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Analysis of Peak-Holding Optimalizing Control

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SUMMARY

The peak-holding optimalizing control is analyzed under the assumption of first-order input linear group and output linear group. Design charts are constructed for determining the required input drive speed and the consequent hunting loss with specified time constants of the input and output linear groups, the hunting period, and the critical indicated difference for input drive reversal.

INTRODUCTION

OPTIMALIZING CONTROL WAS INVENTED BY C. S. Draper, Y. T. Li, and H. Laning, Jr.[1, 2] Their basic idea can be summarized as follows: In almost all engineering systems, within the restrictions of operation, there is an optimum state of the system for performance. For instance, in an internal combustion engine, within the restriction of producing the load torque at the specified speed, there are optimum settings for the manifold pressure and the ignition timing for minimum fuel consumption. Another example is an airplane under cruising condition; then under the restriction of engine cruising r.p.m. and assigned altitude, there is an optimum combination of trim setting and engine throttle for maximum fuel economy or maximum miles per gallon of fuel. But more important than the existence of an optimum operating state is the fact that the optimum operating state cannot be exactly predicted in advance because of the natural changes in the environment of the engineering system: In the case of the internal combustion engine, it is the changes in the temperature and the humidity of the air; in the case of the airplane, it is unavoidable changes in the aerodynamic properties of the airplane and the engine performance with age. Therefore if the purpose is to operate always near the optimum state in spite of the "drift" of the system, then the control device for the engineering system must be so designed as to search out automatically the optimum state of operation and to confine the operation close to this state. This is the basic idea of optimalizing control.

The application of Draper's optimalizing control to the general cruise control of airplanes was discussed by Shull.[3] Shull emphasized the possible elimination of extensive flight testing of new airplanes for performance determination, because the optimalizing control will automatically measure the performance whenever the airplane is flown. This in itself would constitute a great saving. But moreover, in critical circumstances such as flight through icing atmosphere, the ability of the optimalizing control to extract the best performance of a radically changed system (through ice deposition on the airplane) could be of utmost importance.

There are two fundamental problems in the theory of optimalizing control. One of the problems is the dynamic effects of the controlled system on the performance of the control. The other problem is the elimination of the noise interference. The two problems are somewhat interrelated, because if large deviations from the optimum state or the optimum operating point and hence large loss can be tolerated, then the noise interference will not be critical. The basic design aim of optimalizing control is to have the smallest loss or to operate as close to the optimum state as possible without the danger of having the control misled by the noise interference. Both of these problems were considered by the original inventors of optimalizing control. The noise problem is essentially the problem of detection of a sinusoidal variation under heavy random interference, a subject of much current research. The purpose of the present paper is to solve completely the first problem of dynamic effects under the assumption that the dynamic properties of the controlled system can be approximated by a first-order linear system. We shall begin with the brief review of the operating principles of an optimalizing control of the peak-holding type—a type least affected by the noise interference.[1, 2]

PRINCIPLE OF OPERATION

The heart of an optimalizing control system is the nonlinear component that characterizes the optimum operating condition of the controlled system. For simplicity of discussion, it is assumed that this basic component has a single input and a single output. For the time being the dynamic effects will be neglected and the output is assumed to be determined by the instantaneous value of the input. Since there is an optimum point, output as a function of input has a maximum at the output y_0 at the input x_0, as shown in Fig. 1. It is convenient to refer the output and the input to the optimum point and put the physical input as $x + x_0$ and the physical output as $y^* + y_0$. The optimum point is then the point $x = y^* = 0$. The purpose of an optimalizing control is then to search out this optimum point and to keep the system in the immediate neighborhood of this point. In this neighbor-

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hood, the relation between x and y^* can be represented as

$$y^* = -kx^2 \tag{1}$$

where k is a characteristic constant of the controlled system.

The operation of a peak-holding optimalizing control, neglecting the dynamic effects, then would be as follows: Say the input x is below the optimum value and is thus negative. The input drive is then set to increase the input at a constant rate. At the time instant 1 (Fig. 2) the input changes from negative to positive and passes through the optimum point. The output y^* is thus maximum at the time instant 1 and is decreasing after the instant 1. Now if an output sensing instrument is so designed as to follow the output exactly when the output is increasing, but hold to the maximum value after the maximum is passed and the output starts to decrease; then there will be a difference between the reading of this output sensing instrument and the output itself after the time instant 1. This difference is shown in the lower graph of Fig. 2. When this difference is built up to a critical value c at the time instant 2, the input drive is tripped and the direction of the input drive is reversed, but still at the same constant rate as before. After the instant 2 then, the input decreases and the output increases till a maximum in output is again reached at the time instant 3. At time instant 3, the input, of course, again passes from positive to negative, and the indicated difference between the output sensing instrument and the output itself again builds up. At the time instant 4, the difference reaches the critical value c again, and the input drive direction is again reversed. At the time instant 5, the input x becomes zero again and another maximum of the output is reached. The period of input variation is thus the time interval from the instant 1 to the instant 5, and the input, when plotted as a function of time, consists of a series of straight line segments forming a saw-tooth variation. The period of output variation is the time interval from the instant 1 to the instant 3, and the output, when plotted as a function of time, consists of a series of parabolic arcs. The periodic variations of input and output are called the hunting of the system, and the period of output variation is called the hunting period T. The period of input variation is thus $2T$.

The extreme variation of output Δ (Fig. 2) is called the hunting zone. If a is the amplitude of the saw-tooth variation of the input (Fig. 2), then due to Eq. (1),

$$\Delta = ka^2 \tag{2}$$

The difference between the maximum output and the average output of the hunting system is called the hunting loss D (Fig. 2). Because of the fact that the output is a series of parabolic arcs,

$$D = (1/3)\Delta = (1/3)ka^2 \tag{3}$$

For this idealized case, the critical indicated difference c between the output sensing instrument and the output itself is equal to Δ, the hunting zone. It is then clear from this discussion that in order to reduce the hunting loss for better efficiency of the system, one must try to reduce the hunting zone or the amplitude of input variation. Unfortunately the critical indicated difference is also reduced by such modification, and a limit is set by the noise interference on the proper tripping operation of the input drive.

The dynamic effects are so far neglected. But in any physical system, this is not possible because of the ever present inertial and damping forces. The output y^* given by Eq. (1) has to be considered then as the fictitious "potential output" but not the actual output y measured by the output indicating and sensing instrument. y^* is equal to y only when the period T of hunting becomes extremely long. The relation between y^* and y is determined by the dynamical effects. For the conventional engineering systems, these dynamical effects are determined by a linear relation. For instance, in the case of an internal combustion engine, the potential output is essentially the corrected effective pressure generated in the engine cylinders, while the actual output is the brake mean effective pressure of the engine. The dynamical effects are here mainly due to the inertia of the piston, the crankshaft, and other moving parts of the engine. For small changes in the operating conditions of the engine, such dynamical effects can be represented as a linear differential equation with constant coefficients.

