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MODELS IN AEROTHERMOCHEMISTRY

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I. - INTRODUCTION.

As the result of a tremendous increase in activity, our understanding of many simple combustion processes has advanced considerably in recent years ⁽¹⁻⁴⁾. Nevertheless, successfully operating engines cannot yet be designed on the basis of fundamental data and require rather extensive development work ⁽⁵⁾. An outstanding example of the practical problems associated with engine scaling is provided by the occurrence of uncontrolled combustion oscillations in liquid-fuel rocket engines, a problem which has been studied experimentally and which has also received considerable attention from some of the ablest theoreticians concerned with jet propulsion developments ⁽⁶⁾.

In view of the practical importance of engine scaling, it is of obvious interest to examine the possibility of utilizing similarity studies as a guide to the scaling problem. Related studies are concerned with corresponding investigations on chemical reactors and on simplified combustion processes. The pioneering investigation on similarity analysis and scaling of chemical

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(1) B. LEWIS and G. VON ELBE, *Combustion, Flames and Explosions of Gases*, Academic Press, New York 1951.

(2) W. JOST, *Explosion and Combustion Processes in Gases*, Croft Translation, McGraw-Hill Book Co., New York 1946.

(3) *Selected Combustion Problems—Fundamentals and Aeronautical Applications*, Butterworths Scientific Publications, London 1954.

(4) *Combustion Researches and Reviews*, 1955, Butterworths Scientific Publications, London 1955.

(5) For recent surveys of engine combustion problems, see the group of papers on engine combustion problems in *Fifth Symposium (International) on Combustion*, Reinhold Publishing Corp., New York 1955.

(6) For an exhaustive summary of this work, see L. CROCCO and S. I. CHENG, *Agarograph on Combustion Instability*, Butterworths Scientific Publications. London 1955.

reactors (of which engines are a special class) is due to G. Damköhler (7). For a recent discussion of similarities in combustion, which includes a treatment on the methods of dimensional analysis and similarities, as well as an extensive literature survey, a report issued by the Battelle Memorial Institute should be consulted (8).

II. - SIMILARITY PARAMETERS.

Scaling criteria for homogeneous and heterogeneous chemical reactors may be obtained from the conservation equations (9) for reacting gas mixtures. The complete set of similarity parameters has been shown to include the following dimensionless groups (10):

- (1) Reynolds number = $Re = \rho_o v_o L / \eta_o$,
- (2) Schmidt number = $Sc = \eta_o / \rho_o D_o$,
- (3) Prandtl number = $Pr = c_{p, o} \eta_o / \lambda_o$,
- (4) Mach number = $M = \sqrt{\rho_o v_o^2 / \gamma_o p_o}$,
- (5) Froude number = $Fr = v_o^2 / gL$,
- (6) Damköhler's first similarity group = $D_I = LU_{i, o} / v_o$,
- (7) Damköhler's third similarity group = $D_{III} = q' U_{i, o} L / v_o c_{p, o} \vartheta_o$,
- (8) $\varphi = \frac{1}{2} v_o^2 / (c_{p, o} \gamma_o) \vartheta_o$,
- (9) $\gamma_o = c_{p, o} / c_{v, o}$

Here ρ = density, v = linear flow velocity, L = length, η = viscosity coefficient, D = diffusion coefficient, c_p = specific heat at constant pressure, λ = thermal conductivity, γ = ratio of the specific heat at constant pressure (c_p) to the specific heat at constant volume (c_v), g = constant gravitational acceleration, U_i = characteristic reaction frequency of an important chemical species, q' = standard specific enthalpy difference between reaction pro-

(7) G. DAMKÖHLER, "Zeitschrift für Elektrochemie", 42, 846 (1936).

(8) A. E. WELLER, R. E. THOMAS, and B. A. LANDRY, *Similarities in Combustion*, Battelle Memorial Institute, Technical Report No. 15038-1, 505 King Ave., Columbus 1, Ohio, June 1954.

(9) TH. VON KÁRMÁN, *Sorbonne Lectures 1952-53*, Paris, France; S. S. PENNER, *Introduction to the Study of Chemical Reactions in Flow Systems*, Chapter II, Butterworths Scientific Publications, London 1955.

(10) S. S. PENNER, *Similarity Analysis and the Scaling of Liquid-Fuel Rocket Engines*, Chapter 12 in reference 4.

ducts and reactants, \mathfrak{D} = temperature of the gas mixture; the subscript o identifies suitably selected upstream reference conditions.

The characteristic reaction frequency U_i is defined by the relation

$$(10) \quad U_i = \omega_i / \rho Y_i$$

where ω_i represents the net rate of mass change per unit volume, produced by chemical reaction, of the important chemical species i and Y_i denotes the corresponding weight or mass fraction. The parameter D_I measures the ratio of the rate of change of i , produced by chemical reaction, to the rate of change of i produced by convection. Similarly, D_{III} expresses the ratio of the rate of heat addition resulting from chemical reaction to the rate of heat addition associated with enthalpy flux by convection.

For low-velocity flow problems involving fixed chemical reactants in the absence of external forces, results equivalent to Damköhler's five similarity criteria are obtained (viz., only (1), (2), (3), (6), and (7) must remain invariant in order to maintain thermal, dynamic, and reaction-kinetic similarity). Maintenance of complete chemical and reaction-kinetic similarity in the gas stream automatically satisfies the boundary conditions for heat transfer to the chamber walls (i.e., the Nusselt heat transfer number is constant for fixed values of the Reynolds and Prandtl numbers) except in the case where surface-catalyzed chemical reactions play an important role. For surface-catalyzed reactions, the boundary conditions remain similar if two new groups, analogous to those described by Eqs. (6) and (7), are introduced for the important chemical species reacting at the wall.

