(U_{S-}^{S,t}\cdot U_{t+}^{S-1,t})(x_i,y_j)]_{i,j=1,\dots,N}.
\end{aligned}
$$

Our claim reduces to verifying the equality

$$
U_{t+}^{S,t}\cdot U_{S-}^{S,t+1}=U_{S-}^{S,t}\cdot U_{t+}^{S-1,t}.
$$

Note that this will also imply the coincidence of normalization constants, since all matrices under consideration are stochastic.

A straightforward computation yields

$$
U_{t+S-}^{S,t}=U_{t+}^{S,t}\cdot U_{S-}^{S,t+1}=U_{S-}^{S,t}\cdot U_{t+}^{S-1,t},
$$

where

$$
U_{t+S-}^{S,t}(x,y)=\begin{cases}u_1, & \text{if } y=x+1,\\ u_0, & \text{if } y=x,\\ u_{-1}, & \text{if } y=x-1,\\ 0, & \text{otherwise,}\end{cases}
$$

and

$$u_1 = q^{T+N-1-t}(1-q^{x-S-N+1})(1-q^{x-S-N+2})\frac{(1-\kappa^2 q^{x-S-t+1})(1-\kappa^2 q^{x-T+1})}{(1-\kappa^2 q^{2x-t-S+2}))1-\kappa^2 q^{2x-t-S+1})},$$

$$u_0 = -(1-q^{x-S-N+1})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}$$
$$\times\left(q^{T-S-t}(1-q^{x+1})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+2}} + (1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x-S-t}}{1-\kappa^2 q^{2x-t-S}}\right),$$

$$u_{-1} = q^{-(S+N-1)}(1-q^{x+T-t-S})(1-q^{x})\frac{(1-\kappa^2 q^{x+N-t})(1-\kappa^2 q^{x+N-t-1})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S})}.$$

□

6 Perfect sampling algorithm

6.1 Definition of transition matrices

In this section we aim to define two new stochastic matrices

$$P^S_{S\mapsto S+1}(X,Y), \quad X\in\Omega(N,T,S), \quad Y\in\Omega(N,T,S+1)$$

and

$$P^S_{S\mapsto S-1}(X,Y), \quad X\in\Omega(N,T,S), \quad Y\in\Omega(N,T,S-1)$$

that preserve the measures $\mu(N,T,S,q,\kappa)$. Both $P^S_{S\mapsto S+1}$ and $P^S_{S\mapsto S-1}$ depend on parameters N, T, q, κ but we omit these parameters from the notation.

Suppose we are given a sequence $X=(X(0),X(1),\dots,X(T))\in\Omega(N,T,S)$ (recall that $X(t)\in\mathcal{X}^{S,t}$). Below we construct a random sequence $Y=(Y(0),\dots,Y(T))\in\Omega(N,T,S+1)$ and therefore define the transition probability (or, equivalently, stochastic matrix) $P^S_{S\mapsto S+1}(X,Y)$.

First note that $Y(0)\in\mathcal{X}^{S+1,0}$ and $|\mathcal{X}^{S+1,0}|=1$. Thus, $Y(0)$ is uniquely defined. We will perform a *sequential update*. Suppose $Y(0),Y(1),\dots,Y(t)$ have been already defined. Define the conditional distribution of $Y(t+1)$ given X, $Y(0),Y(1),\dots,Y(t)$ by

$$\begin{aligned}\mathrm{Prob}\{Y(t+1)=Z\} &= \frac{P^{S+1,t}_{t+}(Y(t),Z)\cdot P^{S+1,t+1}_{S-}(Z,X(t+1))}{(P^{S+1,t}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1))}\\ &= \frac{P^{S,t+1}_{S+}(X(t+1),Z)\cdot P^{S+1,t+1}_{t-}(Z,Y(t))}{(P^{S,t+1}_{S+}P^{S+1,t+1}_{t-})(X(t+1),Y(t))}. \quad (13)\end{aligned}$$

(The second equality follows from $\rho_{S+1,t+1}(X)P^{S+1,t+1}_{t-}(X,Y)=\rho_{S+1,t}(Y)P^{S+1,t}_{t+}(Y,X)$.)

This definition follows the idea of [DF, Section 2.3], see also [BF].

Observe that $(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))>0$ (here and below see [BG] for more details).

One could say that we choose $Y(t+1)$ using conditional distribution of the middle point in the successive application of $P_{t+}^{S+1,t}$ and $P_{S-}^{S+1,t+1}$ (or $P_{S+}^{S,t+1}$ and $P_{t-}^{S+1,t+1}$), provided that we start at $Y(t)$ and finish at $X(t+1)$ (or start at $X(t+1)$ and finish at $Y(t)$).

After performing T updates we obtain the sequence Y.

Equivalently, define $P_{S\mapsto S+1}^{S}$ by

$$P_{S\mapsto S+1}^{S}(X,Y)=\begin{cases}\displaystyle\prod_{t=0}^{T-1}\frac{P_{t+}^{S+1,t}(Y(t),Y(t+1))\cdot P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))}, \\ \qquad \text{if} \displaystyle\prod_{t=0}^{T-1}(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))>0, \\ 0, \ \text{otherwise.}\end{cases}$$

Theorem 6.1. *The matrix $P_{S\mapsto S+1}^{S}$ on $\Omega(N,T,S)\times\Omega(N,T,S+1)$ is stochastic. The transition probabilities $P_{S\mapsto S+1}^{S}(X,Y)$ preserve the measures $\mu(N,T,S,q,\kappa)$:*

$$\mu(N,T,S+1,q,\kappa)(Y)=\sum_{X\in\Omega(N,T,S)}P_{S\mapsto S+1}^{S}(X,Y)\mu(N,T,S,q,\kappa)(X).$$

Proof. See [BG]. □

Similarly to $P_{S\mapsto S+1}$, one defines a transition matrix

$$P_{S\mapsto S-1}^{S}(X,Y),\quad X\in\Omega(N,T,S),\quad Y\in\Omega(N,T,S-1),$$

by

$$P_{S\mapsto S-1}^{S}(X,Y)=\begin{cases}\displaystyle\prod_{t=0}^{T-1}\frac{P_{t+}^{S-1,t}(Y(t),Y(t+1))\cdot P_{S+}^{S-1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S-1,t}P_{S+}^{S-1,t+1})(Y(t),X(t+1))}, \\ \qquad \text{if} \displaystyle\prod_{t=0}^{T-1}(P_{t+}^{S-1,t}P_{S+}^{S-1,t+1})(Y(t),X(t+1))>0, \\ 0, \ \text{otherwise.}\end{cases}$$

Similarly to (13) there is another way to write $P_{S\mapsto S-1}^{S}$ because of the equality

$$\frac{P_{t+}^{S-1,t}(Y(t),Y(t+1))\cdot P_{S+}^{S-1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S-1,t}P_{S+}^{S-1,t+1})(Y(t),X(t+1))} \\ =\frac{P_{S-}^{S,t+1}(X(t+1),Y(t+1))\cdot P_{t-}^{S-1,t+1}(Y(t+1),Y(t))}{(P_{S-}^{S,t+1}P_{t-}^{S-1,t+1})(X(t+1),Y(t))}$$

Similarly to Theorem 6.1 one proves the following claim.

Theorem 6.2. *The matrix $P^S_{S\mapsto S-1}$ on $\Omega(N,T,S)\times\Omega(N,T,S-1)$ is stochastic. The transition probabilities $P^S_{S\mapsto S-1}(X,Y)$ preserve the measures $\mu(N,T,S,q,\kappa)$:*

$$\mu(N,T,S-1,q,\kappa)(Y) = \sum_{X\in\Omega(N,T,S)} P^S_{S\mapsto S-1}(X,Y)\mu(N,T,S,q,\kappa)(X).$$

Remark. The above construction performs a sequential update from $t=0$ to $t=T$. One can equally well update from $t=T$ to $t=0$ by suitably modifying the definitions. The resulting Markov chains also preserve the measures $\mu(N,T,S,q,\kappa)$, and they are different from the Markov chains defined above.

6.2 Algorithm for the $S\mapsto S+1$ step.

Now we suggest an algorithmic description of the Markov chain from the previous section.

Denote

$$p(x,t,q,\kappa,S,T) = \frac{1-q^{x+T-t-S-1}}{q^{T-t-S-1}(1-q^{x+1})}\,\frac{1-\kappa^2q^{x-S-t-1}}{1-\kappa^2q^{x-T+1}}\,\frac{1-\kappa^2q^{2x-t-S+1}}{1-\kappa^2q^{2x-t-S-1}}$$

and

$$\begin{aligned} P(x,t,q,\kappa,S,T;k) &= \prod_{i=1}^{k} p(x+i-1,t,q,\kappa,S,T)\\ &= \frac{(q^{x+T-t-S-1};q)_k}{q^{k(T-t-S-1)}(q^{x+1};q)_k}\,\frac{(\kappa^2q^{x-S-t-1};q)_k}{(\kappa^2q^{x-T+1};q)_k}\,\frac{(\kappa^2q^{2x-t-S+1};q^2)_k}{(\kappa^2q^{2x-t-S-1};q^2)_k}. \end{aligned}$$

Denote by $D(x,t,S;n)$ (it also depends on q,κ,T, but we omit these parameters) the probability distribution on $\{0,1,\dots,n\}$ given by

$$\mathrm{Prob}(\{k\}) = D(x,t,S;n)\{k\} = \frac{P(x,t,q,\kappa,S,T;k)}{\sum_{j=0}^{n} P(x,t,q,\kappa,S,T;j)}. \tag{14}$$

Suppose we are given $X=(X(0),X(1),\dots,X(T))\in\Omega(N,T,S)$. We want to construct $Y=(Y(0),Y(1),\dots,Y(T))\in\Omega(N,T,S+1)$.

In the first place we note that $Y(0)$ is uniquely defined,

$$Y(0) = (0,1,\dots,N-1).$$

Then we perform T sequential updates, i.e., for $t=0,1,\dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t)=(y_1<y_2<\dots<y_N)$ and $X(t+1)=(x_1<x_2<\dots<x_N)$. We are going to construct $Y(t+1)=(z_1<z_2<\dots<z_N)$.

Recall that

$$z_i\in\mathfrak{X}^{S+1,t+1} = \{x\in\mathbb{Z}\mid \max(0,t+S-T+2)\le x\le \min(t+N,S+N)\}.$$

Observe that $Y(t)$ and $X(t+1)$ satisfy $(P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t), X(t+1)) > 0$. This implies that $x_i - y_i$ is equal to either -1, 0 or 1 for every i.

• First, consider all indices i such that $x_i - y_i = 1$. For every such i we set $z_i = x_i$.

• Second, consider all indices i such that $x_i - y_i = -1$ and set $z_i = y_i$.

• Finally, consider all remaining indices, i.e., all i such that $x_i = y_i$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a (k,l)-block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i = y_i = k, \quad x_{i+1} = y_{i+1} = k+1, \quad \dots, \quad x_{i+l-1} = y_{i+l-1} = k+l-1$$

and

$$y_{i-1} < k-1, \quad y_{i+l} > k+l.$$

For each (k,l)-block we perform the following procedure: consider a random variable ξ distributed according to $D(k,t,S;l)$ (ξ's corresponding to different (k,l)-blocks are independent). Set $z_i = x_i$ for the first ξ integers of the block (their coordinates are $k, k+1, \dots, k+\xi-1$) and set $z_i = x_i + 1$ for the rest of the block.

At Figure 4 we provide an example of constructing $Y(t+1)$ using $X(t+1)$ and $Y(t)$: there is only one (k,l)-block and it splits into two groups, here $\xi = 2$.

