Astronautica Acta. Vol. 13, pp. 369–377. Pergamon Press Ltd., 1967. Printed in Great Britain

Flow Induced by the Impulsive Motion of an Infinite Flat Plate in a Dusty Gas

J. T. C. Liu[1]

Daniel and Florence Guggenheim Jet Propulsion Center, von Kármán Laboratory of Fluid Mechanics and Jet Propulsion, California Institute of Technology, Pasadena, California

(*Received* 23 *December* 1966)

Abstract—Flow Induced by the Impulsive Motion of an Infinite Flat Plate in a Dusty Gas. The problem of flow induced by an infinite flat plate suddenly set into motion parallel to its own plane in an incompressible dusty gas is of considerable physical interest in its own right as well as because of its close relation to the non-linear, steady (constant-pressure) laminar boundary layer. Its solution provides complete and exact information about modifications of the boundary layer growth and skin friction due to particle–fluid interaction. Moreover, it provides a basis for judging the accuracy of approximations which have been employed in more complex problems of viscous fluid–particle motion. The uncoupled thermal Rayleigh problem for small relative temperature differences is directly inferred and this answers questions about the modifications of the surface heat transfer rate and about the possibility of similarity with the velocity boundary layer. Similarity is possible when, in addition to a Prandtl number of unity, the streamwise relaxation processes are also similar.

Translated abstracts appear at the end of this article.

1. Introduction

THE RECENT use of high-energy solid rocket propellants has stimulated the study of rocket-nozzle exhaust gases containing finely distributed solid particles [1, 2], thereby motivating several interesting and related fluid mechanical problems. The present paper illustrates how physical insight to the flow of such a gas in a constant-pressure viscous laminar boundary layer could be gained through the solution of a relatively simpler problem.

The problem of the flow induced by the impulsive motion of an infinite flat plate parallel to its own plane, first considered by Stokes [3], has proven a convenient means to illustrate in an elementary manner some fundamental problems in boundary layer theory. Moreover, the relationship between this problem and that of the constant-pressure laminar boundary-layer problem led Rayleigh [4] to infer the results of one from the other. Although approximate, this led to the result for the shear stress at the wall for the boundary layer on a flat plate that differs only by a numerical factor from the exact calculations of Blasius [5]. The Blasius problem for a dusty gas was treated by Marble [6] and his analysis was extended by Singleton [7] to the corresponding problem for compressible fluid. In order to examine more closely some effects that were observed in Marble's analysis, the present paper considers the corresponding Rayleigh problem for a viscous, incompressible dusty gas. It is physically clear that there are two elementary limiting situations. For "small" times, the viscous diffusion layer grows parabolically like $(\nu t)^{\frac{1}{2}}$ as if the particle phase were absent. For "large" times, the viscous diffusion layer also grows parabolically, but as $(\bar{\nu} t)^{\frac{1}{2}}$, where $\bar{\nu}$ is the kinematic viscosity based upon the viscosity of the gas but the total density of combined gas and solid phases. The relaxation from one behavior to another is then the main feature of the problem.

2. Solution for Velocity and Shear

Consider a flat plate of infinite extent located on the x-axis; the geometry and notation are shown in Fig. 1. Let the gas be incompressible and assume the particles to be spherical and uniform in size. The appropriate momentum equations from Marble [6], Liu [8], or Saffman [9] are, for the gas phase,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\varkappa}{\tau}(u_p - u) \tag{2.1}$$

[1] Present address: Division of Engineering and Center for Fluid dynamics, Brown University, Providence, Rhode Island 02912, U.S.A.

and the particle phase,

$$\frac{\partial u_p}{\partial t} = -\frac{1}{\tau}(u_p - u), \tag{2.2}$$

where $\varkappa = \rho_p/\rho$, the particle-to-gas density ratio. The Stokes' form of the interaction force between the two phases is assumed and $\tau = m/6\pi\mu r_p$ is the particle velocity relaxation time, r_p the particle

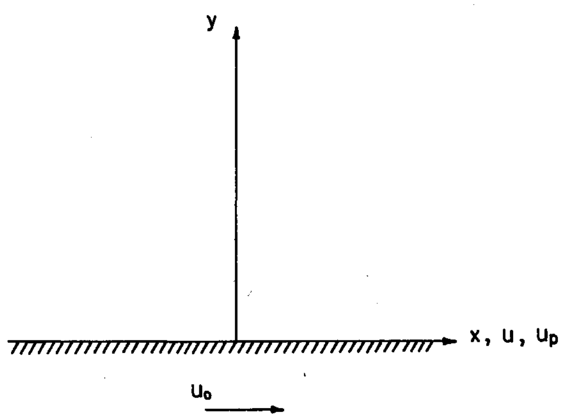

FIG. 1. Geometry of the problem.

radius, and m the mass of a single particle. Equations (2.1) and (2.2) can be combined into a single equation for u (or for u_p)

$$\frac{\tau}{1+\varkappa}\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial y^2}\right)}_{\text{Frozen diffusion}} + \underbrace{\left(\frac{\partial u}{\partial t} - \bar{\nu}\frac{\partial^2 u}{\partial y^2}\right)}_{\text{Equilibrium diffusion}} = 0. \tag{2.3}$$

In this form, Eq. (2.3) exhibits the relaxation from a frozen diffusion process to an equilibrium diffusion process, which was previously mentioned.

