Rotational Axisymmetric Mean Flow and Damping of Acoustic Waves in a Solid Propellant Rocket

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Solution for Axisymmetric Rotational Mean Flow

ALTHOUGH for many purposes the one-dimensional approximation to the steady flow in a rocket chamber is adequate, there are occasions when more precise information is required. For example, analysis of the stability of pressure oscillations involves knowledge of the streamlines. It has been common practice to use the solution for potential flow subject to the boundary conditions of no flow through the head end and uniform speed normal to the burning surface. Since the Mach number generally is very small, one may assume the density to be constant; the result for the Mach number in a cylindrical chamber is

$$\bar{\mathbf{M}} = M_r\hat{e}_r + M_z\hat{e}_z = M_b(2z\hat{e}_z - r\hat{e}_r)$$

where M_b is the Mach number at the burning surface, $\hat{e}_z$, $\hat{e}_r$ are unit vectors, and the coordinates r, z are normalized with respect to the port radius.

However, at the boundary $r = 1$, this solution exhibits a component of flow parallel to the surface. It seems that a better approximation, more consistent with the burning process, should satisfy the condition that the velocity is

Table 1 Values of Λ_c for the seventeen lowest modes, $l = 0$[a]

m	n = 0	1	2
0	0 . . .	0 0.166	0 0.212
1	−0.418 −0.154	−0.0366 +0.1954	−0.0140 +0.2110
2	−0.768 −0.334	−0.0978 +0.1892	−0.0422 +0.2038

[a] Upper values are for potential mean flow and lower values are for rotational mean flow.

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normal to the surface. Vorticity is produced at the boundary and transported along streamlines; the field is rotational. A solution meeting these conditions may be determined without difficulty. For a reason that is given below, the damping of waves due to the convection of wave energy by the mean flow is greater for this case than when the average flow is assumed to be irrotational. A quantitative comparison of the two possibilities for the first 17 modes in a cylindrical chamber is shown in Tables 1 and 2.

The equation for Stokes' stream function ψ in incompressible flow is

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = r\zeta = r^2 f(\psi)$$

where $f(\psi)$ is initially an unspecified function of ψ related to the boundary values set on the vorticity. Only the azimuthal component of vorticity ζ is nonzero, and is given by

$$\zeta = (\partial M_r/\partial z) - (\partial M_z/\partial r)$$

The boundary conditions are specified on the velocity, or Mach number,

$$M_z = 0 \qquad (z = 0)$$

$$\left.\begin{aligned} M_z &= 0 \\ M_r &= -M_b \end{aligned}\right\} \quad (r = 1)$$

with ψ so defined that

$$M_r = (1/r)(\partial\psi/\partial z) \qquad M_z = -(1/r)(\partial\psi/\partial r)$$

For potential flow, $f = 0$, and one finds the formula for Mach number given previously. It is verified easily that the solution desired here is obtained when $f = (2n + 1)\pi\psi$ with n an integer; the stream function is then

$$\psi = -M_b z \frac{\sin(2n + 1)(\pi r^2/2)}{\sin(2n + 1)(\pi/2)}$$

There are other solutions for f proportional to ψ, but they do not satisfy the boundary conditions proposed here. Even this solution evidently is not unique, but only for the case $n = 0$ is the radial flow nonzero everywhere in the chamber, except on the axis. Thus, for example, if $n = 1$, $M_r = 0$ on $r = \frac{2}{3}$; and all of the flow from the boundary is contained in the annulus $\frac{2}{3} < r \leq 1$. For $r < \frac{2}{3}$, $M_r > 0$, and $M_z > 0$ for $0 \leq r < \frac{1}{3}$; but $M_z < 0$ for $\frac{1}{3} < r < \frac{2}{3}$. Hence, about the axis, there is a region for any $n \neq 0$ in which the vorticity is not determined by the boundary conditions assumed. On physical grounds, then, the correct solution is that for $n = 0$,

$$\psi = -M_b z \sin(\pi r^2/2)$$

$$M_r = -M_b \frac{\sin(\pi r^2/2)}{r}$$

Table 2 Values of Λ_c for the seventeen lowest modes, $l \neq 0$[a]

m	n = 0	1	2
0	1	1	1
	1	1.332	1.424
1	0.682	0.963	0.986
	1.220	1.427	1.436
2	0.232	0.902	0.958
	1.100	1.442	1.450

[a] Upper values are for potential mean flow and lower values are for rotational mean flow.

