AIAA'87

AIAA-87-1873

Third-Order Nonlinear Acoustic Waves and Triggering of Pressure Oscillations in Combustion Chambers, Part I: Longitudinal Modes

Vigor Yang and Seung-Ill Kim
The Pennsylvania State University
University Park, PA

Fred E. C. Culick
California Institute of Technology
Pasadena, CA

AIAA/SAE/ASME/ASEE 23rd Joint Propulsion Conference

June 29-July 2, 1987/San Diego, California

THIRD-ORDER NONLINEAR ACOUSTIC WAVES AND TRIGGERING OF PRESSURE OSCILLATIONS IN COMBUSTION CHAMBERS, PART I: LONGITUDINAL MODES

Vigor Yang* and Seung-Ill Kim**
The Pennsylvania State University
University Park, Pennsylvania 16802

Fred E. C. Culick+
California Institute of Technology
Pasadena, California 91125

ABSTRACT

An analytical analysis has been developed to investigate the behavior of unsteady motions in combustion chambers. The model accommodates the third-order nonlinear acoustics and a second-order combustion response. The influences of various linear and nonlinear parameters on the limit cycles and triggering of pressure oscillations are discussed in detail. Results indicate that the third-order acoustics has little influence on the triggering of instability. It only affects the limiting amplitudes and the stability domains of limit cycles. The nonlinear combustion response plays an essential role in determining the characteristics of triggering.

1. INTRODUCTION

Unsteady motions excited and sustained by combustion processes are a fundamental problem in the development of high performance propulsion systems. The essential cause is the high rate of energy release confined to a volume in which energy losses are relatively small. Only a very small amount of chemical energy needs to be transformed to mechanical energy of time-varying fluid motions to produce unacceptable excursion of pressure oscillations. The ensuing vibrations of the structure may lead to failure of the structure itself or of equipment and instrumentation.

Two types of nonlinear instabilities have been commonly observed. They are classified as spontaneous and pulsed oscillations according to the mechanisms of initiation. Spontaneous instabilities require no external disturbances and arise from causes entirely to the system. Typically, a small unstable initial disturbance grows exponentially for some time, eventually reaching a limiting amplitude. Pulsed oscillation, also known as triggered instability, refers to initiation of instabilities by finite amplitude disturbance in a system which is otherwise stable to small perturbations. Instabilities occur only if the amplitude of initial disturbance exceeds certain critical value. Both spontaneous and pulsed instabilities necessarily involve nonlinear processes. It is impossible for a disturbance to be triggered to a limiting amplitude by linear processes alone. However, the form and order of nonlinearities for each case may by quite different.

There are two important questions of practical concern: What amplitudes will unstable oscillations reach, and what sort of initial disturbances will cause a linearly stable system to exhibit oscillations? Both are related to the fundamental behavior of a nonlinear system and can be translated as theoretical problems of general nature. The first deals with the conditions for the existence and stability of limit cycles, a matter previously addressed by the authors.[1,2] The second is related to the triggering of combustion instabilities. It appears that in order to answer this question, nonlinear influences higher than second order must be considered. The purpose of this paper is to develop a third-order nonlinear acoustic model within which a broad study of triggering of combustion instabilities can be conducted.

Several analyses of nonlinear combustion instabilities have been carried out. Powell[3] considered pressure oscillations in liquid propellant rockets. With the aid of the methods of normal mode expansion and spatial averaging, he was able to derive a system of ordinary differential equations for the amplitude of each mode, which was then solved numerically. Kooker and Zinn[4] studied triggering in solid propellant rockets by solving numerically the conservation equations with various combustion response functions included. Powell, et al.[5] also employed a similar technique described in Ref. 3 for investigation of pressure oscillations in solid propellant rockets. Recently, Levine and Baum[6-8] conducted extensive numerical studies of pulse triggered instability in solid rocket motors. The scheme is capable of describing multi-shock, steep-fronted type of instabilities in various tactical motors. Very good comparison between calculated and measured[9] wave motions was obtained.

In spite of the significant progress made so far, these numerical works suffer a common deficiency. They provide detailed results for each special situation. Many cases must be calculated to perceive trend and to draw conclusions concerning general behavior. As an alternative, we resort to analytical approximate methods here. In the following sections, a general framework accommodating various linear and

* Assistant Professor, Mechanical Engineering, Member AIAA
** Graduate Student, Mechanical Engineering
+ Professor, Jet Propulsion and Applied Physics; Fellow AIAA

nonlinear processes is first developed. The influences of the third-order acoustics and nonlinear combustion response are then discussed in detail.

2. CONSTRUCTION OF THE NONLINEAR WAVE EQUATION

The nonlinear wave equation for the pressure can be constructed by suitably manipulating the conservation equations and the equation of state. The method extends the previous model for the second-order nonlinear acoustics[10] and accommodates third-order nonlinearities. In brief outline, the conservation equations for a two-phase mixture of gas and solid particles are first written in the following form.

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = -\rho \nabla \cdot \vec{u} + W \qquad (2.1)$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p + \vec{F} \qquad (2.2)$$

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot \vec{u} = -\vec{u} \cdot \nabla p + P \qquad (2.3)$$

The function W represents the mass conversion rate of condensed phases to gas per unit volume, $\vec{F}$ is the force of interaction between the gas and condensed phases, and P is the sum of the heat release associated with chemical reaction and the energy transfer between two phases. The next step is based on the decomposition of flow variables into mean and time-varying parts.

$$\begin{aligned} \rho &= \bar{\rho} + \rho'(t,\vec{r}) \\ \vec{u} &= \bar{\vec{u}}(\vec{r}) + \vec{u}'(t,\vec{r}) \\ p &= \bar{p} + p'(t,\vec{r}) \end{aligned} \qquad (2.4)$$

Now substitute (2.4) in (2.1)-(2.3), collect coefficients of like powers, and rearrange the results to produce the nonlinear wave equation valid to third order.