Since the reference level of input and output is taken to be the optimum input x_0 and the optimum output y_0, the physical potential output is $y^* + y_0$ and the physical actual output is $y + y_0$. Thus the relation between the physical potential output and the physical actual output can be written as an operator equation

$$y + y_0 = F_0(d/dt)\,(y^* + y_0) \tag{4}$$

where F_0 is generally the quotient of two polynomials in the time differential operator d/dt. In the language of the Laplace transform then $F_0(s)$ is the transfer function. Let the linear system which transforms the potential output to actual output be called the output linear group. Then $F_0(s)$ is, specifically, the transfer function of the output linear group. By implication however, when the dynamical effects are negligible or when $s = 0$, the potential output is equal to the actual output. Therefore

$$F_0(0) = 1 \tag{5}$$

Since the optimum output y_0 only varies extremely slowly by the drift of the controlled system, during a time interval of many hunting periods y_0 can be taken as a constant. Then the condition of Eq. (5) simplifies Eq. (4) to

$$y = F_0(d/dt)y^* \tag{6}$$

In a similar manner, let x^* be the "potential input" that is actually the forcing function generated by the optimalizing control system but not the actual input

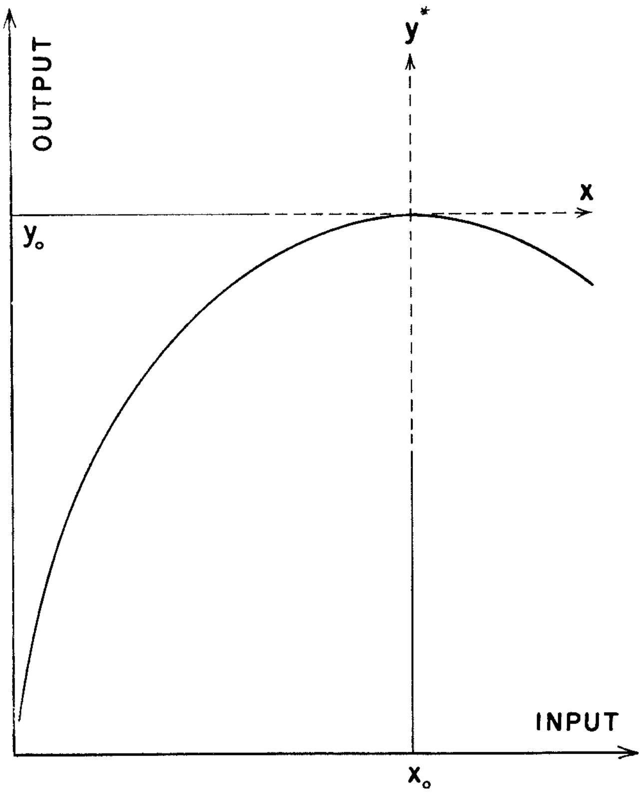

FIG. 1. Input-output characteristic of controlled system.

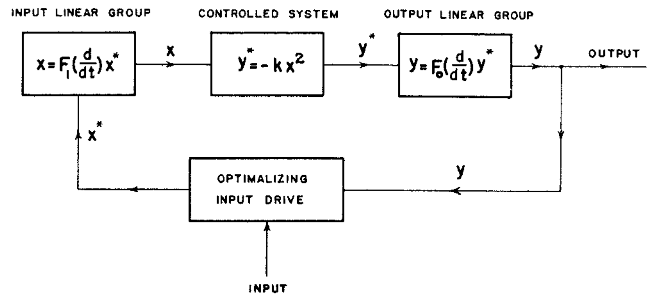

FIG. 3. Block diagram of a complete peak-holding optimalizing control system.

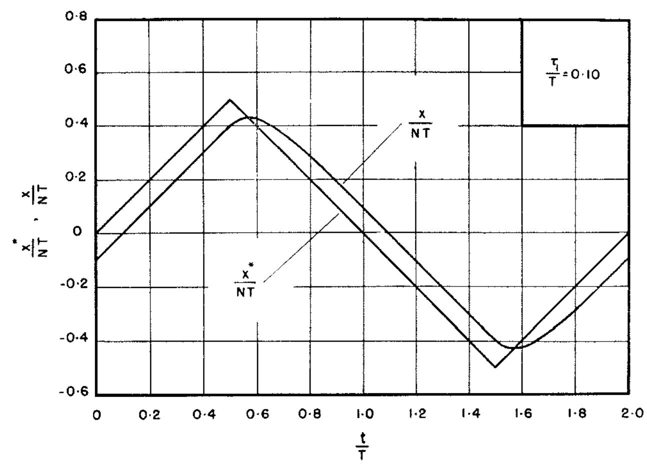

FIG. 4. "Potential" input and actual input for value of $\tau_i/T = 0.1$.

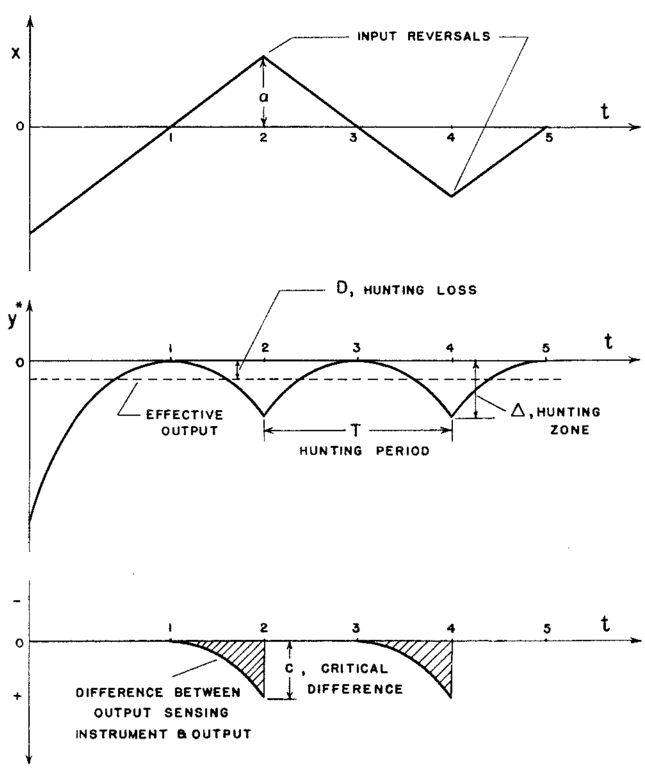

FIG. 2. Typical performance diagram for an ideal peak-holding optimalizing control system.

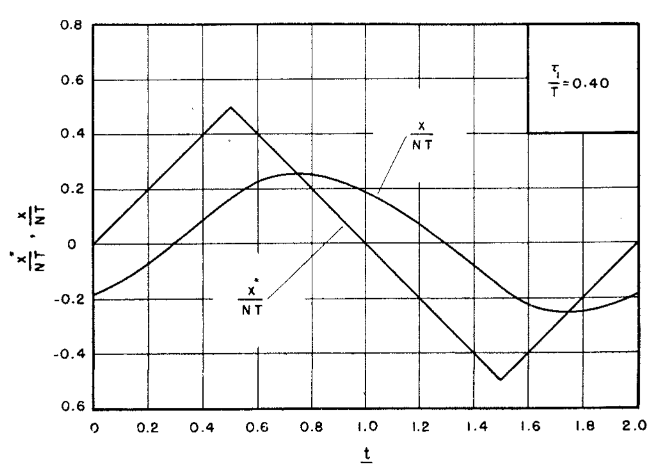

FIG. 5. "Potential" input and actual input for value of $\tau_i/T = 0.4$.

