Probabilistic aspects of isoscaling

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The phenomenon of nuclear isoscaling is studied under the premise of probability sampling. We show that isoscaling is expected in all cases of disassembly through fair sampling, obtain exact expressions for the yield ratio R_{21} and further show that the previously power law observed in experimental data is only an approximation to exact expression. Comparison to percolation and experimental data is presented.

Keywords: Nuclear reactons; nuclear fragmentation; isoscaling; isospin.

El fenómeno de isoescala nuclear es estudiado bajo la premisa de muestreo probabilístico. Demostramos que la isoescala es esperada en todos los casos donde haya un desensamblamiento a través de muestreo balanceado, obtenemos expresiones exactas para la proporción R_{21}, y también mostramos que la ley de potencia que ha sido observada en los datos experimentales es sólo una aproximación a la expresión exacta. Se presenta una comparación con los resultados obtenidos por el método de percolación y así como con datos experimentales.

Descriptores: Reacciones nucleares; fragmentación nuclear; isoescala; isospín.

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1. Introduction

The experimentally observed phenomenon of isoscaling relates fragmenting collisions with similar mass and energies but different isospin. Through these reactions the ratio of isotope yields in two different systems, 1 and 2, has been seen to follow, approximately, the so-called isoscaling law [1–3]: $R_{21} = Y_2(n,z)/Y_1(n,z) = C \exp(\alpha n + \beta z)$, where n and z are the number of neutrons and protons of the isotopes, α and β are fitting parameters, and C is a normalization constant.

Justification to this law has been provided by several models of nuclear reactions under different assumptions. Ref. [4], for instance, relates the parameters α and β to the symmetry term of the nuclear equation of state (EOS): $\alpha = 4C_{sym}[(Z_1/A_1)^2 - (Z_2/A_2)^2]/T$, where Z_i and A_i are the proton and nucleon components of colliding nuclei i, and T is the assumed temperature of both reactions at fragmentation time. This, of course, motivates the study of isoscaling as it has the potential of providing information about the elusive nuclear EOS.

Other studies, however, have shown that isoscaling can be expected in classical disassembling systems in- and out-of thermal equilibrium [5], in nonthermal percolating grids [6,7], and in systems disassembling under simple fair samplings assumptions [8]. These last studies have all connected the fitting parameters to the neutron, proton and nucleon abundances: $\alpha = \ln(q_2/q_1)$ and $\beta = \ln(p_2/p_1)$, with $q_i = N_i/A_i$ and $p_i = Z_i/A_i$.

These results indicate that even though isoscaling can be connected to the EOS, it may also have a component produced by the mere disassembling of the nucleus. Thus the motivation of the present study: to quantify the effect of the non-thermal probabilistic component of isoscaling by expanding the work of Ref. [6] studying the effect in the framework of an "urn problem."

In the following section the problem at hand is properly introduced and solved for the cases of distinguishable and undistinguishable particles. Then a comparison to experimental and percolation results is presented in Sec. 3. The paper closes with a summary of our findings and perspectives for future studies.

2. Urn sampling and isoscaling

In this section we will show that the isoscaling property is present in the simplest case of sampling particles at random out of an urn.

Consider an urn containing a total number of "nucleons" A composed of Z "protons" and N "neutrons", and proceed to partition such system into "fragments" of varying sizes by randomly grabbing nucleons out of the urn. Restating the procedure, a fragment of a given size a will be formed by grabbing at once a nucleons from the urn containing $A = N + Z$ nucleons. To eventually find an expression for the yield ratio R_{21}, we now proceed to find the probability a fragment of mass a is composed by n neutrons and z protons, *i.e.* that $a = n + z$. In turn we do this in the case of distinguishable particles and indistinguishable.

2.1. Distinguishable particles

In the disassembling process a bunch of nucleons are grabbed from the urn containing $A = N + Z$ nucleons, the resulting fragment can be of sizes 1 to A each with probabilities depending on the "grabbing" mechanism; let us say that the probability for grabbing a nucleons is an unspecified $P(a)$. Next, focusing on a resulting fragment of size a, its composition can range from $(n, z) = (0, a) \cdots (a, 0)$ and, in general, the number compositions of a is given by the binomial

$$\binom{N + Z}{n + z}.$$

Finally, since each of such compositions will have a different probability, the number of ways in which n neutrons can be picked up out of a total of N neutrons and z protons out of a total amount of Z protons is

$$\binom{N}{n} \times \binom{Z}{z}.$$

With these arguments, the probability of having a fragment of mass a composed by n neutrons and z protons and sampled out of an urn containing N neutrons and Z protons is:

$$Y(n, N, z, Z) = P(a)P(n, N, z, Z)$$

$$= P(a)\frac{\binom{N}{n}\binom{Z}{z}}{\binom{N+Z}{n+z}} = P(a)\frac{\binom{N}{n}\binom{Z}{z}}{\binom{A}{a}}$$