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h \qquad (2.5)$$

the function h contains all linear and nonlinear influences of acoustic motions, mean flow, and combustion. Care must be exercised in treating the nonlinear terms. If the Mach number of the mean flow and the amplitude of the oscillation have the same order of magnitude, the contributions from the nonlinear acoustics can be treated separately. For convenience, the source term h is further written as

$$h = h_\mu + h_\varepsilon + h_\nu + h_{\varepsilon\varepsilon} \qquad (2.6)$$

The subscripts μ, ε, and ν represent respectively the linear processes, the second-order gasdynamics and all nonlinear contributions for $\vec{F}'$ and P'. Their explicit expressions are given in Ref. 10. The third-order acoustic term $h_{\varepsilon\varepsilon}$ is shown to be

$$h_{\varepsilon\varepsilon} = -\nabla \cdot \left[\frac{p'}{\bar{a}^2}(\vec{u}' \cdot \nabla)\vec{u}'\right] + \frac{\gamma-1}{2\bar{\rho}\,\bar{a}^4}\nabla \cdot \left(p'^2 \frac{\partial u'}{\partial t}\right) \qquad (2.7)$$

The couplings between mean flow variations and acoustic motions are not included, but can be treated in the same manner.

The boundary condition associated with (2.5) is found by taking the component $\nabla p'$ normal to the boundary:

$$\vec{n} \cdot \nabla p' = -f \equiv -(f_\mu + f_\varepsilon + f_\nu + f_{\varepsilon\varepsilon}) \qquad (2.8)$$

where the third-order term is

$$f_{\varepsilon\varepsilon} = \left[\frac{p'}{\bar{a}^2}(\vec{u}' \cdot \nabla)\vec{u}' - \frac{\gamma-1}{2\bar{\rho}\,\bar{a}^4}p'^2\frac{\partial u'}{\partial t}\right] \cdot \vec{n} \qquad (2.9)$$

The solution of the wave equation (2.5) is approximated by a synthesis of the normal modes of the chamber, but with unknown time-varying amplitudes.

$$p'(\vec{r},t) = \bar{p}\sum_n \eta_n(t)\,\psi_n(\vec{r}) \qquad (2.10a)$$

$$\vec{u}'(\vec{r},t) = \sum_n \frac{\dot{\eta}_n(t)}{\gamma k_n^2}\nabla\psi_n(\vec{r}) \qquad (2.10b)$$

where ψ_n is the normal mode function satisfying

$$\nabla^2\psi_n + k_n^2\psi_n = 0 \qquad (2.11)$$

$$\vec{n} \cdot \nabla\psi_n = 0 \qquad (2.12)$$

The basis of this expansion is that many of the observed combustion instabilities are composed of several harmonic motions with mode shapes corresponding to the normal modes of the system. It can be applied to describe a wide variety of motions, including shock-like oscillations.

After substituting (2.10) in (2.5), multiplying the result by ψ_n, and integrating over the entire volume, we obtain the set of ordinary differential equations for the amplitude of each mode.

$$\ddot{\eta}_n + \omega_n^2\eta_n = F_n \qquad (2.13)$$

The forcing function F_n is

$$F_n = -\frac{\bar{a}^2}{E_n^2}\left[\iiint \psi_n h\,dv + \iint \psi_n f\,ds\right] \qquad (2.14)$$

and

$$E_n^2 = \iiint \psi_n^2\,dv \qquad (2.15)$$

ultimately it is important that missing are terms of the form $\dot\eta\eta$, $\dot\eta\dot\eta\eta$ & $\dot\eta\eta\eta$

Thus, the problem comes down to solving the set (2.13) for the time-dependent amplitude, to give the evolution of the system subject to a specified initial condition.

If only the nonlinear contributions from acoustic wave motions are considered, F_n can be expressed more conveniently as

$$
\begin{aligned}
F_n = & - \sum_i \left[D_{ni}\dot{\eta}_i + E_{ni}\eta_i \right] \\
& - \sum_i \sum_j \left[A_{nij}\dot{\eta}_i\dot{\eta}_j + B_{nij}\eta_i\eta_j \right] \\
& - \sum_i \sum_j \sum_m \left[R_{nijm}\dot{\eta}_i\dot{\eta}_j\dot{\eta}_m + S_{nijm}\eta_i\eta_j\eta_m \right]
\end{aligned}
\tag{2.16}
$$

The coefficients D_{ni} and E_{ni} arise from linear processes, and the nonlinear coefficients are defined as follows.

$$A_{nij} = \frac{I_{nij}}{4\gamma k_i^2 k_j^2} \left[(k_i^2 + k_j^2)^2 - k_n^4 - 4\gamma k_i^2 k_j^2 \right] \tag{2.17a}$$

$$B_{nij} = \frac{I_{nij}}{2\gamma} (\gamma - 1)\, \bar{a}^2 \, (k_i^2 + k_j^2) \tag{2.17b}$$

$$
\begin{aligned}
R_{nijm} = \frac{1}{2\gamma^2 k_i^2 k_j^2 E_n^2} & \left[k_n^2 \int \psi_n \psi_m \nabla\psi_i \cdot \nabla\psi_j dv \right. \\
& \left. - \int (\nabla\psi_i \cdot \nabla\psi_j)(\nabla\psi_n \cdot \nabla\psi_m) dv \right]
\end{aligned}
\tag{2.17c}
$$

$$S_{nijm} = \frac{(\gamma-1)\, \bar{a}^2 k_n^2}{6\, \gamma^2\, E_n^2} \int \psi_i \psi_j \psi_m \psi_n dv \tag{2.17d}$$

and

$$I_{nij} = \int \psi_i \psi_j \psi_n dv / E_n^2 \tag{2.17e}$$

Within the present representation of unsteady motions, the amplitude of acoustic wave is assumed to have the same order of magnitude of the Mach number of the mean flow. Consequently, the influences of the mean flow do not appear directly in the nonlinear terms. The nonlinear interactions in (2.16) simply cause energy exchange between modes which then generates harmonics in the acoustic field.

