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NONLINEAR ENERGY TRANSPORT BETWEEN LONGITUDINAL ACOUSTIC MODES IN CYLINDRICAL COMBUSTION CHAMBERS

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Abstract

The second order nonlinear longitudinal acoustics in a cylindrical combustion chamber are studied for the case of an unstable second mode. A modal analysis is undertaken and a continuation method is used to determine the limit cycle behavior of the time dependent amplitudes of the acoustic modes as functions of the linear stability of the unstable acoustic mode. It is shown that if an insufficient number of modes are included in the truncated system, bifurcations of the primary limit cycle occur. The energy in the limit cycles is analyzed and the bifurcations are shown to occur as a means of increasing the amount of energy transfer out of the unstable acoustic mode and into the stable acoustic modes through the nonlinear terms.

Introduction

Because combustion instabilities arise normally as linearly unstable motions, nonlinear processes must be present to prevent the instabilities from growing without limit. Experimentally, therefore, nonlinear behavior is always observed. Serious analysis of nonlinear combustion instabilities began with work by Crocco, Sirignano, Mitchell and Zinn[1,2,3,4] at Princeton in the 1960's. The results reported here are the most recent from a continuing investigation begun in the early 1970's, using a form of Galerkin's method[5,6,7,8].

This approach is based on expressing any unsteady motion in a combustion chamber as a synthesis of normal modes,

$$P'(\mathbf{x},t) = \bar{P} \sum_{n=1}^{\infty} \eta_n(t)\, \psi_n(\mathbf{x}),$$

$$u'(\mathbf{x},t) = \sum_{n=1}^{\infty} \frac{\dot{\eta}_n(t)}{\gamma\, k_n^2}\, \nabla \psi_n(\mathbf{x}),$$

where $\psi_n(\mathbf{x})$ are normal modes for the combustion chamber geometry in question. Spatial averaging converts the problem of solving the system of nonlinear partial differential equations to the much simpler problem of solving a system of nonlinearly coupled ordinary differential equations for the time-dependent amplitudes of the normal modes of the form

$$\ddot{\eta}_n + \omega_n^2\, \eta_n = F_n,$$

where F_n is a nonlinear function of η_n, $\dot{\eta}_n$ and time. Various tests have confirmed that accurate results can be obtained with this procedure for a broad range of conditions[9]. Hence this system of equations, representing a collection of nonlinear oscillators, seems to be an acceptable formulation for studying various aspects of observed behavior understood poorly or not at all.

There are two main classes of nonlinear problems in this subject: determining the conditions for

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existence and stability of limit cycles; and determining the conditions under which a linearly stable system may become unstable when subjected to an appropriate disturbance. As a practical matter, two approximations have commonly been used to simplify the analysis and to try to obtain simpler methods for routine applications: (1) time averaging converts the second-order equations to a first order system governing the slowly changing amplitudes and phases of the modes; (2) in any case the expansion must be truncated at a finite number of modes.

Previous work by Jahnke and Culick[10] examined the effect of truncation at a finite number of modes on the limit cycle behavior of the system. For systems with an unstable first mode it was shown that the stability boundaries predicted by the two-mode approximation are artifacts of the truncation. Stability boundaries do not occur for systems of four or more modes. For systems with an unstable second mode, bifurcations were found to occur for moderately unstable systems. In particular, a pitchfork bifurcation was found to lead to a second branch of limit cycles and a subsequent Hopf bifurcation was found to cause quasi-periodic behavior. The work presented here is a further analysis of a system with an unstable second mode and shows that bifurcations occur as a result of insufficient energy transfer between acoustic modes due to truncation.

Nonlinear Acoustic Equations

The equations analyzed in this paper represent the time evolution of the amplitudes of the longitudinal acoustic modes in a cylindrical combustion chamber. Linear contributions from the combustion processes, gas/particle interactions, boundary conditions, and the interaction between the steady and unsteady flow fields are included along with nonlinear contributions from the gas dynamics. The equations representing the time evolution of the amplitudes of the longitudinal acoustic modes were obtained from Paparizos and Culick[11] and have the form

$$\begin{aligned}\ddot{\eta}_n + \omega_n^2 \eta_n = {} & 2\alpha_n \dot{\eta}_n + 2\theta_n \omega_n \eta_n \\ & - \sum_{i=1}^{n-1} \left(C_{ni}^{(1)} \dot{\eta}_n \dot{\eta}_{n-i} + D_{ni}^{(1)} \eta_i \eta_{n-i} \right) \\ & - 2 \sum_{i=1}^{\infty} \left(C_{ni}^{(2)} \dot{\eta}_n \dot{\eta}_{n+i} + D_{ni}^{(2)} \eta_i \eta_{n+i} \right)\end{aligned} \tag{1}$$

where

$$\begin{aligned}C_{ni}^{(1)} &= \frac{-1}{2\gamma i (n-i)} \left[n^2 + i(n-i)(\gamma-1) \right] \\ C_{ni}^{(2)} &= \frac{1}{2\gamma i (n+i)} \left[n^2 - i(n+i)(\gamma-1) \right] \\ D_{ni}^{(1)} &= \frac{(\gamma-1)\omega_1^2}{4\gamma} \left[n^2 - 2i(n-i) \right] \\ D_{ni}^{(2)} &= \frac{(\gamma-1)\omega_1^2}{4\gamma} \left[n^2 + 2i(n+i) \right].\end{aligned}$$

This system of equations has the form of a system of nonlinearly coupled oscillators. The parameters α_n and θ_n account for the linear processes mentioned above and represent the linear damping and frequency shift of each mode, respectively. Parameter values used in this study were obtained from Paparizos and Culick[11] and are listed in Table I.

