

Droplet Agglomeration in Rocket Nozzles Caused by Particle Slip and Collision

Frank E. Marble¹

California Institute of Technology

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Abstract—Droplet Agglomeration in Rocket Nozzles Caused by Particle Slip and Collision. The development of the particle mass spectrum in a rocket nozzle is investigated under the assumption that droplet growth by collision and agglomeration is the dominant mechanism subsequent to initial appearance of particles in the rocket chamber. Collisions are calculated on the basis of linearized particle slip theory and a spectral integral equation is derived describing the development of particle mass spectrum during the flow process along the nozzle. This agglomeration process continues until the droplet temperature falls below the freezing point of the material.

A solution is obtained for the approximate growth in the average particle size during the expansion process. The results show that, according to this model, the particle size is strongly dependent on the initial pressure in the rocket chamber and is independent of nozzle geometry.

These results suggest that the collision-agglomeration process is at least one of the critical factors that accounts for the size of solid particles in rocket exhausts.

Translated abstracts appear at the end of this article.

Introduction

THE MECHANICS of gas-particle mixtures flowing in in rocket nozzles is relatively well understood, both in principle and in magnitude, and the significant features of such flows may be analysed as described, for example, by Rannie [1] and by Marble [2]. The quantity which contributes most heavily to the inaccuracy of results based on such analyses is the particle size. The origins of the particle size are poorly understood and reliable empirical information exists for only a small class of propellant additives. The size of aluminum oxide particles produced by aluminized solid propellants has been studied more extensively than others; the careful experiments of Sehgal [3] and of Dobbins [4] are especially noteworthy. The disparity of their results, however, stresses the need for a more thorough understanding of the growth mechanism. At conventional chamber pressures, Sehgal's results show average particle radii in the order of 2μ ; Dobbins' results give radii nearer to $\frac{1}{4} \mu$. The former is closer to the value found from various less carefully controlled experiments. On

the other hand, the radii measured by Sehgal show a strong increase with chamber pressure, whereas those of Dobbins are strictly independent of pressure. Sehgal's data are, in fact, the only published results that show any systematic variations at all. In general, the observed sizes of aluminum oxide particles in rocket exhaust are remarkably independent of initial particle size of aluminum in the propellant, residence time in the combustion chamber, size of the rocket motor, detailed configuration of the nozzle, and changes in propellant composition that do not affect the chamber temperature seriously. On the other hand, there is some indication that zirconium and boron oxides may be of quite different size. Hence, it is tempting to assume that the observed particle sizes depend solely upon the properties of the condensing substance, the details of the phase diagram accounting in some way for any pressure dependence. This is a generally unacceptable picture, however, because the condensation nuclei are much smaller than the observed particles and a mechanism of growth so independent of the various physical conditions is not easily conceived.

The agglomeration of small liquid droplets to form larger ones is a process of great interest in

¹Professor, Guggenheim Jet Propulsion Center, Karman Laboratory of Fluid Mechanics and Jet Propulsion.

meteorology, and two mechanisms studied extensively in this connection are of importance here. These mechanisms describe droplet growth caused by (i) collisions arising from turbulent diffusion and by (ii) collisions resulting from differential mean velocity of droplet of different sizes. Much of the work concerning these extensive studies is reviewed by Fuks [5] in his excellent monograph on aerosols. The relative importance of these two mechanisms depends to a large extent upon the strength of the acceleration field that produces the relative motion of different sized droplets. In the rocket chamber the turbulent diffusion is the more important, but in the rocket nozzle, where accelerations of $10^4 g$ are common, the collisions from differential particle velocity probably dominate the entire agglomeration mechanism. A problem could arise when all nuclei formed in the chamber are so nearly the same size that no differential mean velocity exists among the particles as they flow through the nozzle. The corresponding problem for droplet growth in clouds has been discussed by Saffman and Turner [6] where it is suggested that the initial spread in the droplet spectrum results from turbulent shear motions.

For the rocket nozzle, these differential slip velocities for particles of different radii, and the resulting collision processes, have been examined in detail by this author [7] in connection with their effect upon the gas-particle dynamics. In the present paper, it is proposed to utilize these results as a starting point and to estimate the rate of droplet growth during the acceleration of the mixture through the nozzle. Although sufficient knowledge of some physical details is not available to make a reliable absolute calculation of droplet growth, it is readily possible to check the order of magnitude, the dependence of the growth process upon chamber pressure, nozzle geometry, and other factors pertinent to existing and future experiments.