III. — SCALING OF CHEMICAL REACTORS (7, 10, 11, 11 a).

Scaling of chemical reactors with complete similarity is impossible. For scaling with loss of geometric similarity, as well as for scaling with loss of geometric and dynamic similarity, a distinction must be drawn between similarity criteria for two extreme types of reactors, viz., (a) flow systems without heat transfer to the chamber walls and without wall-catalyzed chemical reactions; (b) flow systems with conductive heat transfer occurring only to the chamber walls and, for heterogeneous reactors, diffusive transport limited to lengths corresponding to the pore size of a catalyst bed. For case (a) obvious results are obtained for both homogeneous and heterogeneous reactors with loss of geometric similarity. Case (b) has been treated before by Damköhler, Bosworth⁽¹¹⁾, and others, and leads to scaling criteria which can be met in practice but whose utility has not yet been demonstrated convincingly. Since the problem of scaling for chemical reactors is adequately described elsewhere^(7, 8, 10, 11), we shall not present a detailed discussion here.

(11) R. C. L. BOSWORTH, *Heat Transfer Phenomena*, Chapter IX, John Wiley and Sons, New York 1952.

(11 a) F. SCHULTZ-GRUNOW, «Forschung auf dem Gebiete des Ingenieurwesens», 9, 41 (1938).

IV. — SCALING OF MONOPROPELLANT LIQUID-FUEL ROCKET ENGINES.

In order to illustrate the analysis of scaling procedures for a relatively simple engine, we present a brief discussion of scaling for a monopropellant liquid-fuel rocket engine

A) *Formulation of Scaling Procedure* ⁽¹⁰⁾.

Consider a monopropellant liquid-fuel engine utilizing an injector with orifice diameter h . The injection orifices are spaced at a distance d' . The total number of injection orifices on a plate of diameter d is proportional to d^2/d'^2 where we assume that $h \ll d'$. If the flow velocity v_o is chosen to be the flow velocity in the orifices (the injection velocity), then the total mass flow rate in the large-scale motor H will be n times the flow rate in the model M if

$$(11) \quad (v_o h^2 d^2 / d'^2)_H = n (v_o h^2 d^2 / d'^2)_M.$$

The quantity n is thus the ratio of the thrust of the large motor to the thrust of the small motor. For a given monopropellant with fixed physico-chemical parameters and for fixed chamber pressure, constancy of Pr and Sc is assured. Furthermore, q' is fixed and, for the same injection conditions, Φ_o is also fixed; therefore, constancy of D_{III} is implied by constancy of D_I . In order to have similarity, only constancy of Re and D_I needs to be enforced. Hence

$$(12) \quad (v_o d')_H = (v_o d')_M$$

and

$$(13) \quad (lU_{i,o}/v_o)_H = (lU_{i,o}/v_o)_M$$

where l , the chamber length, is used as the typical length. Geometric similarity is maintained if we impose the condition

$$(14) \quad d_H/d_M = l_H/l_M = d'_H/d'_M = n'.$$

Equations (12) and (14) assure constant Re based on all important chamber dimensions. From Eqs. (12) and (14) it follows that

$$(15) \quad v_{o,H} = v_{o,M}/n'$$

and, from Eqs. (11), (14), and (15), that

$$(16) \quad h_H^2 = nn' h_M^2.$$

Finally, in order to satisfy Eq. (13), we have by Eqs. (14) and (15) the relation

$$(17) \quad (U_{i,o})_H = \frac{1}{n'^2} (U_{i,o})_M.$$

It has been emphasized previously⁽¹⁰⁾ that a prescription for rational motor scaling requires experimental determination of $U_{i,o}$, as a function of injector design and of the physico-chemical parameters, before Eqs. (14) to (17) can be used for the determination of the relation between n and n' . In the absence of quantitative information concerning the functional form of $U_{i,o}$, it is nevertheless of interest to illustrate the method of analysis by a simple example based on a plausible but oversimplified picture of the combustion processes.

B) *Uniform Combustion Throughout the Chamber Volume*⁽¹²⁾.

Observations of the interior of rocket combustion chambers during operation show that the hot gases are under strong agitation and turbulence with eddy sizes of the order of the chamber dimensions. Hence it is reasonable to approximate the combustion chamber as a homogeneous reactor. The mass rate of conversion of propellant for steady burning, per unit volume, is then equal to mass of propellant injected per unit time, divided by the volume of the combustion chamber, i.e.,

$$(18) \quad \omega_i \propto \rho_o \frac{v_o h^2 d^2}{d'^2} \frac{1}{ld^2} = \frac{\rho_o v_o h^2}{ld'^2}.$$

Since combustion occurs uniformly throughout the chamber volume, it follows that

$$(19) \quad U_{i,o} \propto \frac{v_o h^2}{ld'^2 Y_{i,o}}.$$

For a given propellant system the weight fraction $Y_{i,o}$, is fixed. Therefore, in view of Eqs. (13) to (19),

$$(20) \quad n' = n.$$

To summarize, for uniformly distributed combustion reactions, rational scaling of the engine involves increasing of all linear dimensions proportionately with the increase in mass flow rate; furthermore, the linear flow velocity in the injector orifices must be reduced by the same factor.

Reference to Eqs. (17) and (20) shows that the success of the entire scaling procedure rests on our ability of reducing the characteristic reaction frequency proportionately with the square of the engine thrust⁽¹³⁾. In this connection it is of interest to note that a microscopic analysis, which is based on the assumption that the mass burning rate of monopropellant droplets

(12) The suggestion to treat the combustion chamber as a homogeneous reactor is due to H. S. Tsien, to whom the author is indebted for helpful criticism and advice concerning the contents of this section as well as of Section V-D.