Owing to the nonlinearities and the couplings between modes, the exact analytical approach to (2.13) is formidable. However, considerable simplification can be achieved by taking advantage of the fact that in most practical situations the wave amplitudes vary quite slowly, the fractional change being small in one cycle of the oscillation. There are certainly situations in which this assumption may not hold, but as the limit cycle is approached, changes in amplitude obviously take place much more slowly on a time scale associated with the normal mode oscillations. A convenient procedure to follow this observation is based on the method of time averaging. In the following remarks, an approximate solution technique using the method of time averaging is developed to solve the set (2.13) for the time behavior of each acoustic mode.

Not true if cycle of ... should be shown

Method of Time Averaging

The solutions based on the time-averaging procedure start with the method of variation of parameters which transform the time-varying amplitude η_n to two variables A_n and B_n.

$$
\begin{aligned}
\eta_n(t) &= A_n(t)\sin\omega_n t + B_n(t)\cos\omega_n t \\
&= r_n(t)\sin[\omega_n t + \phi_n(t)]
\end{aligned}
\tag{2.18}
$$

where A_n and B_n are slowly varying functions of time. After substituting in (2.13) imposing the condition

$$\dot{A}_n \sin\omega_n t + \dot{B}_n \cos\omega_n t = 0 \tag{2.19}$$

and integrating the result over the period of the fundamental mode, we obtain the approximate equations for A_n and B_n. For longitudinal mode of oscillations, this procedure eventually leads to the following equations.

$$
\begin{aligned}
\frac{dA_n}{dt} = & \ \alpha_n A_n + \theta_n B_n \\
& + \frac{\beta n}{2} \sum_i \sum_j \left[A_i A_j (\delta_{n,i+j} - \delta_{n,i-j} - \delta_{n,j-i}) \right. \\
& \qquad \left. - B_i B_j (\delta_{n,i+j} + \delta_{n,i-j} + \delta_{n,j-i}) \right] \\
& + \epsilon n \sum_i \sum_j \sum_m \left[A_i A_j B_m (\delta_{n,m-i-j} + \delta_{n,i+j-m} \right. \\
& \qquad + \delta_{n,i+j+m} - \delta_{n,j-m-i} - \delta_{n,m-i+j} \\
& \qquad - \delta_{n,i-j-m} - \delta_{n,i+m-j} + 2\delta_{n,m}\delta_{i,j}) \\
& \qquad - \frac{1}{3} B_i B_j B_m (\delta_{n,m-i-j} + \delta_{n,i+j-m} + \delta_{n,i+j+m} \\
& \qquad + \delta_{n,j-m-i} + \delta_{n,m-i+j} + \delta_{n,i-j-m} \\
& \qquad + \delta_{n,i+m-j} - 6\delta_{n,m}\delta_{i,j}) \\
& \qquad \left. - \frac{4(\gamma-1)}{\gamma+1} (A_i A_j B_m + B_i B_j B_m)\, \delta_{n,m}\delta_{i,j} \right]
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\frac{dB_n}{dt} = & - \theta_n A_n + \alpha_n B_n \\
& + \frac{\beta n}{2} \sum_i \sum_j 2A_i B_j (\delta_{n,i+j} + \delta_{n,i-j} - \delta_{n,j-i}) \\
& + \epsilon n \sum_i \sum_j \sum_m \left[B_i B_j A_m (\delta_{n,m-i-j} - \delta_{n,i+j-m} \right. \\
& \qquad + \delta_{n,i+j+m} - \delta_{n,j-m-i} + \delta_{n,m-i+j} \\
& \qquad - \delta_{n,i-j-m} + \delta_{n,i+m-j} - 2\delta_{n,m}\delta_{i,j}) \\
& \qquad - \frac{1}{3} A_i A_j A_m (\delta_{n,m-i-j} - \delta_{n,i+j-m} + \delta_{n,i+j+m} \\
& \qquad + \delta_{n,j-m-i} - \delta_{n,m-i+j} + \delta_{n,i-j-m} \\
& \qquad - \delta_{n,i+m-j} + 6\delta_{n,m}\delta_{i,j}) \\
& \qquad \left. + \frac{4(\gamma-1)}{\gamma+1} (A_i A_j A_m + B_i B_j A_m)\, \delta_{n,m}\delta_{i,j} \right]
\end{aligned}
\tag{2.21}
$$

$\bar{u}\varepsilon \ll \bar{u}$, $\bar{u}^2 \ll \bar{u}$

Order of terms in wave eqn.

$-\varepsilon- = \varepsilon,\ \bar{u}\varepsilon,\ \varepsilon^2,\ \varepsilon^3,\ \bar{u}\varepsilon^2,\ \bar{u}^2\varepsilon \Rightarrow O(1) = 1,\ \bar{u},\ \varepsilon,\ \varepsilon^2,\ \bar{u}\varepsilon,\ \bar{u}^2$

2nd order — 3rd order — ignored

where β and ε are coefficients arising from the second- and third-order nonlinear acoustics, respectively, and defined as

$$\beta = \frac{\gamma+1}{8\gamma}\,\omega_1 \quad \text{and} \quad \varepsilon = \frac{\gamma+1}{64\gamma^2}\,\omega_1$$

