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Compressibility Effects on Secondary Flows*

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SUMMARY

The method of W. R. Hawthorne for the calculation of the secondary vorticity is generalized for compressible flow. It is shown that in the linearized theory (small vorticity) the influence of compressibility upon the secondary vorticity is due to (1) the entropy gradient in the approaching flow and (2) the compression of the fluid during the turning of the flow. The analysis is applied to the secondary vorticity which generates in a cascade or bend if the approaching flow has a boundary layer with a Prandtl Number equal to unity and has been developed along an insulated wall.

For a cascade with a passage cross section which increases linearly with the turning angle and at the exit is 1.4 times the entrance cross-section area, the compressibility correction, to be applied upon the secondary vorticity, amounts to 24 per cent for a cascade entrance Mach Number of 0.8.

The problem of determination of the secondary flow downstream of the cascade, which is associated with a given secondary vorticity, is the same for compressible and incompressible flow

SYMBOLS

q = velocity vector
ω = vorticity vector
ω_σ = vorticity component in stream direction
ω_n = vorticity component normal to streamline
ρ = density
T = temperature
S = entropy
c_p = specific heat at constant pressure
R = radius of curvature of streamline
ϑ = flow angle
M = Mach Number
γ = ratio of specific heats
ϕ = angle between centripetal acceleration and the vector $q \times (c_p \text{ grad } T - T_0 \text{ grad } S)$
ϵ = small parameter
f_1, f_2 = correction factors

INTRODUCTION

The study of rotational flows is of considerable importance for the understanding and analysis of three-dimensional flows in turbomachines. A most interesting aspect of rotational flows is the development of a vorticity component in the stream direction (secondary vorticity), due to a curvature of the streamlines in the presence of a vorticity component normal to the stream direction. The development of the secondary vorticity has been studied by H. B. Squire and K. G. Winter in reference 1 and by W. R. Hawthorne in reference 2 for incompressible fluids.

In modern turbojet propulsion plants, the Mach Number of the flow entering the compressor is usually so high that a study of the influence of the compressibility on the secondary flow seems desirable. Another case where such an investigation might be of interest is the secondary flow in a turbine, following a combustion chamber. There we can expect appreciable temperature gradients, caused by the combustion process.

In the present investigation, the method of Hawthorne[2] to derive an expression for the secondary vorticity is generalized for compressible flows. An application to cascades, approached by a flow with a boundary layer, developed along an insulated wall and with a Prandtl Number equal to unity, is given, and an estimate of the compressibility effects on the secondary vorticity is made. Some attention is given to the integration problem in compressible flow for the secondary motion far downstream from a cascade with a given vorticity field.

The Linearized Equation for the Secondary Vorticity

Let q denote the velocity vector, ω the vorticity vector, ω_σ the component of the vorticity in the direction of q and ρ the density of the fluid. The vorticity vector may be resolved into its components in the flow direction and perpendicular to it,

$$\omega = \frac{\omega_\sigma}{\rho q}\rho q + \frac{(q \times \omega) \times q}{q^2}$$

The divergence of the vorticity vanishes, hence

$$\frac{\omega_\sigma}{\rho q}\text{div}(\rho q) + \rho q \cdot \text{grad}\frac{\omega_\sigma}{\rho q} + \text{div}\frac{(q \times \omega) \times q}{q^2} = 0 \quad (1)$$

For steady flow, div $(\rho q) = 0$ and Eq. (1) reduces to

$$\rho q \cdot \text{grad}\frac{\omega_\sigma}{\rho q} = -\text{div}\frac{(q \times \omega) \times q}{q^2} \quad (2)$$