Equation for the Droplet Spectrum

Consider the problem as a one-dimensional flow of a gas-droplet mixture through a convergent-divergent nozzle, the state of the gas and the liquid droplets being uniform. If m denotes the mass of a liquid droplet having a radius σ , it is convenient to treat the droplet mass spectrum as continuous, so that

$$f(m) dm \quad (1)$$

represents the number of droplets in a unit volume of space having masses in the range m to $m + dm$.

For the droplet size range of interest in the present problem, the motion of the droplets is predominantly governed by their viscous interaction with the gas. Consequently, a droplet of a fixed mass m has a local velocity $u(m)$ independent from that of other droplets. The mass flow of droplets in the range (m, dm) is then

$$mf(m) dm \cdot u(m)A \quad (2)$$

where A is the local cross-sectional area of the nozzle.

Droplets in the mass range (m, dm) are produced and lost by collision, and it will be assumed that the collision is the only mechanism that is significant. It will be assumed further that all collisions result in the formation of droplets having a mass equal to the sum of the colliding masses, that is, elastic collisions contribute negligibly to the change.

The real nature of the collision process is exceedingly complex, involving factors of relative droplet velocity, surface chemistry, and nature of the medium in which the droplets move. Too little known in detail [5] to warrant the use of a detailed description in the following analysis. The model employed will be about as simple as can be made and still retain the realism required. It will be assumed that every collision that will geometrically occur is a perfectly inelastic one and leads to a stable union of the droplets. This geometric cross section is seriously reduced when the relative velocity of the two droplets is very low. More precisely, the momentum of one particle, when viewed from the other one, must be sufficient to carry it through the Stokes flow field of the latter with negligible change in velocity. It was shown by the present author [7], that this condition may be stated qualitatively as

$$\rho \frac{\sigma |u(m_1) - u(m_2)| \rho_l}{\mu} \gg 1 \quad (3)$$

where ρ and μ are the density and viscosity of the gas, σ and ρ_l the typical droplet radius and liquid density, and $|u(m_1) - u(m_2)|$ the magnitude of the relative velocity between droplets of mass m_1 and m_2 . It is clear from experiments cited by Fuks [5] and from the experiments of Ranz and Wong [8] that this criterion is sufficient when the expression in Eq. (3) is only slightly greater than unity. For droplets of very nearly the same mass, the relative velocity is too small to satisfy this criterion. Because of this small relative velocity, however, the frequency of such collisions is very low and their contribution is unimportant in the present problem.

The number of collisions per unit volume between droplets of mass m_1 and mass m_2 is then

$$f(m_1) dm_1 f(m_2) dm_2 \cdot \pi(\sigma_1 + \sigma_2)^2 |u(m_1) - u(m_2)| \quad (4)$$

where σ_1 and σ_2 are the corresponding droplet radii. A droplet is removed from the range (m, dm) each time a collision with a member of this set takes place. Consequently, the total rate of droplet loss from the range (m, dm) for a length dx of the nozzle is

$$A dx \int_{m'=0}^{\infty} f(m) dm f(m') dm' \pi(\sigma + \sigma')^2 |u(m) - u(m')| \quad (5)$$

where the integration is carried over the entire droplet spectrum because a droplet is removed from the range of intensity by a collision with any other droplet. On the other hand, droplets in the range (m, dm) are produced only by collisions between pairs (m', dm') and (m'', dm'') such that $m' + m'' = m$. Therefore, the total rate of droplet production in the range of interest is

$$\frac{1}{2} A dx \int_{m'+m''=m} f(m') dm' \cdot f(m'') dm'' \pi(\sigma' + \sigma'')^2 |u(m') - u(m'')| \quad (6)$$

where the region of integration, loosely defined here, will be examined in detail subsequently. The factor $\frac{1}{2}$ appears simply because when summing over $m' + m'' = m$ each collision is counted twice.

The rate at which the mass flow of droplets in the range (m, dm) changes along the nozzle is then given by Eqs. (2), (5), and (6) as

$$\begin{aligned} & d/dx \{mf(m) dm \cdot u(m) A\} \\ &= \frac{1}{2} Am \int_{m'+m''=m} f(m) dm' \cdot f(m'') dm'' \\ &\times \pi(\sigma' + \sigma'')^2 |u(m') - u(m'')| \\ &- Am \int_{m'=0}^{\infty} f(m) dm f(m') dm' \\ &\times \pi(\sigma + \sigma')^2 |u(m) - u(m')|. \quad (7) \end{aligned}$$

Equation (7) defines the droplet spectrum as a function of distance along the nozzle and plays roughly the same role here that the Boltzmann equation does in problems of gasdynamics treated from the kinetic theory viewpoint. Strictly speaking, Eq. (7) augments the equations of motion and thermodynamic relations for a heterogeneous mixture such as given in [7] and [8] to determine the local spectrum as well as the local thermodynamic and dynamic variables. This process would determine, among other things, the speeds of individual particle classes $u(m)$ that occur in Eq. (7).