(13) The relation $(U_{i,o})_H = (1/n^2) (U_{i,o})_M$ is reminiscent of the results obtained for homogeneous reactors (compare references (7) and (10)).

is a linear function of the effective droplet diameter⁽¹⁴⁾, can be shown to be consistent with the result expressed in Eq. (19). Furthermore, if the burning rate is proportional to drop diameter, then the drop diameter itself must be increased proportionately with engine thrust, a condition which is difficult or impossible to meet in practice. The preceding statements are proved by the analysis presented in the Appendix.

C) *Concluding Remarks.*

The analysis of combustion similarity, based on the assumption of uniform reaction throughout the chamber volume, has led to unconventional scaling laws. Thus the linear chamber dimensions should be increased by a factor equal to the thrust ratio and, at the same time, the characteristic reaction frequency must be reduced by increasing the size of the propellant droplets, the chamber diameter, and the orifice diameter by the same factor. Large rocket engines should have a smaller rate of energy liberation per unit volume than small rocket engines. For this reason, only the scaling of very compact small rocket engines with intense and smooth chemical reaction will lead to a useful service model. As has been emphasized previously, the principal problem is the control of reaction rate. This, in turn, can be accomplished only if the combustion processes are understood phenomenologically. For example, if the mass burning rate is a linear function of drop diameter, then the drop diameter must be proportional to engine thrust. For an arbitrary engine, the first step is therefore seen to involve an experimental study of the important physicochemical factors influencing overall chemical reaction frequencies for reactants and reaction products.

V. — DIMENSIONAL ANALYSIS AND SIMILARITY STUDIES ON SELECTED COMBUSTION PROCESSES.

From dimensional analysis and similarity studies, useful results can be obtained for correlating combustion data on a number of selected problems, which are simpler to describe and better understood than the complex combustion processes in rocket engines. Examples are: correlations of laminar flame velocities, flash-back and blow-off data, flame stabilization on bluff bodies, burning of (single) fuel droplets in oxidizing atmospheres, and lengths of free and enclosed turbulent jet flames. It should be noted that the utility of dimensional analysis in these cases is to be attributed largely to the fact that the important physical variables have been determined through independent theoretical and or experimental studies.

Similarity analysis refers to transformations of a known equation, from which the important dimensionless groups for modeling are then obtained.

(14) This relation has been shown to hold for the burning of single drops of fuels in oxidizing atmospheres. Its applicability to arrays of monopropellant droplets is unproved.

In dimensional analysis, on the other hand, functional dimensionless groupings are obtained for an equation which is presumed to exist and whose form is approximated by comparison with experimental measurements. According to the basic Buckingham Pi theorem, the number of dimensionless groups is usually equal to the number of variables minus the number of dimensions.

A) *Laminar Flame Propagation.*

(1) Dependence of laminar burning velocity on mixture composition ⁽¹⁵⁾.

For premixed gaseous systems in which the chemical reaction processes are well understood, it is a simple matter to formulate and solve the laminar flame equations and to obtain not only valid functional relations for the laminar burning velocity but also reasonably reliable absolute estimates ⁽¹⁶⁾. For complex flames (e.g., hydrocarbon-air flames), in which the combustion processes are not understood in detail, there are two possible lines of approach. One involves formulation and solution of the flame equations with a hypothetical rate-controlling reaction step, whose activation energy may then be estimated by comparison with experimental data ⁽¹⁷⁾. The other approach, through the use of dimensional analysis, has as its objective the determination of functional relations between dimensionless groups in order to correlate the experimental data ⁽¹⁵⁾. We shall describe here the methods used in dimensional analysis and attempt an evaluation of the utility and significance of the resulting expressions.

Schultz-Grunow describes a two-component premixed gaseous system by 9 variables, viz., the laminar burning velocity u_0 , the initial molar concentrations ($c_{1,0}$, and $c_{2,0}$) of the two reactants, the thermal conductivity λ , density ρ , diffusion coefficient D , heat liberated per mole of mixture by chemical reaction q , and specific reaction rate constant $k = (1/c_1 c_2) dc_1/dt$. For most chemical reactions the overall stoichiometric changes do not correspond with the assumed bimolecular rate-controlling reaction step. In these cases the stoichiometric ratio (i.e., the number of moles of component 1 which have reacted with species 2 when heat liberation is complete) occurs both as a variable and as a dimensionless group. We omit this group since Schultz-Grunow was able to obtain correlation of experimental results without its use.

The variables have five dimensions (e.g., °K for temperature T , sec for time t , cm for length L , cal for heat, and g for mass) whence it follows that

(15) F. SHULTZ-GRUNOW, "Zeitschrift für Physikalische Chemie", 201, 212 (1952); *ibidem*, *Fourth Symposium (International) on Combustion*, pp. 439-443, Williams and Wilkins Co., Baltimore 1953; *ibidem*, "Chemie Ingenieur Technik", 26, 18 (1954).

(16) See, for example, TH. VON KÁRMÁN and S. S. PENNER, reference 3, pp. 5 to 41.