The equations of motion require that

$$\begin{aligned} q \times \omega &= c_p \text{ grad } T_0 - T \text{ grad } S \\ &= c_p \text{ grad } T_0 - T_0 \text{ grad } S + (1/2c_p)q^2 \text{ grad S} \end{aligned}$$

where T_0 is the stagnation temperature and S the entropy. This relation substituted in Eq. (2) gives

$$\begin{aligned} \rho q \cdot \text{grad}(\omega_\sigma/\rho q) = -\text{div}\{(1/q^2)\,[(c_p \text{ grad } T_0 - T_0 \\ \text{grad } S) \times q]\} \\ -(1/2c_p)\text{ div (grad } S \times q) \quad (3) \end{aligned}$$

We use the identities

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$$\begin{aligned}\operatorname{div}(\operatorname{grad} S \times \boldsymbol{q}) &= \operatorname{curl}\operatorname{grad} S\cdot\boldsymbol{q} - \operatorname{grad} S\cdot\operatorname{curl}\boldsymbol{q}\\ &= -\boldsymbol{\omega}\cdot\operatorname{grad} S\end{aligned}$$

$$\operatorname{div}\{(1/q^2)\,[\boldsymbol{q}\times(c_p \operatorname{grad} T_0 - T_0 \operatorname{grad} S)]\} = [\operatorname{grad}(1/q^2)]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] + (1/q^2)\operatorname{div}[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)]$$

and

$$\begin{aligned}\operatorname{div}[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] &= (c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)\cdot\operatorname{curl}\boldsymbol{q} - \boldsymbol{q}\cdot\operatorname{curl}(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S) =\\ &(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)\cdot\boldsymbol{\omega} - c_p\boldsymbol{q}\cdot\operatorname{curl}\operatorname{grad} T_0 + \boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S) + \boldsymbol{q}\cdot(T_0\operatorname{curl}\operatorname{grad} S) =\\ &(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)\cdot\boldsymbol{\omega} + \boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S)\end{aligned}$$

Then Eq. (3) may be written

$$\rho\boldsymbol{q}\cdot\operatorname{grad}\omega_\sigma/\rho q = [\operatorname{grad}(1/q^2)]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] + (1/q^2)\,[(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)\cdot\boldsymbol{\omega} + \boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S)] + (1/2c_p)\boldsymbol{\omega}\cdot\operatorname{grad} S \quad (4)$$

With the identity

$$\operatorname{grad}(1/q^2) = -(1/q^4)\operatorname{grad} q^2 = -(1/q^4)\,[2(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} + 2\boldsymbol{q}\times\operatorname{curl}\boldsymbol{q}]$$

the first term on the right-hand side of Eq. (4) becomes

$$[\operatorname{grad}(1/q^2)]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] = -(1/q^4)\,[2(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} + 2\boldsymbol{q}\times\omega]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] \quad (5)$$

The vector $(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}$ is easily recognized as the acceleration vector for steady flow. For a flow of an inviscid fluid, the stagnation temperature T_0 and the entropy S are constant along streamlines, so that the vector $c_p \operatorname{grad} T_0 - T_0 \operatorname{grad} S$ is perpendicular to the vector $\boldsymbol{q}$. Also, $\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)$ is perpendicular to $\boldsymbol{q}$ so that in the product $[(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)]$ only the centripetal acceleration gives a contribution. If ϕ is the angle between the centripetal acceleration and the vector $\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)$, then we have

$$[(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}]\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] = (q^3/R)\cos\phi\,|c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S|$$

where R is the radius of curvature of the streamline. The product $(\boldsymbol{q}\times\boldsymbol{\omega})\cdot[\boldsymbol{q}\times(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)]$ in Eq. (5) can be expanded as

$$q^2[\boldsymbol{\omega}\cdot(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)] - [\boldsymbol{q}\cdot(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)]\,(\boldsymbol{\omega}\cdot q) = q^2\boldsymbol{\omega}\cdot(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)$$

because $\boldsymbol{q}$ and $c_p \operatorname{grad} T_0 - T_0 \operatorname{grad} S$ are perpendicular. Use of these relations in Eq. (4) yields