$$M_z = M_b \pi z \cos(\pi r^2/2)$$

$$\zeta = -M_b \pi r z \sin(\pi r^2/2)$$

The pressure distribution is computed from the momentum equation

$$\nabla[(p/pa^2) + (M^2/2)] = \mathbf{M} \times \boldsymbol{\zeta}$$

where a is the reference speed of sound. Along the surface $r = 1$, $\mathbf{M} \times \boldsymbol{\zeta} = M_b \zeta \hat{e}_z$. The last equation then is integrated easily to give

$$(p - p_0)/pa^2 = (\pi M_b^2/2)z^2$$

if p_0 is the pressure at the head end. The variation of pressure along the burning surface is, for most propellants, inconsistent with the assumption of uniform burning rate along $r = 1$. However, the error is negligible, being of order M_b^2.

Influence of the Mean Flow on Acoustic Waves

When oscillations of pressure are induced in a rocket chamber, the rate at which they decay or grow is affected significantly by the presence of the mean flow. There are two reasons for this: 1) the interactions between the fluctuating velocity and the mean flow are such as to cause extra work to be done on or by the waves; and 2) there is gross transport of energy by the average flow. Fortunately, calculations are simplified by the coincidence that the two effects turn out to be equal. The net influence is comparable to that of the primary driving provided by the burning process, and, hence, cannot be neglected in a calculation of stability.

If the waves are assumed to have exponential behavior in time with an attenuation constant λ, then the contribution to λ by the mean flow is[1]

$$\frac{\lambda R}{\omega} = M_b \Lambda_c = \frac{1}{2E_N^2} \iint \eta^2 \overline{\mathbf{M}} \cdot \hat{n} dS$$

in which the integral extends over the entire boundary of the chamber, $0 \leq r \leq 1$, $0 \leq z \leq 4R$. The mode shape η of the standing wave† having frequency ω is

$$\eta = \cos(m\phi) \cos(k_l z) J_m(\varkappa_{mn} r)$$

and the normalization constant E_N^2 is

$$E_N^2 = \int_0^{2\pi} d\phi \int_0^1 r\,dr \int_0^{4R} dz \eta^2 =$$

$$\frac{L\pi}{2R}\left[1 + \frac{\sin l\pi}{l\pi}\right]\begin{cases} [J_0^2(\kappa_{on}) + J_1^2(\varkappa_{on})] & m = 0 \\ \frac{1}{2}(1 - m_1^2) J_m^2(\varkappa_{mn}) & m \neq 0 \end{cases}$$

The unperturbed frequency of the mode is $\omega^2 = (a/R)^2 (l^2\pi^2 + \varkappa_{mn}^2)$ where the $\varkappa_{mn}$ are the roots of $(dJ_m/dr)_{r=1} = 0$; $m_1 = m/\varkappa_{mn}$. An extended derivation of the previous results appears in Ref. 1.

The original formula for Λ_c [Eq. (17c) of Ref. 1] was

$$2M_b\Lambda_c E_N^2 = \int \overline{\mathbf{M}} \cdot \nabla(\eta^2/2) dV - \int \eta \mathbf{M}_c' \cdot \hat{n} dS + \int \eta \nabla \cdot \mathbf{M}_c' dV$$

with

$$\mathbf{M}_c' = -1/k^2[\overline{\mathbf{M}} \cdot \nabla(\nabla\eta) + \nabla\eta \cdot \nabla\overline{\mathbf{M}}]$$

An additional factor M_b appears explicitly because, as defined here, $\overline{\mathbf{M}}$ contains M_b. Reduction to the simpler formula for Λ_c proceeds as in Ref. 1, but the apparent restriction $\nabla \times \overline{\mathbf{M}} = 0$ can be removed. Let $\varphi = \overline{\mathbf{M}} \cdot \nabla\eta$, and after some manipulation,

$$\mathbf{M}_c' = -(1/k^2)\nabla\varphi + (1/k^2)(\nabla\eta \times \nabla \times \overline{\mathbf{M}})$$

† The same results for Λ_c are obtained for a wave traveling in the azimuthal direction.