Since this study is restricted to longitudinal acoustic modes in a cylindrical combustion chamber the modal frequencies are related by the relation $\omega_n = n\,\omega_1$. Substituting this relation into Eq. (1) and nondimensionalizing time with the fundamental acoustic frequency ($\widehat{t} = \omega_1 t$, where $\widehat{t}$ is nondimensional time) results in the system

$$\begin{aligned}\ddot{\eta}_n + n^2 \eta_n = {} & 2\widehat{\alpha}_n \dot{\eta}_n + 2n\widehat{\theta}_n \eta_n \\ & - \sum_{i=1}^{n-1} \left(C_{ni}^{(1)} \dot{\eta}_n \dot{\eta}_{n-i} + \frac{1}{\omega_1^2} D_{ni}^{(1)} \eta_i \eta_{n-i} \right) \\ & - 2 \sum_{i=1}^{\infty} \left(C_{ni}^{(2)} \dot{\eta}_n \dot{\eta}_{n+i} + \frac{1}{\omega_1^2} D_{ni}^{(2)} \eta_i \eta_{n+i} \right)\end{aligned} \tag{2}$$

where

$$\widehat{\alpha}_n = \frac{\alpha_n}{\omega_1} \qquad \widehat{\theta}_n = \frac{\theta_n}{\omega_1}.$$

Dynamical systems theory and continuation methods[10] are used to analyze this system so it must be written as a first-order system. This can be done by defining the new variable

$$\xi_n = \dot{\eta}_n. \tag{3}$$

and writing the system as

$$\begin{aligned}\dot{\eta}_n &= \xi_n \\ \dot{\xi}_n &= -n(n - 2\widehat{\theta}_n)\eta_n + 2\widehat{\alpha}_n \xi_n \\ & \quad - \sum_{i=1}^{n-1} \left(\widehat{C}_{ni}^{(1)} \xi_i \xi_{n-i} + \widehat{D}_{ni}^{(1)} \eta_i \eta_{n-i} \right) \\ & \quad - \sum_{i=1}^{\infty} \left(\widehat{C}_{ni}^{(2)} \xi_i \xi_{n+i} + \widehat{D}_{ni}^{(2)} \eta_i \eta_{n+i} \right)\end{aligned} \tag{4}$$

where

$$\widehat{C}_{ni}^{(1)} = \frac{-1}{2\gamma i(n-i)}\left[n^2 + i(n-i)(\gamma-1)\right]$$
$$\widehat{C}_{ni}^{(2)} = \frac{1}{\gamma i(n+i)}\left[n^2 - i(n+i)(\gamma-1)\right]$$
$$\widehat{D}_{ni}^{(1)} = \frac{\gamma-1}{4\gamma}\left[n^2 - 2i(n-i)\right]$$
$$\widehat{D}_{ni}^{(2)} = \frac{\gamma-1}{2\gamma}\left[n^2 + 2i(n+i)\right].$$

Energy in Limit Cycle

An equation for the energy in the limit cycle may be formed in the same manner as the energy equation for a mass-spring mechanical system; multiply Eq. (2) by $\dot{\eta}_n$ and write the system as

$$\begin{aligned} d\left[\tfrac{1}{2}\dot{\eta}_n^2 + \tfrac{1}{2}(n^2 - 2n\widehat{\theta}_n)\eta_n^2\right] &= 2\widehat{\alpha}_n \dot{\eta}_n \dot{\eta}_n \\ &- \dot{\eta}_n \sum_{i=1}^{n-1}\left(\widehat{C}_{ni}^{(1)} \dot{\eta}_n \dot{\eta}_{n-i} + \widehat{D}_{ni}^{(1)} \eta_i \eta_{n-i}\right) \\ &- \dot{\eta}_n \sum_{i=1}^{\infty}\left(\widehat{C}_{ni}^{(2)} \dot{\eta}_n \dot{\eta}_{n+i} + \widehat{D}_{ni}^{(2)} \eta_i \eta_{n+i}\right). \end{aligned} \quad (5)$$

The term $\frac{1}{2}\dot{\eta}_n^2$ represents the kinetic energy of the limit cycle and the term $\frac{1}{2}(n^2 - 2n\widehat{\theta}_n)\eta_n^2$ represents the potential energy of the limit cycle. The total energy in the limit cycle is conserved, as can be seen by integrating Eq. (5) over one period of the limit cycle. Since the left-hand side of the equation is an exact differential, integrating over the limit cycle results in

$$\begin{aligned} 0 =& 2\widehat{\alpha}_n \int_0^{\tau} \dot{\eta}_n^2\, dt \\ &- \int_0^{\tau} \dot{\eta}_n \sum_{i=1}^{n-1}\left(\widehat{C}_{ni}^{(1)} \dot{\eta}_n \dot{\eta}_{n-i} + \widehat{D}_{ni}^{(1)} \eta_i \eta_{n-i}\right) dt \\ &- \int_0^{\tau} \dot{\eta}_n \sum_{i=1}^{\infty}\left(\widehat{C}_{ni}^{(2)} \dot{\eta}_n \dot{\eta}_{n+i} + \widehat{D}_{ni}^{(2)} \eta_i \eta_{n+i}\right) dt. \end{aligned} \quad (6)$$

The terms in Eq. (6) represent energy production/dissipation terms related to the linear damping of the acoustic modes and energy transport terms related to the nonlinear coupling between acoustic modes. Equation (6) holds for each mode, so the energy production/dissipation and the energy transport must balance for each mode.