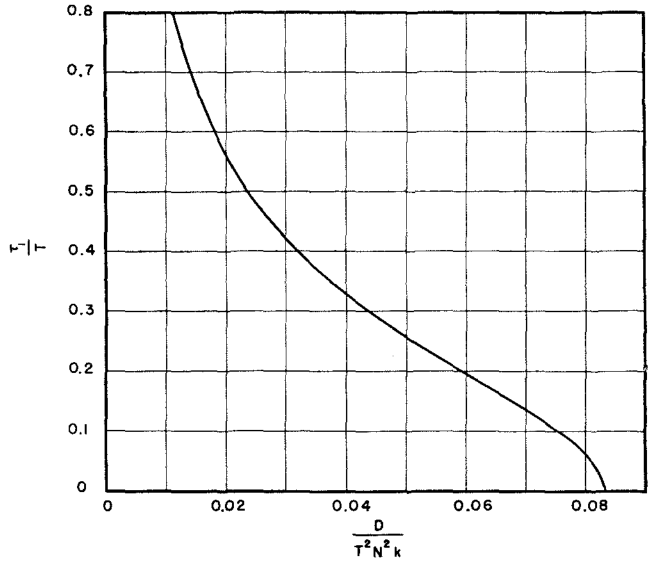

FIG. 6. Variation of dimensionless hunting loss, $D/(T^2N^2k)$ with τ_i/T.

x. It is x^* that has the saw-tooth form shown in Fig. 2, but not x. The relation between x^* and x is determined by the inertial and dynamical effects of the input drive system. This input drive system can be called the "input linear group" of the optimalizing control. The operator equation between the potential input x^* and the actual input x is

$$x = F_i(d/dt)x^* \tag{7}$$

$F_i(s)$ is thus the transfer function of the input linear group. Similar to Eq. (5), the meaning of potential and actual inputs implies

$$F_i(0) = 1 \tag{8}$$

Thus a simple representative block diagram of the complete optimalizing control system can be drawn as shown in Fig. 3. The nonlinear components of the system are thus the optimalizing input drive and the controlled system itself.

Formulation of the Mathematical Problem

The general relation between the input x and the output y is determined by the system of Eqs. (1), (6), and (7), with the potential input x^* specified as a saw-tooth curve with period $2T$ and amplitude a. Let ω_0 be the hunting frequency defined by

$$\omega_0 = 2\pi/T \tag{9}$$

then x^* can be expanded into a Fourier series,

$$\begin{aligned} x^* &= \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin(2n+1)\frac{\omega_0 t}{2} \\ &= \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{1}{2i}\left(e^{[(2n+1)/2]i\omega_0 t} - e^{-[(2n+1)/2]i\omega_0 t}\right) \end{aligned} \tag{10}$$

Therefore by using Eq. (7), the actual input x is given by

$$x = \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2(2i)} \times \left[F_i\left(\frac{2n+1}{2}i\omega_0\right)e^{[(2n+1)/2]i\omega_0 t} - F_i\left(-\frac{2n+1}{2}i\omega_0\right)e^{-[(2n+1)/2]i\omega_0 t}\right] \tag{11}$$

By using Eqs. (11) and (16), the actual output y is given by

$$\begin{aligned} y = \frac{16a^2k}{\pi^4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2} \times \Bigg\{ & F_0[(n+m+1)i\omega_0]F_i\left(\frac{2n+1}{2}i\omega_0\right) \times F_i\left(\frac{2m+1}{2}i\omega_0\right)e^{(n+m+1)i\omega_0 t} - F_0[(n-m)i\omega_0] \times \\ & F_i\left(\frac{2n+1}{2}i\omega_0\right)F_i\left(-\frac{2m+1}{2}i\omega_0\right)e^{(n-m)i\omega_0 t} - F_0[-(n-m)i\omega_0]F_i\left(-\frac{2n+1}{2}i\omega_0\right) \times \\ & F_i\left(\frac{2m+1}{2}i\omega_0\right)e^{-(n-m)i\omega_0 t} + F_0[-(n+m+1)i\omega_0] \times F_i\left(-\frac{2n+1}{2}i\omega_0\right)F_i\left(-\frac{2m+1}{2}i\omega_0\right) \times e^{-(n+m+1)i\omega_0 t}\Bigg\} \end{aligned} \tag{12}$$

By comparing Eqs. (11) and (12), it is seen that the input has half the frequency of the output. This is, of course, to be expected from the basic parabolic relation of input and output as specified by Eq. (1).

The average of the actual output y with respect to time t, being here referred to the optimum output y_0, gives directly the hunting loss D. Equation (12) shows that this average value is the sum of terms with $n = m$ from the second and the third terms of that equation. Therefore, using Eq. (5),

$$D = \frac{32a^2k}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}F_i\left(\frac{2n+1}{2}i\omega_0\right) \times F_i\left(-\frac{2n+1}{2}i\omega_0\right) \tag{13}$$

This equation can be easily checked by observing that when the dynamic effects are absent, $F_i \equiv 1$, then the series can be easily summed and $D = (1/3)a^2k$ as required by Eq. (3). Equation (13) also shows that the average output and hence the hunting loss are independent of the output linear group. This agrees with the one's physical understanding: Only detailed time variation of the output is modified by the dynamics of the output linear group. In the case of an internal combustion engine, the average output specifies the power of the engine. The dynamics of the output linear group is determined by the inertia of the moving parts. The power of the engine is certainly independent of the inertia of the moving parts.

Equations (11) to (13) fully determine the performance of the optimalizing control system once the values of a, k, and ω_0 are specified and the transfer functions $F_i(s)$ and $F_0(s)$ of the input linear group and the output linear group are given. The following sections give the detailed calculations and results for the case of first-order input and output groups.

First-Order Input and Output Groups

The frequency ω_0 of the optimalizing control is usually low, and the important dynamic effects come from the inertia in the input and the output linear groups. Then these linear groups can be closely approximated by first-order systems. In other words, their transfer functions are

$$F_i(i\omega) = 1/(1 + i\omega\tau_i) \tag{14}$$

$$F_0(i\omega) = 1/(1 + i\omega\tau_0) \tag{15}$$

where τ_i and τ_0 are the characteristic time constants of the input linear group and the output linear group, respectively. It is evident that these transfer functions satisfy the conditions of Eqs. (5) and (8).

By substituting Eq. (14) into Eq. (11), the actual output x is given by

$$x = \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2i(2n+1)^2}\left[\frac{e^{[(2n+1)/2]i\omega_0 t}}{1+(2n+1)i(\omega_0\tau_i/2)} - \frac{e^{-[(2n+1)/2]i\omega_0 t}}{1-(2n+1)i(\omega_0\tau_i/2)}\right] \tag{16}$$

When the summation is carried out, Eq. (16) yields the following equations for the input x:

$$x = NT\left[\frac{t}{T} - \frac{\tau_i}{T} + \frac{\tau_i}{T}\frac{e^{-[(t/T)/(\tau_i/T)]}}{\cosh\,(T/2\tau_i)}\right] \quad \text{for } -\frac{1}{2} \leqslant \frac{t}{T} \leqslant \frac{1}{2} \tag{17a}$$

and

$$x = -NT\left[\frac{t}{T} - \left(1 + \frac{\tau_i}{T}\right) + \frac{\tau_i}{T}\frac{e^{(1-t/T)/(\tau_i/T)}}{\cosh\,(T/2\tau_i)}\right] \quad \text{for } \frac{1}{2} \leqslant \frac{t}{T} \leqslant \frac{3}{2} \tag{17b}$$

where N is the constant input drive speed—i.e.,

$$N = 2a/T \tag{18}$$

By using these equations, the variation of actual input x with respect to time can be calculated for any specified data. Examples of such calculations are shown in Figs. 4 and 5 for $\tau_i/T = 0.1$ and $\tau_i/T = 0.4$, respectively. Both show the expected effect of rounding-off of the sharp corners of the saw-tooth curve and a time delay. It is of interest to note that while the delay is almost equal to τ_i itself for small τ_i/T, the delay is less than τ_i for larger τ_i/T.