$$\rho\boldsymbol{q}\cdot\operatorname{grad}(\omega_\sigma/\rho q) = (-2/qR)\cos\phi\,|c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S| - (2/q^2)\boldsymbol{\omega}\cdot(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S) + (1/q^2)\,[(c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S)\cdot\boldsymbol{\omega} + \boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S)] + (1/2c_p)\boldsymbol{\omega}\cdot\operatorname{grad} S$$

and after combining the terms on the right-hand side, one finds

$$\rho\boldsymbol{q}\cdot\operatorname{grad}(\omega_\sigma/\rho q) = (-2/qR)\cos\phi\,|c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S| + (1/c_p)\boldsymbol{\omega}\cdot\operatorname{grad} S + (1/q^2)\boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S) \quad (6)$$

Integration of Eq. (6) along a streamline gives

$$\left.\frac{\omega_\sigma}{\rho q}\right|_1^2 = -\int_1^2 \frac{2}{\rho q^2 R}\cos\phi\,|c_p\operatorname{grad} T_0 - T_0\operatorname{grad} S|\,d\sigma + \int_1^2\left[\frac{1}{\rho q^3}\boldsymbol{q}\cdot(\operatorname{grad} T_0\times\operatorname{grad} S) + \frac{1}{\rho q c_p}\boldsymbol{\omega}\cdot\operatorname{grad} S\right]d\sigma \quad (7)$$

where $d\sigma$ is a line element of the streamline.

If grad T_0 and grad S have the same direction—i.e., if the stagnation temperature and the entropy are constant on the same surfaces—then the last integral in Eq. (7) vanishes.

Eq. (7) does not express the secondary vorticity ω_σ explicitly, because the integrands depend on $\boldsymbol{\omega}$. On the other hand, Eq. (7) lends itself very easily to the application of a perturbation procedure.

Expand all fields of interest in a power series in the small parameter ϵ,

$$\begin{aligned}\boldsymbol{q} &= \boldsymbol{q}^{(0)} + \epsilon\,\boldsymbol{q}^{(1)} + \dots,\\ \boldsymbol{\omega} &= \epsilon\,\boldsymbol{\omega}^{(1)} + \dots,\\ T_0 &= T_0^{(0)} + \epsilon\,T_0^{(1)} + \dots,\\ S &= S^{(0)} + \epsilon\,S^{(1)} + \dots,\\ \rho &= \rho^{(0)} + \epsilon\,\rho^{(1)} + \dots,\\ \phi &= \phi^{(0)} + \epsilon\,\phi^{(1)} + \dots,\\ R &= R^{(0)} + \epsilon\,R^{(1)} + \dots,\end{aligned}$$

where $\boldsymbol{q}^{(0)}$ is chosen such that it fulfills the equations of motion and continuity, that curl $\boldsymbol{q}^{(0)} = 0$, and that the corresponding stagnation temperature and entropy, $T_0^{(0)}$ and $S^{(0)}$, are constant.

Introduction of these power series expansions in Eq. (7) and separation of the different powers of ϵ gives for the first-order perturbation

$$\left.\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}}\right|_1^2 = -\int_1^2 \frac{2}{\rho^{(0)}q^{(0)2}R^{(0)}}\cos\phi^{(0)}\,|c_p\operatorname{grad} T_0^{(1)} - T_0^{(0)}\operatorname{grad} S^{(1)}|\,d\sigma \quad (8)$$

If the zeroth-order field is known together with the first-order perturbations of the flow conditions at point 1, then the first-order solution for the secondary vorticity can be calculated with Eq. (8) for all other points on the same streamline of the zeroth-order flow. This calculation involves the knowledge of grad $T_0^{(1)}$ and grad $S^{(1)}$, which can be derived from the given flow perturbation at point 1 and the fact that both $T_0^{(1)}$ and $S^{(1)}$ are constant along the zeroth-order streamlines.