Define the dimensionless quantities,

$$u' = u/u_0, \quad t' = t/\tau^*, \quad y' = y/(\nu\tau^*)^{\frac{1}{2}}$$

where u_0 is the impulsive velocity initiated at $t = 0$, and $\tau^* = \tau/(1 + \varkappa)$. Substituting into (2.3) gives

$$\frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial t'} - \frac{\partial^2 u'}{\partial y'^2}\right) + \left(\frac{\partial u'}{\partial t'} - \frac{1}{1+\varkappa}\frac{\partial^2 u'}{\partial y'^2}\right) = 0. \tag{2.4}$$

Initially, the gas and particle phases are in equilibrium and are at rest so that $u'(y', 0) = 0$, $\partial u'(y', 0)/\partial t' = 0$. The boundary conditions are $u'(0, t') = 0$ when $t' < 0$; and $u'(0, t') = 1$, $u'(y', t') = 0$ as $y' \to \infty$ when $t \geqslant 0$.

Denote $U(y', s)$ as the Laplace transform of $u'(y', t')$ from the t'-plane to the s-plane. Then the transformation of (2.4) gives

$$\frac{d^2 U}{dy'^2} - W^2(s)U = 0, \tag{2.5}$$

where

$$W(s) = \left[\frac{s(s+1)}{s + 1/(1+\varkappa)}\right]^{\frac{1}{2}}.$$

The boundary conditions become $U(0, s) = 1/s$ and $U(\infty, s) = 0$. Considering the region $y' \geqslant 0$, the appropriate solution is

$$U(y', s) = \frac{1}{s}\,e^{-W(s)y'}. \tag{2.6}$$

The solution in the physical plane follows formally from the inversion integral

$$u'(y', t') = \frac{1}{2\pi i}\int_L U(y^1, s)\,e^{st'}\,ds \tag{2.7}$$

where L is the Bromwich path parallel to the imaginary axis and to the right of all singularities of $U(y', s)$.

Shear stress at the wall. The dimensionless form of the shear stress at the wall is defined as

$$\frac{C_f}{2} = \tau_w/\rho u_0^2 = \left(\frac{\nu}{\tau u_0^2}\right)^{\frac{1}{2}}\left(\frac{\partial u'}{\partial y'}\right)_{y'=0} \tag{2.8}$$

Denote by $\bar{C}_f/2$ the quantity

$$\frac{\bar{C}_f}{2} = \left(\frac{\nu}{\tau u_0^2}\right)^{\frac{1}{2}}\left(\frac{dU}{dy}\right)_{y'=0} \tag{2.9}$$

and note that from (2.6)

$$\left(\frac{dU}{dy'}\right)_{y'=0} = -\left[\frac{s+1}{s\{s + 1/(1+\varkappa)\}}\right]^{\frac{1}{2}}. \tag{2.10}$$

The right-hand side of (2.10) can be written

$$-(s+1)\,G_1(s)\,G_2(s) \tag{2.11}$$

where G_1 and G_2 are defined

$$G_1(s) = 1/[s(s+1)]^{\frac{1}{2}} = \mathscr{L}\{e^{-t'/2}\,I_0(t'/2)\},$$
$$G_2(s) = 1/[s + 1/(1+\varkappa)]^{\frac{1}{2}} = \mathscr{L}\{e^{-t'/(1+\varkappa)}\,(\pi t')^{-\frac{1}{2}}\}. \tag{2.12}$$

I_0 is the modified Bessel function of the first kind and of order zero [10]. Hence, according to standard methods [11]

$$\mathscr{L}^{-1}\{G_1(s)\,G_2(s)\}$$
$$= \frac{1}{\pi^{\frac{1}{2}}}\int_0^{t'}\frac{e^{-(t'-\tau)/(1+\varkappa)}}{(t'-\tau)^{\frac{1}{2}}}\,e^{-\tau/2}\,I_0(\tau/2)\,d\tau \equiv Y \tag{2.13}$$

and

$$\mathscr{L}^{-1}\{(s+1)\,G_1(s)\,G_2(s)\} = \left(\frac{d}{dt'}+1\right)Y =$$

$$\frac{e^{-t'/(1+\varkappa)}}{(\pi t')^{\frac{1}{2}}} + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^{t'}\frac{e^{-(t'-\tau)/(1+\varkappa)}}{(t'-\tau)^{\frac{1}{2}}}\, e^{-\tau/2}\left[I_0(\tau/2)+I_1(\tau/2)\right]d\tau. \quad (2.14)$$

I_1 is the modified Bessel function of the first kind and of order one [10]. Finally, from (2.9), (2.10) and (2.14), the skin friction coefficient (2.8) may be written in terms of known functions as

$$\frac{C_f}{2} = -\left(\frac{\nu}{\tau u_0^2}\right)^{\frac{1}{2}}\left\{\frac{dY}{dt'}+Y\right\}. \quad (2.15)$$

The asymptotic behaviour of the skin friction coefficient for the near-equilibrium regime ($t' \gg 1$) can be directly obtained from the expansion of (2.10) for $s \ll 1$,

$$\frac{C_f}{2} = -\left(\frac{\bar{\nu}}{u_0^2 t}\right)^{\frac{1}{2}}\frac{1+\varkappa}{\pi^{\frac{1}{2}}}\left\{1+\frac{1}{4}\frac{\varkappa}{1+\varkappa}\left(\frac{\tau}{t}\right) + \vartheta\left[\left(\frac{\tau}{t}\right)^2\right]\right\} \quad (2.16)$$

The equilibrium value of the skin friction is

$$\left(\frac{C_f}{2}\right)_E = \frac{(\tau_W)_E}{(1+\varkappa)\,\rho u_0^2} = -\left(\frac{\bar{\nu}}{u_0^2 t}\right)^{\frac{1}{2}}\frac{1}{\pi^{\frac{1}{2}}}, \quad (2.17)$$

which is normalised with respect to the equilibrium density $(1+\varkappa)\,\rho$, and corresponds to the first term in (2.16).