3. LIMIT CYCLES FOR TWO MODES

As a first approach, the coupling between the first two modes of oscillation is considered. This case represents the simplest possible situation and may serve as the basis for analyzing more complicated problems. The equations for A_n and B_n reduce to

$$\frac{dA_1}{dt} = \alpha_1 A_1 + \theta_1 B_1 - \beta(A_1A_2 + B_1B_2) + \varepsilon_1(A_1^2 + B_1^2)B_1 - \varepsilon_2(A_2^2 + B_2^2)B_1 \tag{3.1}$$

$$\frac{dB_1}{dt} = -\theta_1 A_1 + \alpha_1 B_1 - \beta(A_1B_2 - A_2B_1) - \varepsilon_1(A_1^2 + B_1^2)A_1 + \varepsilon_2(A_2^2 + B_2^2)A_1 \tag{3.2}$$

$$\frac{dA_2}{dt} = \alpha_2 A_2 + \theta_2 B_2 + \beta(A_1^2 - B_1^2) + 2\varepsilon_1(A_2^2 + B_2^2)B_2 - 2\varepsilon_2(A_1^2 + B_1^2)B_2 \tag{3.3}$$

$$\frac{dB_2}{dt} = -\theta_2 A_2 + \alpha_2 B_2 + 2\beta A_1B_1 - 2\varepsilon_1(A_2^2 + B_2^2)A_2 + 2\varepsilon_2(A_1^2 + B_1^2)A_2 \tag{3.4}$$

The coefficients ε_1 and ε_2 are associated with the third-order coupling, defined as

$$\varepsilon_1 = \frac{5-3\gamma}{64\gamma^2}\,\omega_1, \quad \varepsilon_2 = \frac{\gamma-1}{16\gamma^2}\,\omega_1 \tag{3.5}$$

To facilitate analysis, A_n and B_n can be written in terms of the amplitude and phase.

$$A_n = r_n(t)\cos\phi_n(t), \quad B_n = r_n(t)\sin\phi_n(t) \tag{3.6}$$

Substitute in (3.1)-(3.4) and rearrange the result to give

$$\frac{dr_1}{dt} = \alpha_1 r_1 - \beta\, r_1 r_2 \cos\Psi \tag{3.7}$$

$$\frac{dr_2}{dt} = \alpha_2 r_2 + \beta\, r_1^2 \cos\Psi \tag{3.8}$$

$$\frac{d\Psi}{dt} = -\beta\left(\frac{r_1^2}{r_2} - 2r_2\right)\sin\Psi + (-2\theta_1 + \theta_2) + 2(\varepsilon_1 + \varepsilon_2)(r_2^2 - r_1^2) \tag{3.9}$$

where Ψ denotes the phase difference between modes, defined as

$$\Psi = 2\phi_1 - \phi_2$$

Note that the third-order acoustic coupling does not appear explicitly in the equations for the wave amplitudes; it only affects the energy exchange rate between modes through its influence on the phase shift Ψ.

In the limit cycles, the wave amplitude of each mode remains constant. This implies from (3.7) and (3.8) that the phase shift Ψ is also a constant. Following some straight-forward manipulations, we obtain the limiting amplitudes.

$$r_{10} = \frac{\sqrt{-\alpha_1\alpha_2}}{\beta\cos\Psi_o} \tag{3.10}$$

$$r_{20} = \frac{\alpha_1}{\beta\cos\Psi_o} \tag{3.11}$$

$$\cos^2\Psi_o = \frac{a_1^2 - 2a_2a_3 \pm a_1\sqrt{a_1^2 - 4a_2a_3 - 4a_3^2}}{2(a_1^2 + a_2^2)} \tag{3.12}$$

where

$$a_1 = (2\alpha_1 + \alpha_2)\beta^2 \tag{3.13a}$$

$$a_2 = (-2\theta_1 + \theta_2)\beta^2 \tag{3.13b}$$

$$a_3 = 2\alpha_1(\alpha_1 + \alpha_2)(\varepsilon_1 + \varepsilon_2) \tag{3.13c}$$

The subscript o denotes values in the limit cycle. Because r_{10} is real, in order for the right-hand-side of (3.10) to be real, we must have

$$\alpha_1\alpha_2 < 0 \tag{3.14}$$

This is the first necessary and sufficient condition for the existence of limit cycle. If one mode is stable, then the other mode must be unstable. The other conditions can be derived from (3.12) since $0 \le \cos^2\Psi_o \le 1$, giving

$$a_1^2 - 4a_3(a_2 + a_3) > 0 \tag{3.15}$$

If the third-order coefficients ε_1 and ε_2 vanish, the results of the second-order model are fully recovered. In practice, ε_1 and ε_2 are very small, the deviation from the second-order model is expected to be negligible.

Stability of Limit Cycle

So far we have determined the conditions for the existence of limit cycles. The remaining task is to determine the conditions under which the limit cycle is stable. The procedure consists of examining the behavior of the system in the neighborhood of the limit cycles. Now set

$$r_1 = r_{10} + r_1', \text{ etc.,} \qquad (3.16)$$

the primed quantity being assumed small, and substitute in (3.7)-(3.9) to derive the variational equations for r_1', r_2', and Ψ'. Following the common practice, we decompose the time- and spatial-dependent parts of each perturbation quantity

$$r_1' = \bar{r}_1 \exp(\lambda t)$$

$$r_2' = \bar{r}_2 \exp(\lambda t) \qquad (3.17)$$

$$\Psi' = \bar{\Psi} \exp(\lambda t)$$

Substitution of the above expressions in the variational equations and rearrangement of the result to give the characteristics equation for λ.