With the first-order transfer function of Eq. (14), the hunting loss given by Eq. (13) becomes

$$D = \frac{32a^2k}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4\{1 + [(2n+1)/2]^2\omega_0^2\tau_i^2\}} \tag{19}$$

By carrying out the summation, Eq. (19) gives the hunting loss as

$$D = (N^2T^2k/12)\,[1 - 12(\tau_i/T)^2 + 24(\tau_i/T)^3\tanh\,(T/2\tau_i)] \tag{20}$$

Fig. 6 shows a dimensionless plot of this equation.

To calculate the actual output y, both Eqs. (14) and (15) have to be substituted into Eq. (12)—i.e.,

$$y = \frac{4T^2N^2k}{\pi^4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2}\times \left\{\frac{e^{i(n+m+1)\omega_0 t}}{[1+(n+m+1)i\omega_0\tau_0]\,[1+(2n+1)i(\omega_0\tau_i/2)]\,[1+(2m+1)i(\omega_0\tau_i/2)]} - \frac{e^{i(n-m)\omega_0 t}}{[1+(n-m)i\omega_0\tau_0]\,[1+(2n+1)i(\omega_0\tau_i/2)]\,[1-(2m+1)i(\omega_0\tau_i/2)]} - \frac{e^{-i(n-m)\omega_0 t}}{[1-(n-m)i\omega_0\tau_0]\,[1-(2n+1)i(\omega_0\tau_i/2)]\,[1+(2m+1)i(\omega_0\tau_i/2)]} + \frac{e^{-i(n+m+1)\omega_0 t}}{[1-(n+m+1)i\omega_0\tau_0]\,[1-(2n+1)i(\omega_0\tau_i/2)]\,[1-(2m+1)i(\omega_0\tau_i/2)]}\right\} \tag{21}$$

By changing the summation indices, Eq. (21) can also be written as

$$y = \frac{4T^2N^2k}{\pi^4}\left(\sum_{s=-\infty}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\times\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2[(2n+1)-2s]^2\,[1+(2n+1)i(\omega_0\tau_i/2)]\,\{1-[(2n+1)-2s]i(\omega_0\tau_i/2)\}} + \sum_{s=-\infty}^{\infty}\frac{(-1)^{s-1}e^{-is\omega_0 t}}{(1-is\omega_0\tau_0)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2[(2n+1)-2s]^2\,[1-(2n+1)i(\omega_0\tau_i/2)]\,\{1+[(2n+1)-2s]i(\omega_0\tau_i/2)\}}\right)$$

or

$$y = \frac{8T^2N^2k}{\pi^4}\left\{-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4[1+(2n+1)^2(\omega_0\tau_i/2)^2]} + \sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\times\right.$$
$$\sum_{n=0}^{\infty}\frac{[(2n+1)^2+4s^2]\,[(1+i\omega_0\tau_i s)+(\omega_0\tau_i/2)^2(2n+1)^2]+8(\omega_0\tau_i/2)^2s^2(2n+1)^2}{(2n+1)^2[(2n+1)^2-4s^2]^2\,[1+(\omega_0\tau_i/2)^2(2n+1)^2]\,[(1+i\omega_0\tau_i s)^2+(\omega_0\tau_i/2)^2(2n+1)^2]} +$$
$$\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{-is\omega_0 t}}{(1-is\omega_0\tau_0)}\sum_{n=0}^{\infty}\times$$
$$\left.\frac{[(2n+1)^2+4s^2]\,[(1-i\omega_0\tau_i s)+(\omega_0\tau_i/2)^2(2n+1)^2]+8(\omega_0\tau_i/2)^2s^2(2n+1)^2}{(2n+1)^2[(2n+1)^2-4s^2]^2\,[1+(\omega_0\tau_i/2)^2(2n+1)^2]\,[(1-i\omega_0\tau_i s)^2+(\omega_0\tau_i/2)^2(2n+1)^2]}\right\} \quad (22)$$

The last two summations in Eq. (22) are complex conjugate of each other, thus

$$y = \frac{8T^2N^2k}{\pi^4}\left\{-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4[1+(2n+1)^2(\omega_0\tau_i/2)^2]} + 2\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\sum_{n=0}^{\infty}\times\right.$$
$$\left.\frac{[(2n+1)^2+4s^2]\,[(1+i\omega_0\tau_i s)+(\omega_0\tau_i/2)^2(2n+1)^2]+8(\omega_0\tau_i/2)^2s^2(2n+1)^2}{(2n+1)^2[(2n+1)^2-4s^2]^2[1+(\omega_0\tau_i/2)^2(2n+1)^2]\,[(1+i\omega_0\tau_i s)^2+(\omega_0\tau_i/2)^2(2n+1)^2]}\right\} \quad (23)$$

where Rl means the real part of the expression following it. In order to carry out the summation with respect to the index n, Eq. (23) is resolved into the following partial fraction form:

$$y = \frac{8T^2N^2k}{\pi^4}\left(-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4[1+(2n+1)^2(\omega_0\tau_i/2)^2]} + 2\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\times\right.$$
$$\left\{\frac{1}{4s^2(1+is\omega_0\tau_i)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \frac{(\omega_0\tau_i/2)^4}{2(1+is\omega_0\tau_i)^2\,[1+is(\omega_0\tau_i/2)]}\sum_{n=0}^{\infty}\frac{1}{[1+(\omega_0\tau_i/2)^2(2n+1)^2]} +\right.$$
$$\frac{(\omega_0\tau_i/2)^4}{2(1+is\omega_0\tau_i)\,[1+is(\omega_0\tau_i/2)]}\sum_{n=0}^{\infty}\frac{1}{[(1+i\omega_0\tau_i s)^2+(\omega_0\tau_i/2)^2(2n+1)^2]} -$$
$$\left.\left.\frac{[1+is\omega_0\tau_i+4(\omega_0\tau_i/2)^2s^2]}{4s^2(1+is\omega_0\tau_i)^2}\sum_{n=0}^{\infty}\frac{1}{[(2n+1)^2+(i2s)^2]} + \frac{2}{(1+is\omega_0\tau_i)}\sum_{n=0}^{\infty}\frac{1}{[(2n+1)^2+(i2s)^2]^2}\right\}\right) \quad (24)$$

By using the summation formulas given in the Appendix, the sums with respect to n can be evaluated and the result is, noting that $\tan \pi s = 0$ for integer values of s,

$$y = \frac{8T^2N^2k}{\pi^4}\left(-\left[\frac{\pi^4}{96} - \frac{\pi^2}{8}\left(\frac{\omega_0\tau_i}{2}\right)^2 + \frac{\pi}{4}\left(\frac{\omega_0\tau_i}{2}\right)^3\tanh\frac{\pi}{\omega_0\tau_i}\right] +\right.$$
$$\left.2\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)}\left\{\frac{\pi^2}{4}\frac{1}{(4s^2)(1+is\omega_0\tau_i)} + \frac{\pi}{4}\frac{(\omega_0\tau_i/2)^3\tanh[\pi/(\omega_0\tau_i)]}{(1+is\omega_0\tau_i)^2\,[1+is(\omega_0\tau_i/2)]}\right\}\right) \quad (25)$$