For the accuracy of the present approximation, however, such a refined calculation is not required.

When the conditions for the linearized calculations [1, 2, 7, 8] are met, as they are for most nozzles of practical interest, the value of $u(m)$ on the left-hand side may be replaced by the gas velocity $u^{(0)}(r)$ and the value of $u(m)$ on the right-hand side is replaced by

$$u(m) = u^{(0)} - \lambda(m) \frac{u^{(0)} du^{(0)}}{a^{(0)} dx} \quad (8)$$

Here, $u^{(0)}$ and $a^{(0)}$ are the mixture velocity and sound speed at any point of the nozzle, calculated in the elementary manner, when the two phases are in dynamic and thermodynamic equilibrium. The characteristic length $\lambda(m)$ is

$$\lambda(m) = \frac{ma_c}{6\pi\sigma(m)\mu_c} \quad (9)$$

as described in [2] and [8], known as the velocity equilibration length, and determines the velocity that the particle of mass m slips with respect to the gas. Here μ_c and a_c are the viscosity and sound speed of the gas at chamber conditions and $\sigma(m)$ denotes the radius of a particle having mass m . It is particularly useful to note that the "zerth order" solution for the mixture is a conventional gasdynamic problem satisfying the relations

$$\rho^0 u^0 A = \text{constant} \quad (10)$$

$$T^{(0)} \left(1 + \frac{\bar{\gamma} - 1}{2} M^{02}\right) = T_c \quad (11)$$

$$p^{(0)}/p_c = (\rho^0/\rho_c)^\gamma \quad (12)$$

$$p^0 = \rho^0 R T^0 \quad (13)$$

where the ρ^0 , M^0 , T^0 are the local density, Mach number, and temperature of the mixture, and

$$\bar{\gamma} = \frac{C_p + \kappa C_l}{C_v + \kappa C_l} \quad (14)$$

where C_l is the specific heat of the liquid and κ is the ratio

$$\kappa = \frac{\text{mass flow of liquid or solid}}{\text{mass flow of gas}} \quad (15)$$

Only these zeroth order variables will appear in the subsequent analysis and, consequently, this will be understood and the superscript omitted. Within this approximation, the spectrum Eq. (7) may be written

$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{\rho_c}{\rho} f(m) dm \right\} = \frac{1}{2} \frac{\rho}{\rho_c} \frac{1}{a} \frac{du}{dx} \int_{m'+m''=m} \left(\frac{\rho_c}{\rho} f(m') dm' \right) \\ & \times \left(\frac{\rho_c}{\rho} f(m'') dm'' \right) \pi(\sigma' + \sigma'')^2 |\lambda(m') - \lambda(m'')| \end{aligned}$$

$$-\frac{\rho}{\rho_c} \frac{1}{a} \frac{du}{dx} \int_{m'=0}^{\infty} \left(\frac{\rho_c}{\rho} f(m) dm \right) \times \left(\frac{\rho_c}{\rho} f(m') dm' \right) \pi(\sigma + \sigma')^2 |\lambda(m) - \lambda(m')|. \quad (16)$$

Before proceeding further, let us examine the range of integration in the production integral, mentioned earlier. If two droplets (m' , dm') and (m'' , dm'') combine to form a single droplet m , dm , it is necessary that

$$m' + m'' = m \quad (17)$$

and

$$dm'' = dm. \quad (18)$$

These permit rewriting the first integral in Eq. (16) in the form

$$\frac{1}{2} \frac{\rho}{\rho_c} \frac{1}{a} \frac{du}{dx} \int_{m'=0}^m \left(\frac{\rho_c}{\rho} f(m') dm' \right) \left(\frac{\rho_c}{\rho} f(m - m') dm \right) \times \pi(\sigma' + \sigma'')^2 |\lambda(m') - \lambda(m - m')| \quad (19)$$

which common interval dm explicit for all terms of Eq. (16).