$$P(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \qquad (3.18)$$

where the coefficients b_1, b_2, and b_3 are functions of the linear and nonlinear parameters. In order for the limit cycle to be stable, all roots of $P(\lambda)$ must have negative real part. From the Routh-Hurwitz criteria, the sufficient and necessary conditions for this to be true are

$$b_1 > 0,\ b_3 > 0,\ b_1b_2 - b_3 > 0 \qquad (3.19)$$

The above inequalities can be written explicitly as

$$\alpha_1 + \alpha_2 < 0 \qquad (3.20a)$$

$$a_1 + 2a_3 \tan\Psi_0 < 0 \qquad (3.20b)$$

$$a_5 - 2a_4 + 4a_4 \cos^2\Psi_0 - 2a_3 \tan\Psi_0 > 0 \qquad (3.20c)$$

where

$$a_4 = (\alpha_1 + \alpha_2)\beta^2 \qquad (3.20d)$$

$$a_5 = \alpha_2^2\beta^2/\alpha_1 \qquad (3.20e)$$

Compared with the second-order acoustic model, the third-order model offers two solutions for the limit cycle, corresponding to the "+" and "-" sign in (3.12). However, it can be shown that only the case with the "-" sign satisfies the condition (3.20b), which is always violated by the other case. Thus, the stable limit cycle appears to be unique. Figures 1 and 2 show the regions in which stable limit cycles exist, based on the second- and third-order models, respectively. The third-order nonlinearity modifies the stability domain, especially in the region $1 \le \alpha_2/\alpha_1 \le 2$. Figure 3 shows an example illustrating the existence of limit cycle, in which the dashed and solid lines represent the results based on the second- and third-order models, respectively. The difference between these two models is quite small.

Triggering

In order to study the triggering phenomenon, the stabilities characteristics in the neighborhood of the trivial solution $r_{10} = r_{20} = 0$ must be examined. Within the present analysis in which only the nonlinearities arising from acoustics is considered, the condition for the existence of limit cycle (3.14) precludes the possibility that a linearly stable system ($\alpha_1 < 0$, $\alpha_2 < 0$) may exhibit finite-amplitude oscillations. Thus, the third-order acoustic nonlinearity is not a sufficient condition for triggering of combustion instabilities. This statement can be further examined from the energy balance point of view. To see this, we first multiply (3.7) and (3.8) by r_1 and r_2, respectively, to obtain the equations for the energy of each mode.

$$\frac{1}{2}\frac{dr_1^2}{dt} = \alpha_1 r_1^2 - \beta r_1^2 r_2 \cos\Psi \qquad (3.21a)$$

$$\frac{1}{2}\frac{dr_2^2}{dt} = \alpha_2 r_2^2 + \beta r_1^2 r_2 \cos\Psi \qquad (3.21b)$$

The nonlinear gasdynamics simply serves as a means of transfer of energy between modes. It does not contribute to the growth or decay of the wave, even with the third-order nonlinearities included.

4. NONLINEAR COMBUSTION RESPONSE

We have so far investigated the influences of nonlinear gasdynamics on the behavior of unsteady motions. Results clearly indicate that the third-order acoustics do not change the fundamental characteristics of the system. It only modifies the limiting amplitudes of the second-order waves. In order to study the triggering phenomenon, one must resort to other nonlinear effects. In this section, a general framework accommodating second-order nonlinear gasdynamics and combustion response is constructed to investigate nonlinear stability. As a specific example, the ad-hoc formula proposed by Levine and Baum[6] is used for the combustion response function. The purposes are to help explain the numerical results obtained Baum and Levine, and to identify some important attributes to nonlinear combustion instabilities.

The formulation follows the same procedure leading to the nonlinear wave equation (2.5) except for the inclusion of the following nonlinear combustion response.

$$W = W_{pc}\,[1 + R_{vc}\,|u'|\,] \qquad (4.1)$$

where W is the instantaneous propellant burning rate, W_{pc} is the burning rate based on the linear pressure-coupled response, and R_{vc} is the velocity-coupled response function. Now substitute (4.1) in (2.6) and (2.8), expand the results to second-order, and combine with (2.14) to produce the forcing function associated with unsteady combustion — S.J. Kim's thesis

$$\{F_n\}_{combustion} = \frac{\gamma}{\bar{\rho}\,E_n^2}\iint \psi_n \frac{\partial}{\partial t}\,[C_1 p' + C_2 p'\,|u'|\,]ds \qquad (4.2)$$

where C_1 and C_2 are associated with the linear and nonlinear combustion processes, respectively. These coefficients are, in general, determined by the propellant properties and can be expressed conveniently by the pressure and velocity coupled response functions. Substitution of (2.10) in (4.2) gives the contribution from the nonlinear combustion response.

$$\{F_n\}_{\text{nonlinear combustion}} = \frac{C_2\bar{a}^{2}}{\gamma E_n^2} \iint \sum_i \dot{\eta}_i \psi_i \left| \sum_j \frac{\dot{\eta}_j}{k_j^2} \nabla\psi_j \right| \nabla\psi_n \, ds \tag{4.3}$$

For two modes of longitudinal oscillation, the above equation can be approximated, within second-order accuracy, by

$$\{F_1\}_{\text{nonlinear combustion}} = \frac{4}{3} G \dot{\eta}_1 \left| \dot{\eta}_1 \right| \tag{4.4a}$$

$$\{F_2\}_{\text{nonlinear combustion}} = \frac{32}{15} G \dot{\eta}_2 \left| \dot{\eta}_1 \right| \tag{4.4b}$$

and

$$G = \left(\frac{\ell^2 R}{\pi}\right) \frac{C_2\bar{a}^2}{\gamma E_n^2} \tag{4.4c}$$

where ℓ and R stand for the combustor length and diameter, respectively. In deriving the above expressions for F_1 and F_2, the wave amplitude of the second mode is assumed to be much less than that of the first mode, an assumption which seems to be valid in many practical cases. The dimensionless coefficient G accommodates the influences of combustion processes and chamber geometry.