Similarly, the asymptotic behavior of the skin friction coefficient in the near-frozen regime ($t' \ll 1$) follows from the expansion of (2.10) for $s \gg 1$,

$$\frac{C_f}{2} = -\left(\frac{\nu}{u_0^2 t}\right)^{\frac{1}{2}}\frac{1}{\pi^{\frac{1}{2}}}\left\{1+\varkappa\left(\frac{t}{\tau}\right)+\vartheta\left[\left(\frac{t}{\tau}\right)^2\right]\right\}. \quad (2.18)$$

The frozen value, which behaves as if the particles were absent, is just

$$\left(\frac{C_f}{2}\right)_F = \frac{(\tau_W)_F}{\rho u_0^2} = -\left(\frac{\nu}{u_0^2 t}\right)^{\frac{1}{2}}\frac{1}{\pi^{\frac{1}{2}}}, \quad (2.19)$$

the first term of (2.18).

When $t/\tau \gg 1$, the particles and the fluid are moving very nearly together. The thickening of the fluid viscous diffusion layer continuously accelerates the adjacent layers of fluid and particles towards the plate velocity. However, because the fluid velocity is continuously accelerating, the particles never attain the actual fluid velocity. The fluid velocity profile responds to this interaction with the slower-moving particles by becoming fuller or thinner than the equilibrium profile. Correspondingly, the shear stress at the wall decreases towards its equilibrium value according to (2.16).

Immediately after the plate has been set into motion, on the other hand, the fluid motion is concentrated in a thin layer very close to the plate. This layer contains very few particles and consequently the very early phase of viscous diffusion takes place without appreciable reaction of the particles. For this reason, the zeroth-order term in the near-frozen solution (2.18) does not contain parameters associated with the particles. When the viscous layer spreads so that the viscous shear is reduced and number of entrained particles has increased, the force which results from the slip between the two phases tends to decelerate the fluid, preventing its velocity profile from growing as rapidly as in the absence of particles. As a result, the shear stress at the wall decreases more slowly than $t^{-\frac{1}{2}}$. This early stage of the interaction is described by (2.18).

It is of interest to apply the Rayleigh transformation to the present relatively simple solution in order to compare with the corresponding solution for a semi-infinite flat plate. If a distance x from the leading edge, the time t from the Rayleigh problem is identified with x/u_0, then $C_f(t/\tau) = -C_f(x/\lambda)_{BL}$, where $\lambda = u_0\tau$ is the particle–fluid relaxation length. Equation (2.16) then becomes

$$\frac{C_f}{2} = \frac{\sqrt{(1+\varkappa)}\,\pi^{-\frac{1}{2}}}{\sqrt{(Re_x)}}\left\{1+\frac{1}{4}\frac{\varkappa}{1+\varkappa}\left(\frac{\lambda}{x}\right)+\vartheta\left[\left(\frac{\lambda}{x}\right)^2\right]\right\}, \quad (2.20)$$

where $Re_x = \rho u_0 x/\nu$ is the local Reynolds number. Multiplying $\pi^{-\frac{1}{2}}$ into the bracket, the zeroth-order term in (2.20) then compares the coefficient $\pi^{-\frac{1}{2}}$ with the Blasius coefficient of 0·332 for the (classical) equilibrium limit. The first-order near-equilibrium coefficient here is $1/(4\pi^{\frac{1}{2}}) = 0{\cdot}141$, which is to be compared with 0·332(0·49) = 0·163 obtained from the particle–fluid boundary layer problem through numerical integration of the first-order perturbation boundary-layer equations [6, 7]. After Rayleigh's transformation, (2.18) becomes

$$\frac{C_f}{2} = \frac{\pi^{-\frac{1}{2}}}{\sqrt{(Re_x)}}\left\{1+\varkappa\left(\frac{x}{\lambda}\right)+\vartheta\left[\left(\frac{x}{\lambda}\right)^2\right]\right\}. \quad (2.21)$$

Again, the zeroth-order term is to be compared with the Blasius coefficient for the (classical) frozen limit. The first-order coefficient, $\pi^{-\frac{1}{2}} = 0{\cdot}564$, is to be compared with (0·332)(3·454) = 1·15. The latter was, again, obtained through numerical integration of the

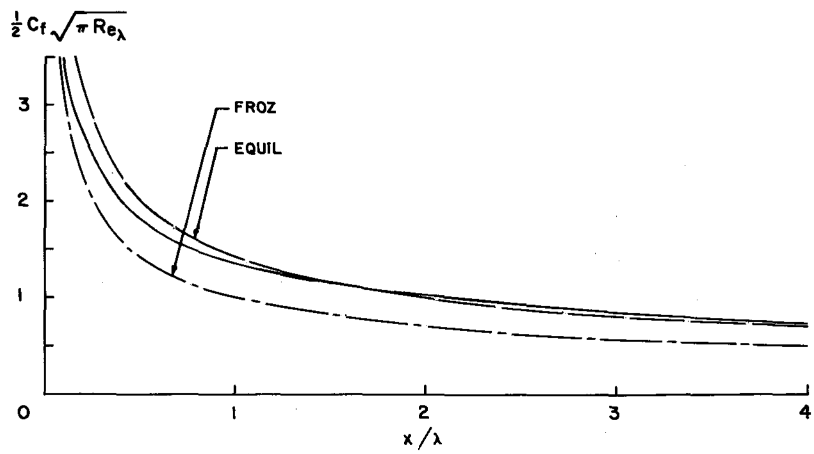

FIG. 2. Shear stress on a flat plate ($\varkappa = 1$).

first-order perturbation boundary-layer equations in the near-frozen regime [7].