It is convenient to introduce a measure for the initial mass of individual particles in the combustion chamber; we choose this as the mass average value m_c so that the total number n_c of droplets per unit volume in the chamber gives

$$\frac{m_c n_c}{\rho_c} = \kappa. \quad (20)$$

Consistently then, the ratio of the local droplet radius to that in the chamber is

$$\frac{\sigma(m)}{\sigma_c} = \left(\frac{m}{m_c} \right)^{\frac{1}{3}} \quad (21)$$

and referring to Eq. (9),

$$\frac{\lambda(m)}{\lambda(m_c)} = \left(\frac{m}{m_c} \right)^{\frac{2}{3}} \quad (22)$$

where $\lambda(m_c)$ is the equilibration length for a droplet of mass m_c and radius σ_0 . Utilizing these quantities, the entire Eq. (16) may be written in dimensionless form, calling

$$\xi = m/m_c \quad (23)$$

and

$$\varphi = \frac{\rho_c}{\rho} f(m) \frac{m_c}{n_c} \quad (24)$$

as

$$\frac{d}{dx} \left\{ \varphi d\xi \right\} = \left(\frac{\rho}{\rho_c} \frac{1}{a} \frac{du}{dx} \right) (\pi n_c \sigma_c^2) \lambda(m_c) \times \left\{ \frac{1}{2} \int_{\xi'=0}^{\xi} \varphi(\xi') d\xi' \varphi(\xi - \xi') d\xi g(\xi', \xi - \xi') - \int_{\xi'=0}^{\infty} \varphi(\xi) d\xi \varphi(\xi') d\xi' g(\xi, \xi') \right\}. \quad (25)$$

Here, the collision function $g(\alpha, \beta)$ is defined

$$g(\alpha, \beta) = (\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}})^2 |\alpha^{\frac{1}{3}} - \beta^{\frac{1}{3}}|. \quad (26)$$

Some observations on Eq. (25) are worthwhile at this point. In the first place, since neither geometric variable x or $A(x)$ occur explicitly in the equation, any variable describing the state of the gas along the nozzle is preferable to the geometric one. In choosing the appropriate variable, it is useful to note that, by employing Eqs. (11), (12), and (13), the product $(\rho/\rho_c)(1/a) du$ defines a thermodynamic variable

$$\frac{\rho}{\rho_c} \frac{1}{a} du = \frac{dM}{[1 + [(\gamma - 1)/2] M^2]^{\gamma/(\gamma-1)}} \equiv d\eta \quad (27)$$

in terms of which it is convenient to treat the problem. Then

$$\eta = \int_0^M \frac{dZ}{[1 + [(\gamma - 1)/2] Z^2]^{\gamma/(\gamma-1)}} \quad (28)$$

so that it appears that the development of the droplet spectrum in the nozzle, from that in the chamber, depends upon the local Mach number and not upon the history of the flow process through the nozzle.

As a second observation on Eq. (25), note that the product $1/\pi n_c \sigma_c^2$ represents a length

$$l = \frac{1}{\pi n_c \sigma_c^2} = \frac{m_c}{\kappa \pi \rho_c \sigma_c^2}. \quad (29)$$

Therefore, this length physically is the length of a tube of gas at density ρ_c which must be swept out by a droplet so that the gas mass is equal to that of the liquid droplet, ρ_l . Thus, the length l is equal to hundreds of particle radii for a reasonable gas state.

If the considerations of the last two paragraphs are applied to Eq. (25), one obtains a convenient form of the droplet spectrum equation

$$\frac{d}{d\eta} \left\{ \varphi d\xi \right\} = \frac{1}{2} \frac{\lambda}{l} \int_{\xi'=0}^{\xi} \varphi(\xi') d\xi' \cdot \varphi(\xi - \xi') d\xi g(\xi', \xi - \xi')$$

$$-\frac{\lambda}{l} \int_{\xi'=0}^{\infty} \phi(\xi) d\xi \cdot \phi(\xi') d\xi' g(\xi, \xi'). \quad (30)$$

The spectral function employed here, it must be kept in mind, is no longer proportional number of droplets per unit volume but, as the result of multiplying by the density ratio ρ_c/ρ , is proportional to the number of droplets per unit mass of gas. Since, within the approximation here, the droplets of various size merely redistribute their mass over size as they flow through the nozzle, the total mass of liquid per unit mass of gas remains constant along the flow path. Thus, the fact that the integral

$$\int_{m=0}^{\infty} mf(m) dm u A$$

representing the total mass flow of liquid through the nozzle, is constant along the nozzle leads to the result

$$\int_{\xi=0}^{\infty} \xi \phi(\xi) d\xi = 1 \quad (31)$$

when the new variables are used. This may be considered a sort of normalization for the spectral function $\phi(\xi)$.