With the aid of the method of time averaging described in Section 2, we finally obtain the equations for the amplitude and phase of each mode

$$\frac{dr_1}{dt} = \alpha_1 r_1 - \beta r_1 r_2 \cos\Psi_1 + \xi_1 r_1^2 \tag{4.5}$$

$$\frac{dr_2}{dt} = \alpha_2 r_2 + \beta r_1^2 \cos\Psi_1 + \frac{\xi_2}{4} r_1 r_2 (15 + \cos\Psi_2) \tag{4.6}$$

$$\frac{d\Psi_1}{dt} = -2\theta_1 + \theta_2 + \beta\left(2r_2 - \frac{r_1^2}{r_2}\right)\sin\Psi_1 + \frac{\xi_2}{4} r_1 \sin\Psi_2 \tag{4.7}$$

$$\frac{d\Psi_2}{dt} = -4\theta_1 + 4\beta r_2 \sin\Psi_1 \tag{4.8}$$

where

$$\Psi_1 = 2\phi_1 - \phi_2 \tag{4.9a}$$

$$\Psi_2 = 4\phi_1 \tag{4.9b}$$

$$\xi_1 = \frac{28}{45\pi} G\omega_1 \tag{4.9c}$$

$$\xi_2 = \frac{32}{35} \xi_1 \tag{4.9d}$$

Note that ξ_1 and ξ_2 arise from nonlinear combustion response, producing a self-coupling term for the first mode and a cross-coupling term for the second mode. For a given propellant, ξ_1 and ξ_2 are mainly functions of combustor geometry and speed of sound.

Special Case with $\phi_1 = \phi_2 = 0$

To aid in understanding and to simplify the analysis, we first consider a special case with all phases set to be zero, $\phi_1 = \phi_2 = 0$. The more general case with nonzero phase will be discussed in a later section. Basically, the phase has a significant influence on the rate of energy exchange between modes, and consequently, affects the limiting amplitudes of oscillations and the stability domain of the system. However, the fundamental behavior of the unsteady motions remains largely unchanged.

For $\phi_1 = \phi_2 = 0$, (4.5) and (4.6) reduce to

$$\frac{dr_1}{dt} - \alpha_1 r_1 = \beta r_1 r_2 + \xi_1 r_1^2 \tag{4.10}$$

$$\frac{dr_2}{dt} = \alpha_2 r_2 + \beta r_1^2 + 4\xi_2 r_1 r_2 \tag{4.11}$$

In limit cycles, r_1 and r_2 are constants, giving the amplitude of each mode,

$$r_{10} = \frac{-(\alpha_2\xi_1 + 4\alpha_1\xi_2) \pm \sqrt{(\alpha_2\xi_1 - 4\alpha_1\xi_2)^2 - 4\alpha_1\alpha_2\beta^2}}{2\beta^2 + 8\xi_1\xi_2} \tag{4.12}$$

$$r_{20} = \frac{\alpha_1 + \xi_1 r_{10}}{\beta} \tag{4.13}$$

Two limit cycles exist in correspondence to the "+" and "-" signs in (4.12). To simplify notation, we use $r_{10}^{(1)}$ and $r_{10}^{(2)}$ to represent the two limiting amplitudes respectively.

The condition for the existence of limit cycle requires that the argument in the square root be positive. Consequently,

$$(\alpha_2\xi_1 - 4\alpha_1\xi_2)^2 - 4\alpha_1\alpha_2\beta^2 > 0 \tag{4.14}$$

The stability behavior of limit cycles can be studied following the same approach described in Section 3. We first decompose the wave amplitudes as

$$r_1 = r_{10} + r_1' \quad \text{and} \quad r_2 = r_{20} + r_2' \tag{4.15}$$

Substitute in (4.10) and (4.11), and linearize the result to get the variational equations for r_1' and r_2'.

$$\frac{dr_1'}{dt} = (\alpha_1 - \beta r_{20} + 2\xi_1 r_{10})\, r_1' - \beta r_{10} r_2' \tag{4.16}$$

$$\frac{dr_2'}{dt} = (2\beta r_{10} + 4\xi_2 r_{20}) r_1' + (\alpha_2 + 4\xi_2 r_{10}) r_2' \tag{4.17}$$

Then set $r_1' = \tilde{r}_1 \exp(\lambda t)$ and $r_2' = \tilde{r}_2 \exp(\lambda t)$, and substitute in (4.16) and (4.17) to obtain the characteristic equation for λ.

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{4.18}$$

where

$$a_1 = -[(\xi_1 + 4\xi_2)r_{10} + \alpha_2] \tag{4.19a}$$

$$a_2 = [4\xi_2\alpha_1 + \xi_1\alpha_2 + 2r_{10}(\beta^2 + 4\xi_1\xi_2)]\, r_{10} \tag{4.19b}$$

For stable limit cycles, the real part of λ must be negative, giving the condition that

$$a_1 > 0 \tag{4.20}$$

$$a_2 > 0 \tag{4.21}$$

Since triggering refers to the excitation of pressure oscillations in a linearly stable system, the requirement of negative linear growth constants together with (4.14), (4.20), and (4.21) constitute the conditions for the existence of triggering in the sense of stable limit cycle. The overall result is sketched in Figure 4. The shaded regions represent the parameter domains in which triggering takes place. The branches with positive and negative ξ_1 lead to the limiting amplitudes $r_{10}^{(1)}$ and $r_{10}^{(2)}$, respectively, in accordance with the condition of (4.21). It is interesting to note that for this special case, the conditions for the existence of triggering depend only on the ratio of linear growth constants, α_2/α_1, and the ratio of the nonlinear acoustic coefficient β to the parameter ξ_1. The system is more susceptible to triggering as the ratio α_2/α_1 increases. Figure 5 shows a typical example illustrating the existence of triggering. If the initial wave amplitudes are less than some critical values, then the disturbances decay and finally vanish completely.

The triggering phenomenon can be best interpreted from the standpoint of energy balance. We now multiply (4.10) by r_1 and (4.11) by r_2 to give the rate of energy change of each mode.

$$\frac{1}{2}\frac{dr_1^2}{dt} = \alpha_1 r_1^2 - \beta r_1^2 r_2 + \xi_1 r_1^3 \tag{4.21}$$

$$\frac{1}{2}\frac{dr_2^2}{dt} = \alpha_2 r_2^2 + \beta r_1^2 r_2 + 4\xi_2 r_1 r_2^2 \tag{4.22}$$

These equations imply that the nonlinear combustion provides energy to the system, which is then dissipated by the linear processes. In order to drive and sustain oscillations, the initial disturbance must be greater than the threshold value to overcome the energy losses. The nonlinear acoustics only serve as a bridge for energy exchange between modes.