For many combustion systems one mode is unstable while the remaining modes are stable. Thus the linearly unstable mode will produce energy that the nonlinear terms will transport from the unstable modes to the stable modes where it is subsequently dissipated. Truncating the system at a finite number of modes alters both the transport of energy between modes and the number of modes available to dissipate energy.

Since the dissipation of energy by a stable acoustic mode is proportional to the time dependent amplitude of the acoustic mode sufficient dissipation can be achieved by simply increasing the amplitude of the acoustic mode. Previous results[10] have shown that the amplitude of the highest frequency acoustic mode in the truncated system is larger than it would be in the non-truncated system. This occurs as the highest frequency mode tends to dissipate much of the energy that would be dissipated by the truncated modes.

Transport of energy from the unstable mode to the stable modes is also affected by truncation of the system. Energy transport between modes can only be accomplished by the nonlinear terms in Eq. (2). Truncation of the system will reduce the ability to transport energy from the unstable to the stable modes. Increasing the modal amplitudes will enhance energy transport, but the phase relationship between modes also plays an important role in the energy transport.

Results and Discussion

Previous results by Jahnke and Culick[10] have shown that complicated behavior occurs for systems with an unstable second acoustic mode and all other modes stable. In particular, bifurcations of the limit cycles representing combustion instabilities occur. Figure 1 shows the maximum time dependent amplitudes of the acoustic modes as a function of α_2 for a six mode approximation. Stable limit cycles are represented by solid lines while unstable limit cycles are represented by dashed lines. Nine branches of limit cycles occur as a result of pitchfork bifurcations. Note that Eq. (2) is invariant to the transformation $\eta_n \mapsto (-1)^n \eta_n$ so the limit cycles arising from the pitchfork bifurcations come in pairs with the odd modes symmetric about zero. For clarity, only one of these pairs is shown in Fig. 1. Hopf bifurcations, which lead to the existence of quasi-periodic motions, also occur. It has not been possible in this work to continue the quasi-periodic solutions that result from the Hopf bifurcations, but it should be kept in mind that these solutions do exist[10].

Bifurcations at which a stable limit cycle becomes unstable are the physically important bifurcations. These bifurcations will result in a change in the qualitative nature of the combustion instability. In Fig. 1 the two physically relevant bifurcations are the pitchfork bifurcation near α_1=80 and the Hopf bifurcation near α_1=150. The pitchfork bifurcation results in a new stable limit cycle in which the odd acoustic modes are excited along with the even acoustic modes. The Hopf bifurcation causes the stable limit cycle to become unstable and results in a stable quasi-periodic motion[10].

Since the limit cycle behavior of systems with an unstable first acoustic mode were found to depend on the number of modes included in the truncated system[10], the limit cycle behavior of a system with a second mode instability was analyzed for systems with up to sixteen modes. The limit cycle behavior for this case was also found to depend on the number of modes included in the truncated system. Figure 2 shows the maximum of the time dependent amplitude of the first acoustic mode in the limit cycle as a function of α_2 for systems composed of from seven to sixteen modes. For small values of α_2 the first acoustic mode is unexcited for all the systems studied. Note that the first acoustic mode being unexcited corresponds to all the odd modes being unexcited (*cf.* Fig. 1).

All systems except the fourteen- and sixteen-mode approximations undergo a bifurcation at a critical value of α_2 which causes the limit cycle with the odd modes unexcited to become unstable. It is striking that the bifurcation not only occurs at different values of α_2 for systems composed of different numbers of modes, but the type of bifurcation also differs. Table II lists the values of α_2 at which the initial bifurcation occurs on the primary branch and the type of bifurcation that occurs. Note that for fourteen- and sixteen-mode systems no bifurcation occurs and the odd modes remain unexcited. Thus it seems that the bifurcations that cause the odd acoustic modes to become excited are artifacts of the truncation of the system.

Figure 3 shows the maximum time dependent amplitudes of the second acoustic mode for the branch of limit cycles with the odd modes unexcited as function of α_2 for various modal truncations. Only solutions for systems with an even number of modes are shown as the amplitude of η_2 does not change if one more mode is added to the system. Thus, the amplitude for η_2 is the same in the six and seven mode systems, the eight and nine mode systems, etc. This is not true for the stability of the limit cycles, as can be seen in Table II. Figure 3 shows that it is necessary to include at least fourteen modes to obtain the correct stability information and the proper amplitude for η_2 for the range of α_2 considered. Recall however that the odd modes are unexcited for the fourteen and sixteen mode approximations, so only seven or eight modes, respectively, are excited.