Similarity Solution of the Spectral Equation

Examination of some properties of the spectral Eq. (30) will assist in the construction of a similarity solution. The central issue here is the collision function $g(\xi, \xi')$ and the manner in which it enters into the droplet production and loss integrals. A schematic diagram of the collision function is shown in Fig. 1.

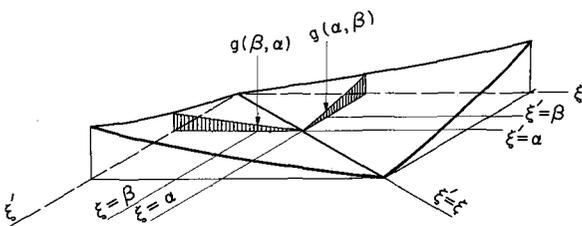


FIG. 1. Schematic diagram of the collision function.

The collision function vanishes along $\xi = \xi'$ and is symmetric about this line. Physically, the line $\xi = \xi'$ represents an encounter between droplets of equal mass, and since they have no relative velocity, the collision function vanishes there. The symmetry about the line $\xi = \xi'$ is equally clear physically. Consider two droplets $\xi = \alpha$ and $\xi = \beta$; $\alpha > \beta$. Then the collision function $g(\alpha, \beta)$ may be interpreted as describing the situation where the larger droplet (α) is being overtaken by the smaller one (β). On the

other hand, the collision function $g(\beta, \alpha)$ describes the situation where the smaller droplet (β) is overtaking the larger one (α). Symmetric points on the two sheets of $g(\xi, \xi')$ represent two ways of looking at the same physical collision, and this fact must be kept in mind when evaluating the droplet production integral.

It is clear that the development of the droplet mass spectrum is from smaller masses to larger ones, since the production term in one size range utilizes smaller droplet sizes and the loss term in that range results in the production of larger droplets. Since the purpose of the present analysis is to trace this general growth rate, it is sufficient to determine the growth rate of the spectral mean droplet mass. The spectral mean mass $\bar{\xi}$ is defined by

$$\bar{\xi} \equiv \frac{\int_0^{\infty} \xi \phi(\xi) d\xi}{\int_0^{\infty} \phi(\xi) d\xi} \quad (32)$$

The initial value of $\bar{\xi}$, the value in the rocket chamber, is unity by virtue of the definition of m_c and Eq. (20). Therefore, $\bar{\xi}$ is in reality the ratio of the local mean droplet mass to the initial mean value in the chamber and consequently $\bar{\xi} \geq 1$. Moreover, since the numerator of Eq. (32) is constant (Eq. (31)), the change in $\bar{\xi}$ is due to the change in the denominator of Eq. (32). Therefore, the change of $1/\bar{\xi}$ may be calculated from Eq. (30). Integrating Eq. (30) over the particle mass, it follows that

$$\begin{aligned} \frac{d}{d\eta} \int_0^{\infty} \phi(\xi) d\xi &\equiv \frac{d}{d\eta} \left(\frac{1}{\bar{\xi}} \right) \\ &= \frac{1}{2} \frac{\lambda}{l} \int_0^{\infty} d\xi \int_0^{\xi} d\xi' \cdot \phi(\xi') \phi(\xi - \xi') g(\xi', \xi - \xi') \\ &\quad - \frac{\lambda}{l} \int_0^{\infty} \phi(\xi) d\xi \int_0^{\infty} \phi(\xi') d\xi' g(\xi, \xi'). \end{aligned} \quad (33)$$

Now the production integral may be transformed by noting that it is an integral covering the whole first quadrant of the ξ, ξ' plane and that, if the transformation

$$\begin{aligned} \alpha &= \xi - \xi' \\ \beta &= \xi' \end{aligned}$$

is introduced, it follows that

$$\begin{aligned} \int_0^{\infty} d\xi \int_0^{\xi} d\xi' \cdot \phi(\xi') \phi(\xi - \xi') g(\xi', \xi - \xi') \\ = \int_0^{\infty} d\beta \int_0^{\infty} d\alpha \phi(\alpha) \phi(\beta) g(\alpha, \beta), \end{aligned} \quad (34)$$

and hence is formally the same as that arising from the loss integral. Therefore,

$$\frac{d}{d\eta} \left(\frac{1}{\bar{\xi}} \right) = -\frac{1}{2} \frac{\lambda}{l} \int_0^{\infty} \phi(\xi) d\xi \int_0^{\infty} \phi(\xi') d\xi' g(\xi, \xi') \quad (35)$$

which is a convenient form from which to pursue an approximate solution.