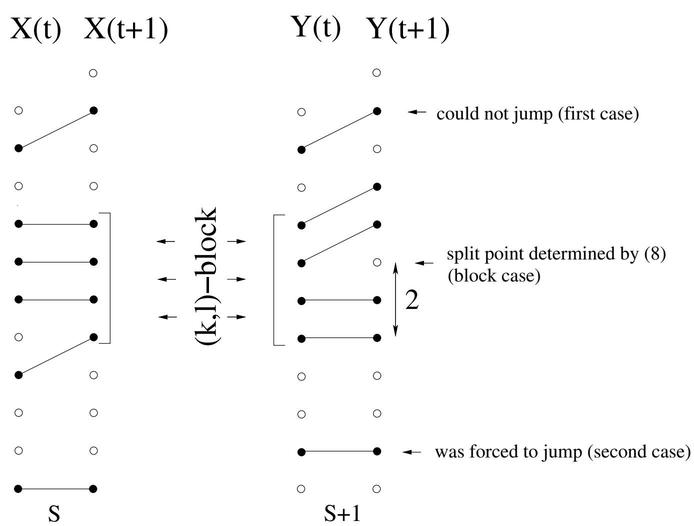


Figure 4. Example of (k,l)-block split, $l = 4$, $\xi = 2$.

Theorem 6.3. *The algorithm described above is precisely the $S \mapsto S+1$ Markov step given by $P_{S \mapsto S+1}^{S}$.*

Proof. Straightforward computations. See [BG] for some details. ∎

Remarks. Setting $\kappa = 0$ in the formulas for the distribution $D(x,t,S;n)$ we obtain the perfect sampling algorithm for boxed plane partitions distributed as $q^{-volume}$.

Sending $q \to 1$ in the formulas for the distribution $D(x,t,S;n)$ as described in Section 2.2, we get a perfect sampling algorithm for the Racah case (recall that in this case the weight of a horizontal lozenge is proportional to a linear function of its vertical coordinate).

6.3 Algorithm for $S \mapsto S-1$ step

Using similar methods we can also obtain $S \mapsto S-1$ Markov step which gives alternative way to sample a random tiling: We start from the case $T = S$ and then perform some amount of $S \mapsto S-1$ steps.

The $S \mapsto S-1$ step algorithm is very similar to the $S \mapsto S+1$ one.

Denote

$$\hat{p}(x,t,q,\kappa,S,T,N) = \frac{q^{t+1-S}(1-q^{x-t-N-1})}{(1-q^{x-S-N+1})}\frac{1-\kappa^2 q^{x+N-t-1}}{1-\kappa^2 q^{x+N-S+1}}\frac{1-\kappa^2 q^{2x-t-S+1}}{1-\kappa^2 q^{2x-t-S-1}}$$

and

$$\begin{aligned}\hat{P}(x,t,q,\kappa,S,T,N;k) &= \prod_{i=1}^{k}\hat{p}(x+i-1,t,q,\kappa,S,T,N)\\ &= \frac{q^{k(t+1-S)}(q^{x-t-N-1};q)_k}{(q^{x-S-N+1};q)_k}\frac{(\kappa^2 q^{x+N-t-1};q)_k}{(\kappa^2 q^{x+N-S+1};q)_k}\frac{(\kappa^2 q^{2x-t-S+1};q^2)_k}{(\kappa^2 q^{2x-t-S-1};q^2)_k}.\end{aligned}$$

Denote by $\hat{D}(x,t,S;n)$ the probability distribution on $\{0,1,\dots,n\}$ given by

$$\mathrm{Prob}(\{k\}) = \hat{D}(x,t,S;n)\{k\} = \frac{\hat{P}(x,t,q,\kappa,S,T,N;k)}{\sum_{j=0}^{n}\hat{P}(x,t,q,\kappa,S,T,N;j)}. \tag{15}$$

Suppose we are given $X = (X(0), X(1), \dots, X(T)) \in \Omega(N,T,S)$. We want to construct $Y = (Y(0), Y(1), \dots, Y(T)) \in \Omega(N,T,S-1)$.

As above, note that $Y(0)$ is uniquely defined,

$$Y(0) = (0,1,\dots,N-1).$$

Then we again perform T sequential updates, i.e., for $t = 0,1,\dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t) = (y_1 < y_2 < \dots < y_N)$ and $X(t+1) = (x_1 < x_2 < \dots < x_N)$. We are going to construct $Y(t+1) = (z_1 < z_2 < \dots < z_N)$.

Recall that

$$z_i \in \mathfrak{X}^{S-1,t+1} = \{x \in \mathbb{Z} \mid \max(0, t+S-T) \le x \le \min(t+N, S+N-2)\}.$$

$Y(t)$ and $X(t+1)$ satisfy $(P_{t+}^{S-1,t}P_{S+}^{S-1,t+1})(Y(t),X(t+1)) > 0$. This implies that $x_i - y_i$ is equal to either 0, 1 or 2 for every i.

• First, consider all indices i such that $x_i - y_i = 0$. For every such i we set $z_i = x_i$.

• Second, consider all indices i such that $x_i - y_i = 2$ and set $z_i = y_i + 1$.

• Finally, consider all remaining indices, i.e., all i such that $x_i = y_i + 1$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a $(k,l)'$-block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i = y_i + 1 = k, \quad x_{i+1} = y_{i+1} + 1 = k + 1, \quad \ldots, \quad x_{i+l-1} = y_{i+l-1} = k + l - 1.$$

For each $(k,l)'$-block we perform the following procedure: consider random variable ξ distributed according to $\hat{D}(k,t,S;l)$ (ξ's corresponding to different $(k,l)'$-blocks are independent). Set $z_i = y_i$ for the first ξ integers of the block (their coordinates are $k-1, k, \ldots, k+\xi-2$) and set $z_i = y_i + 1$ for the rest of the block.

Theorem 6.4. *The algorithm described above is precisely $S \mapsto S-1$ Markov step defined by $P^S_{S\mapsto S-1}$.*

The proof is similar to Theorem 6.3.

6.4 Markov evolution of the top path

The $S \mapsto S+1$ Markov step described in the previous section has the following property: Its projection to the set of topmost horizontal lozenges (or the topmost holes in terms of nonintersecting paths and point configurations) is also a Markov chain. This Markov chain is an exclusion type process. Let us describe it.

The general setting is as follows. The state space of our discrete time Markov chain consists of semi-infinite particle configurations $\{e_1 < e_2 < e_3 < \ldots\}$ in $\mathbb{Z}$. At each time moment every particle either stays or jumps to the left (any distance) avoiding collisions and jumps over neighbors. Jumps are performed sequentially. First, the leftmost particle (e_1) jumps, then the second one and so on. The distribution D of the length of the jump of a particle depends on the number of the particle, moment of time, current position of the particle (e_i) and the distance between the current position of the particle and the position of the previous particle (e_{i-1}) in the next moment of time. At time 0 we have the step initial condition, i.e., $e_i = i + const$.

Now let us turn back to our situation. All particles are enumerated by the parameter t and our time parameter is S that changes from 0 to T. Consider a sequence $\{u^S_t\}_{t=1,\ldots}$, where u^S_t is the vertical coordinate (in our notation - x) corresponding to the topmost hole inside the hexagon for $t \le S$ and $u^S_t = N+t-1$ for $t > S$. (We can also view u^S_t as the vertical coordinate corresponding to the tth hole, if we count all holes, not just the ones inside a hexagon, starting from the line $x = 0$.)

The evolution of $\{u^S_t\}$ is precisely our Markov process. When $S = 0$ the configuration consists of points $N, N+1, N+2, \ldots$. The distribution of the length of jump of the particle with coordinate u_t^S at the time moment S is given by the distribution $D(u_{t-1}^{S+1}+1,t,S;u_t^S-u_{t-1}^{S+1}-1)$ (see (14) for the definition).

Note that when $S=T$ the configuration consists of points $0,1,2,\dots$.

We can also obtain in a similar way a Markov chain for the bottommost holes.

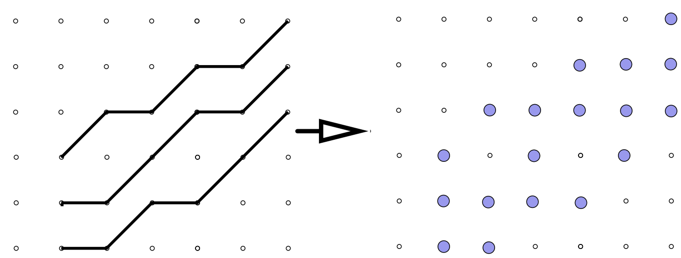
Finally, we may construct two more similar processes using the $S\mapsto S-1$ Markov chain (for this chains the direction of particle jumps changes and distributions D are replaced by distributions $\hat{D}$).

7 Correlation kernel

The aim of this section is to obtain the formulas for the correlation functions of random point configurations in $\mathbb{Z}^2$ obtained from the random tilings we are interested in.

7.1 Expression via orthogonal polynomials

Recall that a tiling of a hexagon corresponds to some family of nonintersecting paths that can be viewed as a point configuration in $\mathbb{Z}^2$. Let us denote this configuration by $\mathbb{M}$.



As above, we denote the horizontal coordinate by t and the vertical coordinate by x.

We want to compute the correlation functions of this random point configuration.

Recall that the nth correlation function is defined by

$$\rho_n(t_1,x_1;\dots;t_n,x_n)=\mathrm{Prob}\{(t_1,x_1)\in\mathbb{M},\dots,(t_n,x_n)\in\mathbb{M}\}$$

for any collection $\{(t_i,x_i)\}_{i=1,\dots,n}$ of distinct points in $\mathbb{Z}^2$.

To compute the correlation functions ρ_n we are going to use a variant of the Eynard-Mehta theorem (see [EM] and [BO, Section 7.4]). Let us state it first.

Proposition 7.1. *Assume that for every time moment t we are given an orthonormal system $\{f_n^t\}_{n\geq 0}$ in $l_2(\{0,1,\dots,L\})$ and a set of numbers $c_0^t,c_1^t,\dots$. Denote*

$$v_{t,t+1}(x,y)=\sum_{n\geq 0}c_n^t f_n^t(x)f_n^{t+1}(y).$$

Assume also that we are given a discrete time Markov process $\mathcal{P}_t$ taking values in N-tuples of elements of the set $\{0,1,\dots,L\}$, with one-dimensional distributions

$$\left(\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\dots,N}\right)^2$$

and transition probabilities

$$\frac{\det\left[v_{t,t+1}(x_i,y_j)\right]_{i,j=1,\dots,N}\det\left[f_{i-1}^{t+1}(y_j)\right]_{i,j=1,\dots,N}}{\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\dots,N}\prod_{n=0}^{N-1}c_n^t}.$$

Then

$$\mathrm{Prob}\{x_1\in\mathcal{P}_{k_1},\dots,x_n\in\mathcal{P}_{k_n}\}=\det\left[K(k_i,x_i;k_j,x_j)\right]_{i,j=1,\dots,n},$$

where

$$K(k,x;l,y)=\sum_{i=0}^{N-1}\frac{1}{c_i^{l,k}}f_i^k(x)f_i^l(y),\ k\geq l;$$
$$K(k,x;l,y)=-\sum_{i\geq N}c_i^{k,l}f_i^k(x)f_i^l(y),\ k<l;$$
$$c_i^{k,k}=1,\ c_i^{k,l}=c_i^k\cdot c_i^{k+1}\cdot\dots\cdot c_i^{l-1}.$$

Theorem 7.2. *The Markov process $X(t)$ meets the assumptions of Proposition 7.1.*

The Markov process $\mathcal{P}_t$ is precisely our Markov process $X(t)$. The orthonormal functions $f_n^t(x)$ are the normalized q-Racah polynomials multiplied by the square root of their weight function (see Section 4 for the definition of q-Racah polynomials, their weight function, and the correspondence between parameters of these polynomials and our parameters t,q,κ,N,T,S):

$$f_n^t(x)=\sqrt{w_{t,S}(x)}\frac{R_n^t(x)}{\sqrt{(R_n^t,R_n^t)}}, \tag{16}$$

where (R_n^t,R_n^t) is the squared norm of the q-Racah polynomials with respect to the weight function $w_{t,S}(x)$. This norm can be obtained from the norm of the q-Racah polynomials provided in [KS] ($w_{t,S}(x)$ differs from the weight function of [KS] by a factor not depending on x). The explicit formula is a little bit different in the four cases of correspondence between parameters of polynomials and t,q,κ,N,T,S. For instance, in the case given by formula (8):

$$\begin{aligned}(R_n^t,R_n^t)=&\frac{(-1)^{t+S}(q^{-2N-T+2},k^{-2}q^{S-N};q)_{t+N-1}}{(\kappa^{-2}q^{-2N+1},q^{S-T-N+1},\kappa^2q^{-t-S+2},q^{-t-N+1};q)_{t+N-1}}\\&\times\frac{(1-q^{-T-2N})(q,q^{-T-N+t+1},\kappa^{-2}q^{-2N+1},q^{S-T-N};q)_n}{(1-q^{-2N-T+2n+1})(q^{-S-N},q^{-2N-T+1},\kappa^2q^{-T},q^{-t-N+1};q)_n}\\&\times\frac{\kappa^{2n}q^{-n(S+t+1)}}{(q;q)_{T-S-t}(q^{-S-N+1};q)_{S+N-1}(\kappa^2q^{-T+1};q)_{T+N-t}}.\end{aligned}$$