Eq. (25) is again resolved into partial fractions in order to carry out the summation with respect to s, viz.,

$$y = \frac{8T^2N^2k}{\pi^4}\left[-\left[\frac{\pi^4}{96} - \frac{\pi^2}{8}\left(\frac{\omega_0\tau_i}{2}\right)^2 + \frac{\pi}{4}\left(\frac{\omega_0\tau_i}{2}\right)^3\tanh\frac{\pi}{\omega_0\tau_i}\right] +\right.$$
$$\frac{\pi}{2}\left(\frac{(\omega_0\tau_0/2)^3}{[(\omega_0\tau_0/2)-(\omega_0\tau_i/2)]}\left\{\frac{2(\omega_0\tau_i/2)^3\tanh[\pi/(\omega_0\tau_i)]}{[(\omega_0\tau_0/2)-(\omega_0\tau_i/2)][\omega_0\tau_0-(\omega_0\tau_i/2)]} - \pi\right\}\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_0)} +\right.$$
$$\frac{(\omega_0\tau_i/2)^3}{[(\omega_0\tau_0/2)-(\omega_0\tau_i/2)]}\left\{\pi - \frac{2(\omega_0\tau_i/2)^2\tanh[\pi/(\omega_0\tau_i)]}{[(\omega_0\tau_0/2)-(\omega_0\tau_i/2)]}\right\}\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_i)} - \frac{\pi}{2}\left(\frac{\omega_0\tau_0}{2}+\frac{\omega_0\tau_i}{2}\right)\times$$
$$\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}ie^{is\omega_0 t}}{s} + \frac{\pi}{4}\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{s^2} - \frac{(\omega_0\tau_i/2)^4\tanh[\pi/(\omega_0\tau_i)]}{[2(\omega_0\tau_0/2)-(\omega_0\tau_i/2)]}\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{[1+is(\omega_0\tau_i/2)]} -$$
$$\left.\left.\frac{2(\omega_0\tau_i/2)^4\tanh[\pi/(\omega_0\tau_i)]}{[(\omega_0\tau_0/2)-(\omega_0\tau_i/2)]}\mathrm{Rl}\sum_{s=1}^{\infty}\frac{(-1)^{s-1}e^{is\omega_0 t}}{(1+is\omega_0\tau_i)^2}\right)\right] \quad (26)$$

The result of carrying out the summations in Eq. (26) and simplifying the expressions is,

$$y = 2T^2N^2k\left[-\left\{\frac{1}{2}\left(\frac{t}{T}\right)^2 - \left(\frac{\tau_i}{T}+\frac{\tau_0}{T}\right)\left(\frac{t}{T}\right) + \left[\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2 + \frac{\tau_i\tau_0}{T^2} + \left(\frac{\tau_0}{T}\right)^2\right]\right\} +\right.$$
$$\frac{1}{2}\left(-\frac{(\tau_0/T)^2}{(\tau_0/T-\tau_i/T)}\left\{\frac{2(\tau_i/T)^3\tanh(T/2\tau_i)}{[(\tau_0/T)-(\tau_i/T)]\,[2(\tau_0/T)-(\tau_i/T)]} - 1\right\}\frac{e^{-(t/T)/(\tau_0/T)}}{\sinh(T/2\tau_0)} +\right.$$
$$\left.\left.\left\{\frac{t}{T} + \frac{(\tau_i/T)^2}{[(\tau_0/T)-(\tau_i/T)]}\right\}\frac{2(\tau_i/T)^2e^{-(t/T)/(\tau_i/T)}}{[(\tau_0/T)-(\tau_i/T)]\cosh(T/2\tau_i)} + \frac{(\tau_i/T)^3}{[2(\tau_0/T)-(\tau_i/T)]}\frac{e^{-(2t/T)/(\tau_i/T)}}{\cosh^2(T/2\tau_i)}\right)\right] \quad (27a)$$

for $-(1/2) \leqslant t/T \leqslant 1/2$ and

$$y = 2T^2N^2k\left[-\left\{\frac{1}{2}\left(\frac{t}{T}\right)^2 - \left(\frac{\tau_0}{T} + \frac{\tau_i}{T} + 1\right)\left(\frac{t}{T}\right) + \left[\frac{1}{2}\left(\frac{\tau_i}{T}\right)^2 + \frac{\tau_i\tau_0}{T^2} + \left(\frac{\tau_0}{T}\right)^2 + \frac{\tau_0}{T} + \frac{\tau_i}{T} + \frac{1}{2}\right]\right\} + \frac{1}{2}\left(-\frac{(\tau_0/T)^2}{[(\tau_0/T) - (\tau_i/T)]}\left\{\frac{2(\tau_i/T)^3 \tanh(T/2\tau_i)}{[(\tau_0/T) - (\tau_i/T)][2(\tau_0/T) - (\tau_i/T)]} - 1\right\}\frac{e^{(1-t/T)/(\tau_0/T)}}{\sinh(T/2\tau_0)} + \left[\frac{t}{T} + \frac{(\tau_i/T)^2}{(\tau_0/T) - (\tau_i/T)} - 1\right]\frac{2(\tau_i/T)^2 e^{(1-t/T)/(\tau_i/T)}}{[(\tau_0/T) - (\tau_i/T)]\cosh(T/2\tau_i)} + \frac{(\tau_i/T)^3 e^{2(1-t/T)/(\tau_i/T)}}{[2(\tau_0/T) - (\tau_i/T)]\cosh^2(T/2\tau_i)}\right)\right] \tag{27b}$$

for $1/2 \leqslant t/T \leqslant 3/2$.

In Eqs. (27a) and (27b), there are apparent singularities whenever $\tau_0/T = \tau_i/T$ and $2\tau_0/T = \tau_i/T$; that is, the value of output y seemingly cannot be determined for these values of time constants. However this is deceptive. By using a simple limit procedure or by direct evaluation of Eq. (25) for these two cases, it can be shown that this is not the case. For example, for $\tau_i/T = \tau_0/T$

$$y = 2T^2N^2k\left\{-\frac{1}{2}\left(\frac{t}{T}\right)^2 + 2\left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) - \frac{5}{2}\left(\frac{\tau_i}{T}\right)^2 + \left[-\left(\frac{\tau_i}{T}\right)^2 - \frac{1}{2}\left(\frac{t}{T}\right)^2 + \left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) + \frac{1}{8}\right]e^{-[(t/T)/(\tau_i/T)]}\operatorname{sech}\frac{T}{2\tau_i} + \frac{3}{2}\left(\frac{\tau_i}{T}\right)e^{-(t/T)/(\tau_i/T)}\operatorname{csch}\frac{T}{2\tau_i} + \frac{1}{2}\left(\frac{\tau_i}{T}\right)^2 e^{-(2t/T)/(\tau_i/T)}\operatorname{sech}^2\frac{T}{2\tau_i}\right\} \tag{28}$$

for $-(1/2) \leqslant t/T \leqslant 1/2$, and, for $2\tau_0/T = \tau_i/T$,

$$y = -2T^2N^2k\left\{\frac{1}{2}\left(\frac{t}{T}\right)^2 - \frac{3}{2}\left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) + \frac{5}{4}\left(\frac{\tau_i}{T}\right)^2 + \left[\left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) + 2\left(\frac{\tau_i}{T}\right)^2 + \frac{1}{2}\left(\frac{\tau_i}{T}\right)\operatorname{ctnh}\frac{T}{\tau_i} + \frac{1}{8}\left(\frac{\tau_i}{T}\right)\operatorname{ctnh}\frac{T}{2\tau_i}\right]\frac{e^{-[(2t/T)/(\tau_i/T)]}}{\cosh^2 T/2\tau_i} + \left(\frac{\tau_i}{T}\right)\left(2\frac{t}{T} - 4\frac{\tau_i}{T}\right)\frac{e^{-[(t/T)/(\tau_i/T)]}}{\cosh T/2\tau_i}\right\} \tag{29}$$

for $-(1/2) \leqslant t/T \leqslant 1/2$.