(17) See, for example, S. S. PENNER and T. H. CROWE, *Correlation of Laminar Flame Velocities for Hydrocarbon-Oxygen-Inert Gas Mixtures*, "Technical Report", No. 12, Contract DA 04-495-Ord-446, California Institute of Technology, Pasadena, May 1955.

there should be a relation between four dimensionless groups involving the laminar burning velocity. Schultz-Grunow chooses ⁽¹⁵⁾ the groups

$$(21) \quad \omega = u_0 / \sqrt{\lambda k c_{1,0} / c_p \rho},$$

$$(22) \quad \Phi_1 = q c_{1,0} / T c_p \rho,$$

$$(23) \quad \Phi_2 = D c_p \rho / \lambda,$$

and

$$(24) \quad \Phi_3 = c_{1,0} / c_{2,0}.$$

For gas mixtures containing inert diluents, which Schultz-Grunow actually used to determine the exponents in his dimensionless correlations, Weller et al ⁽⁸⁾ note that there is an additional variable (namely, the concentration of inert material) and, hence, there should be an additional dimensionless group (e.g., a group such as the mole fraction of inert constituent).

It is evidently possible to vary some of the individual parameters without affecting the laminar burning velocity, as long as the dimensionless groups remain invariant. For fixed values of the dimensionless groups, thermal, dynamic, and reaction-kinetic similarity will presumably be maintained.

Schultz-Grunow ⁽¹⁵⁾ found that relations of the form

$$(25) \quad \omega = \psi(\Phi_1, \Phi_2, \Phi_3) = \Phi_1^\alpha \Phi_2^\beta f(\Phi_3)$$

correlate the observed experimental data for H₂—O₂—inert gas and CH₄—O₂—inert gas flames, unfortunately, however, with different exponents for the dimensionless parameters. Here $f(\Phi_3)$ enters rather than Φ_3 itself because the initial gas composition determines the flammability limits (which, in turn, cannot be described by a simple proportionality).

Weller et al ⁽⁸⁾ have questioned the significance of the correlation on the grounds that it involves, for a given fuel-oxidizer system, practically a correlation between flame speed and fuel concentration. Actually, however, even for a given combustible mixture, we expect specific reaction rate constant to vary with gas composition, whence it follows that Schultz-Grunow's correlations are not entirely fortuitous. Since the exponents to which the dimensionless groups enter into the explicit relation for ω vary from one combustible mixture to another, an optimum choice of dimensionless variables has not been made. The principal difficulty lies probably in the fact that the specific reaction rate constant for the rate-controlling process in a hydrocarbon-air mixture does not correspond to a second-order reaction involving fuel and oxidizer each to the first power. This last conclusion is particularly apparent from an examination of attempts at correlating flame velocities through functional forms derived from a more complete theory on the basis of this same simplified second-order reaction step. Only lean hydrocarbon-air mixtures could be correlated successfully ⁽¹⁷⁾.

(2) Nondimensional correlation of flame propagation at subatmospheric pressure ⁽¹⁸⁾.

On the basis of a thermal theory of laminar flame propagation, in which energy transport to the unburnt gases occurs by thermal conduction alone, Cullen obtained the following relation for the ratio of laminar burning velocity u_o to the laminar burning velocity u_∞ without heat sink at the flame holder (burner with infinite diameter):

$$(26) \quad u_o/u_\infty = 1 - \exp [(-\rho c_p l/\lambda) u_c].$$

Here l is the distance of the heat sink from the position in the flame front corresponding to the ignition temperature (i.e., the temperature above which exothermic chemical reactions occur and below which the chemical reaction rates are assumed to be negligibly small). The other symbols have their usual meaning. The factor

$$(27) \quad \frac{\rho c_p l}{\lambda} u_o = \frac{c_p \eta}{\lambda} \frac{\rho u_o l}{\eta} = (\text{Pr})(\text{Re})$$

is a Peclet number in which the only pressure-dependent terms occur in Re. If it is now assumed that

$$(28) \quad l = 2\beta r_b,$$

where r_b is the diameter of the burner tube and β is a constant, then

$$(29) \quad u_o/u_\infty = 1 - \exp [-\beta (\text{Pr})(\text{R}'_e)] = 1 - \exp [-\beta (\text{P}'_e)]$$

with $(\text{P}'_e) = (\text{Pr})(\text{R}'_e)$ and

$$(30) \quad \text{R}'_e = 2\rho u_o r_b/\eta$$

representing a Reynolds number based on the observed flame speed and the tube diameter. It is evident from Eq. (29) that u_o approaches u_∞ as either ρ or r_b increase without limit. Cullen found that a plot of u_o/u_∞ vs. $\text{P}'_e = 2\rho u_o r_b c_p/\lambda$ gave a satisfactory correlation for all of his observed experimental data for propane-air and ethylene-air flames where u_∞ was chosen to be the limiting value of u_o for large values of P'_e .

Although we regard the statement

$$u_o/u_\infty = f(\text{P}'_e)$$

as significant and useful, the "derivation" of Eq. (29) is of doubtful validity. Thus, in a laminar flame with mass and energy transport by diffusion, u_∞ becomes infinite as the ignition temperature is decreased to the temperature of the unburnt gas. Furthermore, the assumption made in Eq. (28) that l is linear with burner diameter is unjustified, as is also the notion ⁽¹⁹⁾

(18) R. E. CULLEN, "Transactions of the ASME", 75, 48 (1953).

(19) A. A. PUTNAM, "Transactions of the ASME", 75, 48 (1953).

that l should be invariant with burner dimensions because it is essentially a quenching distance.

Putnam⁽¹⁹⁾ has pointed out that Cullen's correlation refers to practically constant jet velocity. For this reason, all of the experimental data could be accounted for without the use of another essential dimensionless parameter, namely, the ratio of jet velocity to flame speed.