To the present time, the laminar boundary layer on a semi-infinite plate has been treated only in the limiting regimes described above. Equation (2.15), after a Rayleigh transformation, may be used as a reasonable approximation for the shear stress at the wall for the complete range of x/λ. This is shown in Fig. 2 for $\varkappa = 1$, where Re_λ is defined as $\rho u_0 \lambda/\nu$. The skin friction coefficient relaxes from the frozen value near the leading edge towards the equilibrium value far downstream.

Vorticity diffusion thickness. An "effective" vorticity thickness may be defined corresponding to that of the classical Rayleigh problem.

$$\delta\,(t) = \int_0^\infty \zeta\, \mathrm{d}y/\zeta(0, t) \tag{2.22}$$

where $\zeta = -\,\partial u/\partial y$ is the fluid vorticity. The integral of the vorticity is simply u_0, and

$$\delta(t) = u_0/\zeta(0, t) = (\nu/u_0)/(-C_f/2). \tag{2.23}$$

It is particularly interesting to observe the relaxation of the vorticity diffusion-layer thickness from the frozen parabolic growth $(\pi\nu t)^{\frac{1}{2}}$ to the equilibrium parabolic growth $(\pi\bar{\nu}t)^{\frac{1}{2}}$. When $t/\tau \gg 1$, the near-equilibrium asymptotic behavior is then

$$\delta = (\pi\bar{\nu}t)^{\frac{1}{2}}\left\{1 - \frac{1}{4}\frac{\varkappa}{1+\varkappa}\left(\frac{\tau}{t}\right) + \mathcal{O}\left[\left(\frac{\tau}{t}\right)^2\right]\right\} \tag{2.24}$$

which approaches the equilibrium layer from below. The near-frozen behavior is

$$\delta = (\pi\nu t)^{\frac{1}{2}}\left\{1 - \varkappa\frac{t}{\tau} + \mathcal{O}\left[\left(\frac{t}{\tau}\right)^2\right]\right\}, \tag{2.25}$$

in which the thinning of the diffusion layer from the frozen growth takes place.

Since $C_f/2$ is available from the Rayleigh problem for all ranges of t/τ, a corresponding Rayleigh transformation then depicts the relaxation of the laminar boundary-layer thickness from the frozen growth

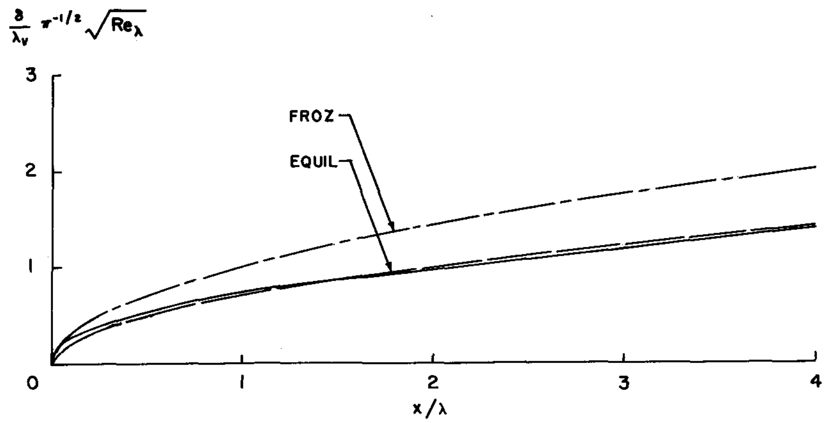

FIG. 3. Boundary layer growth on a flat plate ($\varkappa = 1$).

$(\nu x/u_0)^{\frac{1}{2}}$ near the leading edge towards that of the equilibrium growth $(\bar{\nu} x/u_0)^{\frac{1}{2}}$ far downstream. This is shown in Fig. 3 for $\varkappa = 1$.

Fluid velocity profile. The representation of the solution (2.7) as real definite integrals may be made through application of the usual methods of contour integration [11]. However, the method and expressions are lengthy. The required behavior of the solution here can be deduced directly from (2.6).

In the near-equilibrium regime, $t/\tau \gg 1$, the result may be written as an asymptotic expansion in powers of τ/t,

$$u'(\bar{\eta}, t') = u'_{e_0}(\bar{\eta}) + (\tau/t)u'_{e_1}(\bar{\eta}) + \vartheta[(\tau/t)^2], \quad (2.26)$$

where the first term is the equilibrium Rayleigh solution in terms of the equilibrium similarity variable $\bar{\eta} = y/[2(\bar{\nu}t)^{\frac{1}{2}}]$,

$$u'_{e_0}(\bar{\eta}) = \operatorname{erfc} \bar{\eta}. \quad (2.27)$$

The first-order function is

$$u'_{e_1}(\bar{\eta}) = \frac{\varkappa}{1+\varkappa} \frac{\bar{\eta}\, e^{-\bar{\eta}^2}}{2\pi^{\frac{1}{2}}} H_2(\bar{\eta}), \quad (2.28)$$

where the Hermite polynomial of second degree is

$$H_2(\bar{\eta}) = 2(2\bar{\eta}^2 - 1). \quad (2.29)$$

From (2.28) it is seen that $[(1+\varkappa)/\varkappa]u'_{e_1}(\bar{\eta})$ is a universal function of $\bar{\eta}$, independent of $\varkappa$, and is shown in Fig. 4. The form of this result is entirely similar to the form of the first-order function *assumed* in the expansion for the near-equilibrium limit in the nonlinear, steady, laminar boundary layer on a flat plate [6, 7].