To help understand why bifurcations that excite the odd modes occur, the ratio of linear energy production to linear energy dissipation in the limit was computed. Figure 1h shows a plot of the ratio

$$\Lambda = \frac{2\,\hat{\alpha}_2 \int_0^\tau \dot{\eta}_2^2\, dt}{\sum_{n\neq 2} (2\,\hat{\alpha}_n \int_0^\tau \dot{\eta}_n^2\, dt)} \tag{7}$$

as a function of α_2 for the six mode approximation. This quantity represents the ratio of energy produced by the linearly unstable mode to the energy dissipated by the linearly stable modes. A somewhat surprising result is that the energy dissipated by the linear damping is larger than the energy produced by the unstable acoustic mode. It seems that during the transport of energy from the unstable to the stable modes by the nonlinear terms there is an amplification of the energy. As the system becomes more unstable (i.e. larger α_2) the ratio of linear production to linear dissipation decreases further.

Monitoring the contributions of the terms representing the linear production/dissipation and the nonlinear transport of energy in Eq. (6) shows that energy is produced by the linear term of the unstable mode. This energy is taken out of the unstable mode through the nonlinear terms of the unstable mode and fed into the nonlinear terms of the stable modes where the linear terms dissipate the energy. There always seems to be some amplification of the energy during transport by the nonlinear terms as shown by the fact that linear dissipation is always greater than linear production as shown in Fig. 1h.

The two-mode time averaged results help explain why this is the case. Assuming, as we do in time averaging, that in the limit cycle

$$\eta_n = r_n \sin(2\,\pi\, n\, t + \phi_n) \tag{8}$$

then the linear energy dissipation of each mode is

$$\text{dis}_n = 4\,\pi^2\, n^2\, \alpha_n\, r_n^2. \tag{9}$$

Paparizos and Culick[11] give the energy dissipation as $2\,\alpha_n\, r_n^2$, but the frequency must have some effect on the energy dissipation of each mode. For example, if the maximum amplitude of the η_1 and η_2 are the same, the maximum values of $\dot{\eta}_1$ and $\dot{\eta}_2$ are not the same because the modes have different frequencies. Since the dissipation is dependent on $\dot{\eta}_n$ the frequency will come into the dissipation equation.

The two mode time averaged approximation can be used to examine the energy balance. For the two-mode time averaged approximation[10]

$$\begin{aligned} \dot{r}_1 &= \alpha_1 r_1 - \kappa r_1 r_2 \cos\psi_2 \\ \dot{r}_2 &= \alpha_2 r_2 + \kappa r_1^2 \cos\psi_2 \\ \dot{\psi}_2 &= (\theta_2 - 2\theta_1) + \kappa\left(2 r_2 - \frac{r_1^2}{r_2}\right)\sin\psi_2. \end{aligned} \tag{10}$$

In the limit cycle

$$\frac{r_1}{r_2} = \sqrt{-\alpha_2/\alpha_1}$$

so using Eq. (9) we see that the ratio of linear energy production to linear energy dissipation is

$$\Lambda = \frac{1}{4}.$$

This is the limiting value given by the continuation results for the six-mode approximation for $\alpha_2 \to 0$ as shown in Fig. 1h.

The reason for the decrease in the ratio of linear energy production to linear energy dissipation as α_2 increases can be seen by noting that the phase difference between the two modes, ψ_2, in the limit cycle for the two mode time averaged approximation is

$$\psi_2 = \tan^{-1}\left(\frac{2\theta_1 - \theta_2}{2\alpha_1 + \alpha_2}\right). \tag{11}$$

Nonlinear transport of energy between the first and second acoustic modes depends on $\cos\psi_2$ as shown in Eq. (10). As α_2 is increased $2\alpha_1 + \alpha_2$ decreases and ψ_2 increases causing the nonlinear transport of energy between the modes to decrease. The stability boundary at $\alpha_2 = -2\alpha_1$ for the two mode time averaged approximation[11] results because $\psi_2 = \frac{\pi}{2}$ for this situation and it is no longer possible to transport energy out of the unstable acoustic mode. This causes the amplitude of the unstable acoustic mode in the limit cycle to become infinite. The stability boundary thus has more to do with the transport of energy that with the linear dissipation of energy. The stable modes can always dissipate more energy by increasing their amplitudes, but a bottle-neck can occur in transporting the energy from the unstable to the stable modes through the nonlinear terms.

Figure 1h shows that a stability change occurs at the first pitchfork bifurcation and Λ on the stable branch is larger than on the unstable branch. Recall that the linear production is equal to the nonlinear transport of energy out of the unstable mode and, as we saw for the two mode time averaged result, there may be some bottle-neck in the nonlinear transport such that not enough energy is getting out of the unstable mode causing the limit cycle to become unstable. Since the odd acoustic modes are excited in the new branch of limit cycles it is now possible to transport more energy out of the unstable mode and into the stable modes.