B) *Flash-back and Blow-Off of Burner Flames.*

According to a proposal by Lewis and von Elbe^(20,21), flash-back of flames on burner tubes occurs when the flow velocity at some distance from the burner rim is less than the flame velocity. Similarly, blow-off occurs when the flow velocity is everywhere greater than the flame velocity. Under conditions of stable burning, the curve of stream velocity vs. distance from the burner rim is just tangent to the flame velocity curve. The flame velocity as a function of distance decreases more rapidly to zero than the stream velocity, as the burner rim is approached, because heat loss and removal of free radicals reduce the reaction rates to the point where stable burning can no longer be sustained at some distance from the burner rim (Fig. 1).

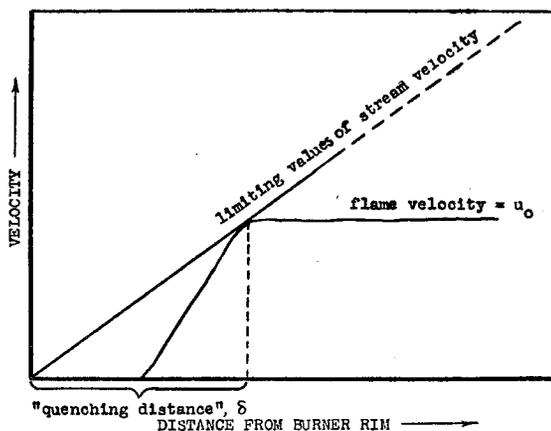


Fig. 1. Schematic diagram showing the stream velocity gradient and flame velocity profile near the burner rim at the critical condition for flash-back and blow-off (after Lewis and von Elbe).

The critical condition for both flash-back and blow-off of flames on burner tubes is then determined, approximately, by the requirement that the stream velocity gradient near the wall equals the ratio of laminar flame velocity u_0 to quenching distance δ . For laminar flow in a tube, the velocity profile is determined by Poiseuille's equation, viz.,

$$(31) \quad U = \frac{2}{\pi} \frac{(r_b^2 - r^2)}{r_b^4}.$$

Here U = stream velocity, r_b = tube radius, r = radial distance from tube axis, V = volume flow rate. The limiting gradient of the stream velocity

(20) B. LEWIS and G. VON ELBE, "Journal of Chemical Physics", 11, 75 (1943).

(21) A. B. PUTNAM and R. A. JENSEN, *Third Symposium (International) on Combustion*, pp. 89-98, The Williams and Wilkins Co., Baltimore 1949.

at the burner rim is, therefore,

$$(32) \quad \lim_{r \rightarrow r_b} \left(\frac{-dU}{dr} \right) = (4/\pi) (V/r_b^3).$$

Hence the critical condition for blow-off and flash-back becomes (see Fig. 1)

$$(33) \quad u_o/\delta = (4/\pi) (V/r_b^3).$$

Markstein and Polanyi⁽²²⁾ have proposed a relation between flame speed and distance from the burner rim which corresponds to the relation

$$(34) \quad u_o \delta = K (\lambda/c_p \rho).$$

Here K is, in first approximation, a constant for a given mixture composition. Equations (33) and (34) lead to the relation

$$(35) \quad (4/\pi) (V/r_b^3) = (u_o^2/K) (c_p \rho/\lambda).$$

The Peclet number based on the flame speed u_o is

$$(36) \quad P_{e,F} = \frac{2 r_b u_o c_p \rho}{\lambda}$$

whereas the Peclet number based on the average stream velocity U_s is

$$(37) \quad P_{e,S} = \frac{2 r_b U_s c_p \rho}{\lambda}.$$

Combination of Eqs. (35) to (37) now leads to the result

$$(38) \quad P_{e,S} = \frac{1}{8K} (P_{e,F})^2$$

or

$$(39) \quad P_{e,S} = 8K (U_s/u_o)^2$$

since $V = \pi r_b^2 U_s$. Equation (38) would be expected to correlate flash-back and blow-off data for $r_b \gg \delta$ (i.e., for sufficiently large values of $P_{e,F}$). It should be noted that a dimensional analysis, without introduction of the physical picture implied by Eq. (33) or Fig. 1, would not yield the explicit form of the correlation shown in Eqs. (38) and (39) but rather an expression such as

$$P_{e,S} = f(P_{e,F}).$$

Putnam and Jensen⁽²¹⁾ have shown that Eq. (38) yields a fair correlation of experimental flash-back data for ethylene-air, acetylene-oxygen, and natural gas-air mixtures, as well as for blow-off data of natural gas-air mixtures on some axial rod flame holders. A refined analysis, using an ex-

(22) G. H. MARKSTEIN and M. POLANYI, *Flame Propagation, A Critical Review of Existing Theories* "Bumblebee Series Report", 61, Cornell Aeronautical Laboratory, April 1947.

plicit relation of the flame speed near the wall, suggests that the functional relation shown in Eq. (38) should break down for sufficiently small values of $P_{e,F}$, a result which is in accord with observations^(8,21).

The preceding discussion of the determination of important similarity parameters for flash-back and blow-off data has been prejudiced by the a priori assumption that heat transfer and free-radical recombination on the burner wall are the determining factors. Actually the Peclet number equals the product of Reynolds number and Prandtl number. The latter is substantially the same for many gases. Hence, for practical purposes, a correlation between Peclet numbers is equivalent to a correlation between Reynolds numbers. Therefore, as far as correlation of the empirical data is concerned, it would hardly be possible to show that similarity of heat transfer (fixed Prandtl number) is really required for flows with similar Reynolds numbers. Finally, it may be noted that the correlation equation does not contain an effective Schmidt number for diffusion of chain carriers to the burner wall. It is therefore not surprising to note, as has been stressed by Wohl⁽²³⁾, that Eqs. (38) and (39) do not really correlate the variations of flash-back and blow-off velocities for a given fuel with fuel concentration, or of different fuels for fixed mixture ratios. Wohl also questions the validity of Eq. (34) and the constancy of β .