In the near-frozen regime, $t/\tau \ll 1$, the result may be written as an asymptotic expansion in powers of t/τ,

$$u'(\eta, t') = u'_{f_0}(\eta) + (t/\tau)u'_{f_1}(\eta) + \vartheta[(t/\tau)^2], \quad (2.30)$$

where

$$u'_{f_0}(\eta) = \operatorname{erfc} \eta \quad (2.31)$$

is the frozen Rayleigh solution in terms of the frozen similarity variable $\eta = y/[2(\nu t)^{\frac{1}{2}}]$. The first-order function is

$$u'_{f_1}(\eta) = -\frac{\varkappa}{2} 4\eta[i \operatorname{erfc} \eta], \quad (2.32)$$

where the integral of the complementary error function

$$i \operatorname{erfc} \eta = \int_{\eta}^{\infty} \operatorname{erfc} \eta' \, d\eta' = \pi^{-\frac{1}{2}} e^{-\eta^2} - \eta \operatorname{erfc} \eta$$

is explicitly tabulated [10]. From (2.32) it follows that

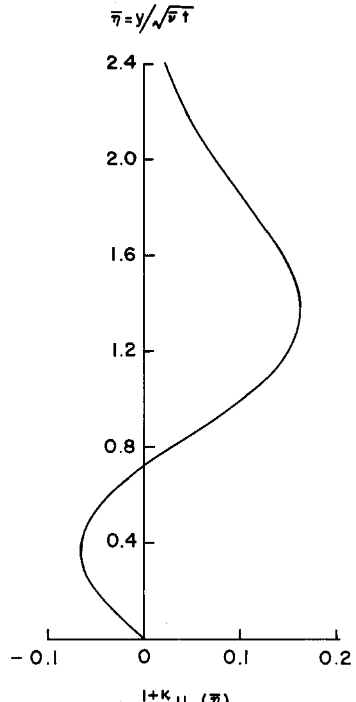

FIG. 4. First-order fluid velocity universal function (t/τ large).

$u'_{f_1}(\eta)/\varkappa$ is a universal function of η, independent of $\varkappa$, and is shown in Fig. 5. Again, the form of the first-order function is analogous to the corresponding assumed expansion in the laminar boundary layer on a semi-infinite flat plate [7].

Particle velocity profile. The momentum equation for the dimensionless particle velocity $u'_p = u_p/u_0$

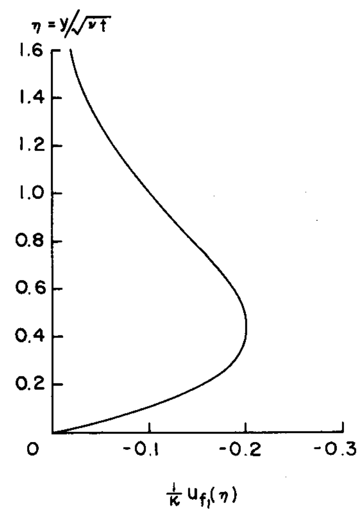

FIG. 5. First-order fluid velocity universal function (t/τ small).

from (2.2) appears in the form

$$(1 + \varkappa)\frac{\partial u_p'}{\partial t'} = -(u_p' - u'). \tag{2.33}$$

Initially the particle cloud is at rest, so that $u_p'(y', 0) = 0$. The Laplace transform of equation (2.33) then gives

$$\mathscr{L}\{u_p'\} \equiv U_p(y', s) = \frac{e^{-W(s)y'}}{s[1 + (1 + \varkappa)s]}, \tag{2.34}$$

and the near-equilibrium ($t/\tau \gg 1$) behavior of the particle velocity may be obtained in a manner similar to that employed for (2.26). It is convenient to discuss the particle–fluid slip velocity $(u' - u_p')$ rather than the particle velocity $u_p'(y', t')$. Hence,

$$u' - u_p' = (\tau/t)g_{e_1}(\bar{\eta}) + \vartheta\,[(\tau/t)^2], \tag{2.35}$$

where

$$g_{e_1}(\bar{\eta}) = \pi^{-\frac{1}{2}}\bar{\eta}\, e^{-\bar{\eta}^2} \tag{2.36}$$

is a universal function of $\bar{\eta}$ and is shown in Fig. 6.

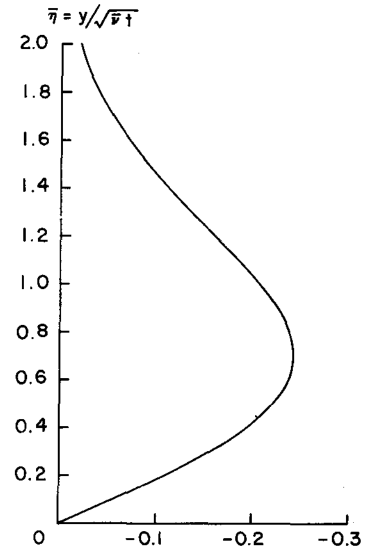

FIG. 6. First-order particle–fluid slip velocity universal function (t/τ large).