An analysis for the continuity of Eqs. (27a) and (27b) at $t/T = 1/2$ shows that the values of y and its derivative with respect to t are the same at $t/T = 1/2$ whether they are computed from Eq. (27a) or from Eq. (27b).

Now the computation of the potential output y^* can be accomplished by letting $\tau_0/T = 0$ in Eqs. (27a) and (27b). Thus,

$$y^* = 2T^2N^2k\left[-\frac{1}{2}\left(\frac{t}{T}\right)^2 + \left(\frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) - \frac{1}{2}\left(\frac{\tau_i}{T}\right)^2 - \left(\frac{t}{T} - \frac{\tau_i}{T}\right)\left(\frac{\tau_i}{T}\right)\frac{e^{-[(t/T)/(\tau_i/T)]}}{\cosh(T/2\tau_i)} - \frac{1}{2}\left(\frac{\tau_i}{T}\right)^2\frac{e^{-[(2t/T)/(\tau_i/T)]}}{\cosh^2(T/2\tau_i)}\right] \tag{30a}$$

for $-(1/2) \leqslant t/T \leqslant 1/2$ and

$$y^* = 2T^2N^2k\left\{-\frac{1}{2}\left(\frac{t}{T}\right)^2 + \left(1 + \frac{\tau_i}{T}\right)\left(\frac{t}{T}\right) - \frac{1}{2}\left(1 + \frac{\tau_i}{T}\right)^2 - \left(\frac{\tau_i}{T}\right)\left[\frac{t}{T} - \left(\frac{\tau_i}{T} + 1\right)\right]\frac{e^{(1-t/T)/(\tau_i/T)}}{\cosh(T/2\tau_i)} - \frac{1}{2}\left(\frac{\tau_i}{T}\right)^2\frac{e^{[2(1-t/T)]/(\tau_i/T)}}{\cosh^2(T/2\tau_i)}\right\} \tag{30b}$$

for $1/2 \leqslant t/T \leqslant 3/2$.

These expressions check with the result of direct calculation of y^* by Eqs. (1) and (17).

Figures 7 and 8 show the dimensionless plots of actual output y and potential output y^* for the particular values of τ_0/T and τ_i/T. In these figures it is clearly seen that the dynamic effects not only decrease the output of the system but also introduce a time lag and lower the maximum output of the system. Figure 8 with $\tau_i/T = 0.4$, $\tau_0/T = 0.6$, has the maximum value of y almost at the very instants of input drive reversal points, $t/T = n + (1/2)$. This is indeed an extreme case.

Design Charts

From the principle of operation of the peak-holding optimalizing control, it is seen that the most important quantity to be specified for its design is the critical indicated difference c between the reading of the special output sensing instrument and the output itself. By definition, c is the difference of the maximum of the actual output y and the value of y at the tripping instant of the input drive. The instant of reversing the input drive is typified by $t/T = 1/2$. If the corresponding instant of maximum y is t^*, then the critical indicated difference c is calculated as

$$c = y(t^*/T) - y(1/2) \tag{31}$$

by using any one of Eqs. (27), (28), or (29). Since the instant of input drive reversal must come after the instant of maximum output, $t^*/T < 1/2$.

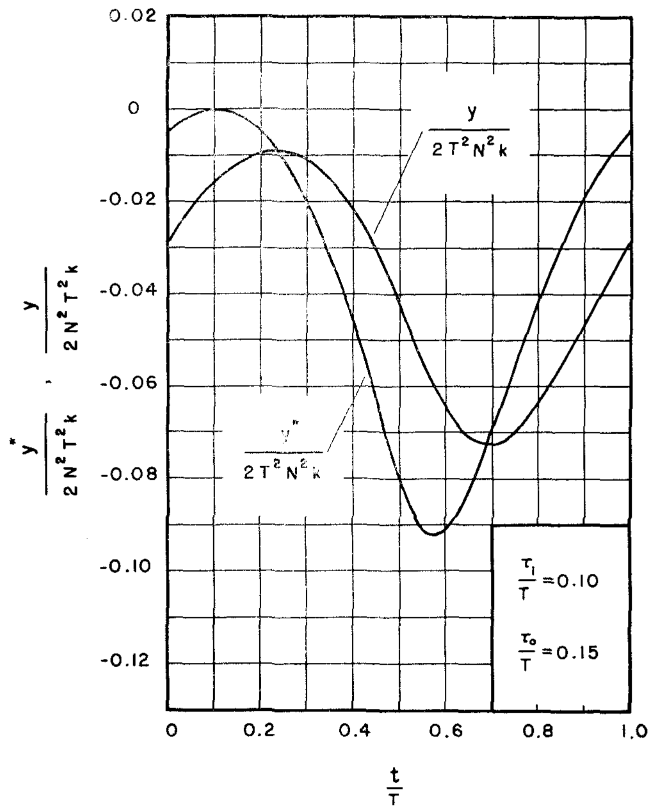

FIG. 7. "Potential" output and indicated output for values of $\tau_i/T = 0.1$ and $\tau_0/T = 0.15$.

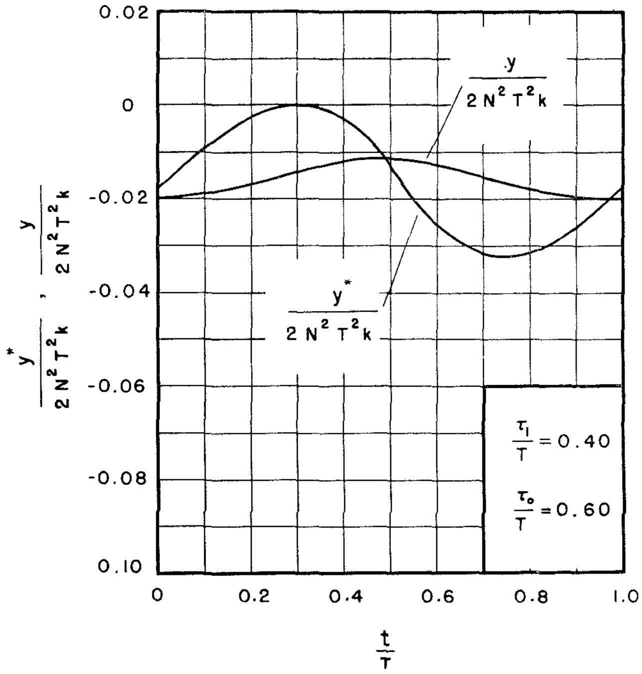

FIG. 8. "Potential" output and indicated output for values of $\tau_i/T = 0.4$ and $\tau_0/T = 0.6$.