If the infinitesimal change in flow angle $d\vartheta = d\sigma/R$ is introduced, Eq. (8) can be written in a somewhat more convenient form,

$$\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}} = -2\int_1^2 \frac{\cos\phi^{(0)}}{\rho^{(0)}q^{(0)^2}} \left| c_p \text{ grad } T_0^{(1)} - T_0^{(0)} \text{ grad } S^{(1)}\right| d\vartheta \quad (9)$$

A comparison with the formula of Hawthorne[2] shows that the compressibility has two effects: (1) the entropy distribution of the approaching flow is in general nonuniform and this gives rise to an additional secondary vorticity generation; and (2) the streamwise density variations in the region where the secondary vorticity develops modifies this vorticity generation.

Eq. (9) is valid when grad T_0 and grad S are small. It is possible, however, to apply the small perturbation technique under less restrictive assumptions. When $|c_p \text{ grad } T_0 - T_0 \text{ grad } S|$ is small, but if not both grad T_0 and grad S are small, then the angle between the vectors grad T_0 and grad S should be small, or one of these vectors is small. In both cases, (grad $T_0 \times$ grad S) in Eq. (7) is a small vector and $\boldsymbol{\omega}\cdot$grad S is a small number, so that ω_σ is also small and the perturbation procedure is applicable if the second integral in Eq. (7) is properly taken into account. As in most cases where the perturbation procedure is at all applicable, both grad T_0 and grad S are small; we shall only consider this case and use Eq. (9) as a linearized equation. We can express this equation in a more convenient form by writing

$$|c_p \text{ grad } T_0 - T_0 \text{ grad } S| = |c_p \text{ grad } T_0 - T \text{ grad } S - (T_0 - T_1) \text{ grad } S|$$

and

$$(T_0 - T)/\gamma RT = (1/2c_p)\,(q^2/c^2) = (1/2c_p)M^2$$

or

$$T_0 - T = [(\gamma - 1)/2]M^2T$$

so that

$$|c_p \text{ grad } T_0 - T_0 \text{ grad } S| = |c_p \text{ grad } T_0 - T \text{ grad } S - [(\gamma-1)/2]M^2T \text{ grad } S| = |\mathbf{q}\times\boldsymbol{\omega} - [(\gamma-1)/2]M^2T \text{ grad } S|$$

Therefore, we may write

$$\left.\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}}\right|_1^2 = -2\int_1^2 \frac{\cos\phi^{(0)}}{\rho^{(0)}q^{(0)^2}} \left|\mathbf{q}\times\boldsymbol{\omega} - \frac{\gamma-1}{2} M^{(0)^2}T^{(0)^2} \text{ grad } S^{(1)}\right| d\vartheta \quad (10)$$

Application to Compressible Boundary-Layer Flow

Consider a three-dimensional boundary layer with free flow streamlines which are almost straight over a considerable length and show an appreciable lateral curvature in a fairly small length interval. Then, in the region where the free flow streamlines are almost straight, the boundary-layer flow may be considered as two-dimensional. In the region where the turning of the flow takes place, however, there is a certain part of the boundary layer where inertial forces dominate the viscous forces. This part of the boundary layer is the outer part, where the gradient and the curvature of the velocity profile is small. Also, in about the same "layer," the conditions for applicability of the linearized theory hold. Close to the wall the results of the linearized theory for inviscid flow have no significance because of the large viscous forces, the no-slip condition for the cross flow, and the magnitude of the velocity gradients.

If the boundary layer is not too thick we may assume that $\partial p/\partial y = 0$ if y is the distance to the wall. Then

$$|\text{grad } S| = \left|\frac{\partial S}{\partial y}\right| = \frac{c_p}{T}\left|\frac{\partial T}{\partial y}\right|$$

so that the correction term in Eq. (10) can be written

$$\frac{\gamma-1}{2} M^{(0)^2}T^{(0)}|\text{grad } S^{(1)}| = \frac{\gamma-1}{2} M^{(0)^2}c_p\left|\frac{\partial T}{\partial y}\right|$$

Hence we may write

$$\left|\mathbf{q}\times\boldsymbol{\omega} - \frac{\gamma-1}{2} M^{(0)^2}T^{(0)} \text{ grad } S^{(1)}\right| = q\omega_n + \frac{\gamma-1}{2} M^{(0)^2}c_p\left|\frac{\partial T}{\partial y}\right|$$

where ω_n is the component of the vorticity normal to $\mathbf{q}$.