C) *Flame Stabilization on Bluff Bodies.*

As the result of recent experimental studies⁽²⁴⁾, the physical processes responsible for flame stabilization behind bluff bodies have become greatly clarified. The conditions for maximum blow-off velocity for a given fuel correspond to maximum wake temperature and, therefore, to stoichiometric mixture composition in the wake.

For low Reynolds numbers, molecular transport processes are of dominant importance and the parameters on which the wake composition depends are the fuel mole fraction (initial mixture composition), Schmidt number, Reynolds number, and Prandtl number. The Prandtl number would be expected to play a vital role since reignition of gases entering the mixing zone behind the bluff body must occur for a stabilized flame. However, for a given fuel-oxidizer system, the Prandtl and Schmidt numbers are effectively constant. Hence the maximum blow-off velocity should correspond to a unique representation of mixture ratio (M.R.) in the approach stream as a function of Reynolds number based on the dimensions of the bluff-body flame holder, i.e.,

$$(40) \quad (\text{M.R.})_{\text{maximum blow-off velocity}} = f(\text{Re}).$$

(23) K. WOHL, *Third Symposium (International) on Combustion*, p. 98, Williams and Wilkins Co., Baltimore 1949.

(24) E. E. ZUKOSKI and F. E. MARBLE, *The Role of Wake Transition in the Process of Flame Stabilization on Bluff Bodies* in reference 4, Chapter 14.

The detailed functional form of the resulting expression has been obtained experimentally by Zukoski and Marble for gasoline-air and methane-air mixtures⁽²⁴⁾.

For sufficiently high Reynolds numbers, turbulent transport processes determine the composition in the mixing zone; the chemical reaction rates for the stoichiometric mixture ratio, which were relatively fast in the case of laminar diffusion transport to the wake region, now become the slow and rate-controlling steps. Under these conditions the approach stream mixture ratio for maximum blow-off velocity becomes stoichiometric and remains invariant with increasing Reynolds number.

In another publication Zukoski and Marble discuss the important similarity group for scaling of single bluff-body flame holders in large ducts⁽²⁵⁾. The ratio of wake temperature T_w to approach stream temperature T_o is independent of the Reynolds number of the approach stream (Re) for fixed mixture ratio (M.R). The ratio of wake length (L) to flame holder diameter (L_F) depends only on Re and (weakly for turbulent flow) on T_w/T_o . For a given fuel, the experimentally important similarity group for correlating data on the blow-off velocity u_{bo} is a quantity corresponding to Damköhler's first similarity parameter, namely,

$$(41) \quad D_I = \frac{L}{u_{bo} \tau}$$

In Eq. (41), τ represents a chemical reaction time in the mixing region which is determined by fuel type, M.R., and upstream conditions such as T_o and pressure. The physical significance of constancy of the group D_I is the statement that the critical conditions corresponding to blow-off are determined by the ratio of a residence time in the mixing zone of the wake for the gases, L/u_{bo} , to the chemical reaction time τ . It is apparent that the use of Eq. (41) permits a straightforward scaling procedure for a single flame holder in a duct since a change of flame holder dimensions affects only L and not τ . In fact, for two different flame holders 1 and 2, with arbitrary fuel-oxidizer systems and upstream conditions,

$$(42) \quad (u_{bo})_2 = (u_{bo})_1 \frac{L_{F,1} g[(Re)_1, (T_w/T_o)_1]}{L_{F,2} g[(Re)_2, (T_w/T_o)_2]}$$

since

$$(43) \quad L/L_F = g[(Re), (T_w/T_o)].$$

The functional relation between L/L_F and Re has been determined experimentally for different ratios T_w/T_o ⁽²⁵⁾.

(25) E. E. ZUKOSKI and F. E. MARBLE, *Flame Stabilization on Bluff-Bodies - The Mechanism and Similarity Properties*, paper presented at the Gas Dynamics Symposium, Northwestern University, Evanston, Illinois, August 22-24, 1955, and to be published in *Jet Propulsion*. The author is indebted to Dr. Marble for a helpful discussion concerning data contained in this manuscript.

D) *Burning Rate of Stationary Liquid Droplets in Oxidizing Atmospheres.*

It is commonly believed that a good first approximation to the heterogeneous burning of fuel droplets in oxidizing media may be obtained on the assumption that the burning rate is controlled by mass transport of fuel and oxidizer in stoichiometric proportions to the flame front, and by heat transport from the flame front to the fuel droplet. Although a successful molecular description of this process is available for the burning of single droplets⁽²⁶⁻²⁸⁾, it may be of interest to examine the problem by the use of dimensional analysis in order to clarify the role of important variables in the burning of droplet arrays.

(1) The burning of single droplets.