Under this approach, denoting the yield of the reaction i with N_i neutrons by $Y_i(n, N_i, z, Z_i)$, the isoscaling ratio can be constructed by

$$R_{21} = \frac{Y_2(n, N_2, z, Z_2)}{Y_1(n, N_1, z, Z_1)} = \frac{P(a)_2}{P(a)_1}\frac{P(n, N_2, z, Z_2)}{P(n, N_1, z, Z_1)}$$

$$= \frac{P(a)_2}{P(a)_1}\frac{\binom{N_2}{n}\binom{Z_2}{z}}{\binom{N_2+Z_2}{n+z}}\frac{\binom{N_1+Z_1}{n+z}}{\binom{N_1}{n}\binom{Z_1}{z}}. \qquad (1)$$

This is an exact expression of the yield ratio for the case of sampling distinguishable nucleons out of urns. Notice that since, presumably, the same disassembling mechanisms are at action in both reactions, it can be safely assumed that $P(a)_2 \approx P(a)_1$.

To illustrate the behavior of this R_{12} we use the previous expression to calculate the ratio of the yield of reaction 1: $(N_1, Z_1) = (20, 20)$, to the yield of reaction 2: $(N_2, Z_2) = (32, 20)$ for values of n and p, ranging from $0 \le n \le 20$, and $0 \le z \le 20$, and under the assumption of equal probabilities for production of equal size fragments, $P(a_1) = P(a_2)$. In this case, since $Z_1 = Z_2 = Z$, expression (1) becomes

$$R_{21} = \frac{\binom{N_2}{n}\binom{N_1+Z}{n+z}}{\binom{N_2+Z}{n+z}\binom{N_1}{n}}.$$

Figure 1 shows the values of R_{21} produced by this expression as a function of n and presented in curves with the same value of p.
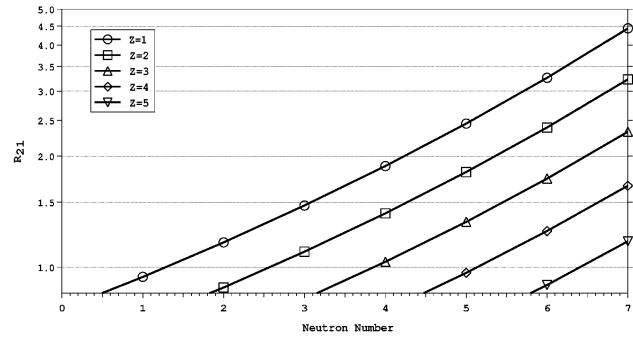

FIGURE 1. R_{21} obtained with expression (1) as a function of n for the cases of $N_1 = 20$, $N_2 = 32$, and $Z_1 = Z_2 = 20$.

As it can be seen, the curves present the characteristic isoscaling behavior. It is important to remark that the incipient deviation from a straight power law that can be detected has also been observed experimentally [9, 10].

2.2. Indistinguishable particles

We now turn to the case of sampling indistinguishable particles out of an urn. If fragments are produced by grabbing a particles, a fragment of size a will be composed by $n + z$ with a probability

$$P_a(n, z) = \mathbb{C}p^z q^n = \mathbb{C}p^z(1 - p)^{a - z}$$

where q and p are the probabilities of extracting a neutron and a protons, respectively: $q = N/A$ and $p = Z/A$, and $\mathbb{C}$ is an overall normalization which can be calculated to give $\mathbb{C} = (p - q)/(p^{a+1} - q^{a+1})$. With this, the previous expression becomes,

$$P_a(n, z) = \frac{(p - q)}{p^{a+1} - q^{a+1}}p^z q^n ,$$

and retaking the probability for grabbing a nucleons as an unspecified $P(a)$, the isoscaling ratio can be written as

$$R_{21} = \frac{Y_2(n, z)}{Y_1(n, z)} = \frac{P_a(n, z)_2 P(a)_2}{P_a(n, z)_1 P(a)_1}$$

$$= \frac{P(a)_2}{P(a)_1}\frac{p_2^z q_2^n}{p_1^z q_1^n}\left[\frac{(p_2 - q_2)\left(-q_1^{a+1} + p_1^{a+1}\right)}{(p_1 - q_1)\left(-q_2^{a+1} + p_2^{a+1}\right)}\right].$$

Noticing that the term in the right bracket is independent of (n, z) and denoting it by $C(a)$, it yields

$$R_{21} = C(a)\frac{P(a)_2}{P(a)_1}\left[\frac{q_2}{q_1}\right]^n\left[\frac{p_2}{p_1}\right]^z \qquad (2)$$

which, again, is an exact expression of the yield ratio for the case of sampling indistinguishable nucleons out of urns.