If the Bernoulli surfaces are at a constant distance from each other (for instance, parallel planes or coaxial cylinders), then ω_n and $\partial T/\partial y$ are constant in the inviscid flow, so that the factor

$$\left(q\omega_n + \frac{\gamma-1}{2} M^{(0)^2}c_p\left|\frac{\partial T}{\partial y}\right|\right)$$

can be put in front of the integral in Eq. (10). The angle ϕ between the centripetal acceleration and the vector $\mathbf{q} \times (c_p \text{ grad } T_0 - T_0 \text{ grad } S)$ is 0 or π, depending on the orientation of the centripetal acceleration. More precisely, $\cos\phi = 1$, if the curvature vector, $\mathbf{q}$, and the vector normal to the wall pointing towards the fluid form a right-hand system, and $\cos\phi = -1$ if they form a left-hand system. The sign of $\cos\phi$ can also be taken into account by giving $d\vartheta$ a sign. Let $d\vartheta$ be positive if the streamline curves away from the vorticity vector and negative if it curves toward the vorticity vector. Then, Eq. (10) may be expressed as

$$\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}}\Bigg|_1^2 = -2\left(q\omega_n + \frac{\gamma-1}{2}M^{(0)^2}c_p\left|\frac{\partial T_1}{\partial y}\right|\right)\times \int_1^2 \frac{d\vartheta}{\rho^{(0)}q^{(0)^2}} \quad (11)$$

where the index 1 refers to the position just upstream of the cascade.

For a laminar or turbulent boundary layer with Prandtl Number equal to unity along an insulated wall, the energy in the boundary layer is constant.

Thus, $$c_p\left|\frac{\partial T_1}{\partial y}\right| = \left|-q_1\frac{\partial q_1}{\partial y}\right| = q_1\,\omega_{n1}$$

and Eq. (11) becomes

$$\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}}\Bigg|_1^2 = -2q_1^{(0)}\,\omega_{n1}^{(0)}\left(1+\frac{\gamma-1}{2}M^{(0)^2}\right)\times \int_1^2 \frac{d\vartheta}{\rho^{(0)}q^{(0)^2}} \quad (12)$$

If the mass flow density $\Theta = \rho q/\rho^* q^*$ is introduced, where the star refers to the critical condition, we have

$$\int_1^2 \frac{d\vartheta}{\rho q^2} = \frac{1}{\rho_1 q_1^2}\int_1^2 \frac{d\vartheta}{(\Theta/\Theta_1)\,(q/q_1)}$$

The ratios's q/q_1 and Θ/Θ_1 can be related to the Mach Number

$$\frac{q}{q_1} = \frac{M}{M_1}\left\{\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M^2}\right\}^{1/2},$$

$$\frac{\Theta}{\Theta_1} = \frac{M}{M_1}\left\{\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M^2}\right\}^{\frac{\gamma+1}{2(\gamma-1)}}$$

From the last equation it follows that

$$\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M^2} = \left(\frac{\Theta}{\Theta_1}\frac{M}{M_1}\right)^{\frac{2(\gamma-1)}{\gamma+1}}$$

so that the equation for q/q_1 becomes

$$q/q_1 = (M/M_1)\ [(\Theta/\Theta_1)\ (M/M_1)]^{\frac{\gamma-1}{\gamma+1}}$$

Thus,

$$\int_1^2 \frac{d\vartheta}{\rho q^2} = \frac{1}{\rho_1 q_1^2}\times \int_1^2 \frac{d\vartheta}{(\Theta/\Theta_1)\ (M/M_1)\ [(\Theta/\Theta_1)\ (M/M_1)]^{\frac{\gamma-1}{\gamma+1}}} = \frac{1}{\rho_1 q_1^2}\int_1^2 \frac{d\vartheta}{[(\Theta/\Theta_1)^\gamma\ (M/M_1)]^{\frac{2}{\gamma+1}}} \quad (13)$$