At the wall, $g_{e_1}(0) = 0$, and hence the particle cloud follows the "no-slip condition" at the wall in this approximation. Away from the wall, however, the fluid velocity is continuously changing due to viscous diffusion. Generally speaking, in the near-equilibrium regime, the particle cloud continuously lags behind, but equilibrates with, the fluid velocity. Referring to (2.35), it is noted that the local acceleration of the zeroth-order equilibrium flow $u_{e_0} = u_0 \operatorname{erfc} \bar{\eta}$ is given by

$$-\frac{u_0}{t}\pi^{-\frac{1}{2}}\bar{\eta}\, e^{-\bar{\eta}^2}. \tag{2.37}$$

This implies that the first-order slip velocity is proportional to the local acceleration of the zeroth-order flow in the near-equilibrium approximation. These useful concepts, including that of zero particle–fluid slip velocity at the wall, have been utilized by Marble [6] in the asymptotic treatment of the laminar boundary layer on a semi-infinite flat plate far downstream from the leading edge. Here, this result is recovered from the solution of the problem itself, whereas in [6], it is obtained from expansion of the differential equations.

In the near-frozen regime, $t/\tau \ll 1$, the appropriate expansion for the particle velocity is

$$u_p' = (t/\tau)u_{p_{f(1)}}'(\eta) + \vartheta\,[(t/\tau)^2], \tag{2.38}$$

where the first-order universal function

$$u_{p_{f(1)}}'(\eta) = \operatorname{erfc}\eta - 2\eta\left(\frac{e^{-\eta^2}}{\pi^{\frac{1}{2}}} - \eta \operatorname{erfc}\eta\right) \tag{2.39}$$

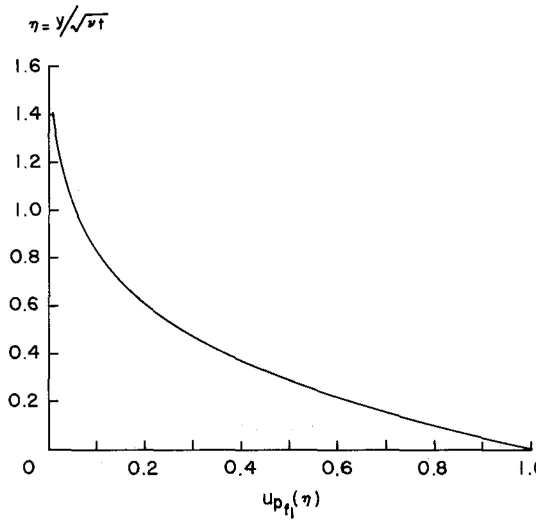

FIG. 7. First-order particle velocity universal function (t/τ small)

is shown in Fig. 7. At the wall, $u_{p_{f(1)}}'(0) = 1$ and $u_p'(0, t/\tau) = (t/\tau) + \vartheta[(t/\tau)^2]$, which indicates an acceleration of the particle cloud at the wall toward the velocity of the fluid there. The qualitative behavior of $g_{e_1}(\bar{\eta})$ and $u_{p_{f(1)}}'(\eta)$ are also similar to the corresponding particle velocity functions in the boundary layer problem [6, 7].

3. The Thermal Rayleigh Problem

Since the momentum and energy equations are uncoupled in the incompressible flow, the thermal Rayleigh problem for small relative temperature differences may be treated directly. The mixture of fluid and particle cloud is initially in equilibrium with the infinite flat plate at temperature T_∞. The plate is then discontinuously heated to a constant temperature T_w. The energy equations can be obtained in a similar manner as (2.1) and (2.2) from Marble [6]. For the fluid,

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial y^2} + \frac{\varkappa c/c_p}{\tau_T}(T_p - T), \tag{3.1}$$

and for the particle phase

$$\frac{\partial T_p}{\partial t} = -\frac{1}{\tau_T}(T_p - T), \tag{3.2}$$

where T is the fluid temperature and T_p that of the particle phase. The viscous dissipation and the work done on the fluid due to particle–fluid momentum interactions are neglected in (3.1) for the heat transfer problem. The thermal diffusivity of the fluid is defined as $\chi = k/(\rho c_p)$, where k is the fluid thermal conductivity, c_p the heat capacity per unit mass of the fluid, c the heat capacity per unit mass of the solid particles. The temperature relaxation time of the particles is $\tau_T = (3/2)\,Pr\,(c/c_p)\tau$ for particles obeying the Stokes law; $Pr = c_p\mu/k$ is the Prandtl number of the fluid. Here, $\varkappa c/c_p$ is the thermal equilibration parameter and is a measure of the relative temperature change in the two phases.