2.3. Connection to isoscaling power law

The expression (1) for distinguishable particles can be cast in the usual isoscaling law form, $R_{21} = C \exp(\alpha n + \beta z)$, by the use of the Stirling approximation on the binomial coefficients.

$$\frac{P(n, N_2, z, Z_2)}{P(n, N_1, z, Z_1)} = \frac{\binom{N_2}{n}\binom{Z_2}{z}}{\binom{N_2+Z_2}{n+z}} \frac{\binom{N_1+Z_1}{n+z}}{\binom{N_1}{n}\binom{Z_1}{z}}$$

$$= \exp\{N_2 \ln N_2 + Z_2 \ln Z_2 + (A_2 - a)\ln(A_2 - a)$$
$$+ A_1 \ln A_1 + (N_1 - n)\ln(N_1 - n) + (Z_1 - z)$$
$$\times \ln(Z_1 - z) - (N_2 - n)\ln(N_2 - n) - (Z_2 - z)$$
$$\times \ln(Z_2 - z) - A_2 \ln A_2 - (A_1 - a)\ln(A_1 - a)$$
$$- N_1 \ln N_1 - Z_1 \ln Z_1\}.$$

Assuming that the set of nucleon, neutron and proton values are much smaller than the total numbers, *i.e.* $a \ll A_i$, $n \ll N_i$ and $z \ll Z_i$, respectively, and using $q_i = N_i/A_i$ and $p_i = Z_i/A_i$, we arrive at

$$\frac{P(n, N_2, z, Z_2)}{P(n, N_1, z, Z_1)} = \exp\left\{ n \ln\left(\frac{N_2}{A_2}\right) + z \ln\left(\frac{Z_2}{A_2}\right) \right.$$
$$\left. - n \ln\left(\frac{N_1}{A_1}\right) - z \ln\left(\frac{Z_1}{A_1}\right) \right\}$$
$$= \exp\left\{ n \ln\left(\frac{q_2}{q_1}\right) + z \ln\left(\frac{p_2}{p_1}\right) \right\},$$

from which we readily recover the expected expression for isoscaling:

$$R_{21} = \frac{P(a)_2}{P(a)_1} \exp\{\alpha n + \beta z\}.$$

Again, as in previous studies [6,7], the fitting parameters are also given by $\alpha = \ln(q_2/q_1)$ and $\beta = \ln(p_2/p_1)$ and, as mentioned before, the term $P(a)_2/P(a)_1$ is dictated by the sampling process and it is expected to be equal to unity in the case of normalized yields, or to an overall norm in the case of unnormalized yields.

It is important to emphasize that, in this case, the expected power law is only an approximation to the real relationship between yields, whereas in the case of indistinguishable particles, expression (2) immediately yields the usual power law
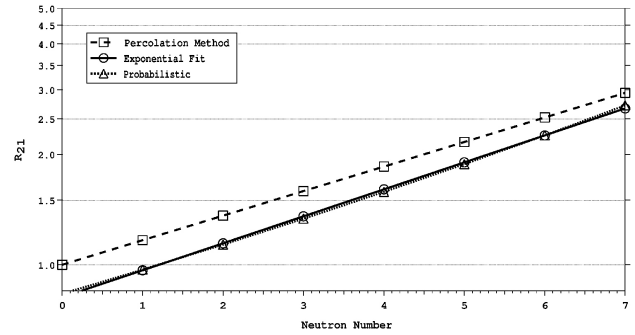

FIGURE 2. Plot of R_{21} for reactions $(N_1, Z_1) = (40, 40)$ and $(N_2, Z_2) = (56, 40)$ drawn with the percolation expression (3), the probabilistic result (1) and its averaged best fit.

with the same α and β. Next we evaluate this expression for a study case and compare to experimental and percolation results.

3. Comparison to percolation and experimental data

The previous results can be easily compared to those obtained by percolation [6]. Using the method of partitioning two-component three-dimensional grids with varying numbers of "protons" and "neutrons" through the method of percolation, Dávila *et al.* determined analytically that the ratio of of yields expected in such case is

$$R_{21}(n, z) = \left(\frac{p_2}{p_1}\right)^z \left(\frac{q_2}{q_1}\right)^n = \exp\left(\alpha n + \beta z\right), \quad (3)$$

again with $p_i = Z_i/A_i$ and $q_i = 1 - p_i$ and a clear microscopic interpretation for the isoscaling parameters: $\alpha = \ln q_2/q_1$ and $\beta = \ln p_2/p_1$. We now turn to a comparison between the percolation result (3) and the probabilistic expression (1).