Returning now to the spectral equation, Eq. (30), we inquire whether there exist similarity solutions in the sense that the spectrum $\varphi(\xi, \eta)$ may be expressed as a function of a single variable. This question may be answered by introducing

$$\omega \equiv \xi/\bar{\xi} \quad (36)$$

as a similarity variable and seeking $\bar{\xi}(\eta)$ so that the conditions of similarity may be satisfied. Then the spectral function will be written

$$\varphi(\xi, \eta) = k(\bar{\xi}) \psi(\omega) \quad (37)$$

and Eq. (31) permits writing that

$$\int_{\omega=0}^{\infty} \omega \psi(\omega) d\omega = 1 \quad (38)$$

$$k(\bar{\xi}) = (\bar{\xi})^{-2} \quad (39)$$

and from the definition of $\bar{\xi}$, Eq. (32), that

$$\int_{\omega=0}^{\infty} \psi(\omega) d\omega = 1. \quad (40)$$

Substitution of Eqs. (36), (37), and (39) into the general spectral equation shows that a spectrum with the appropriate similarity properties exists if

$$\bar{\xi} = \left(1 - C \frac{\lambda}{l} \eta \right)^{-3} \quad (41)$$

and

$$\begin{aligned} & -\frac{1}{3} C \psi(\omega) d\omega \\ & = \frac{1}{2} \int_0^{\infty} \psi(\omega') d\omega' \psi(\omega - \omega') g(\omega', \omega - \omega') d\omega \\ & - \int_0^{\infty} \psi(\omega') d\omega' \psi(\omega) g(\omega, \omega') d\omega, \end{aligned} \quad (42)$$

the constant C to be determined as an eigenvalue of Eq. (42). The similarity variable

$$\omega = \xi \left(1 - C \frac{\lambda}{l} \eta \right)^3 \quad (43)$$

is of central interest since it shows the manner in which the spectrum spreads in mass distribution as η increases, that is, as the flow develops along the nozzle. Equation (41), on the other hand, provides the

desired growth rate of the mean droplet mass along the flow path. The spectrum is of the form

$$\varphi(\xi, \eta) = \left(1 - C \frac{\lambda}{l} \eta \right)^6 \psi \left\{ \xi \cdot \left(1 - C \frac{\lambda}{l} \eta \right)^3 \right\} \quad (43')$$

so that as the value of η increases the breadth of the spectrum grows and the amplitude is rapidly diminished.

It is not proposed here to investigate the shape of the spectrum. Rather, we shall be content with an estimate of the eigenvalue C , the value of which will be obtained in the following manner. Integrate Eq. (42) over ω , make the transformation of the production integral indicated in Eq. (34), and utilize Eq. (40) to evaluate the integral appearing on the left-hand side of Eq. (42). The value of C is therefore

$$C = \frac{3}{2} \int_0^{\infty} \psi(\omega) d\omega \int_0^{\infty} \psi(\omega') g(\omega, \omega') d\omega'. \quad (44)$$

Because C is given by integrals of the function ψ , it is possible to obtain a reasonable approximate value for C by utilizing an approximate droplet mass spectrum subject to the appropriate constraints. One such spectral distribution is

$$\begin{aligned} \psi(\omega) &= 0; & \omega < \omega_0 \\ &= a/\omega^4; & \omega > \omega_0 \end{aligned} \quad (45)$$

where a and ω_0 are constants. The distribution (45) must satisfy conditions (38) and (40), and these constraints determine the constants as

$$\begin{aligned} \omega_0 &= \frac{2}{3} \\ a &= 2 \cdot \left(\frac{2}{3} \right)^2. \end{aligned} \quad (46)$$

The value of C follows from Eq. (44) as

$$\begin{aligned} C &= 4 \cdot \left(\frac{2}{3} \right)^3 \int_{\frac{2}{3}}^{\infty} \frac{d\omega}{\omega^4} \int_{\frac{2}{3}}^{\infty} \frac{d\omega'}{(\omega')^4} \\ &\quad \times (\omega^{\frac{1}{3}} - \omega'^{\frac{1}{3}})^3 |\omega^{\frac{1}{3}} - \omega'^{\frac{1}{3}}| \end{aligned} \quad (47)$$

which, after some calculating, yields

$$C = \frac{124}{105} = 1.181. \quad (48)$$

In particular, we find that the ratio of the mean droplet radius at any point, divided by the mean droplet radius in the chamber, is

$$\frac{\bar{\sigma}}{\sigma_0} \equiv (\bar{\xi})^{\frac{1}{3}} = \frac{1}{1 - \frac{1.181}{0.134} (\lambda/l) \eta} \quad (49)$$