Table 3 Values of $\beta = \dfrac{\int_0^1 \cos\left(\frac{\pi}{2} r^2\right) J_m^2(x_{mn}r)rdr}{\int_0^1 J_m^2(x_{mn}r)rdr}$

	n		
m	0	1	2
0	$2/\pi$	0.743	0.772
1	0.805	0.784	0.780
2	0.913	0.809	0.793

so that

$$\int \eta\nabla\cdot \mathbf{M}_c' dV = -\frac{1}{k^2}\int \eta\nabla^2\varphi dV + \frac{1}{k^2}\int \eta\nabla\cdot(\nabla\eta \times \nabla \times \bar{\mathbf{M}})dV$$

By Green's theorem and use of the properties $\nabla^2\eta = -k^2\eta$, $\nabla\eta\cdot\hat{n}dS = 0$, one finds

$$\int \eta\nabla^2\varphi dV = -k^2\int \bar{\mathbf{M}}\cdot\nabla\frac{\eta^2}{2}\, dV + \int \eta\nabla\varphi\cdot\hat{n}dS$$

Thus,

$$2M_b\Lambda_c E_N{}^2 = \int \bar{\mathbf{M}}\cdot\nabla\eta^2 dV + \int \frac{\eta}{k^2} \times (\nabla\varphi - \nabla\eta\times\nabla\times\bar{\mathbf{M}})\cdot\hat{n}dS - \int \frac{\eta}{k^2}\nabla\varphi\cdot\hat{n}dS + \frac{1}{k^2}\int \eta\nabla\cdot(\nabla\eta\times\nabla\times\bar{\mathbf{M}})dV = \int \bar{\mathbf{M}}\cdot\nabla\eta^2 dV + \frac{1}{k^2}\int \nabla\eta\cdot(\nabla\eta\times\nabla\times\bar{\mathbf{M}})dV$$

But the second integrand vanishes because $\nabla\eta$ is normal to $\nabla\eta\times\nabla\times\bar{\mathbf{M}}$, and

$$2M_b\Lambda_c E_N{}^2 = \int \bar{\mathbf{M}}\cdot\nabla\eta^2 dV = \int \eta^2\bar{\mathbf{M}}\cdot\hat{n}dS$$

Consequently, the very simple result for Λ_c is true for both rotational and irrotational mean flow.

With the expressions for $\bar{\mathbf{M}}$ already given, the surface integrals in the formula for Λ_c are performed easily. One finds that $(\Lambda_c)_{pot}$ for potential mean flow and $(\Lambda_c)_{rot}$ for rotational mean flow can be put in the forms valid for all l, m, n,

$$(\Lambda_c)_{pot} = [1 - (\sin l\pi/l\pi)] - m_1{}^2/(1 - m_1{}^2)$$

$$(\Lambda_c)_{rot} = \frac{2[(\pi/2)\beta - 1]}{[1 + (\sin l\pi/l\pi)]} + (\Lambda_c)_{pot}$$

where

$$\beta = \int_0^1 \cos\left(\frac{\pi r^2}{2}\right) J_m{}^2 r\, dr \Big/ \int_0^1 J_m{}^2 r\, dr$$

The numerator of β has been calculated numerically, and the values of β for a few values of m, n are shown in Table 3. Finally, the corresponding values of Λ_c, for both potential and rotational mean flowfields, are shown in Tables 1 and 2.

In all modes, the damping associated with rotational mean flow is greater than that for potential flow. The most interesting results, however, are those for $l = 0$ modes not exhibiting axial vibrations. If the mean flow is assumed to be a potential flow, then Λ_c is never positive. The waves are, in fact, driven by the mean flow for $m \neq 0$ and all values of n; but in rotational flow, the waves are driven only if $n = 0$.

The reason that the damping is greater for rotational flow is explained conveniently in terms of the convection of wave energy (proportional to η^2) through the exit plane. Convection of energy in at the burning surface is the same for the two flowfields; it appears in the term $-m_1{}^2/(1 - m_1{}^2)$ in Λ_c. When the axial speed is forced to be zero at the burning surface, its value near the axis is greatly increased. Compared to potential flow, the speed is $\pi/2$ greater on the axis. Thus, the rate at which energy is transported out of the chamber is increased, evidently enough to exceed the reduction in axial convection near the burning surface.

Of course, one can assess the stability of the waves only by considering all contributions to the attenuation. In particular, the response of the burning surface to fluctuations in pressure leads to a term which is proportional also to M_b and thus combines directly with Λ_c.[1] If Λ_c is increased, then the surface response, generally expressed as an admittance function, must be greater if the waves are to be maintained. Put another way, for a given propellant, and, hence, for a particular admittance function, the tendency for unstable oscillations appears to be greater if the mean flow is assumed to be irrotational.

The actual flowfield, as a result of turbulence, for example, probably lies somewhere between the rotational and irrotational cases. Thus, the results given here may be regarded perhaps as upper and lower bounds on Λ_c.

Reference

[1] Culick, F. E. C., "Acoustic oscillations in solid propellant rocket chambers," Astronaut. Acta. (to be published).