Since the ability to transport energy out of the unstable mode to the stable modes is dependent on the number of modes included in the truncated system, Λ was calculated for systems of up to 16 acoustic modes. Figure 4 shows Λ as a function of α_2 for systems of seven to sixteen modes. The figure shows that the ratio of linear energy production to linear energy dissipation decreases for more unstable systems(i.e increasing values of α_2). Thus there is more than enough ability to dissipate energy by the stable modes. The limiting factor is the ability to transport energy out of the unstable mode to the stable modes. As a way of getting around the artificial restrictions on energy flow caused by truncation of the system, bifurcations occur that cause the odd acoustic modes to become excited, thus opening up more avenues for energy transfer out of the unstable acoustic mode. Figures 4b and 4d show that Λ is larger on the stable branch arising form the pitchfork bifurcation than on the unstable branch. This is a result of the odd modes being excited on the stable branch but not on the unstable branch.

Conclusions

This analysis has shown that the limited energy transport provided in the truncated system is responsible for the bifurcations of the limit cycles seen in previous work on systems with an unstable second mode. If a sufficient number of acoustic modes is included in the approximate system no bifurcations of the limit cycle occur. Since the transport of energy is seen to be the limiting factor in the truncated system, the inclusion of higher order nonlinearities in the system may be necessary to model combustion instabilities for highly unstable systems.

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TABLE I: Damping and Frequency Shift

$\omega_1 = 2827.435$		
n	$\alpha_n\,(s^{-1})$	$\theta_n\,(rad/s)$
1	−84.9	−66.7
2	$0 \rightarrow 300$	12.9
3	−161.0	108.2
4	−279.4	46.8
5	−392.7	8.8
6	−520.2	−29.3
7	−664.4	0.0
8	−826.0	0.0
9	−1005.0	0.0
10	−1199.4	0.0
11	−1408.0	0.0
12	−1617.0	0.0
13	−1836.0	0.0
14	−2066.0	0.0
15	−2306.0	0.0
16	−2555.0	0.0

TABLE II: Primary Bifurcation Locus

# Modes	$\alpha_2\,(s^{-1})$	Bif. Type
4	107	Pitchfork
5	110	Turning Pt.
6	82	Pitchfork
7	113	Torus
8	156	Pitchfork
9	173	Torus
10	149	Pitchfork
11	200	Torus
12	274	Torus
13	235	Torus
14	− − −	None
15	274	Torus
16	− − −	None