The burning of a single stationary fuel droplet in an oxidizing atmosphere involves the following 23 variables: mass burning rate of fuel droplet \dot{m}^F ; stoichiometric ratio of mass of O_2 burnt for unit mass of fuel burnt, ν ; constant pressure p ; specific latent heat of evaporation of fuel droplet Δl ; standard specific enthalpy difference between reactants and reaction products q ; droplet diameter d_l ; specific heats at constant pressure $c_{p,l}$, $c_{p,p}$, $c_{p,f}$ and $c_{p,o}$; densities ρ_l , ρ_f , and ρ_A ; temperatures T_l , T_c , and $T_O = T_A$; thermal conductivities λ_f and λ_O ; viscosity coefficients η_f , η_O , and η_A ; diffusion coefficients D_f and D_O . Here the subscripts l , P , F , O , and A identify liquid fuel, combustion products, gaseous fuel, oxygen and air, respectively. The temperature T_c is the temperature at the reaction surface. In our physical model the system is characterized by average diffusion, viscosity, and thermal conduction coefficients on the fuel and air sides of the flame front. A suitable set of 18 dimensionless groups is the following:

$$\begin{aligned} & \eta_f/\rho_f D_f \quad ; \quad \lambda_f/\eta_f c_{p,f} \quad ; \quad \rho_l/\rho_f \quad ; \quad \rho_A/\rho_f \quad ; \quad c_{p,f}(T_c - T_l)/\Delta l \quad ; \quad D_O/D_f \quad ; \\ & \lambda_O/\lambda_f \quad ; \quad \eta_O/\eta_f \quad ; \quad \eta_A/\eta_f \quad ; \quad c_{p,o}/c_{p,f} \quad ; \quad c_{p,p}/c_{p,f} \quad ; \quad c_{p,l}/c_{p,f} \quad ; \quad \nu \quad ; \quad q/\Delta l \quad ; \\ & (T_c - T_O)/(T_c - T_l) \quad ; \quad p/\rho_A R_A T_O, \end{aligned}$$

where R_A is the gas constant per gram of air; the Grashof number⁽²⁹⁾ $Gr = d_l^3 \rho^2 g \beta' (T_c - T_O) \eta_A^{-2}$ where g is the acceleration of gravity and β' the

(26) G. A. E. GODSAVE, *Studies of the Combustion of Drops in a Fuel Spray - The Burning of Single Drops of Fuel in Fourth Symposium (International) on Combustion*, pp. 818-830, Williams and Wilkins Co., Baltimore 1953.

(27) D. B. SPALDING, *The Combustion of Liquid Fuels in Fourth Symposium (International) on Combustion*, pp. 847-864, Williams and Wilkins Co., Baltimore 1953.

(28) M. GOLDSMITH and S. S. PENNER, "Jet Propulsion", 24, 245 (1954).

(29) The Grashof number may well involve the diameter of the combustion surface, $2r_c$, rather than $d_l = 2r_l$. Since r_c and r_l are related, this change will not affect the following arguments.

coefficient of expansion of air; and K'/D_F where $K' = -d(d_l^2)/dt$ is the evaporation constant.

From the conservation of mass equations it is apparent that

$$\dot{m}_F = -\frac{d}{dt} \left(\frac{\pi}{6} d_l^3 \rho_l \right) = \frac{\pi}{4} \rho_l d_l K'$$

whence

$$(44) \quad K'/D_F = 4 \dot{m}_F / \pi \rho_l d_l D_F.$$

From the analytic solution⁽²⁷⁾ for \dot{m}_F (using constant average values for specific heats and thermal conductivities) it is easily shown that

$$(45) \quad \frac{K'}{D_F} = 8 (S_{c,F}) (P_{r,F})^{-1} \left(\frac{\rho_F}{\rho_l} \right) \frac{\ln [1 + c_{p,F} (T_c - T_l) / \Delta t]}{[1 - (r_l/r_c)]}$$

where r_l/r_c represents the ratio of the radius of the fuel droplet to the radius of the flame surface. The ratio r_l/r_c is a complicated function of several of the dimensionless groups listed previously.

It has been found that K' is constant during burning and is therefore independent of the instantaneous value of d_l . Reference to the 18 dimensionless groups shows that only Gr , which determines the magnitude of the convection currents, is a function of d_l . The constancy of K' during burning therefore suggests that the convection currents induced during burning have no important effect on K' . This observation is in agreement with the result that absolute values calculated for K' , without considering the influence of convection, are in fair agreement with experimental measurements⁽²⁷⁾. The functional relation given in Eq. (45) also involves the assumption that convection effects may be neglected in first approximation.

Reference to Eq. (45) shows that K' is a linear function of $\rho_F D_F$ and of other terms which are pressure-independent, except possibly r_l/r_c . It has been found experimentally⁽³⁰⁾ that K' varies as $p^{1/4}$. Hence, in terms of the simple picture used in the derivation of Eq. (45), it follows that $[1 - (r_l/r_c)]$ must vary as $p^{-1/4}$ in order to account for the observed variation of K' with pressure. In a more general analysis, which does not involve the assumption that chemical reactions are fast compared with the transport processes, a dependence of K' on pressure could be introduced through pressure-sensitive chemical reaction rates. There is an obvious difficulty in attempting to account for the pressure variation of K' through the influence of the Grashof number and free convection currents. Thus, since K' is constant during burning, it must be independent of droplet or combustion surface diameter. But Gr is a more sensitive function of this length than of pressure. Therefore, the pressure dependence of K' should not be attributed to the influence of Gr .

(30) A. R. HALL and J. DIEDERICHSEN, *An Experimental Study of the Burning of Single Drops of Fuel in Air at Pressures up to Twenty Atmospheres* in *Fourth Symposium (International) on Combustion*, pp. 837-46, Williams and Wilkins Co., Baltimore 1953.

(2) Two droplets burning in close proximity ⁽³¹⁾.

For two droplets of equal size, whose centers are separated by a distance c_0 , we have the additional variable c_0 and the new dimensionless group d_i/c_0 , where d_i is the initial diameter of the burning droplets. It has been found experimentally that K' for each of the two droplets is again independent of the instantaneous value of d_i . Hence, as for single droplets, it seems likely that convection effects are of secondary importance. However, K' appears to be a function not only of d_i/c_0 but of the initial drop diameter d_i^0 itself (with the exact dependence varying with fuel type). This result is difficult to understand since it suggests that the entire burning process is influenced by the initial configuration unless, of course, K' is not really constant during burning but varies so little that its change with time (i.e., the change of the second derivative in a plot of d_i^2 vs. time) cannot be detected with certainty. If K' is really constant during burning, then we are dealing again with a fundamental problem similar to that involved in the pressure dependence of K' for single droplets. Questions of this sort are not answered by dimensional studies alone.