If $M_1 \ll 1$, we find as an approximation

$$\frac{M}{M_1} \simeq \frac{\Theta}{\Theta_1}\left\{1+\frac{\gamma+1}{4}M_1^2\left[\left(\frac{\Theta}{\Theta_1}\right)^2-1\right]\right\}$$

which, inserted in the integral (13) gives

$$\int_1^2 \frac{d\vartheta}{\rho q^2} = \frac{1}{\rho_1 q_1^2}\left\{\left(1+\frac{1}{2}M_1^2\right)\int_1^2 \frac{d\vartheta}{(\Theta/\Theta_1)^2} - \frac{1}{2}M_1^2(\vartheta_2-\vartheta_1)\right\} \quad (14)$$

If A is the normal cross-section area of the passage, the continuity equation may be written

$$\overline{\Theta A} = \overline{\Theta_1 A_1}$$

where $\overline{\Theta}$ denotes the average mass flow density through the corresponding cross section of the passage. If furthermore, the relative variation $[(\Theta - \overline{\Theta})/\overline{\Theta}]$ is small, we may replace Θ by $\overline{\Theta}$ in Eq. (14) so that

$$\int_1^2 \frac{d\vartheta}{\rho q^2} = \frac{1}{\rho_1 q_1^2}\left\{\left(1+\frac{1}{2}M_1^2\right)\int_1^2\left(\frac{A}{A_1}\right)^2 d\vartheta - \frac{1}{2}M_1^2(\vartheta_2-\vartheta_1)\right\}$$

and the expression (12) for the secondary vorticity becomes

$$\frac{\omega_\sigma^{(1)}}{\rho^{(0)}q^{(0)}}\Bigg|_1^2 = -2\frac{\omega_{n1}^{(1)}}{\rho_1^{(0)}q_1^{(0)}}\left(1+\frac{\gamma-1}{2}M_1^{(0)^2}\right)\times \left\{\left(1+\frac{1}{2}M_1^{(0)^2}\right)\int_1^2\left(\frac{A}{A_1}\right)^2 d\vartheta - \frac{1}{2}M_1^{(0)^2}(\vartheta_2-\vartheta_1)\right\} \quad (15)$$

It is clear that in a diverging and curved passage, the form between the brackets in Eq. (15) is greater than $\int_1^2 \left(\frac{A}{A_1}\right)^2 d\vartheta$ while the reverse is true for a converging channel. This means that in a compressor cascade the compressibility effects on the secondary flow are more severe than in a turbine cascade.

In order to evaluate the influence of compressibility on the generation of secondary vorticity, we will assume that $\omega_{\sigma_1} = 0$ and write Eq. (15) in the form

$$\omega_{\sigma_2} = -2\frac{A_1}{A_2}\omega_{n_1}\left[\int_1^2\left(\frac{A}{A_1}\right)^2 d\vartheta\right] f_1 f_2 \quad (16)$$

where

$$f_1 = 1 + [(\gamma-1)/2]M_1^2 \quad (17)$$

and

$$f_2 = 1 + \frac{1}{2}M_1^2\left[1 - \frac{\vartheta_2-\vartheta_1}{\int_1^2\left(\frac{A}{A_1}\right)^2 d\vartheta}\right] \quad (18)$$