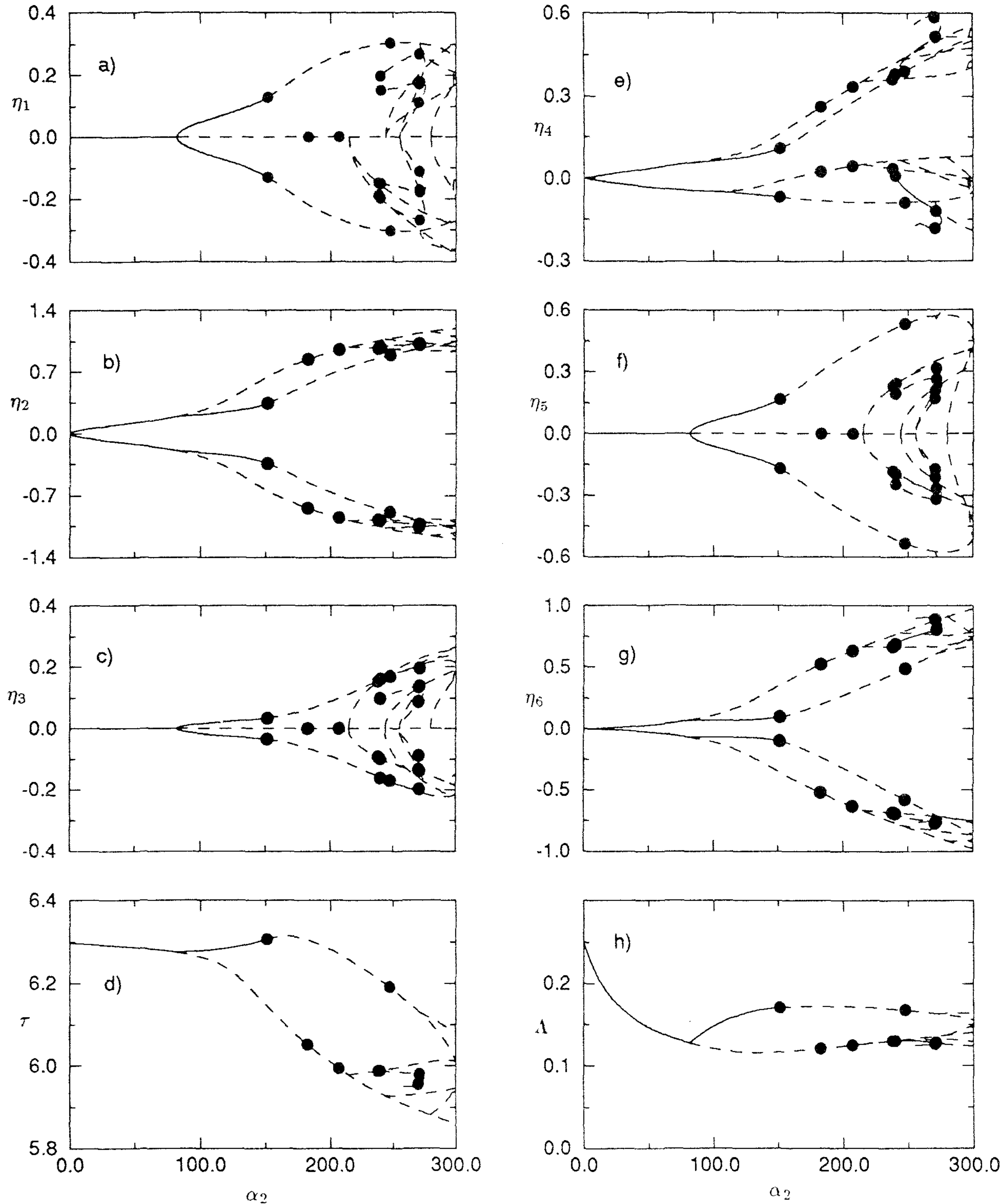

Figure 1: Limit cycle behavior for six-mode approximation as a function of α_2, •—Hopf bifurcation; a) η_1, b) η_2, c) η_3, d) Period, e) η_4, f) η_5, g) η_6, h) Λ.

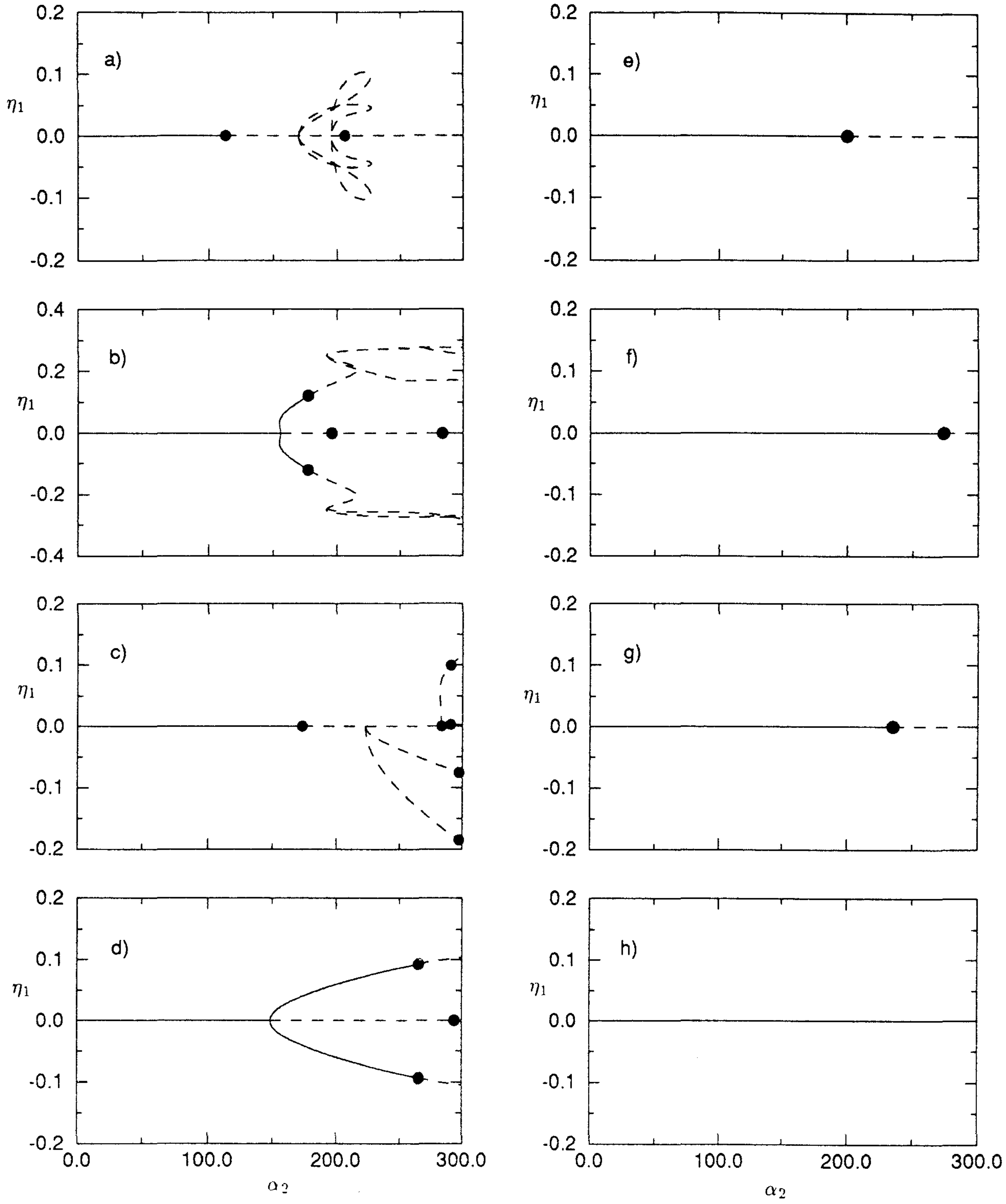

Figure 2: Maximum amplitude of η_1 in limit cycle as a function of α_2 for different modal approximations, $\bullet$—Hopf bifurcation; a) Seven mode, b) Eight mode, c) Nine mode, d) Ten mode, e) Eleven mode, f) Twelve mode, g) Thirteen mode, h) Fourteen mode, i) Fifteen mode, j) Sixteen mode.