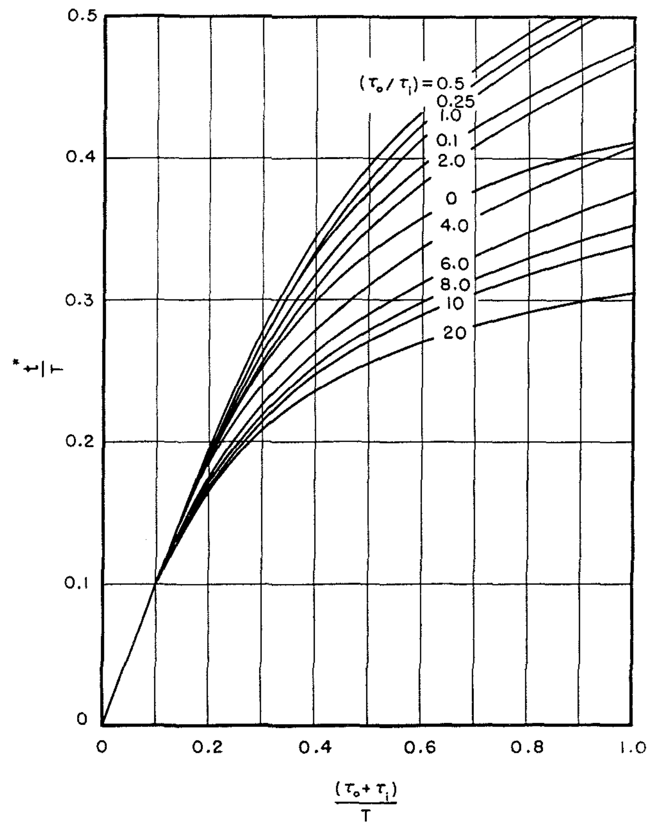

FIG. 9. Maximum output occurrence instant, t^*/T in interval $(0 \leq t/T \leq 1/2)$ versus $(\tau_0 + \tau_i)/T$ with τ_0/τ_i as parameter.

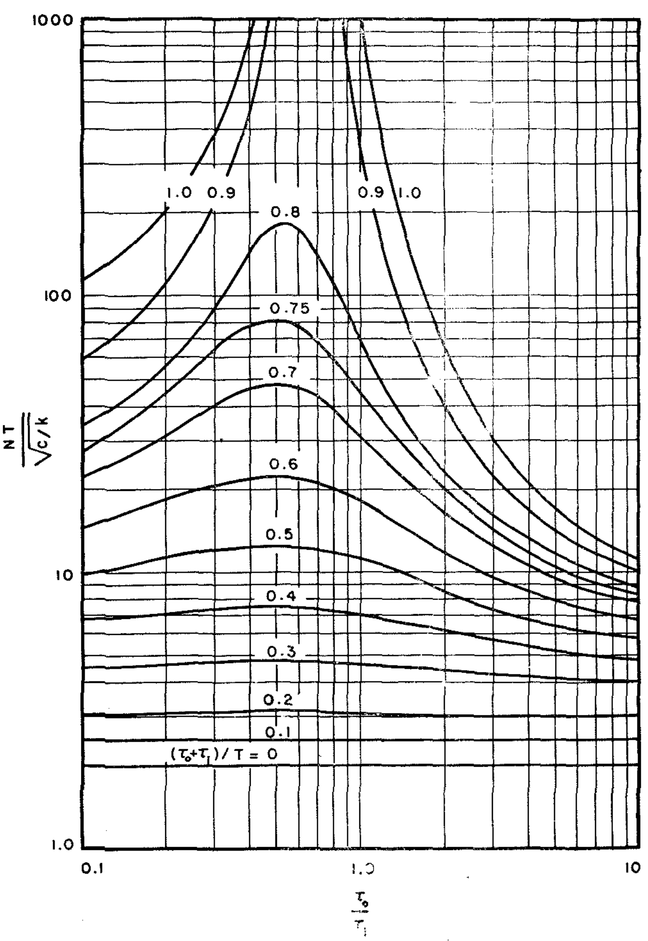

FIG. 10. Critical indicated difference parameter, $TN/\sqrt{c/k}$ versus τ_0/τ_i with $(\tau_0 + \tau_i)/T$ as parameter.

To determine t^*, one may use the condition of zero slope—i.e., $dy/dt = 0$. Then Eq. (27a) gives

$$-\left[\frac{t^*}{T} - \left(\frac{\tau_0}{T} + \frac{\tau_i}{T}\right)\right] + \frac{(\tau_0/T)}{2[(\tau_0/T) - (\tau_i/T)]}\left\{\frac{2(\tau_i/T)^3 \tanh(T/2\tau_i)}{[(\tau_0/T) - (\tau_i/T)]\,[2(\tau_0/T) - (\tau_i/T)]} - 1\right\}\frac{e^{-[(t^*/T)/(\tau_0/T)]}}{\sinh(T/2\tau_0)} + \left\{1 - \left(\frac{t^*}{T}\right)\left(\frac{T}{\tau_i}\right) - \frac{(\tau_i/T)}{[(\tau_0/T) - (\tau_i/T)]}\right\}\frac{(\tau_i/T)^2\, e^{-(t^*/T)/(\tau_i/T)}}{[(\tau_0/T) - (\tau_i/T)]\cosh(T/2\tau_i)} - \frac{(\tau_i/T)^2\, e^{-[(2t^*/T)/(\tau_i/T)]}}{[2(\tau_0/T) - (\tau_i/T)]\cosh^2(T/2\tau_i)} = 0 \quad (32)$$

This transcendental equation for t^*/T may be solved by iteration. For instance, for small τ_0/T and τ_i/T, only terms within the first brackets are of importance, then $t^*/T \simeq (\tau_0 + \tau_i)/T$. This is already recognized by Draper and co-workers.[1,2] The complete results of calculation are shown in Fig. 9, which shows that t^*/T is almost only a function of $(\tau_0 + \tau_i)/T$ with minor modifications from the parameter τ_0/τ_i, the ratio of characteristic times of the output linear group and the input linear group. Values of t^*/T beyond 1/2 are not shown, as clearly then the maxima of the output will occur after the corresponding input drive reversal points and proper operation of the control will be difficult if not impossible.

With t^*/T determined, Eq. (31) gives c by substituting Eq. (27a). However the specified quantities of an optimalizing control are k, the characteristics of the controlled system, and τ_i, τ_0, the characteristics of the linear group. From considerations on the noise interference, the designer can make an appropriate choice of the period T and the critical indicated difference c for input drive reversal. Therefore the quantities that the designer wishes to know, after he has the values of k, τ_i, τ_0, T, and c, are N, the input drive speed, and D, the hunting loss. Thus the result of calculation with Eq. (31) should be written as follows:

$$\frac{TN}{\sqrt{c/k}} = \left(\left[\frac{1}{4} - \left(\frac{t^*}{T}\right)^2\right] + 2\left(\frac{\tau_i}{T}\right)\left(\frac{\tau_0}{\tau_i} + 1\right)\left(\frac{t^*}{T} - \frac{1}{2}\right) - \frac{(\tau_0/\tau_i)^2\,(\tau_i/T)}{[(\tau_0/\tau_i) - 1]\sinh(\tau_i/\tau_0)(T/2\tau_i)} \times \left\{\frac{2(\tau_i/T)\tanh(T/2\tau_i)}{[(\tau_0/\tau_i) - 1]\,[2(\tau_0/\tau_i) - 1]} - 1\right\}\left(e^{-[(t^*/T)/(\tau_0/\tau_i)\,(\tau_i/T)]} - e^{-\{1/[2(\tau_0/\tau_i)\,(\tau_i/T]\}}\right) + \frac{2(\tau_i/T)}{[(\tau_0/\tau_i) - 1]\cosh(T/2\tau_i)}\left\{\left[\frac{t^*}{T} + \frac{(\tau_i/T)}{(\tau_0/\tau_i) - 1}\right]e^{-[(t^*/T)/(\tau_i/T)]} - \left[\frac{1}{2} + \frac{(\tau_i/T)}{(\tau_0/\tau_i) - 1}\right]e^{-(T/2\tau_i)}\right\} + \frac{(\tau_i/T)^2(e^{-[(2t^*/T)/(\tau_i/T)]} - e^{-(T/\tau_i)]}}{[2(\tau_0/\tau_i) - 1]\cosh^2(T/2\tau_i)}\right)^{-(1/2)} \quad (33)$$

When N is determined, Eq. (20) then gives the hunting loss D.