However, this long formula is not important for us, since factors involving it always cancel out. In particular, in the case given by (8) the quotient $(R_n^{t+1}, R_n^{t+1})/(R_n^t, R_n^t)$, that is crucial for us, is simply

$$\frac{(R_n^{t+1}, R_n^{t+1})}{(R_n^t, R_n^t)} = -\frac{(1-q^{T+N-t-n-1})(1-q^{-t-N+n})}{(1-q^{-t-N})^2}.$$

The constants c_n^t are given by

$$c_n^t = \sqrt{(1-q^{-N-t+n})(1-q^{T+N-t-n-1})}. \tag{17}$$

Proof. Theorem 4.1 yields

$$\operatorname{Prob}\{X(t) = (x_1, x_2, \dots, x_N)\} = \frac{1}{Z}\prod_{i<j}(\mu_{t,S}(x_i) - \mu_{t,S}(x_j))^2 \prod_{i=1}^N w_{t,S}(x_i).$$

On the other hand

$$\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\dots,N} = \mathit{const}\cdot\prod_{i=1}^N\sqrt{w_{t,S}(x_i)}\det\left[R_{i-1}(x_j)\right]_{i,j=1,\dots,N}.$$

The last determinant is a Vandermonde determinant in variables $\mu_{t,S}(x_j)$, hence

$$\left(\det\left[f_{i-1}^t(x_j)\right]_{i,j=1,\dots,N}\right)^2 = \frac{1}{Z'}\prod_{i<j}(\mu_{t,S}(x_i) - \mu_{t,S}(x_j))^2 \prod_{i=1}^N w_{t,S}(x_i).$$

Coincidence of the constants ($Z = Z'$) follows from the fact that the left-hand side in the last equality defines a probability distribution.

Thus, the one-dimensional distributions of our process have the required form.

Next, we need the following standard facts.

Lemma 7.3. *The following relation for the basic hypergeometric function holds:*

$$(c-w)(1-d)\,{}_4\phi_3\left(\begin{matrix}a,b,c,qd\\ u,v,qw\end{matrix}\,\middle|\, q;q\right) + (w-d)(1-c)\,{}_4\phi_3\left(\begin{matrix}a,b,qc,d\\ u,v,qw\end{matrix}\,\middle|\, q;q\right)$$
$$= (c-d)(1-w)\,{}_4\phi_3\left(\begin{matrix}a,b,c,d\\ u,v,w\end{matrix}\,\middle|\, q;q\right). \tag{18}$$

In terms of q-Racah polynomials, this relation can be rewritten as

$$(q^{-x} - q\gamma)(1-\gamma\delta q^{x+1})R_n\Big(\mu(x);\alpha,\beta,q\gamma,\delta\mid q\Big)$$
$$+ (q\gamma - \gamma\delta q^{x+1})(1-q^{-x})R_n\Big(\mu(x-1);\alpha,\beta,q\gamma,\delta\mid q\Big)$$
$$= (q^{-x} - \gamma\delta q^{x+1})(1-q\gamma)R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big) \tag{19}$$

or as

$$(q^{-x}-q\alpha)(1-\gamma\delta q^{x+1})R_n\Big(\mu(x);q\alpha,q^{-1}\beta,\gamma,q\delta\mid q\Big)$$
$$+(q\alpha-\gamma\delta q^{x+1})(1-q^{-x})R_n\Big(\mu(x-1);q\alpha,q^{-1}\beta,\gamma,q\delta\mid q\Big)$$
$$=(q^{-x}-\gamma\delta q^{x+1})(1-q\alpha)R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big),\quad (20)$$

where R_n is given by (7).

For the balanced terminating ${}_4\phi_3\left(\begin{smallmatrix}a,b,c,d\\u,v,w\end{smallmatrix}\middle|q;q\right)$ (i.e., one of a, b, c or d equals q^{-n} and $a\cdot b\cdot c\cdot d=u\cdot v\cdot w$) we also have the following relation:

$$(c-u)(1-vc^{-1})(wq^{-1}-1)\,{}_4\phi_3\left(\begin{matrix}a,b,q^{-1}c,d\\u,v,q^{-1}w\end{matrix}\middle|q;q\right)$$
$$+(u-d)(1-vd^{-1})(wq^{-1}-1)\,{}_4\phi_3\left(\begin{matrix}a,b,c,q^{-1}d\\u,v,q^{-1}w\end{matrix}\middle|q;q\right)$$
$$=(c-d)(wq^{-1}-b)(1-aqw^{-1})\,{}_4\phi_3\left(\begin{matrix}a,b,c,d\\u,v,w\end{matrix}\middle|q;q\right).\quad (21)$$

In terms of q-Racah polynomials, the last relation can be rewritten as

$$(q^{-x}-q\gamma)(1-\beta\delta q^{x+1})(\alpha-1)R_n\Big(\mu(x+1);q^{-1}\alpha,q\beta,\gamma,q^{-1}\delta\mid q\Big)$$
$$+(q\gamma-\gamma\delta q^{x+1})(1-q^{-x}\beta\gamma^{-1})(\alpha-1)R_n\Big(\mu(x);q^{-1}\alpha,q\beta,\gamma,q^{-1}\delta\mid q\Big)$$
$$=(q^{-x}-\gamma\delta q^{x+1})(\alpha-\alpha\beta q^{n+1})(1-\alpha^{-1}q^{-n})R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big)\quad (22)$$

or as

$$(q^{-x}-q\alpha)(1-\beta\delta q^{x+1})(\gamma-1)R_n\Big(\mu(x+1);\alpha,\beta,q^{-1}\gamma,\delta\mid q\Big)$$
$$+(q\alpha-\gamma\delta q^{x+1})(1-q^{-x}\beta\gamma^{-1})(\gamma-1)R_n\Big(\mu(x);\alpha,\beta,q^{-1}\gamma,\delta\mid q\Big)$$
$$=(q^{-x}-\gamma\delta q^{x+1})(\gamma-\alpha\beta q^{n+1})(1-\gamma^{-1}q^{-n})R_n\Big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\Big),\quad (23)$$

Proof. To prove the first relation for the basic hypergeometric function we expand ${}_4\phi_3$ into series in q and perform straightforward computations in every term.

To obtain (19) and (20) we simply rewrite the relation (18) in terms of q-Racah polynomials using their definition (7).

Next, we observe that q-Racah polynomials form an orthogonal basis in the corresponding l_2 space. Consequently, we can write dual relations for (19) and (20) and these are precisely (22) and (23). It is easily seen that two last relations are equivalent to just one relation for basic hypergeometric function (21). $\square$

Using the last lemma we obtain the following one:

Lemma 7.4.

$$v_{t,t+1}(x,y) = \sum_{n\geq 0} c_n^t f_n^t(x) f_n^{t+1}(y) = \sqrt{\frac{w_{t,S}(x)}{w_{t+1,S}(y)}} \left(\delta_{x+1}^y w_1(x) + \delta_x^y w_0(x)\right),$$

where

$$w_0(x) = -(1-q^{x+T-t-S}) \frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}},$$

$$w_1(x) = q^{T+N-1-t}(1-q^{x-S-N+1}) \frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}},$$

while $w_{t,S}(x)$ stands for the weight function corresponding to the parameters t, q, κ, N, T, S (see Section 4 and Theorem 4.1 for details).

Proof. First, we substitute the parameters of q-Racah polynomials given by formulas (8)-(11) into the statement of Lemma 7.3.

We use (19), (20) in cases (8), (9), and we use (22), (23) in cases (10), (11).

In all 4 cases we rewrite the corresponding relation in terms of orthogonal functions $f_n^t(x)$ and get the following

$$c_n^t f_n^{t+1}(y) = \sqrt{\frac{w_{t,S}(y-1)}{w_{t+1,S}(y)}} f_n^t(y-1) w_1(y-1) + \sqrt{\frac{w_{t,S}(y)}{w_{t+1,S}(y)}} f_n^t(y) w_0(y) \quad (24)$$

Multiply the last relation by $f_n^t(x)$ and sum over all meaningful n.

Since functions $f_n^t(y)$ form an orthonormal basis in the corresponding l_2 space,

$$\sum_n f_n^t(x) f_n^t(y) = \delta_x^y,$$

and the needed relation follows. $\square$

Proposition 5.2 implies that the transition probabilities $P_{t+}^{S,t}(X,Y)$ have a determinantal form. The last lemma yields that this form is exactly the one required for the application of Proposition 7.1. Thus, the theorem is proved. $\square$

Applying Proposition 7.1 for the process $X(t)$ we obtain the following statement.

Theorem 7.5.

$$\rho_n(t_1, x_1; \dots; t_n, x_n) = \det\left[K(k_i, x_i; k_j, x_j)\right]_{i,j=1,\dots,n},$$

where

$$K(k,x;l,y) = \sum_{i=0}^{N-1} \frac{1}{c_i^{l,k}} f_i^k(x) f_i^l(y),\ k \ge l;$$
$$K(k,x;l,y) = -\sum_{i \ge N} c_i^{k,l} f_i^k(x) f_i^l(y),\ k < l;$$
$$c_i^{k,k} = 1,\ c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1}$$

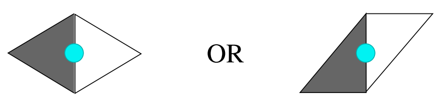
and functions $f_i^k(x)$ and numbers c_i^t are given by the formulas (16) and (17).

7.2 Inverse Kasteleyn matrix

Let us present another way to view the correlation kernel derived in the previous section.

Recall that we deal with lozenge tilings of a hexagon. Divide every lozenge into two unit triangles and color the resulting triangles into black and white (west triangle is black). In this way a tiling turns into a perfect matching of the part of the dual hexagonal lattice that fits in our hexagon. Correlation functions of the perfect matchings can be computed using Kasteleyn's theorem (see [Ka]). Let us describe it.

Associate to every triangle the midpoint of its vertical side. Note that in this way both black and white triangles can be parameterized by the points of the two-dimensional lattice. Thus, we can use our usual coordinates (t, x) for the triangles.



The Kasteleyn matrix $\mathrm{Kast}(t, x; r, y)$ is a weighted adjacency matrix. Here (t, x) stand for the coordinates of a white triangle and (r, y) stand for the coordinates of a black triangle. In our case,

$$\mathrm{Kast}(t,x;r,y) = \begin{cases} \kappa q^{-S/2+x-t/2+1/2} - \dfrac{1}{\kappa q^{-S/2+x-t/2+1/2}}, & (t,x) = (r,y), \\ 1, & (t,x) = (r-1, y-1), \\ 1, & (t,x) = (r-1, y), \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$g(t,x) = \frac{1}{\sqrt{w_{t,S}(x)}} \frac{(-1)^{t+x} \kappa^{-t} q^{x(T+N-t-1)+t(S/2-1/2)+t(t+1)/4} (1 - \kappa^2 q^{2x-t-S+1})}{(q^{-1}; q^{-1})_{S+N-1-x} (q;q)_{T-S+x-t} (\kappa^2 q^{x-T+1}; q)_{T+N-t}}. \tag{25}$$

Theorem 7.6. *Consider n lozenges enumerated by pairs of triangles $((t_i, x_i), (r_i, y_i))$. The probability that a random tiling contains these lozenges equals*

$$\prod_{i=1}^{n} \mathrm{Kast}(t_i, x_i; r_i, y_i) \cdot \det \left[K^{ext}(r_i, y_i; t_j, x_j) \right]_{i,j=1,\dots,n},$$

where

$$K^{ext}(r, y; t, x) = \frac{g(r, y)}{g(t, x)} \left(\delta_{(r,y)}^{(t,x)} - K(r, y; t, x) \right),$$

the function g is given by (25), and $K(r, y; t, x)$ is given in Theorem 7.5.