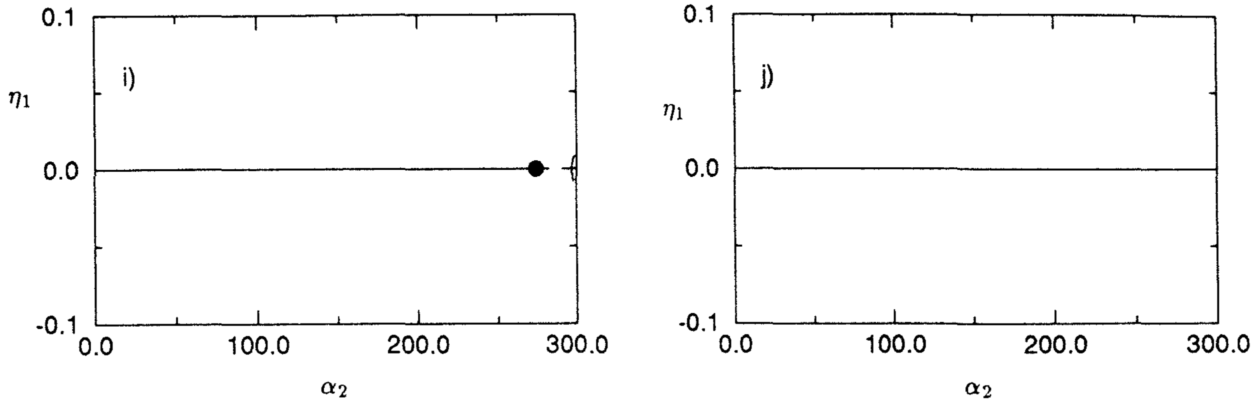

Figure 2: Continued.

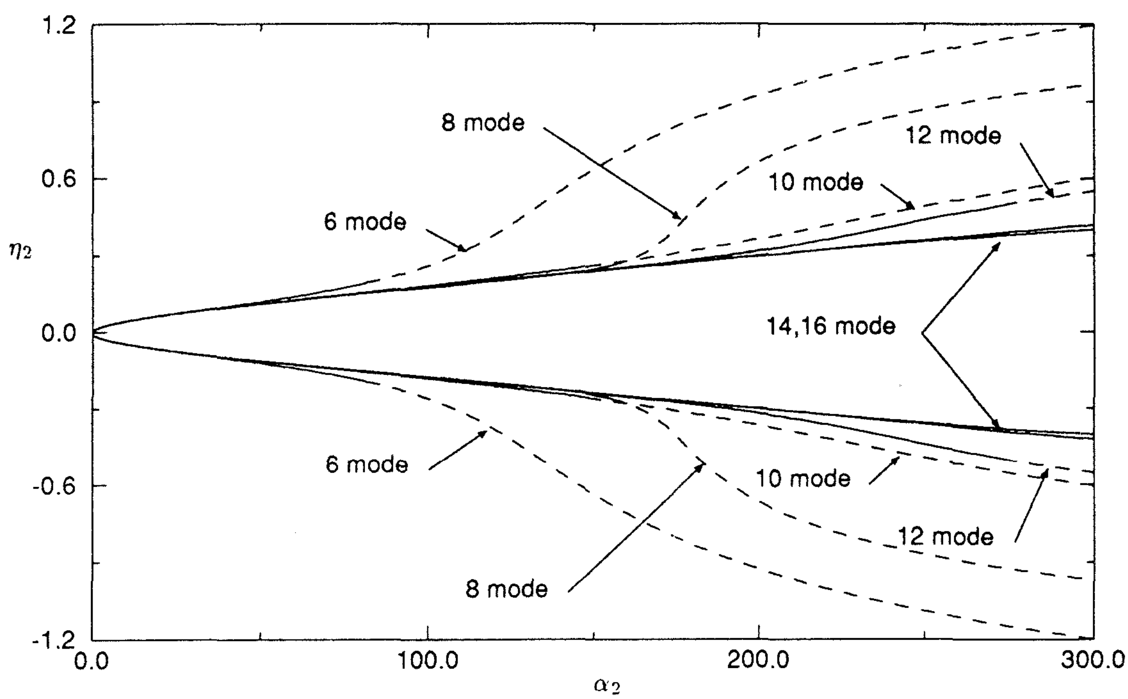

Figure 3: Maximum amplitude of η_2 in limit cycle as a function of α_2 for different modal approximations.