A detailed comparison between the probabilistic expression (1) and the percolation result (3) can be obtained through the coefficients α and β. To compare the probabilistic results to those of percolation, an averaged best fit was used to determine the α and β coefficients fitting the curves obtained from (1). Since the coefficients for each isocurve (different z)

TABLE I. Comparison of coefficients obtained with the percolation expression (3), an averaged best fit to the probabilistic expression (1), and to experimental data [11].

Reaction	E (MeV/A)	Experiment			Percolation			Probabilistic		
		α	β	α/β	α	β	α/β	α	β	α/β
^{86}Kr+^{124}Sn/^{86}Kr+^{112}Sn	25	0.43	-0.51	-0.84	0.043	-0.059	-0.73	0.044	-0.060	-0.73
^{58}Fe+^{58}Fe/^{58}Ni+^{58}Ni	30	0.37	-0.39	-0.95	0.065	-0.074	-0.87	0.068	-0.078	-0.88
^{60}Ca+^{60}Ca/^{40}Ca+^{40}Ca	35	1.83	-2.31	-0.79	0.29	-0.41	-0.71	0.32	-0.46	-0.69
^{48}Ca+^{48}Ca/^{40}Ca+^{40}Ca	35	1.03	-1.22	-0.84	0.15	-0.18	-0.84	0.17	-0.20	-0.83
^{48}Ca+^{48}Ca/^{40}Ca+^{40}Ca	25	0.3	-0.36	-0.82	0.15	-0.18	-0.84	0.17	-0.20	-0.83

are different, and the exponential approximation is good only for small values of n, we took the average of coefficients for $z = 1, 2, 3, 4$ from $0 \leq n \leq 7$. Table I compares the values of the α and β obtained through these methods for nuclear systems for which experimental data exists [11].

In Table I it can be observed that the values obtained with these two different methods yield very similar results. This is illustrated graphically in Fig. 2 for the ratio of yields of reactions $N_1 = 40$ and $Z_2 = 40$ with $N_2 = 56$, and $Z_2 = 40$; the figure shows the percolation results of Eq. (3), the probabilistic result (1), and the averaged best fit.

4. Discussion and outlook

The comparison made on Table I and Fig. 2 shows an excellent agreement between the the percolation expression (3) and the probabilistic result (1). Since the derivation of this last result was based only on the assumption of fair sampling from an urn, it stands to reason that isoscaling is a general phenomenon that should be expected in any random partitioning of systems, as in a percolating grid, an urn being sampled, or a nucleus being fragmented.

Having said that, however, the comparison to experimental nuclear data shows that this probabilistic isoscaling is not large enough as to account for the entire phenomenon as observed in nuclear reactions. Caution should be exerted, nevertheless, in assigning the whole isoscaling to nuclear originated phenomenon, and the ever-present background isoscaling found in this work should always be taken into account.

Next lies an obvious task, that is to determine how this sampling isoscaling affects the nuclear one, and how it can be removed from the whole to leave the nuclear part alone. This task will prove to be necessary in trying to extract information about the symmetry term, C_{sym}, of the nuclear EOS.

1. M.B. Tsang *et al.*, *Phys. Rev. C* **64** (2001) 054615.
2. M.B. Tsang *et al.*, *Phys. Rev. Lett.* **86** (2001) 5023.
3. A.R. Raduta, *Eur. Phys. J. A* **24** (2005) 85.
4. A.S. Botvina, O.V. Lozhkin, and W. Trautmann, *Phys. Rev. C* **65** (2002) 044610.
5. C.O. Dorso, C.R. Escudero, M. Ison, and J.A. López, *Phys. Rev. C* **73** (2006) 044601.
6. A. Dávila, C. Escudero, J.A. López, and C.O. Dorso, *Physica A* **374** (2007) 663.
7. C.O. Dorso, *Phys. Rev. C* **73** (2006) 034605.
8. C.O. Dorso, C.M. Hernández, J.A. López, and J.A. Muñoz, *Phys. Rev. C* **78** (2008) 034613.
9. M. Mocko, *Ph.D. thesis*, (Michigan State University 2006).
10. G. Chaudhuri, S. Das Gupta, and M.Mocko, arXiv:0711.4992v1 [nucl-th] (2007).
11. D.V. Shetty, S.J. Yennello, and G.A. Souliotis, *Phys. Rev. C* **75** (2007) 034602.