Now the function η depends only upon the local Mach number and the ratio of specific heats for the mixture, Eq. (28). This integral, giving this state variable η , is shown in Fig. 2 as a function of the local

Mach number or of the ratio of local gas temperature to chamber temperature. The ratio λ/l is

$$\frac{\lambda}{l} = \frac{1}{6} \frac{\rho_c a_c \sigma_0}{\mu_0} \cdot \kappa \quad (50)$$

For a fixed chamber temperature, which essentially depends only upon the composition of the propellant, λ/l increases linearly with the chamber pressure p_c and linearly with the initial droplet radius.

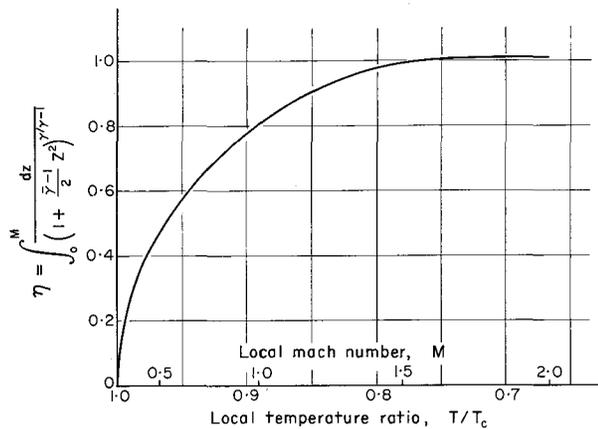


FIG. 2. Values of the state variable η as a function of local nozzle temperature ratio or of local nozzle Mach number.

If the agglomeration mechanism were controlling, therefore, the mean particle radius observed in experiments would be determined by the value of η at which the droplets solidify and the value of λ/l determined from the initial state of the mixture in the chamber. In the case of aluminum oxide droplets, solidification occurs when the droplet temperature falls below 2323°K, and for a reasonable chamber temperature of 3500°K, this implies solidification at a value $T/T_c = 0.667$. This, in turn, corresponds to the value of $\eta \equiv \eta_s \cong 1.0$ at solidification. The value of η_s is very insensitive to the exact values of solidification temperature and chamber temperature, as can be seen in Fig. 2. Moreover, it appears that η_s is also insensitive to a particle temperature lag, even of several hundred degrees. Hence, it appears that, for nozzles carrying aluminized propellants, $\eta_s \cong 1.0$ is a rather good approximation.

This observation is a very important one, and it may be well to elaborate here on some of its consequences. In the first place, note that the nozzle geometry or scale never enter the problem explicitly. This results from the fact that the droplet slip velocity is proportional to du/dx . Physically then, as the nozzle size increases, the time available for droplet agglomeration increases directly as the length. On the

other hand, the droplet slip velocities, and consequently the droplet collision frequency, varies as du/dx , or inversely as the length. Therefore, the number of collisions, which determine the final droplet size, does not depend upon nozzle scale. This observation is valid for usual conventional nozzle sizes and will be violated only for very small nozzles or very abrupt expansion where the slip relation given in Eq. (8) is no longer applicable. The fact that η_s is also insensitive to chamber temperature and droplet temperature lag is not a general result, but follows from the region of Fig. 2 appropriate to the solidification temperature ratio of aluminum oxide. A different material or a very cold propellant need not show this insensitivity to temperature. For aluminum oxide, however, the independence of final particle size from nozzle geometry, chamber temperature, and particle temperature lag follow from the agglomeration mechanism.

If one concludes then that the value of η appropriate to final droplet size in Eq. (49) is about unity, the details of the observed particle sizes must depend only upon the ratio λ/l . In particular, this quantity is directly proportional to the chamber pressure p_c , so that it appears that, as the chamber pressure rises, the final droplet radius increases also, in some cases quite rapidly. This is in agreement with the trend shown by the experiments of [3]. The detailed dependence of particle size upon chamber pressure corresponds less closely to experimental observations, but many factors in both the proposed mechanism and in the experiments may account for this.