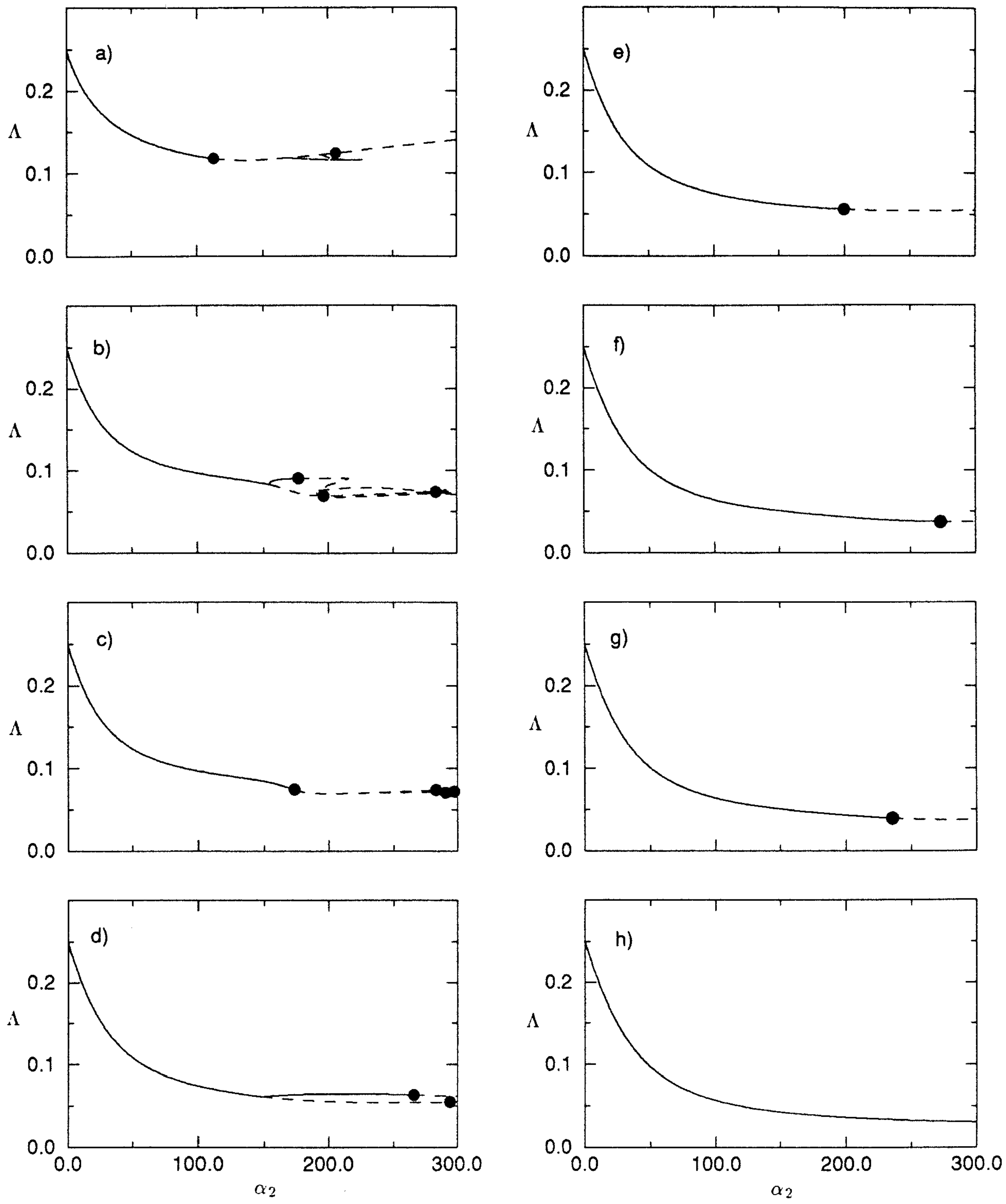

Figure 4: Ratio of linear energy production to linear energy dissipation as a function of α_2 for different modal approximations, •–Hopf bifurcation; a) Seven mode, b) Eight mode,c) Nine mode, d) Ten mode, e) Eleven mode, f) Twelve mode, g) Thirteen mode, h) Fourteen mode, i) Fifteen mode, j) Sixteen mode.

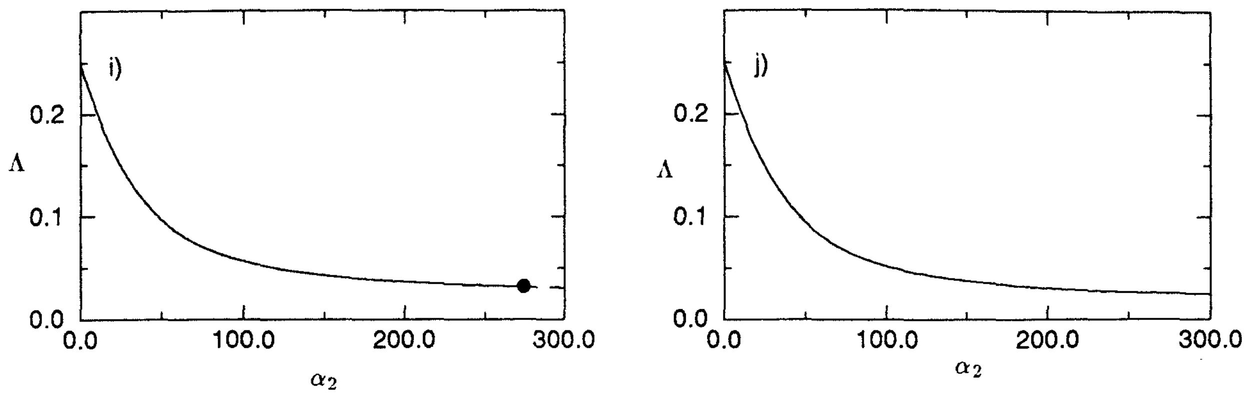

Figure 4: Continued.