Proof. Kasteleyn's theorem states that the probability to find lozenges

$$((t_1, x_1); (r_1, y_1)), \dots, ((t_n, x_n); (r_n, y_n))$$

can be expressed via the inverse of the Kasteleyn matrix:

$$\prod_{i=1}^{n} \mathrm{Kast}(t_i, x_i; r_i, y_i) \cdot \det \left[\mathrm{Kast}^{-1}(r_i, y_i; t_j, x_j) \right]_{i,j=1,\dots,n}$$

We can compare this statement with Theorem 7.5. Note that Theorem 7.5 describes the correlation functions of the particles. Consequently the correlation kernel

$$\hat{K}(t, x; r, y) = \delta_{(t,x)}^{(r,y)} - K(t, x; r, y)$$

(where $K(t, x; r, y)$ is the correlation kernel of Theorem 7.5) is the correlation kernel of holes or, equivalently of horizontal lozenges. (See, e.g., [BOO, Appendix A.3] for some details on the particle-hole involution.)

Since the correlation kernel appears only in a determinant, it is only determined up to conjugation: the transformation

$$\hat{K}(t, x; r, y) \mapsto \frac{g(t, x)}{g(r, y)} \hat{K}(t, x; r, y)$$

does not change correlation functions. We conclude that

$$\frac{\hat{K}(t, x; r, y)}{\kappa q^{-S/2+x-t/2+1/2} - (\kappa q^{-S/2+x-t/2+1/2})^{-1}}$$

should be (perhaps, after some conjugation) the inverse Kasteleyn matrix.

Let us verify this fact and find the appropriate conjugation factor.

We have

$$\sum_{(h,z)} \frac{g(t, x)}{g(h, z)} \frac{\hat{K}(t, x; h, z)}{\kappa q^{-S/2+x-t/2+1/2} - (\kappa q^{-S/2+x-t/2+1/2})^{-1}} \mathrm{Kast}(h, z; r, y) = \delta_{(t,x)}^{(r,y)}. \tag{26}$$

First, suppose that $t < r - 1$. In this case the $((t, x); (r, y))$ matrix element of the right-hand side is zero while the one of the left-hand side of (26) is

$$\frac{g(t,x)}{\kappa q^{-S/2+t-x/2+1/2} - (\kappa q^{-S/2+t-x/2+1/2})^{-1}} \sum_{i \geq N} f_i^t(x) c_i^{t,r-1}$$
$$\times \left[\frac{\left(\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+y-r/2+1/2}}\right) c_i^{r-1} f_i^r(y)}{g(r,y)} + \frac{f_i^{r-1}(y)}{g(r-1,y)} + \frac{f_i^{r-1}(y-1)}{g(r-1,y-1)} \right].$$

Let us find such function g that for every i

$$\frac{\left(\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+y-r/2+1/2}}\right) c_i^{r-1} f_i^r(y)}{g(r,y)} + \frac{f_i^{r-1}(y)}{g(r-1,y)} + \frac{f_i^{r-1}(y-1)}{g(r-1,y-1)} = 0. \tag{27}$$

We know that (see Lemma 7.4)

$$c_i^t f_i^{t+1}(y) - \sqrt{\frac{w_{t,S}(y-1)}{w_{t+1,S}(y)}} w_1^t(y-1) f_i^t(y-1) - \sqrt{\frac{w_{t,S}(y)}{w_{t+1,S}(y)}} w_0^t(y) f_i^t(y) = 0, \tag{28}$$

where

$$w_0^t(x) = -(1 - q^{x+T-t-S}) \frac{1 - \kappa^2 q^{x+N-t}}{1 - \kappa^2 q^{2x-t-S+1}},$$

$$w_1^t(x) = q^{T+N-1-t}(1 - q^{-(S+N-1-x)}) \frac{1 - \kappa^2 q^{x-T+1}}{1 - \kappa^2 q^{2x-t-S+1}},$$

while $w_{t,S}(x)$ stands for the weight function corresponding to the parameters t, q, κ, N, T, S (see Section 4 and Theorem 4.1 for details).

For every pair (r, y) we get the following three equations defining g

$$g(r,y) \propto \frac{\kappa q^{-S/2+y-r/2+1/2} - \frac{1}{\kappa q^{-S/2+z-r/2+1/2}}}{\sqrt{w_{r,S}(y)}},$$

$$g(r-1,y-1) \propto -\frac{1}{\sqrt{w_{r-1,S}(y-1)} w_1^{r-1}(y-1)},$$

$$g(r-1,y) \propto -\frac{1}{\sqrt{w_{r-1,S}(y)} w_0^{r-1}(y)},$$

where the proportionality coefficient is the same for all three equations (but it may depend on the pair (r, y)). One checks that g given by (25) satisfies these relations and after the conjugation with g the $((t, x); (r, y))$ matrix element of the left-hand side of (26) is zero.

Next, suppose that $t > r$. In this case the $((t, x); (r, y))$ matrix element of the left-hand side of (26) is zero by the similar reasoning.

If either $t = r$ or $t = r - 1$ the argument becomes a little more involved, but the computation requires no new ideas.

□

8 Bulk limits. Limit shapes.

8.1 Bulk limit theorem

In this section we compute so-called "bulk limits" of the correlation functions introduced in the previous section.

We are interested in the following limit regime. Fix positive numbers S, T, N, t, x, q. Introduce a small parameter $\varepsilon \ll 1$, and set

$$S = \mathsf{S}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \mathsf{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \mathsf{N}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad q = \mathsf{q}^{\varepsilon + o(\varepsilon)}.$$

Consider also integer valued functions $t_i = t_i(\varepsilon)$ and $x_i = x_i(\varepsilon)$, $i = 1, \ldots, n$, such that

$$\lim_{\varepsilon\to 0} \varepsilon t_i(\varepsilon) = \mathsf{t}, \quad \lim_{\varepsilon\to 0} \varepsilon x_i(\varepsilon) = \mathsf{x}, \quad i = 1, \ldots, n,$$

and pairwise differences $t_i - t_j$, and $x_i - x_j$ do not depend on ε.

Then the correlation functions ρ_n computed in Theorem 7.5 tend to a limit $\hat{\rho}_n$ which depends on the parameters of the limit regime q, S, T, N, t, x and the original parameter κ.

We consider the region where the limit correlation functions are nontrivial. This region is commonly referred to as the "bulk", sometimes also called the "liquid region". This is a simply connected domain inside the hexagon.

The main result of this section is Theorem 8.1.

Note that the first limit correlation function allows us to predict *the limit shape* which appears in our model.

Theorem 8.1. *We have*

$$\lim_{\varepsilon\to 0} \rho_n(t_1, x_1; \ldots; t_n, x_n) = \det\left[\hat{K}(t_i, x_i; t_j, x_j)\right]_{i,j=1,\ldots,n},$$

where

$$\hat{K}(x, s; y, t) = \frac{1}{2\pi i} \oint_{e^{-i\phi}}^{e^{i\phi}} (1 + cw)^{t-s}\, w^{x-y-1} dw.$$

Here the integration is to be performed over the right side of the unit circle when $s \geq t$ and over the left side otherwise,

$$c = \left(\frac{\mathsf{q}^{\mathsf{T}-2\mathsf{t}}(1-\mathsf{q}^{-(\mathsf{S}+\mathsf{N}-\mathsf{x})})(1-\mathsf{q}^{\mathsf{x}})}{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})} \frac{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{T}})}{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})} \right)^{\frac{1}{2}},$$

and ϕ is given by the formula:

$$\phi = \arccos \frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B}{2\sqrt{AB}},$$

where

$$A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}}),$$

$$B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}).$$

If the expression under arccos *is greater than 1, then we set $\phi = 0$. If the expression is less than -1, then $\phi = \pi$.*

Setting $v = -cw$ (and omitting some "conjugation factors" again) we get the incomplete beta-kernel form of the integral, cf. [OR],

$$K(x,s;y,t) = \frac{1}{2\pi i}\oint_{-c\cdot e^{-i\phi}}^{-c\cdot e^{i\phi}} (1-v)^{t-s}\, v^{x-y-1} dv.$$

Here the contour of integration intersects $(-\infty, 0)$, if $s \geq t$, and intersects $(0,1)$ otherwise. For an explanation of the relation of the incomplete beta-kernel and Gibbs measures see [KOS], [BS].

It is not hard to compute that $z = -ce^{i\phi}$ has the form

$$z = \frac{1}{2}\,\frac{\mathsf{q}^{\mathsf{T}+\mathsf{N}-\mathsf{t}}}{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})}$$
$$\cdot\Bigg[\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B$$
$$+\, i\sqrt{4AB - \left(\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2 + A + B\right)^2}\,\Bigg],$$

with

$$A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}})$$

and

$$B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}).$$

Proposition 8.2. *The parameter z defined above coincides with the one defined in Theorem 2.1.*

Proof. The quadratic equation satisfied by z is

$$Z^2 - (z+\bar{z})Z + z\bar{z} = 0.$$

Substituting the expression for z given above, one obtains a relation equivalent to the one in Theorem 2.1. ☐

8.2 Proof of the bulk limit theorem

In this section we prove Theorem 8.1.

Recall that the correlation kernel before the limit is given by

$$K(x,k;y,l)=\sum_{i=0}^{N-1}\frac{1}{c_i^{l,k}}f_i^k(x)f_i^l(y),\ k\geq l;$$
$$K(x,k;y,l)=-\sum_{i\geq N}c_i^{k,l}f_i^k(x)f_i^l(y),\ k<l;$$
$$c_i^{k,k}=1,\ c_i^{k,l}=c_i^k\cdot c_i^{k+1}\cdot\dots\cdot c_i^{l-1}.$$

The functions $f_i^k(x)$ and the coefficients c_i^k were defined in Section 7.

First, let us consider the case $k=l$. We want to find a limit of the projection kernel

$$\mathcal{P}_t(x,y)=\sum_{n=0}^{N-1}f_n^t(x)f_n^t(y).$$

In order to find a limit of $\mathcal{P}_t(x,y)$ we use the spectral projection method proposed by G. Olshanski, see [BO2] and [O].

We want to consider $\mathcal{P}_t(x,y)$ as a matrix element of the operator $\mathcal{P}_t$. It turns out that finding the limit of the operator is easier than computing the limit of the matrix elements. Note that functions $f_n^t(x)$ are eigenvectors of some difference operator (it will be explicitly given below). The projection operator can be regarded as the spectral projection on the segment containing the first N eigenvalues of this difference operator. Now, to find the limit of the spectral projection operators we will take the limit of the difference operators. Note that both the difference operator and the spectral segment are varying simultaneously.

To justify the limit transition we use some facts from functional analysis.

Consider the set $l_2^0(\mathbb{Z})$ of the finite vectors from $l_2(\mathbb{Z})$ (i.e., the algebraic span of the basis elements δ_x) as a common essential domain of all considered difference operators. It will be clear from the following that the difference operators strongly converge on this domain. It follows that the operators converge in the strong resolvent sense (see [RS], Theorem VIII.25). The last fact, continuity of the spectrum of the limit operator, and Theorem VIII.24 from [RS] imply that the spectral projections associated with the difference operators strongly converge on the set of finite vectors to the limit spectral projection associated with the limit difference operator.

Now we present some details and computations.

Note that since q-Racah polynomials are eigenfunctions of a certain difference operator (see [KS]), the same is true for the functions $f_n^t(x)$. The difference operator is

$$q^{-n}(1-q^n)(1-\alpha\beta q^{n+1})f_n^t(x) = B(x)f_n^t(x+1)\sqrt{\frac{w_{t,S}(x)}{w_{t,S}(x+1)}}$$
$$- [B(x)+D(x)]f_n^t(x) + D(x)f_n^t(x-1)\sqrt{\frac{w_{t,S}(x)}{w_{t,S}(x-1)}}, \quad (29)$$

where $w_{t,S}(x)$ is the weight function corresponding to the parameters t, q, κ, N, T, S (see Theorem 4.1), and

$$B(x) = \frac{(1-\alpha q^{x+1})(1-\beta\delta q^{x+1})(1-\gamma q^{x+1})(1-\gamma\delta q^{x+1})}{(1-\gamma\delta q^{2x+1})(1-\gamma\delta q^{2x+2})},$$

$$D(x) = \frac{q(1-q^x)(\alpha-\gamma\delta q^x)(\beta-\gamma q^x)(1-\delta q^x)}{(1-\gamma\delta q^{2x})(1-\gamma\delta q^{2x+1})}.$$

(Here α, β, γ, δ are the corresponding parameters of q-Racah polynomials.)