It should be emphasized that since there is no adequate knowledge of initial droplet radii, Eq. (49) cannot be treated quantitatively. If we accept the assumption that the collision accommodation coefficient is unity, then the pressure dependence of $\bar{\sigma}$ may be employed to estimate the value of σ_0 that is appropriate. If one takes the data from [3] at 100 psi and 500 psi, it turns out that Eq. (49) describes the variation adequately for $\sigma_0 \sim 0.05 \mu$. It is relevant that the results of [3] always show a background of residual particles having a radius less than 0.1 μ .

Concluding Remarks

The fact that the foregoing theoretical analysis, based upon a particular idealized mechanism, agrees in so many respects with the general trends of experimental results encourages the further examination of the assumptions underlying the proposed mechanism. The most critical of these, aside from the initial droplet radius mentioned earlier, is the collision process and the formation of new droplets. This simple collision and union is certainly not valid for

droplets whose approach Reynolds number is very small, and the results of the collision are probably not a single droplet when the droplets are large. Therefore, the form of the collision function $g(\xi, \xi')$ differs from that employed both below and above some range of ξ . The modification in the lower range may alter somewhat the initial growth from the original nuclei present in the rocket chamber. The modification in the higher range, however, may be very instrumental during the latter period of growth and consequently in determination of the final particle spectrum. Finally, it is possible that, when Eq. (49) leads to very large droplet radii, the final radius may be determined by hydrodynamic instability, a value that will depend upon nozzle scale. Although certain analytical studies will shed light upon these effects, the major problem resolves to a difficult and important experimental investigation.

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Note added in proof: Subsequent to submitting the present paper, a numerical calculation of agglomeration in rocket nozzles was presented by CROWE, C. T. and WILLOUGHBY, P. G., A mechanism for particle growth in a rocket nozzle, *AIAA Jl.* **4**, 1677-78 (1966).

F. E. Marble: *Guggenheim Jet Propulsion Center, Karman Laboratory of Fluid Mechanics and Jet Propulsion, California Institute of Technology, Pasadena, California, USA.*

Résumé—Agglomération de gouttelettes sur les tuyères de fusées, causée par glissement et par collision de particules. Le développement du spectre de masse des particules dans une tuyère de fusée est étudié dans l'hypothèse que le grossissement des gouttelettes par collision et agglomération est le mécanisme principal qui suit l'apparence initiale des particules dans la chambre de fusée. Les collisions sont calculées sur la base de la théorie de glissement linéaire et une équation spectrale intégrale est dérivée, qui décrit le développement du spectre de masse de particule durant le processus de flux au long de la tuyère. Le mécanisme d'agglomération se poursuit jusqu'à ce que la température des gouttelettes tombe au dessous du point de congélation du matériau.

Une solution est trouvée pour le grossissement approximatif de la dimension des particules durant le procédé d'expansion. Le résultat montre que d'après le modèle, la dimension de la particule dépend en majeure partie de la pression initiale dans la chambre de fusée et est indépendante de la géométrie de la tuyère.

Ces résultats suggèrent que le processus collision-agglomération est au moins l'un des facteurs critiques qui influencent la dimension des particules solides de l'échappement des fusées.

Zusammenfassung—Tröpfchenanhäufung in Raketendüsen verursacht durch Teilchen und Stöße. Die Entwicklung der Verteilung der Masse der Teilchen in einer Raketendüse wird untersucht, unter der Voraussetzung, dass das Wachsen der Tröpfchen durch Stöße eine Anhäufung die hauptsächlichsten Vorgänge sind, die dem ursprünglichen Erscheinen der Teilchen in der Raketenkammer folgen. Die Stöße werden auf Grund der linearisierten Theorie der Teilchen-Abgleitung berechnet und eine spektrale Integralgleichung wird abgeleitet, die die Entwicklung der Massen-Verbeilung der Teilchen während des Tröpfchenflusses entlang der Raketendüse beschreibt.

Die Tröpfchenbildung läuft weiter, bis die Tröpfchentemperatur unter den Gefrierpunkt des Materiales fällt.

Eine Lösung wird erhalten, die das annähernde Wachstum der Durchschnittsteilchengröße während des Expansionsvorganges gibt. Die Resultate zeigen, dass entsprechend diesen Beispielen die Teilchengröße stark vom Anfangsdruck in der Raketenkammer abhängt und von der Düsengeometrie unabhängig ist.

Die Resultate deuten darauf hin, dass die Stoss- Anhäufungsvorgänge zumindest teilweise einen kritischen Faktor bilden, der die Größe der Festteilchen im Raketenauspuß bestimmt.