The factor f_1 is independent of the cascade geometry and is plotted in Fig. 1. The factor f_2 depends not only on the entrance Mach Number, but also upon the cascade geometry. As an example, let us consider a cascade with a passage cross section which varies linearly with the turning angle, such that $A_2/A_1 = 1.4$. For this diverging passage, we have

$$\frac{A}{A_1} = 1 + 0.4\frac{\vartheta-\vartheta_1}{\vartheta_2-\vartheta_1}$$

Then the factor f_2 becomes

$$f_2 = 1 + (1/2)\ (1 - 0.69)M_1^2 = 1 + 0.16M_1^2$$

The factor f_2, together with the product $f_1 f_2$ is plotted in Fig. 1. The value of $f_1 f_2$ is 1.24 for $M_1 = 0.8$ so that for this entrance Mach Number the compressibility correction for the secondary vorticity is 24 per cent.

The Secondary Flow far Downstream of the Cascade

The next question to consider concerns the effect of compressibility on the integration problem of the secondary flow—i.e., the problem of determination of the secondary flow associated with a given secondary vorticity field. Because of the mathematical complications involved in the calculation of a three-dimensional secondary flow, the analysis is restricted to the flow far downstream of the cascade. There, we have a secondary vorticity ω_σ, which, in the linearized theory, is independent of the streamwise coordinate x and varies in a known way with the perpendicular coordinates y, z.

If the components of the secondary flow in x, y, z directions are u, v, w respectively, and u/U, v/U, w/U are small compared with unity, where U denotes the main downstream flow velocity, then it is clear that only u gives a first-order contribution to the pressure field, while v and w only contribute to the second-order pressure perturbation. The flow far downstream of the cascade has "settled down," so that, in this region, the first-order pressure field is constant. Hence, the first-order continuity equation reduces to

$$(\partial v/\partial y) + (\partial w/\partial z) = 0$$

for positions far downstream of the cascade. Then, for given secondary vorticity ω_σ, the velocity components v, w must fulfill the equation

$$(\partial w/\partial y) - (\partial v/\partial z) = \omega_\sigma$$

Clearly, the problem of determination of v and w is the same as for incompressible flow. It should be emphasized, however, that this is only true for the linearized analysis and far downstream of the cascade.

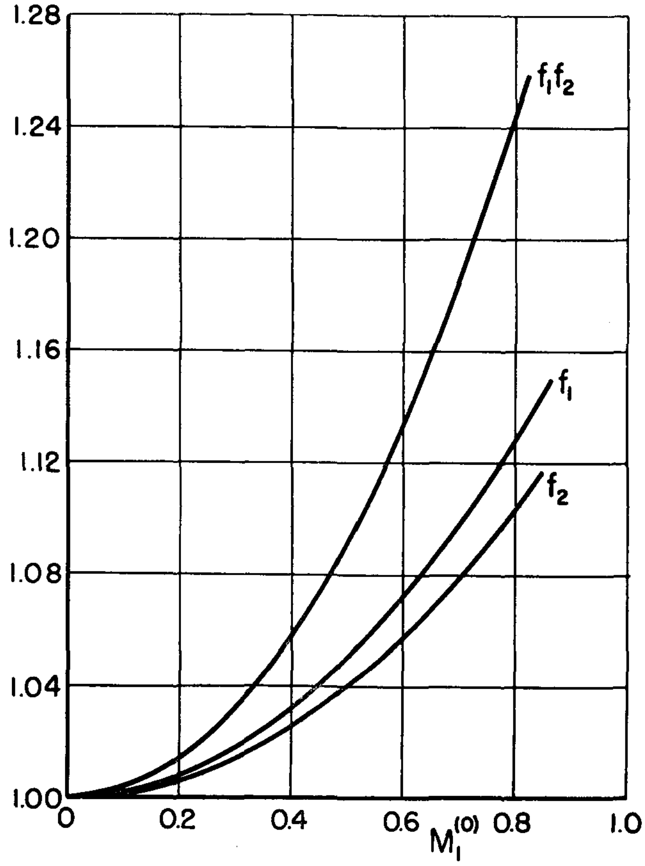

Fig. 1. Compressibility correction factors for secondary vorticity.

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