We find

$$w_{t,S}(x+1)/w_{t,S}(x) = \frac{q^{2N+T-1}(1-\kappa^2 q^{2x-t-S+3})}{1-\kappa^2 q^{2x-t-S+1}}$$
$$\times \frac{(1-q^{x-t-N+1})(1-q^{x-S-N+1})(1-\kappa^2 q^{x-T+1})(1-\kappa^2 q^{x-t-S+1})}{(1-q^{x+1})(1-q^{T-S-t+x+1})(1-\kappa^2 q^{x+N-t+1})(1-\kappa^2 q^{x+N-S+1})},$$

$$w_{t,S}(x-1)/w_{t,S}(x) = \frac{q^{-2N-T+1}(1-\kappa^2 q^{2x-t-S-1})}{1-\kappa^2 q^{2x-t-S+1}}$$
$$\times \frac{(1-q^x)(1-q^{T-S-t+x})(1-\kappa^2 q^{x+N-t})(1-\kappa^2 q^{x+N-S})}{(1-q^{x-t-N})(1-q^{x-S-N})(1-\kappa^2 q^{x-T})(1-\kappa^2 q^{x-t-S})}.$$

Substituting the q-Racah parameters of our model (see Section 4) into $B(x)$ and $D(x)$ one computes

$$B(x) = \frac{(1-q^{-S-N+x+1})(1-\kappa^2 q^{-T+x+1})(1-q^{-t-N+x+1})(1-\kappa^2 q^{-t-S+x+1})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{-t-S+2x+2})},$$

$$D(x) = \frac{q(1-q^x)(q^{-S-N}-\kappa^2 q^{-t-S+x})(q^{S-T-N}-q^{-t-N+x})(1-\kappa^2 q^{-S+N+x})}{(1-\kappa^2 q^{-t-S+2x})(1-\kappa^2 q^{-t-S+2x+1})}$$
$$= q^{1-2N-T}\frac{(1-q^x)(1-\kappa^2 q^{-t+N+x})(1-q^{-t-S+T+x})(1-\kappa^2 q^{-S+N+x})}{(1-\kappa^2 q^{-t-S+2x})(1-\kappa^2 q^{-t-S+2x+1})}.$$

It follows that

$$
\begin{aligned}
&\frac{B(x)}{\sqrt{w_{t,S}(x+1)/w_{t,S}(x)}}\\
&= \left(q^{1-2N-T}(1-q^{x+1})(1-q^{T-S-t+x+1})(1-q^{-S-N+x+1})(1-q^{-t-N+x+1})\right)^{\frac{1}{2}}\\
&\times\left(\frac{(1-\kappa^2 q^{-T+x+1})(1-\kappa^2 q^{-t-S+x+1})(1-\kappa^2 q^{x+N-t+1})(1-\kappa^2 q^{x+N-S+1})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{2x-t-S+3})(1-\kappa^2 q^{-t-S+2x+2})^2}\right)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{D(x)}{\sqrt{w_{t,S}(x-1)/w_{t,S}(x)}}\\
&\quad= \left(q^{1-2N-T}(1-q^{x})(1-q^{x-t-N})(1-q^{x-S-N})(1-q^{x-t-S+T})\right)^{\frac{1}{2}}\\
&\quad\times\left(\frac{(1-\kappa^2 q^{-t+N+x})(1-\kappa^2 q^{-S+N+x})(1-\kappa^2 q^{x-T})(1-\kappa^2 q^{x-t-S})}{(1-\kappa^2 q^{-t-S+2x+1})(1-\kappa^2 q^{2x-t-S-1})(1-\kappa^2 q^{-t-S+2x})^2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

The eigenvalues in the left-hand side of (29) become

$$
q^{-n}(1-q^n)(1-q^{-T-2N+n+1}), \qquad n = 0, \ldots, N-1
$$

Note that in the "bulk limit" regime, $q \to 1$, $N, T, S \to \infty$ as $\varepsilon \to 0$ in such a way that q^N, q^T, q^S have finite limits.

In the limit $\frac{B(x)}{\sqrt{w(x+1)/w(x)}}$ tends to some constant and $\frac{D(x)}{\sqrt{w(x-1)/w(x)}}$ tends to the very same constant. After dividing by twice this constant the limit operator becomes

$$
f(x) \mapsto \frac{f(x+1)+f(x-1)}{2} - f(x)\frac{A+B}{2\sqrt{AB}},
$$

where

$$
A = (1-\mathsf{q}^{-\mathsf{S}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{T}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{x}})
$$

and

$$
B = \mathsf{q}^{-2\mathsf{N}-\mathsf{T}}(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}+\mathsf{N}+\mathsf{x}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{S}+\mathsf{T}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{-\mathsf{S}+\mathsf{N}+\mathsf{x}}),
$$

while the spectral interval becomes

$$
\left[\frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2}{2\sqrt{AB}}, 0\right]
$$

Since the spectrum of the operator $f \to \frac{f(x+1)+f(x-1)}{2}$ is $[-1,1]$, while $\frac{\sqrt{A+B}}{2\sqrt{AB}} > 1$, one can equivalently write the operator in the form

$$
f \to \frac{f(x+1)+f(x-1)}{2}
$$

with spectral interval

$$\left[\frac{\mathsf{q}^{-\mathsf{N}}(1-\mathsf{q}^{\mathsf{N}})(1-\mathsf{q}^{-\mathsf{T}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{-\mathsf{t}-\mathsf{S}+2\mathsf{x}})^2+A+B}{2\sqrt{AB}},1\right]. \tag{30}$$

If we perform the Fourier transform $l_2(\mathbb{Z}) \to L_2(S^1)$, where S^1 is the unit circle in $\mathbb{C}$, we get an operator of the projection on the part of the unit circle with x-coordinate varying over precisely the spectral interval (30).

If we do the inverse Fourier transform we will get the discrete sine kernel, cf. the end of Section 3.2 in [Gor].

Next, let us consider the case $k < l$. The prelimit correlation kernel is given by

$$K(x,k;y,l) = -\sum_{i\geq N} c_i^{k,l} f_i^k(x) f_i^l(y), \qquad k<l;$$
$$c_i^{k,k}=1, \qquad c_i^{k,l} = c_i^k \cdot c_i^{k+1}\cdot\dots\cdot c_i^{l-1}.$$

Let us decompose the correlation kernel (i.e., the operator given by it) into the product of the static projection kernel:

$$\mathcal{P'}_t(x,y) = -\sum_{i\geq N}^{N-1} f_i^t(x) f_i^t(y),$$

and a collection of transition operators (or their inverses) $U_h(x,y)$ with

$$U_h(x,y) = \begin{cases} \sum\limits_{i\geq 0} c_i^h f_i^h(x) f_i^{h+1}(y), & x\in\mathfrak{X}_h, y\in\mathfrak{X}_{h+1},\\ 0 & \text{for other } x,y. \end{cases}$$

For the operators $\mathcal{P'}_t(x,y)$ we can use the same methods as for $\mathcal{P}_t(x,y)$. We get minus the operator of projection on the part of the unit circle complementary to the spectral interval (30).

Let us turn to the transition operators U_t.

By virtue of already proved facts (see Section 7), we obtain

$$U_t(x,y) = const\cdot\sqrt{\frac{w_{t,S}(x)}{w_{t+1,S}(y)}}\left[w_1(x)\delta_{x+1}^y + w_0(x)\delta_x^y\right],$$

where

$$w_{t,S}(x) = \frac{q^{x(2N+T-1)}(1-\kappa^2q^{2x-t-S+1})}{(q;q)_x(q;q)_{T-S-t+x}(q^{-1};q^{-1})_{t+N-x-1}(q^{-1};q^{-1})_{S+N-x-1}} \cdot\frac{1}{(\kappa^2q^{x-T+1};q)_{T+N-t}(\kappa^2q^{x-t-S+1};q)_{N+t}},$$

$$w_0(x) = -(1-q^{x+T-t-S})\frac{1-\kappa^2 q^{x+N-t}}{1-\kappa^2 q^{2x-t-S+1}}\,,$$

and

$$w_1(x) = q^{T+N-1-t}(1-q^{-(S+N-1-x)})\frac{1-\kappa^2 q^{x-T+1}}{1-\kappa^2 q^{2x-t-S+1}}.$$

Thus,

$$U_t(x,y) = const\cdot\left[\tilde w_1(x)\delta^y_{x+1} + \tilde w_0(x)\delta^y_x\right],$$

where

$$\tilde w_0(x) = \left(\frac{(1-q^{x+T-t-S})(1-q^{-t-N+x})(1-\kappa^2 q^{x-t-S})(1-\kappa^2 q^{x+N-t})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S})}\right)^{\frac12},$$

and

$$\tilde w_1(x) = \left(\frac{q^{T-t-1}(1-q^{x-S-N+1})(1-q^{x+1})(1-\kappa^2 q^{x+N-S+1})(1-\kappa^2 q^{x-T+1})}{(1-\kappa^2 q^{2x-t-S+1})(1-\kappa^2 q^{2x-t-S+2})}\right)^{\frac12}.$$

Passing to the limit we get the operator

$$\mathsf{U}(x,y) = const\cdot\left[\mathsf{U}_1\delta^y_{x+1} + \mathsf{U}_0\delta^y_x\right],$$

where

$$\mathsf{U}_0 = \left(\frac{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})}{(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})\ (1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac12}$$

and

$$\mathsf{U}_1 = \left(\frac{\mathsf{q}^{\mathsf{T}-2t}(1-\mathsf{q}^{\mathsf{x}-\mathsf{S}-\mathsf{N}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{T}})(1-\mathsf{q}^{\mathsf{x}})(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})}{(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{2\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac12}.$$

Equivalently,

$$\mathsf{U}(x,y) = const\cdot\left[\mathsf{u}\delta^y_{x+1} + \delta^y_x\right],$$

where

$$\mathsf{u} = \frac{\mathsf{U}_1}{\mathsf{U}_0} = \left(\frac{\mathsf{q}^{\mathsf{T}-2\mathsf{t}}(1-\mathsf{q}^{\mathsf{x}-\mathsf{S}-\mathsf{N}})(1-\mathsf{q}^{\mathsf{x}})}{(1-\mathsf{q}^{\mathsf{x}+\mathsf{T}-\mathsf{t}-\mathsf{S}})(1-\mathsf{q}^{-\mathsf{t}-\mathsf{N}+\mathsf{x}})}\,\frac{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{S}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{T}})}{(1-\kappa^2\mathsf{q}^{\mathsf{x}+\mathsf{N}-\mathsf{t}})(1-\kappa^2\mathsf{q}^{\mathsf{x}-\mathsf{t}-\mathsf{S}})}\right)^{\frac12}.$$

Fourier transform gives us the operator of multiplication by $const(1+\mathsf{u}/w)$, where w is the coordinate on the circle $|w|=1$.

If we now multiply all necessary operators, perform inverse Fourier transform and substitute $w\to 1/w$ in the resulting integral, we get the desired limit kernel. Note that the constant prefactor in U can be omitted since it corresponds to the conjugation of the kernel that does not affect the correlation functions.