In a manner similar to that in which the momentum equations were treated, (3.1) and (3.2) can be combined into a single equation for T:

$$\tau_T^* \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t} - \chi\frac{\partial^2 T}{\partial y^2}\right) + \left(\frac{\partial T}{\partial t} - \bar{\chi}\frac{\partial^2 T}{\partial y^2}\right) = 0, \tag{3.3}$$

where $\tau_T^* = \tau_T/(1 + \varkappa\, c/c_p)$, $\chi = \nu/Pr$ is the frozen thermal diffusivity, and $\bar{\chi} = \bar{\nu}/\overline{Pr}$ is the equilibrium thermal diffusivity. Here, $\overline{Pr} = (1 + \varkappa\, c/c_p)Pr/(1 + \varkappa)$ is the equilibrium Prandtl number and is modified through the heat capacity per unit mass of the equilibrium mixture. Define the dimensionless quantities

$$\theta = (T - T_\infty)/(T_W - T_\infty), \quad t'' = t/\tau_T^*, \quad y'' = y/(\nu\tau_T^*/Pr)^{\frac{1}{2}},$$

where the dimensionless time and distance normal to the plate are related to the corresponding quantities of the momentum problem as $t'' = (\tau^*/\tau_T^*)t'$ and $y'' = (\tau^*\,Pr/\tau_T^*)^{\frac{1}{2}}y'$, respectively. The dimensionless temperature function now satisfies the equation

$$\frac{\partial}{\partial t''}\left(\frac{\partial \theta}{\partial t''} - \frac{\partial^2\theta}{\partial y''^2}\right) + \left(\frac{\partial \theta}{\partial t''} - \frac{1}{1 + \varkappa\, c/c_p}\frac{\partial^2\theta}{\partial y''^2}\right) = 0. \tag{3.4}$$

The initial conditions are $\theta(y'', 0) = 0$, $\partial\theta(y'', 0)/\partial t'' = 0$. The boundary conditions are $\theta(0, t'') = 0$ when $t'' < 0$; and $\theta(0, t'') = 1$, $\theta(y'', t'') = 0$ as $y'' \to \infty$ when $t'' \geqslant 0$. At the wall, the particle cloud adjacent to the plate has a temperature slip with respect to the fluid; the temperature rise of the particle cloud there is solely due to the heat received from the fluid.

The form of (2.4) for u' and its associated initial and boundary conditions is mathematically similar to that of (3.4) for θ and its initial and boundary conditions. That is,

$$u'(y', t'; \varkappa) = \theta(y'', t''; \varkappa c/c_p). \tag{3.5}$$

For the two profiles to be identical, however, $u'(y, t, = \theta(y. t)$, it would be required that $\tau/\tau_T = 1$, $\varkappa = \varkappa c/c_p$, and at the same time, $Pr = 1$. For particles obeying the Stokes' law, $\tau_T/\tau = (3/2)\,Pr\,(c/c_p)$, these conditions cannot be simultaneously satisfied, although they are very nearly satisfied for metal particles in a gaseous medium.

From (3.5), the solution for $\theta(y'', t'')$ can be obtained by replacing $\varkappa$ by $\varkappa c/c_p$ and (y', t') by (y'', t'') in the results of $u'(y', t')$. The surface heat-transfer rate is

$$\dot{q}_W = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{k\,Pr^{\frac{1}{2}}\,(T_W - T_\infty)}{(\nu\tau_T^*)^{\frac{1}{2}}}\frac{\partial\theta(0, t'')}{\partial y''}, \tag{3.6}$$

where $\partial\theta(0, t'')/\partial y''$ is obtained from $\partial u'(0, t')/\partial y'$, a result which is valid for all ranges of t/τ_T.

After the Rayleigh transformation of $t = x/u_0$, defining $\lambda_T = u_0\tau_T$, (3.6) may then be used to provide an approximate formula for the surface heat-transfer rate for laminar flow over a cooled or heated semi-infinite flat plate. The behavior of the dimensionless surface heat-transfer rate may be obtained from Fig. 2 by setting

$$\frac{\lambda_T \dot{q}_W}{k(T_W - T_\infty)}\left(\frac{\pi}{Re_{\lambda_T}\,Pr}\right)^{\frac{1}{2}} = \frac{C_f}{2}(\pi\,Re_\lambda)^{\frac{1}{2}} \tag{3.7}$$

and replacing x/λ by x/λ_T. Since Fig. 2 is calculated for a value of $\varkappa = 1$ in the preceding section, the heat-transfer result is then for a value of $\varkappa c/c_p = 1$. The heat-transfer rate relaxes from the frozen behavior of $(\dot{q}_W)_F = [k(T_W - T_0)\pi^{-\frac{1}{2}}\,Pr^{\frac{1}{2}}]/(\nu x/u_0)^{\frac{1}{2}}$ towards that of the equilibrium behavior

$$(\dot{q}_W)_E = [k(T_W - T_\infty)\pi^{-\frac{1}{2}}\,\overline{Pr}^{\frac{1}{2}}]/(\bar{\nu}x/u_0)^{\frac{1}{2}}.$$

An "effective" heat diffusion thickness may be defined

$$\delta_T = \int_0^\infty \dot{q}\, dy/\dot{q}_W \qquad (3.8)$$

where $\dot{q} = -k\partial T/\partial y$. The integral is simply $k(T_W - T_\infty)$ and

$$\delta_T = k(T_W - T_\infty)/\dot{q}_W. \qquad (3.9)$$

After applying the Rayleigh transformation, we write the ratio of the velocity to the thermal boundary-layer thickness

$$\delta/\delta_T = Pr^{\frac{1}{2}} \{1 + [\varkappa(c/c_p)(x/\lambda_T) - \varkappa(x/\lambda)] + \vartheta\,[(x/\lambda_T)^2] + \vartheta\,[(x/\lambda)^2]\} \qquad (3.10)$$

for the near-frozen regime, and

$$\frac{\delta}{\delta_T} = \overline{Pr}^{\frac{1}{2}} \left\{1 + \left[\frac{1}{4}\frac{\varkappa c/c_p}{1 + \varkappa\, c/c_p}\left(\frac{\lambda_T}{x}\right) - \frac{\varkappa}{1+\varkappa}\left(\frac{\lambda}{x}\right)\right] + \vartheta\,[(x/\lambda_T)^{-2}] + \vartheta\,[(x/\lambda)^{-2}]\right\} \qquad (3.11)$$

for the near-equilibrium regime. The ratios of the two boundary layer thicknesses exhibit the dependence on the history of the relaxation processes. Clearly, when $\varkappa = \varkappa c/c_p$ and $\lambda = \lambda_T$, the ratio δ/δ_T reduces to a similar behavior as in the classical boundary-layer theory.