Effects of Frequency Shift

The analysis of the general case with nonzero frequency shift is much more involved. The major difficulty is associated with interpretation of the complicated formulas for the stability conditions of limit cycles. As an alternative, an extensive numerical study was performed to investigate the effects of frequency shift on triggering. Results indicate that for a given combustion response function and chamber geometry, the frequency shift may change the limiting amplitudes of oscillations and the parameter domain in which triggering takes place. However, it has minor influence on the global behavior of unsteady motions. Figures 5 and 6 show two typical examples illustrating the existence of triggering. For the general case, if the initial disturbances are sufficiently large, then the wave amplitudes increase and eventually reach the limit cycle with amplitudes greater than those of the special case.

5. CONCLUSION

An analytical analysis has been constructed to investigate the behavior of unsteady motions in combustion chambers. The model extends the previous model for second-order acoustics and accommodates the third-order acoustic nonlinearities and a second-order combustion response. Results indicate that while the third-order acoustics affects the limiting amplitudes and the stability domain of limit cycles, it has little influence on the triggering of combustion instability. The nonlinear combustion response plays an essential role in determining the characteristics of triggering.

ACKNOWLEDGEMENTS

This work was supported partly by the Office of Naval Research, Grant No. 00014-84-K-0434, and partly by The Pennsylvania State University. The authors are greatly indebted to Dr. H. S. Sun for his assistance in formulating the model of the third-order nonlinear acoustic model.

REFERENCES

1. Awad, E. and Culick, F. E. C., "On the Existence and Stability of Limit Cycles for Longitudinal Acoustic Modes in a Combustion Chamber," Combustion Science and Technology, Vol. 46, 1986, pp. 195-222.

2. Yang, V. and Culick, F. E. C., "On the Existence and Stability of Limit Cycles for Transverse Acoustic Modes in a Combustion Chamber," to be published.

3. Powell, E. A., "Application of the Galerkin Method in the Solution of Combustion Instability Problems," Ph.D. Thesis, Georgia Institute of Technology, 1970.

4. Kooker, D. E. and Zinn, B. T., "Triggering of Axial Instabilities in Solid Rockets: Numerical Predictions," AIAA Paper 73-1298, 1973.

5. Powell, E. A., Padmanabhan, M. S., and Zinn, B. T., "Approximate Nonlinear Analysis of Solid Rocket Motors and T-Burners," AFRPL-TR-77, Vol. 2, 1977.

6. Levine, J. N. and Baum, J. D., "A Numerical Study of Nonlinear Instabilities in Solid Rocket Motors," AIAA Journal, Vol. 21, 1982, pp. 557-564.

7. Baum, J. D., Lovine, R. L. and Levine, J. N., "Pulsing Techniques for Solid-Propellant Rocket Motors: Modeling and Cold-Flow Testing, Journal of Spacecraft, Vol. 20, 1983, pp. 150-157.

8. Lovine, R. L., Baum, J. D., and Levine, J. N., "Ejecta Pulsing of Subscale Solid Propellant Rocket Motors," AIAA Journal, Vol. 23, 1985, pp. 416-423.

9. Lovine, R. L. and Micheli, P. L., "Nonlinear Stability for Tactical Motors," AFRPL-TR-85-017, 1985.

10. Culick, F. E. C., "Nonlinear Behavior of Acoustic Waves in Combustion Chamber," Acta Astronautica, Vol. 3, 1976, pp. 715-734.

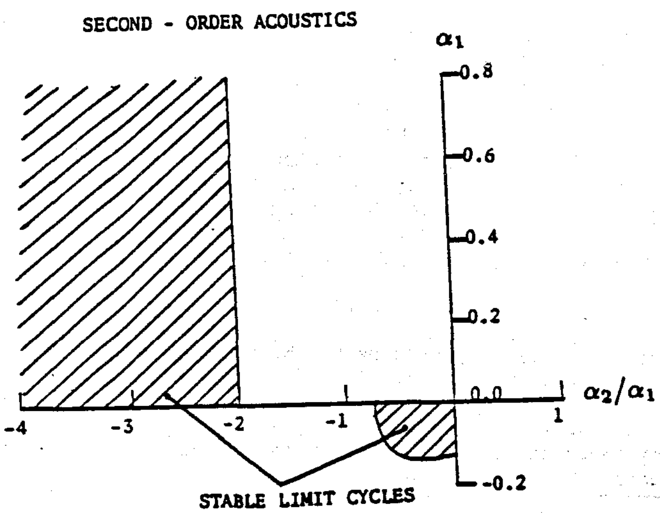

Fig. 1 Illustration of the Necessary and Sufficient Conditions for Stable Limit Cycles Based on the Second-Order Acoustic Model, $-2\theta_1 + \theta_2 = 0.25$.

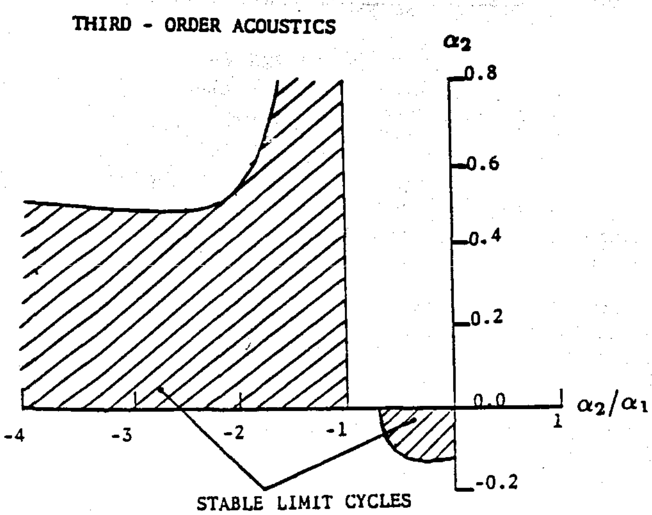

Fig. 2 Illustration of the Necessary and Sufficient Conditions for Stable Limit Cycles Based on the Third-Order Acoustic Model, $-2\theta_1 + \theta_2 = 0.25$.