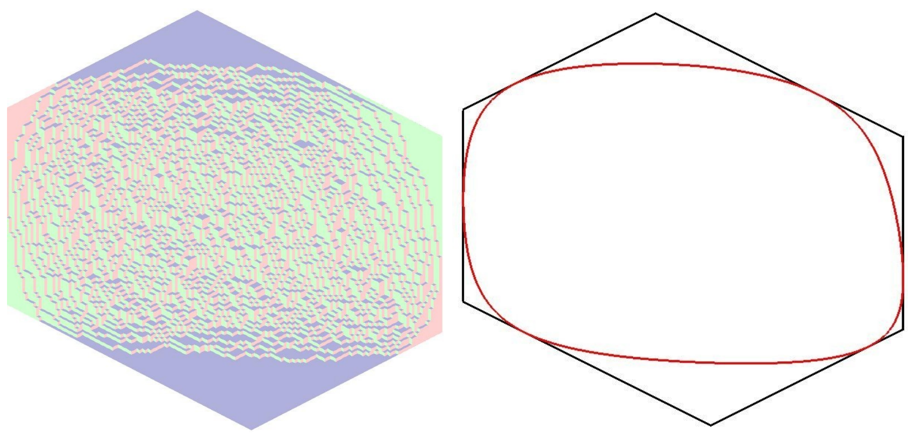
The final case $k>l$ is similar. The interested reader can find some details in [Gor] where similar computations (with Hahn polynomials instead of q-Racah polynomials) were performed.

9 Computer simulations. Different limit regimes.

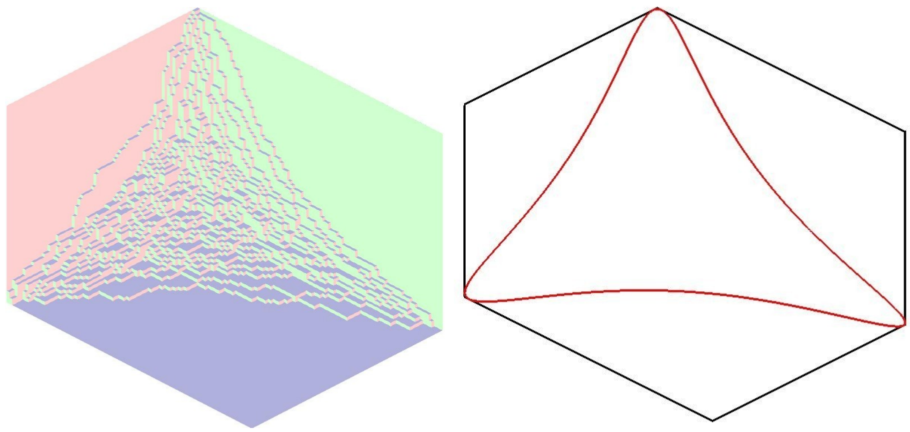
Using the perfect sampling algorithm described in Section 6, we performed some computer simulations; the program that we used can be found at `http://www.math.caltech.edu/papers/Borodin-Gorin-Rains.exe`. We are mostly interested in the case when the hexagon is large, since in this case we can see some limit shapes appearing. In all pictures we color the three types of lozenges in three different colors (as in Figure 1). When we draw big pictures, we erase the borders between lozenges and get some coloring of a hexagon in three colors which can also be viewed as a stepped surface in $\mathbb{R}^3$.

Although we mostly show not very big pictures, our algorithm can generate random tilings of a $1000 \times 1000 \times 1000$ hexagon in a reasonable amount of time.

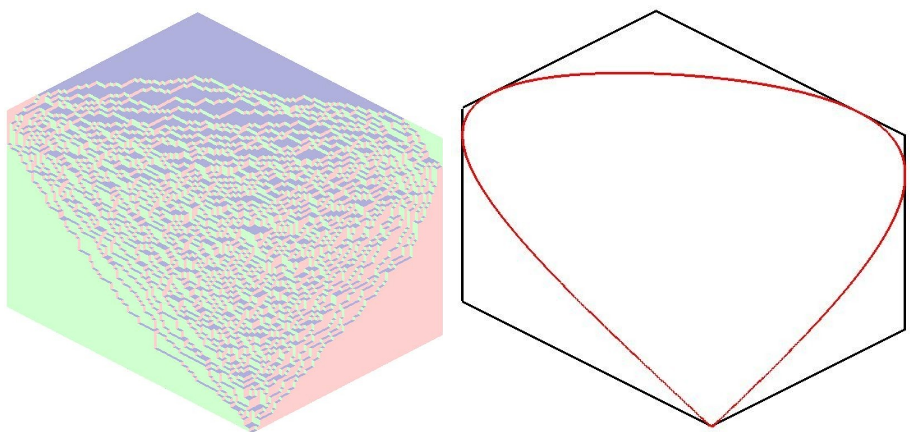
The following picture shows a plane partition in a $70 \times 90 \times 70$ box sampled from the distributions with parameters $q = 0.97$, $\kappa = 1$. The formation of a limit shape with frozen regions is clearly visible on the picture, and the next picture shows the border of the frozen region as predicted by Theorem 2.1.



By the appropriate limit transition, we get plane partitions distributed as q^{volume}. The following picture shows a random plane partitions in a $70 \times 90 \times 70$ box and the corresponding theoretical frozen boundary with $q = 1.04$.

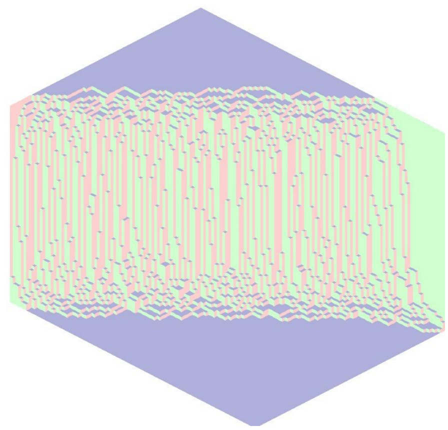


As was explained in Section 4, if we send $q \to 1$ in the original model, then the weight of a horizontal lozenge becomes a linear function in the vertical coordinate. If we tune the parameters in such a way that our linear function has a zero at the bottommost point of a hexagon, then we get the following random plane partition in a $70 \times 90 \times 70$ box, cf. the corresponding theoretical frozen boundary.

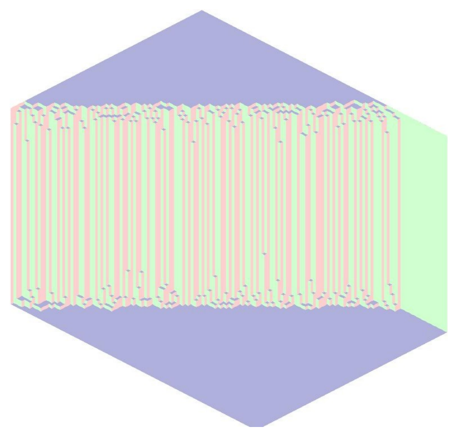


We see that the border of frozen region has a node near the bottommost point of a hexagon.

Another interesting case to consider is when q does not tend to 1 as the size of the hexagon tends to infinity. Look at the following picture where the random plane partition with parameters $q = 0.9$, $\kappa = 1$ in a $70 \times 90 \times 70$ box is shown. We see that the surface becomes different from the ones shown above. We call this new class of surfaces *waterfalls*.

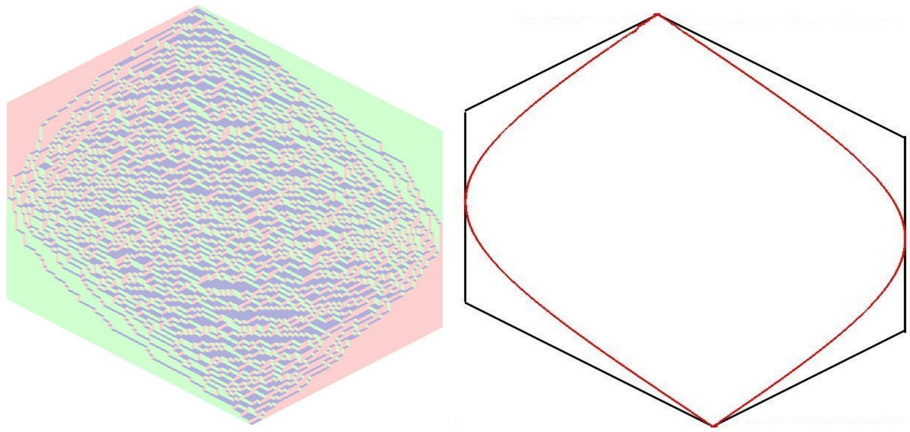


The next picture shows an even more degenerate case $q = 0.7$.



We hope to study the asymptotic behavior of these cases in a later publication.

Finally, let us turn to the trigonometric q-Racah case. In this case the weight of a horizontal lozenge is $\sin(\alpha(j-(S+1)/2)+\beta)$. One can tune the parameters α and β in such a way that this weight becomes zero at both the topmost and the bottommost points of the hexagon. The boundary of the frozen region has two nodes in this case.



10 Appendix. Lozenge tilings and biorthogonal functions

We were led to consider the q-Racah and Racah cases via the realization that, in much the same way that uniform lozenge tilings are related to Hahn polynomials, so were q-weighted lozenge tilings related to q-Hahn polynomials. This naturally led to the question of whether more general (discrete) hypergeometric orthogonal polynomials could arise in this way. In fact, as we will consider in this section, one can generalize even further, to the elliptic analogue, certain biorthogonal elliptic functions due to Spridonov and Zhedanov [SZ].

One way to derive the required lozenge weights (essentially equivalent weights were considered by Schlosser in [Sch], although the connection to plane partitions or lozenge tilings was not made explicit there) is via the following desiderata:

First, we want the total weight of all tilings of a hexagon to be “nice”, in the sense that it should be expressible as a product of simple theta functions

$$\theta_p(x) := \prod_{0\le k}(1-p^k x)(1-p^{k+1}/x);$$

the same should apply to the individual weights as well. Note that, as is traditional in much recent work on elliptic special functions, we use the multiplicative form for our elliptic curves and theta functions. This can be translated to the usual doubly-periodic form by composing with the singly-periodic function $x \mapsto \exp(2\pi i x)$, but the multiplicative form makes certain degenerations simpler to obtain.

Next, the sums that arise should be hypergeometric, in the sense that any parameters should vary along geometric progressions as one moves around in the hexagon.

Finally, the form of the weight of a given cube (i.e., the ratio of the weights of the two ways to tile any given unit hexagon) should be invariant under the symmetry group of the triangular lattice, except that certain reflections should invert the weight. Note that the weights of cubes are gauge-invariant, and any set of choices of cube weights corresponds to a unique gauge-equivalence class.

We may rewrite these criteria in terms of plane partitions, noting that the requirement that the cube weights come from lozenge weights places the additional restriction that if $w(x,y,z)$ is the weight of a cube with centroid (x,y,z), then

$$w(x,y,z) = w(x+1,y+1,z+1).$$

If we consider plane partitions inside a $1\times 1\times n$ box, we thus require that

$$\sum_{0\le l\le n}\prod_{0\le m<l} w(1/2,1/2,m+1/2)$$

should be nice, and should indeed correspond to a hypergeometric sum. The isotropy condition then implies more generally that

$$\sum_{0\le l\le n}\prod_{0\le m<l} w(x,y,z+m)$$

is a hypergeometric sum for any half-integer vector (x,y,z). In particular, this sum is doubly-telescoping, in the sense that the sum over any subinterval is nice. In particular, if we assume that the sum should be a special case of the Frenkel-Turaev sum [FT], we find that it should have the form (up to some convenient changes of parameters)

$$\sum_{0\le l\le n} \frac{q^l\theta_p(abq^{2l-1})}{\theta_p(q^{l-1}a,q^la,q^{l-1}b,q^lb)}\frac{\theta_p(a/q,a,b/q,b)}{\theta_p(ab/q)} = \frac{\theta_p(q^{n-1}ab,q^{n+1},a,b)}{\theta_p(ab/q,q,q^na,q^nb)},$$

where a and b depend on (x,y,z). In particular, we find

$$w(x,y,z+m) = \frac{q\theta_p(q^{m-1}a,q^{m-1}b,abq^{2m+1})}{\theta_p(q^{m+1}a,q^{m+1}b,abq^{2m-1})} = \frac{q^3\theta_p(q^{m-1}a,q^{m-1}b,q^{-2m-1}/ab)}{\theta_p(q^{m+1}a,q^{m+1}b,q^{1-2m}/ab)},$$

where a,b depend on (x,y,z); consistency then implies that $q^{-z}a$ and $q^{-z}b$ are independent of z.