Acknowledgements—The author is indebted to Professor Frank E. Marble for stimulating his interest and for the many helpful and pleasant discussions. The interest and advice of Professor W. D. Rannie were very much appreciated. This work was supported by the National Science Foundation under Grant GP-713.

References

1. HOGLUND, R. F., Recent advances in gas-particle nozzle flows, *ARS J.* **32**, 662–671 (1962).
2. KOMOV, G. A., Issledovanie techeniia zapylennogo potoka v sople Lavalia, *Inz. Fiz. Zh.* **10**, 368–373 (1966).
3. STOKES, G. G., On the effect of internal friction of fluids on the motion of pendulums, *Trans. Cam. Phil. Soc.* **9**, 8–106 (1851); *Mathematics and Physics Papers,* Vol. 3, p. 1, Cambridge University Press, Cambridge (1901).
4. LORD RAYLEIGH, On the motion of solid bodies through a viscous liquid, *Phil. Mag.* **21**, Ser. 6, 697–711 (1911); *Scientific Papers*, Vol. 6, p. 29, Cambridge University Press, Cambridge (1920).
5. BLASIUS, H., Grenzschichten in Flussigkeiten mit kleiner Reibung, *Z. Math. Phys.* **56**, 1–37 (1908). Translated as NACA TM 1256.
6. MARBLE, F. E., Dynamics of a gas containing small solid particles, in *Proceedings of the 5th AGARD Combustion and Propulsion Colloquium,* p. 175, Pergamon Press, New York (1963).
7. SINGLETON, R. E., Fluid mechanics of gas–solid particle flow in boundary layers, Ph.D. Thesis, California Institute of Technology (1964).
8. LIU, J. T. C., Flow induced by an oscillating infinite flat plate in a dusty gas, *Physics Fluids* **9**, 1716–1720 (1966).
9. SAFFMAN, P. G., On the stability of laminar flow of a dusty gas, *J. Fluid Mech.* **13**, 120–128 (1962).
10. *Handbook of Mathematical Functions*, edited by M. Abramowitz and L. A. Stegun, Dover, New York (1966).

J. T. C. Liu: *Assistant Professor of Engineering, Brown University, Providence, Rhode Island.*

Résumé—Ecoulement induit par le mouvement d'impulsion d'une plaque plate infinie dans un gaz charge de poussière. Le problème de l'écoulement induit par une plaque plate infinie soumise soudain à un mouvement parallèle à son propre plan dans un gaz incompressible chargé de poussière présente un intérêt considérable de par lui-même du point de vue physique aussi bien que par suite de son rapport étroit avec la couche limite laminaire stable (pression constante) non linéaire. Sa solution fournit des renseignements complets et exacts sur les modifications du la croissance de la couche limite et sur le frottement superficien dûs à l'action réciproque paticule-fluide. En outre, il fournit une base d'après laquelle juger les approximations qui ont été utilisées pour des problèmes plus complexes du mouvement visqueux fluide-particule. Le problème thermique non couplé de Rayleigh pour des différences de température relative réduites est supposé directement et ceci donne les réponses aux questions sur les modifications du taux de transfert de la chaleur à la surface et sur la possiblité d'une similitude avec la couche limite de vitesse. Une similitude est possible lorsque, en plus d'un chiffre Prandtl d'unité, les procédés de relâchement du flot sont aussi semblables.

Zusammenfassung—Strömung die durch stossweise Bewegung einer unendlichen flachen Platte in staubigem Gas entsteht. Das Problem der Strömung die entsteht, wenn eine unendliche flache Platte parallel zur eigenen Ebene in Bewegung gesetzt wird ist an sich von grossem Interesse, ausserdem auch wegen der Ähnlichkeit mit einer nichtlinearen laminaren Grenzschicht mit stetigem Druck. Die Lösung gibt völlige und genaue Auskunft über Änderungen des Grenzschicht-Wachstums und den Oberflächenwiderstand als Folge der Wechselwirkung von Teilchen und Flüssigkeit. Weiters gibt die Lösung eine Grundlange zur Beurteilung der Genauigkeit der Annäherungen die in komplizierten Problemen von zähen Teilchen-

Flüssigkeiten angewandt wurden. Das ungekoppelte Rayleigh Problem der kleinen relativen Temperatur-Unterschiede wird direkt gefolgert und dies beantwortet Fragen über die Änderungen der Oberflächen-Wärmeübertragung sowie über die Möglichkeit einer Ähnlichkeit mit der Geschwindigkeits-Grenzschicht. Eine Ähnlichkeit ist möglich wenn neben der Prandtl'schen Einheitsnummer die Strömungs-Entspannungsvorgänge einander auch ähnlich sind.