Figures 10 and 11 are the design charts for peak-holding optimalizing control computed from the equations of the preceding analysis. Figure 10 gives $TN/\sqrt{c/k}$ as a function of τ_0/τ_i with $(\tau_0 + \tau_i)/T$ as parameter. Figure 11 gives relative hunting loss D/c again as a function of τ_0/τ_i with $(\tau_0 + \tau_i)/T$ as parameter. The peaks of curves near $\tau_0/\tau_i = 1$ indicate a sort of resonant effect between the input linear group and output linear group. The hunting loss for fixed $(\tau_i + \tau_0)/T$ and c is smaller for τ_0/τ_i away from unity. For fixed τ_i, τ_0, and c, clearly the way to reduce the hunting loss is by increasing the period T.

Concluding Remarks

The present analysis gives the necessary input drive speed N and the hunting loss D for any specified hunting period T, time constants τ_i and τ_0 for the input linear group and the output linear group, and the chosen critical indicated difference c. T and c are fixed by considerations on the noise interference. The analysis shows that whenever the hunting period is relatively short with respect to the time constants τ_i, τ_0, or whenever $(\tau_i + \tau_0)/T$ is relatively large, the hunting loss will be large, especially when τ_i and τ_0 are nearly equal. To avoid such unfavorable condition, the designer should improve his input drive system so as to reduce the constant τ_i. τ_0 is, however, a constant of the intrinsic characteristic of the controlled system, due to, say, the inertia of the moving parts of the system. τ_0 is thus not at the disposal of the designer of the control system. However, suppose there is a compensating circuit between the output y and optimalizing input drive unit (Fig. 3), such that the effects of the output linear group is completely compensated. Then the effective signal for input drive reversal is not the actual output y, but the potential output y^*. In other words, the value of τ_0 is made to be effectively zero. Even if complete compensation is not achieved, the effective value of τ_0 can still be greatly reduced. For difficult cases then, such a compensating unit should certainly be added to reduce the hunting loss. This will be just a minor complication when compared with the additional equipment required for satisfactory noise filtering.

Appendix

Typical Summation Formulas

Rl and Im mean, respectively, the "real part of" and the "imaginary part of" the expression following it.

(1) $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

(2) $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

(3) $$\sum_{n=0}^{\infty} \frac{1}{[1+(2n+1)^2z^2]} = \frac{\pi}{4z} \tanh \frac{\pi^2}{2z}$$

(4) $$\text{Rl} \sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{1+i2sb} = \frac{1}{2} - \frac{\pi}{4b} \frac{e^{-[(\omega_0 t)/(2b)]}}{\sinh \pi/2b}$$

when $-\pi < \omega_0 t < \pi$

$$= \frac{1}{2} - \frac{\pi}{4b} \frac{e^{\pi/b}}{\sinh \pi/2b} \times e^{-[(\omega_0 t)/(2b)]}$$

when $\pi < \omega_0 t < 3\pi$, etc.

(5) $$\text{Im} \sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{s} = \frac{\omega_0 t}{2}$$ when $-\pi < \omega_0 t < \pi$

$$= \frac{\omega_0 t}{2} - \pi$$

when $\pi < \omega_0 t < 3\pi$, etc.

(6) $$\text{Rl} \sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{s^2} = \frac{\pi^2}{12} - \left(\frac{\omega_0 t}{2}\right)^2$$

when $-\pi < \omega_0 t < \pi$

$$= 2\pi\left(\frac{\omega_0 t}{2}\right) - \left(\frac{\omega_0 t}{2}\right)^2 - \frac{11}{12}\pi^2$$

when $\pi < \omega_0 t < 3\pi$, etc.

(7) $$\text{Rl} \sum_{s=1}^{\infty} \frac{(-1)^{s-1} e^{is\omega_0 t}}{(1+i2sa)^2} = \frac{1}{2} - \frac{\pi}{4a^2} \frac{e^{-[(\omega_0 t)/(2a)]}}{\sinh^2(\pi/2a)} \times \left(\frac{\omega_0 t}{2} \sinh \frac{\pi}{2a} + \frac{\pi}{2} \cosh \frac{\pi}{2a}\right)$$

when $-\pi < \omega_0 t < \pi$

$$= \frac{1}{2} - \frac{\pi}{4a^2} \frac{e^{\pi/a} e^{-[(\omega_0 t)/2a)]}}{\sinh^2(\pi/2a)} \times \left[\left(\frac{\omega_0 t}{2} - \pi\right) \sinh \frac{\pi}{2a} + \frac{\pi}{2} \cosh \frac{\pi}{2a}\right]$$

when $\pi < \omega_0 t < 3\pi$, etc.

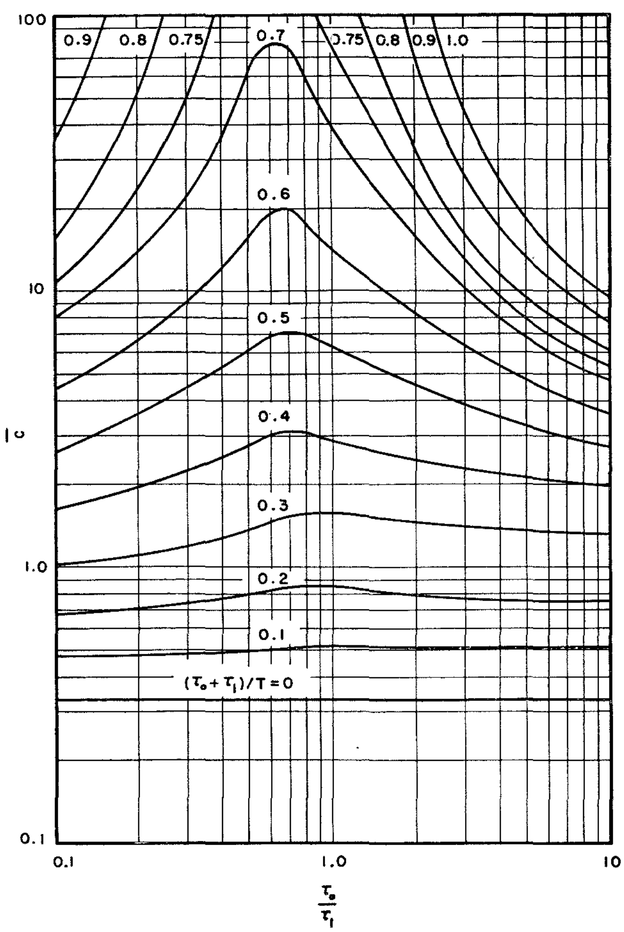

FIG. 11. Relative hunting loss, D/c versus τ_0/τ_i with $(\tau_0 + \tau_i)/T$ as parameter.

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