Rotating the picture by 120 degrees gives a similar expression for $w(i,j,k)$ with the dependence on i or j factored out; comparing the results leads us to an expression of the form

$$w(x,y,z) = \frac{q^3\theta_p(q^{y+z-2x-1}u_1,q^{x+z-2y-1}u_2,q^{x+y-2z-1}u_3)}{\theta_p(q^{y+z-2x+1}u_1,q^{x+z-2y+1}u_2,q^{x+y-2z+1}u_3)},$$

where $u_1=a$, $u_2=b$, $u_3=1/ab$; i.e., u_1, u_2, u_3 are generic such that $u_1u_2u_3=1$. We will see that this indeed gives rise to a factored sum over plane partitions in a cube. Note that if we rewrite this expression in terms of new variables $\tilde u_1 = q^{y+z-2x}u_1$, $\tilde u_2=q^{x+z-2y}u_2$, $\tilde u_3=q^{x+y-2z}u_3$, then

$$w(x,y,z) = \frac{q^3\theta_p(\tilde u_1/q,\tilde u_2/q,\tilde u_3/q)}{\theta_p(q\tilde u_1,q\tilde u_2,q\tilde u_3)},$$

which can be described in a canonical way: w is the value at q of the unique elliptic function with simple zeros at $\tilde{u}_i$, simple poles at $\tilde{u}_i^{-1}$, and the value 1 at 1. It follows, in particular, that w is invariant under shifting any parameter by p, as well as under all modular transformations.

To write this in terms of lozenge weights, there are, of course, a number of choices one could make. One convenient choice is to allow only horizontal lozenges to have nonzero weights. We must thus have

$$\frac{w(i,j+1)}{w(i,j)} = \frac{q\theta_p(q^{j-3i/2-1}u_1, q^{j+3i/2-1}u_2, q^{2j+1}u_1u_2)}{\theta_p(q^{j-3i/2+1}u_1, q^{j+3i/2+1}u_2, q^{2j-1}u_1u_2)}$$

where $w(i,j)$ denotes the weight of a horizontal lozenge with upper corner at (i,j) (recall that in terms of 3-D coordinates, $i = x-y$, $j = z-(x+y)/2$). This recurrence is straightforward to solve, and one obtains

$$w(i,j) = C(i)\frac{q^{j-1/2}(u_1u_2)^{1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)},$$

where $C(i)$ is an arbitrary (non-vanishing) function of i.

If we view (with a mind to applying Kasteleyn's theorem) the lozenge weight as a matrix indexed by a right-pointing and a left-pointing triangle, we find that, coordinatizing triangles by their upper corners,

$$\begin{gathered}
w((i,j),(i,j)) = C(i)\frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)} \\
w((i,j),(i+1,j+1/2)) = 1, \qquad w((i,j),(i+1,j-1/2)) = 1,
\end{gathered}$$

and all other values are 0.

Let $\Pi^{y_0,y_1}_{x_0,x_1}$ represent the parallelogram

$$x_0 \le i \le x_1, y_0 \le j + i/2 \le y_1,$$

and observe that the restriction of w to triangles in $\Pi^{y_0,y_1}_{x_0,x_1}$ is a square matrix, and we can thus attempt to invert w in such a region. In fact, not only can we explicitly invert w in such parallelograms, but the result is independent of the choice of parallelogram.

Theorem 10.1. *The inverse transpose of w in $\Pi^{y_0,y_1}_{x_0,x_1}$ has the form*

$$\begin{aligned}
W((i_0,j_0),(i_1,j_1)) &= \delta_{i_0<i_1} \prod_{i_0\le k<i_1} C(k)\cdot(u_1u_2)^{(i_1-i_0-1)/2} \\
&\times \delta_{j_1+i_1/2\le j_0+i_0/2}(-1)^{j_0-i_0/2-j_1+i_1/2-1}q^{(i_1-i_0-1)(i_1-i_0+4j_1-2)/4} \\
&\times \frac{\theta_p(q^{j_0+i_0/2-j_1-i_1/2+1}, q^{j_0+i_0/2+j_1-i_1/2}u_1u_2;q)_{i_1-i_0-1}}{\theta_p(q, q^{j_0-i_0/2-i_1}u_1, q^{j_1-3i_1/2+1}u_1, q^{j_1+i_0+i_1/2}u_2, q^{j_0+3i_0/2+1}u_2;q)_{i_1-i_0-1}},
\end{aligned}$$

where

$$\theta_p(x;q)_k := \prod_{0\le i<k}\theta_p(q^ix).$$

Proof. Note that by Kasteleyn's theorem, W is the total weight (up to sign) of all lozenge tilings of the parallelogram that omit the two given triangles. In particular, $W((i_0,j_0),(i_1,j_1))$ must vanish if $i_0 \ge i_1$ or $j_0 + i_0/2 < j_1 + i_1/2$ simply because then no such tiling exists.

Now, the claim that W is the inverse transpose of w reduces to the statement

$$\begin{aligned}W((i_0,j_0),(i_1,j_1))w(i_1,j_1) + W((i_0,j_0),(i_1+1,j_1-1/2))&\\ + W((i_0,j_0),(i_1+1,j_1+1/2)) &= \delta_{(i_0,j_0),(i_1,j_1)}.\end{aligned}$$

This holds trivially for $i_0 > i_1$, since all three terms vanish, and similarly for $i_0 = i_1$, $j_1 > j_0$. When $i_0 = i_1$, $j_0 = j_1$, the claim reduces to

$$W((i_0,j_0),(i_0+1,j_0-1/2)) = 1,$$

while for $i_0 = i_1$, $j_0 \ge j_1$, we have

$$\begin{aligned}W((i_0,j_0),(i_0+1,j_1-1/2)) + W((i_0,j_0),(i_0+1,j_1+1/2))&\\ = (-1)^{j_0-j_1-1} + (-1)^{j_0-j_1} &= 0,\end{aligned}$$

as required.

It remains to consider the case $i_0 < i_1$. If $j_1 + i_1/2 > j_0 + i_0/2$, then all three terms again vanish, while if $j_1 + i_1/2 = j_0 + i_0/2$, the third term vanishes, and

$$W((i_0,j_0),(i_1,j_1))w(i_1,j_1) + W((i_0,j_0),(i_1+1,j_1-1/2)) = 0$$

as required. Finally, when $j_1 + i_1/2 < j_0 + i_0/2$, so that all three terms survive, if we divide by

$$\frac{W((i_0,j_0),(i_1,j_1))w(i_1,j_1)}{\theta_p(q^{i_1-i_0}, q^{j_0-i_0/2-i_1-1}u_1, q^{j_0+i_0/2+i_1}u_2, q^{2j_1-1}u_1u_2)},$$

we simply obtain a special case of the addition law for θ_p, in the form

$$\begin{aligned}\theta_p(a_0z, a_1z, a_2z, a_0a_1a_2/z) - \theta_p(a_0a_1, a_0a_2, a_1a_2, z^2)&\\ + \theta_p(z/a_0, a_1/z, a_2/z, a_0a_1a_2z)za_0 &= 0\end{aligned}$$

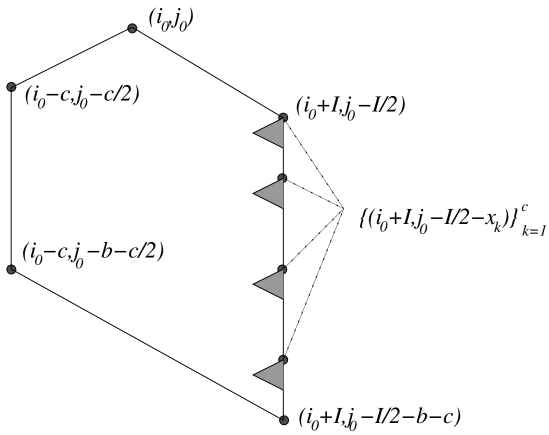
with

$$\begin{aligned}a_0 &= q^{-j_0/2-3i_0/4+j_1/2+3i_1/4}, \quad a_1 = q^{j_0/2-i_0/4+j_1/2-5i_1/4-1}u_1,\\ a_2 &= q^{j_0/2+3i_0/4+j_1/2+3i_1/4}u_2, \quad z = q^{j_0/2-i_0/4-j_1/2+i_1/4}.\end{aligned}$$

□

Remark. One major source of guidance regarding the form of W is that it corresponds to an enumeration of plane partitions in a rectangular parallelepiped with dimensions $m \times n \times 1$, say; i.e., a sum over ordinary partitions. If we first sum over the first part of the partition, we find that W should look like the term of a (singly) telescoping hypergeometric sum. One can also, of course, compute "small" values of W in the case $p = 0$, and look for patterns in the resulting factorizations.

Lemma 10.2. *Consider the domain*



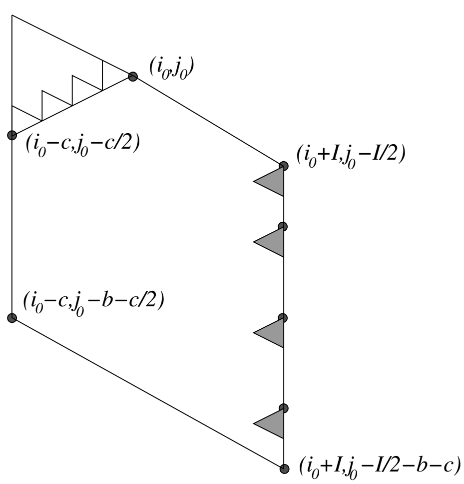
The total weight of lozenge tilings of this domain is equal to a constant independent of $\{x_k\}$ *times*

$$\prod_{1\le l\le c} \frac{(-1)^{x_l}\theta_p(q^{I+1}, q^{I-j_0+3i_0/2}/u_1, q^{-I+2-j_0-3i_0/2}/u_2, q^{I+1-2j_0}/u_1u_2;q)_{x_l}}{\theta_p(q, q^{2I-j_0+3i_0/2}/u_1, q^{2-j_0-3i_0/2}/u_2, q^{1-2j_0}/u_1u_2;q)_{x_l}}$$

times

$$\frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le l\le c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1, q^{x_l-j_0-3i_0/2+2}/u_2;q)_{c-1}}.$$

Proof. The problem is equivalent to computing the weight of tilings of the domain



According to Kasteleyn's theorem, cf. [Ka], we obtain

$$\det\left[W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))\right]_{k,l=1}^{c}.$$

Now, we can write

$$\begin{aligned}&\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\&\qquad=\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\&\qquad\qquad\times\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))},\end{aligned}$$

noting that (for generic values of the parameters) $W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))\neq 0$. Since

$$\begin{aligned}&\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l-1))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\\&=-\frac{\theta_p(q^{x_l}q^{I+1},q^{x_l}q^{I-j_0+3i_0/2}/u_1,q^{x_l}q^{-I+2-j_0-3i_0/2}/u_2,q^{x_l}q^{I+1-2j_0}/u_1u_2)}{\theta_p(q^{x_l}q,q^{x_l}q^{2I-j_0+3i_0/2}/u_1,q^{x_l}q^{2-j_0-3i_0/2}/u_2,q^{x_l}q^{1-2j_0}/u_1u_2)},\end{aligned}$$

we find

$$\begin{aligned}&\frac{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}\\&\quad=(-1)^{x_l}\frac{\theta_p(q^{I+1},q^{I-j_0+3i_0/2}/u_1,q^{-I+2-j_0-3i_0/2}/u_2,q^{I+1-2j_0}/u_1u_2;q)_{x_l}}{\theta_p(q,q^{2I-j_0+3i_0/2}/u_1,q^{2-j_0-3i_0/2}/u_2,q^{1-2j_0}/u_1u_2;q)_{x_l}}.\end{aligned}$$

For the other factor, we similarly have

$$\begin{aligned}&\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\\&\qquad\qquad\propto\frac{\theta_p(q^{-x_l},q^{1+x_l-2j_0+I}/u_1u_2;q)_{k-1}}{\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1,q^{x_l-j_0-3i_0/2+2}/u_2;q)_{k-1}},\end{aligned}$$

where we have removed factors independent of x_l, and observed that the right-hand side vanishes when $0\leq x_l<k$ as required. As a function of