(3) Five droplets burning in close proximity.

Some experimental data have been obtained also for the burning of an array of five n-heptane droplets in air. Four of the droplets were located at the corners of a square and the fifth in the center of the square. The square of the diameter for the droplet surrounded by four other droplets was again found to decrease linearly with time. The work has not yet progressed to the point where the variables determining the absolute value of K' are properly defined.

E) *Combustion Length of Enclosed Turbulent Jet Flames.*

Following important fundamental studies, ^(32,33) useful procedures have been developed for modeling turbulent jet flames ^(34,35). Since the procedures for modeling are well described in the literature, particularly in the paper by Thring and Newby ⁽³⁵⁾, no details concerning this work will be presented here.

(31) J. REX, "Aeronautical Engineer's Thesis", California Institute of Technology, Pasadena, May 1955.

(32) W. R. HAWTHORNE, D. S. WEDDELL, and H. C. HOTTEL, *Mixing and Combustion in Turbulent Gas Jets in Third Symposium (International) on Combustion*, pp. 266-288, Williams and Wilkins Co., Baltimore 1949.

(33) K. WOHL, C. GAZLEY, and N. KAPP, *Diffusion Flames in Third Symposium (International) on Combustion*, pp. 288-300, Williams and Wilkins Co., Baltimore 1949.

(34) S. YAGI and K. SAJI, *Problems of Turbulent Diffusion and Flame Jet in Fourth Symposium (International) on Combustion*, pp. 771-781, Williams and Wilkins Co., Baltimore 1953.

(35) M. W. THRING and M. P. NEWBY, *Combustion Length of Enclosed Turbulent Jet Flames in Fourth Symposium (International) on Combustion*, pp. 789-796, Williams and Wilkins Co., Baltimore 1953.

APPENDIX:

MICROSCOPIC CONSIDERATIONS RELATING TO THE SCALING OF MONO-
PROPELLANT ROCKET ENGINES

We wish to obtain a scaling law for the effective average droplet diameter \bar{D} on the assumption that the mass burning rate \dot{m}_F varies linearly with \bar{D} . Here \bar{D} is, in general, a function of the pressure drop across the injector orifices, of orifice diameter, and of the properties of the injected material. A calculation of $U_{i,o}$ from microscopic considerations is not feasible without introduction of a residence time t_r . We proceed by determining the value of t_r which will make the microscopic considerations consistent with the result given in Eq. (19). The characteristic reaction frequency is again given by the defining relation of Eq. (10) where

$$(A-1) \quad \omega_i \propto \left[\frac{\bar{\lambda} \bar{D}}{\bar{c}_p} f(T_c - T_l, \Delta l, \bar{c}_p) \right] \left[\frac{d^2}{d'^2} \frac{v_o h^2}{\bar{D}^3} t_r \right] \left[\frac{1}{l d^2} \right].$$

The first factor on the right-hand side of Eq. (A-1) is the mass rate of burning of a single drop of diameter \bar{D} , which is proportional to the first power of \bar{D} ; $f(T_c - T_l, \Delta l, \bar{c}_p)$ is a known function of the difference between the combustion temperature T_c and the droplet temperature T_l ; Δl equals the heat of evaporation of the propellant; the bar over λ and c_p indicates that suitable average values are to be used. The second bracket is proportional to the number of droplets injected per second, multiplied by the residence time t_r , i.e., it equals the total number of droplets in the chamber at any time. The third bracket equals the reciprocal of the chamber volume. Equations (19) and (A-1) are seen to be consistent only if

$$(A-2) \quad t_r \propto \left[\frac{\rho_o \bar{c}_p}{\bar{\lambda} f(T_c - T_l, \Delta l, \bar{c}_p)} \right] \bar{D}^2.$$

In other words, the macroscopic point of view is consistent with the microscopic point of view if the residence time t_r is proportional to \bar{D}^2 . But t_r will, in fact, be proportional to \bar{D}^2 if the assumed burning law applies in practice. Thus, if the mass rate of burning of a droplet of diameter \bar{D} is proportional to \bar{D} , the time required for complete combustion of the droplet, with mass proportional to \bar{D}^3 , is proportional to \bar{D}^2 . The time required for complete burning should be equal to the residence time t_r for a well designed combustion chamber.

The microscopic point of view permits a detailed statement about the required changes in drop diameter on scaling. Thus the residence time t_r is proportional to a length divided by a velocity, i.e., $t_r \propto l/v_o$. It therefore follows from Eqs. (14), (15), and (20), and from the relation

$$l/v_o \propto \bar{D}^2$$

that

$$(A-3) \quad \bar{D}_H / \bar{D}_M \alpha n' = n,$$

that is, the drop diameter, for the assumed burning law specified by Eq. (A-1), must be increased linearly with the motor thrust. For fixed values of the physico-chemical parameters, it is seen from Eq. (19) and from the relation $t_r \propto l/v_o \propto \bar{D}^2$ that

$$(A-4) \quad U_{i,o} \propto \frac{h^2}{\bar{D}^2 d'^2}.$$

In other words, the characteristic reaction frequency varies as the square of the ratio of orifice diameter to the square of the product of drop diameter and orifice spacing for a mass burning rate proportional to \bar{D} .