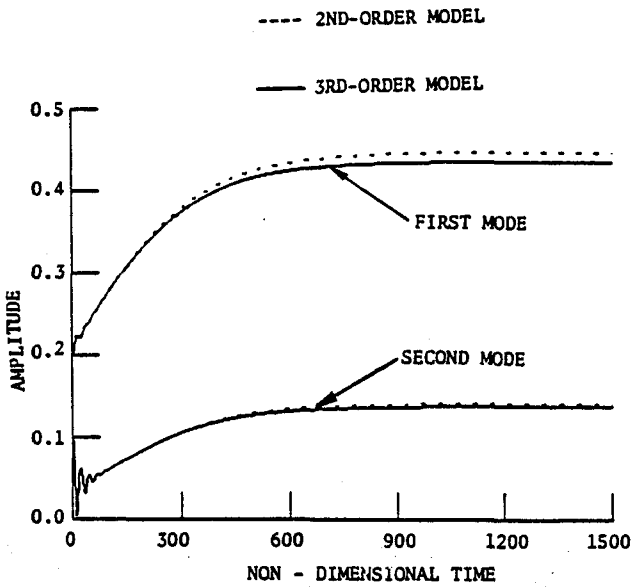

Fig. 3 An Example Showing the Existence of a Stable Limit Cycle.

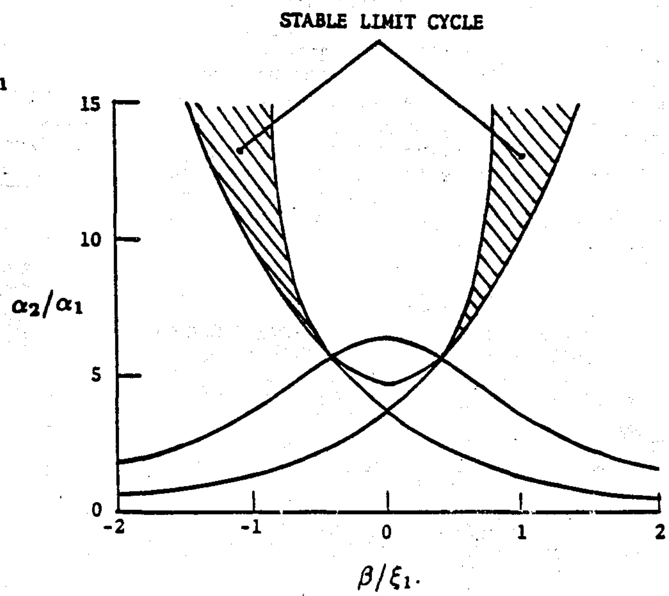

Fig. 4 Illustration of the Necessary and Sufficient Conditions for the Existence of Triggering.

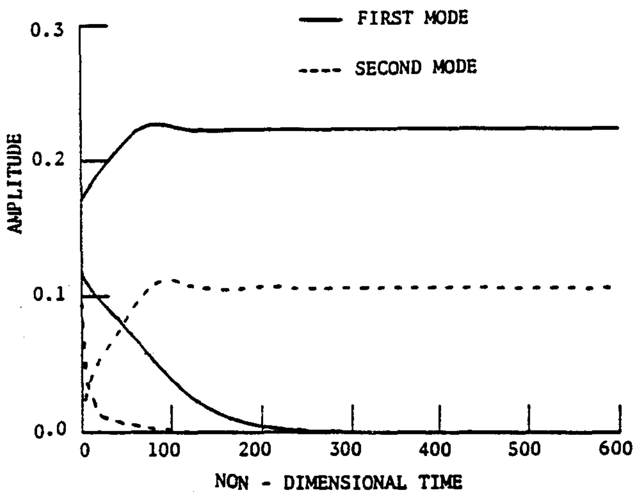

Fig. 5 An Example Showing the Existence of Triggering, $\Psi_1(0) = \Psi_2(0) = 0$.

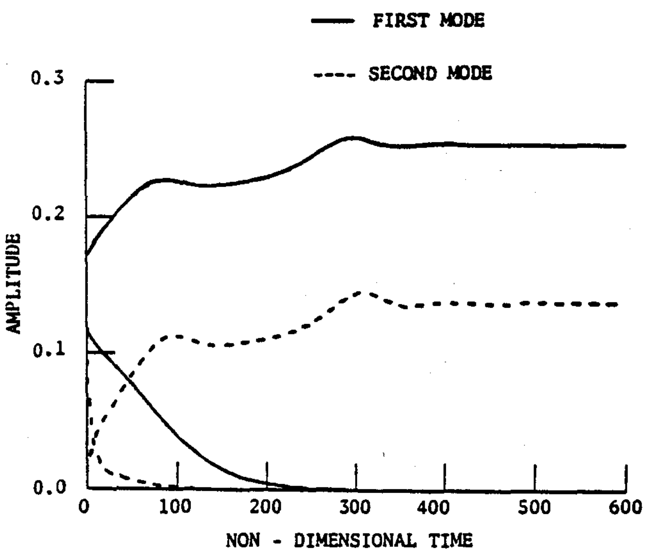

Fig. 6 An Example Showing the Existence of Triggering, $\Psi_1(0) = \Psi_2(0) = 0.1$.