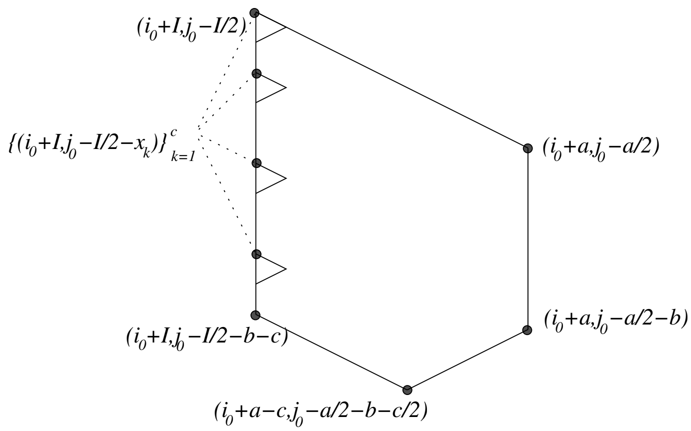
$$z_l=q^{x_l+I/2-j_0+1/2}(u_1u_2)^{-1/2},$$

this is invariant under $z_l\mapsto 1/z_l$, and we may thus apply Corollary 5.4 of [Wa] to conclude that

$$\begin{aligned}&\det\left[\frac{W((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{W((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\right]_{k,l=1}^{c}\\&\qquad\propto\frac{\prod_{1\leq k<l\leq c}q^{-x_k}\theta_p(q^{x_k-x_l},q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\leq l\leq c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1,q^{x_l-j_0-3i_0/2+2}/u_2;q)_{c-1}}.\end{aligned}$$

□

Lemma 10.3. *Similarly, the total weight of lozenge tilings of the domain*



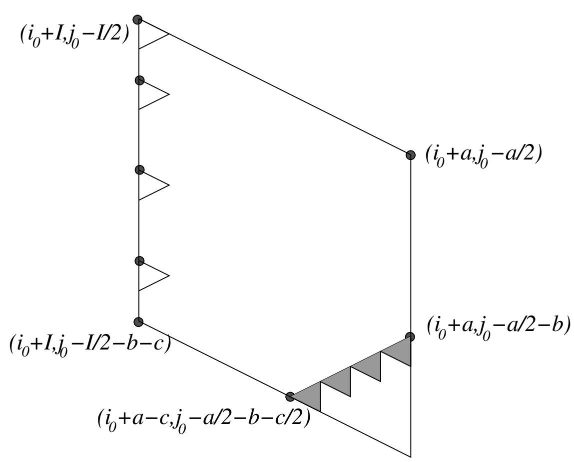
is equal to a constant independent of $\{x_k\}$ *times*

$$\prod_{1\le k\le c}\frac{(-1)^{x_k}\theta_p(q^{1-b-c},q^{2I-j_0+2+3i_0/2}/u_1,q^{-a+c-j_0-3i_0/2}/u_2,q^{-2j_0+a+b+1}/u_1u_2;q)_{x_k}}{\theta_p(q^{I-a-b+1},q^{I+2-j_0+3i_0/2+a-c}/u_1,q^{-I-j_0-3i_0/2}/u_2,q^{I-2j_0+b+c+1}/u_1u_2;q)_{x_k}}$$

times

$$\frac{\prod_{1\le k<l\le c}q^{-x_k}\theta_p(q^{x_k-x_l},q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le k\le c}\theta_p(q^{x_k+I-j_0+3i_0/2+2+a-c}/u_1,q^{j_0+3i_0/2+a+1-c-x_k}u_2;q)_{c-1}}$$

Proof. The problem is equivalent to computing the weight of tiling of the domain



Again, by the Kasteleyn theorem, we need to compute

$$\det\left[W((i_0+I,j_0-I/2-x_k),(i_0+a-c+l,j_0-a/2-b-c/2+l/2))\right].$$

Again,

$$W((i_0+I,j_0-I/2-x_k),(i_0+a-c+1,j_0-a/2-b-c/2+1/2)) \neq 0,$$

allowing us to factor the weights accordingly:

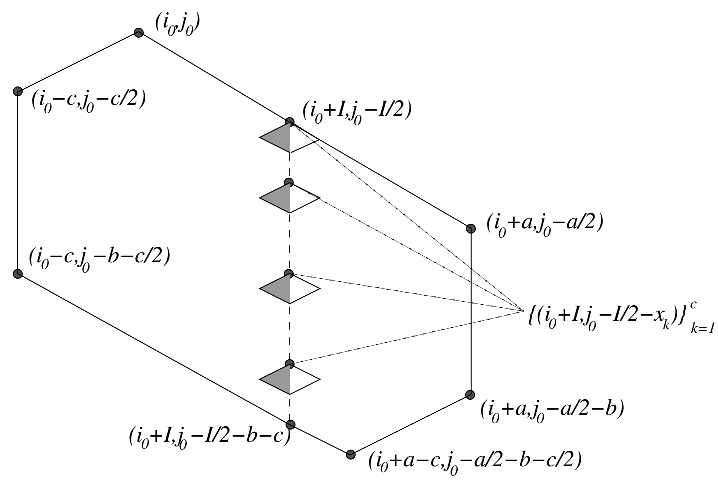
$$\frac{W((i_0+I,j_0-I/2-x_k),(i_0+a-c+1,j_0-a/2-b-c/2+1/2))}{W((i_0+I,j_0-I/2),(i_0+a-c+1,j_0-a/2-b-c/2+1/2))}$$
$$= (-1)^{x_k}\frac{\theta_p(q^{1-b-c},q^{2I-j_0+2+3i_0/2}/u_1,q^{-a+c-j_0-3i_0/2}/u_2,q^{-2j_0+a+b+1}/u_1u_2;q)_{x_k}}{\theta_p(q^{I-a-b+1},q^{I+2-j_0+3i_0/2+a-c}/u_1,q^{-I-j_0-3i_0/2}/u_2,q^{I-2j_0+b+c+1}/u_1u_2;q)_{x_k}}$$

and

$$\frac{W((i_0+I,j_0-I/2-x_k),(i_0+a-c+l,j_0-a/2-b-c/2+l/2))}{W((i_0+I,j_0-I/2-x_k),(i_0+a-c+1,j_0-a/2-b-c/2+1/2))}$$
$$\propto \frac{\theta_p(q^{x_k-b-c+1},q^{2j_0-I-b-c-x_k}u_1u_2;q)_{l-1}}{\theta_p(q^{I-j_0+3i_0/2+2+a-c}/u_1,q^{j_0+3i_0/2+a+1-c-x_k}u_2;q)_{l-1}}.$$

□

Lemma 10.4. *The weight of all tilings of the hexagon*



such that the horizonal tiles along the line $i = i_0+I$ are as specified, is a constant independent of $\{x_k\}$ times

$$\prod_{1\le k\le c}\frac{\theta_p(q^{2x_k+I+1-2j_0}/u_1u_2)}{\theta_p(q^{I+1-2j_0}/u_1u_2)}\cdot\frac{q^{x_k}\theta_p(q^{I+1},q^{1-b-c})}{\theta_p(q,q^{I-a-b+1})}$$
$$\times\frac{\theta_p(q^{I-j_0+3i_0/2}/u_1,q^{-a+c-j_0-3i_0/2}/u_2,q^{I+1-2j_0}/u_1u_2,q^{-2j_0+a+b+1}/u_1u_2;q)_{x_k}}{\theta_p(q^{I+2-j_0+3i_0/2+a-c}/u_1,q^{2-j_0-3i_0/2}/u_2,q^{1-2j_0}/u_1u_2,q^{I-2j_0+b+c+1}/u_1u_2;q)_{x_k}},$$

times

$$\prod_{1\le k<l\le c} q^{-2x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)^2$$
$$\times \prod_{1\le k\le c} \frac{1}{\theta_p(q^{x_k+I-j_0+3i_0/2+2+a-c}/u_1, q^{I-j_0+3i_0/2+2+a-c}/u_1;q)_{c-1}}$$
$$\times \prod_{1\le k\le c} \frac{1}{\theta_p(q^{j_0+3i_0/2+a+1-c-x_k}u_2, q^{j_0+3i_0/2+a+1-c-x_k}u_2;q)_{c-1}}.$$

Remark. The first "univariate" factor is just the product over the variables of a special case of the weight function considered by Spiridonov and Zhedanov in [SZ], and the two sets of elliptic hypergeometric biorthogonal functions constructed there are triangular with respect to the functions appearing in the two determinants. We could thus replace the second "cross-term" factor by a product of Vandermonde-style determinants in which the k-th row consists of the $(k-1)$st biorthogonal function evaluated at x_1 through x_c, analogously to our calculations for the q-Racah case above.

Using either lemma, we can in particular obtain a formula for the total weight of all tilings of a hexagon. Written in terms of plane partitions, we obtain the following elliptic analogue of MacMahon's identity.

Theorem 10.5. *Let p, q, u_1, u_2, u_3 be generic parameters such that $|p|<1$, $u_1u_2u_3=1$. Then*

$$\sum_{\Pi\subset a\times b\times c} \prod_{(i,j,k)\in\Pi} \frac{q^3\theta_p(q^{j+k-2i-1}u_1, q^{i+k-2j-1}u_2, q^{i+j-2k-1}u_3)}{\theta_p(q^{j+k-2i+1}u_1, q^{i+k-2j+1}u_2, q^{i+j-2k+1}u_3)}$$
$$= q^{abc} \prod_{1\le i\le a, 1\le j\le b, 1\le k\le c} \frac{\theta_p(q^{i+j+k-1}, q^{j+k-i-1}u_1, q^{i+k-j-1}u_2, q^{i+j-k-1}u_3)}{\theta_p(q^{i+j+k-2}, q^{j+k-i}u_1, q^{i+k-j}u_2, q^{i+j-k}u_3)}.$$

Proof. The term corresponding to a specific plane partition Π is easily seen by induction to be the ratio of the weight of the corresponding tiling to the weight of the tiling associated to the empty plane partition. The claim follows by simplifying the corresponding determinant of W using Corollary 5.4 of [Wa]. $\square$

Although it is possible to arrange for the elliptic weights to be positive, there are difficulties in the analysis. For one thing, the algebra required to replace orthogonal polynomials by biorthogonal functions in constructing the kernel has not been fully developed. A further complication in computing the limit kernel is that, although the biorthogonal functions do satisfy reasonably simple difference equations, they are not eigenfunctions of any difference operators. In addition, the corresponding variational problem is more difficult; while we can indeed solve the associated PDE, we have so far been unable to *derive* the solution. This is why we focus on a limiting case in the present paper.

Fix $u_1u_2 = pq\zeta^2$, let $u_1, u_2 = \Omega(\sqrt{p})$ as $p \to 0$, and similarly let $C(i) \sim p^{1/2}$. In this limit, one has

$$\lim_{p\to 0} w(i,j) = C'(i)\left(\zeta q^j - \frac{1}{\zeta q^j}\right),$$

or in other words the q-Racah weight(s) discussed above. In the notations of Sections 2-9, $\zeta = \kappa q^{-(S+1)/2}$. The corresponding limit of the elliptic MacMahon identity is

$$\sum_{\Pi\subset a\times b\times c}\prod_{(i,j,k)\in\Pi}\frac{q^{2k+1}\zeta^2 - q^{i+j-1}}{q^{2k}\zeta^2 - q^{i+j}}$$
$$= \prod_{1\le i\le a, 1\le j\le b, 1\le k\le c}\frac{(1-q^{i+j+k-1})(\zeta^2 - q^{i+j-k-2})}{(1-q^{i+j+k-2})(\zeta^2 - q^{i+j-k-1})},$$

or equivalently (performing the products over k and simplifying)

$$\sum_{\Pi\subset a\times b\times c} q^{|\Pi|}\prod_{1\le i\le a, 1\le j\le b}\frac{\zeta^2 - q^{i+j-2\Pi_{ij}-2}}{\zeta^2 - q^{i+j-c-2}} = \prod_{1\le i\le a, 1\le j\le b}\frac{1-q^{i+j+c-1}}{1-q^{i+j-1}}.$$

This, of course, becomes the usual MacMahon identity upon taking the limit $\zeta \to \infty$, cf. Section 7.21 in [St].

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