

DIFFUSION IN NEUTRAL AND IONIZED GASES WITH EXTREME
PRESSURE GRADIENTS

by

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ABSTRACT

Diffusion in vortex flows is considered as a simple case of the more general problem of diffusion in flows with large pressure gradients normal to the principal flow direction. Two examples are considered. In the first the two gases are assumed electrically neutral, and pressure and concentration diffusion are equally important. In the second, diffusion of the electrons of an ionized gas is studied. Diffusion due to electromagnetic body forces is of equal importance with pressure diffusion in this case, while concentration diffusion is negligible.

It is found in the first example that the ratio of the radial mass flow of one species to the total radial mass flow is a characteristic value of the diffusion equation. The rates of diffusion are such that significant separation of the isotopes of uranium should be possible in vortices with supersonic tangential velocities.

The radial pressure gradient leads to a radial electric field in the second example. A solution is obtained for the case of zero currents. By means of a perturbation technique, the solution is then extended to the case of small currents and induced fields.

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INTRODUCTION

Studies of fluid mechanical systems in which molecular diffusion plays a significant role have, for the most part, been confined to circumstances where diffusion due to pressure gradients and external body forces has been unimportant compared to diffusion due to concentration gradients. Pressure diffusion has usually been negligible because the flows have been such that the principal pressure gradients were in the direction of flow. Thus, for example in shock waves, the pressure gradients and resulting diffusion velocities are large, but negligible compared to the flow velocity.

In contrast, the present study deals with flows in which diffusion resulting from a pressure gradient is a dominant factor because the pressure gradient, and the resulting diffusion, are normal to the principal flow velocities. Although there are many examples of flows in which such conditions occur, the simplest is the vortex flow, in which strong radial pressure gradients may be produced by large tangential velocities, while the radial velocity is small. To illustrate the effects of pressure diffusion without involving excessive mathematical complexity, this study will be restricted to two-dimensional vortex flows.

Two examples of diffusion in such flows are considered. In the first, the two diffusing gases are assumed electrically neutral, so that there is no diffusion due to body forces. Pressure and concentration diffusion are of nearly equal importance. Previous studies of this type of diffusion have been largely experimental, and directed toward isotope separation. The principal references to this work are given in Ref. (1), which also presents a brief analytical treatment with emphasis on three dimensional effects. The aims of the present study are to formulate the equations governing binary diffusion in two dimensional vortex flows, and to give some analytical solutions for the case where the two gases are of nearly equal molecular weight.

In the second example, diffusion of the electrons of an ionized gas is studied. The pressure gradient resulting from the tangential velocity causes a corresponding electron pressure gradient, which in turn leads to a radial electric field. Concentration diffusion is limited by the tendency of the gas to neutrality, so that in this case, pressure diffusion and diffusion due to electromagnetic body forces are of principal importance. Because of the complexity of the magneto-gasdynamics equations, solutions are readily obtainable only for small currents and magnetic fields, however these serve to indicate the general characteristics of such flows.

I. BINARY DIFFUSION OF NEUTRAL GASES

A physical model for the type of flow which is to be analyzed is sketched in Fig. (1). A mixture of two gases is introduced to the annular region with a tangential velocity, v_t , through a porous cylinder of radius r_1 . It flows spirally inward (or outward) and leaves through another porous cylinder which rotates, in general, with an angular velocity different from that of the first cylinder.

A more complicated flow can be constructed by combining two such as the above, for example by specifying that the radial flow between the two porous cylinders be inward for $r < r_f$, some radius between those of the two porous cylinders, and outward for $r > r_f$, as in Fig. (2). There are then two distinct regions, which must be joined by appropriate boundary conditions.

Initially, the flow in one region only will be considered.

General Formulation

Since the two gases diffuse relative to the mean motion of the mixture, two variables are added to the usual fluid-mechanical set. These may be chosen as the density and velocity of one species, in which case the continuity equation for this species, and the relation connecting its diffusion velocity to the pressure and concentration gradients, must be added to the usual conservation relations.

The conservation equation for the mixture is simply,

$$\rho u r = A, \quad \text{I(1)}$$

a constant, where ρ is the density of the gas mixture and u is the radial mass averaged velocity. Similarly, the conservation equation for one species, say that of lower molecular weight for definiteness, is

$$\rho_i (u + u_i) r = B, \quad \text{I(2)}$$

a second constant, where ρ_i is the density of the lighter species and u_i is its diffusion velocity.

The diffusion velocity is related to concentration and total pressure gradients by,

$$u_i \left(\frac{\rho}{\rho - \rho_i} \right) = -D_{12} \left\{ \frac{n}{n - n_i} \frac{d \log(n_i/n)}{dr} + \frac{n(m_2 - m_1)}{\rho} \frac{d \log p}{dr} \right\}, \quad \text{I(3)}$$

where n and n_i are the molecular concentrations of the mixture and lighter gas, and m_2 and m_1 are the masses of the heavier and lighter molecules.

By use of Eqs. (1) and (2), u_i may be written in terms of the densities, as $u_i = (A/\rho_i r)(B/A - \rho_i/\rho)$, and Eq. (3) may then be put in the form,

$$\left(\frac{B}{A} - \frac{\rho_i}{\rho} \right) = -\frac{1}{Re Sc} \frac{\rho_i}{\rho} \left(1 - \frac{\rho_i}{\rho} \right) \left\{ \frac{r n}{n - n_i} \frac{d \log(n_i/n)}{dr} + \frac{r n (m_2 - m_1)}{\rho} \frac{d \log p}{dr} \right\}, \quad \text{I(3a)}$$

where $Re \equiv \rho u r / \mu = A / \mu$ is a Reynolds number based on the radial flow velocity and $Sc \equiv \mu / \rho D_{12}$ is a Schmidt number for the mixture. Since ρ_i and ρ are expressible in terms of n_i and n , Eq. (3a) is sufficient to determine n_i/n if the term involving the pressure gradient can be determined.

It is interesting to note that B/A , the ratio of the mass flow of light gas to that of the mixture, is a characteristic value of Eq. (3a). The

way in which it enters the boundary conditions will be discussed later.

With the radial velocity small compared to the tangential velocity, the radial momentum equation may be used to write the last term of Eq. (3a) as

$$\frac{r(m_2 - m_1)h}{\rho} \frac{d \log p}{dr} = \frac{m_2 - m_1}{k} \frac{v^2}{T} = \delta(\gamma M^2), \quad \text{I(4)}$$

since it is just the difference of γM^2 for the two gases, γ and M being those for the individual gases, with the common values of v and T .

If the viscosity is assumed constant, the tangential momentum equation is

$$r \frac{dv}{dr} + v = \frac{1}{Re} \left(r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} - v \right)$$

This is an Euler equation, and has the solution,

$$vr = Cr^{2+Re} + D, \quad \text{I(5)}$$

where C and D are constants of integration.

By making use of the fact that u is small compared to v , and using Eq. (4) to eliminate the radial pressure gradient, a first integral of the energy equation may be written as,

$$C_p T + \frac{v^2}{2} = -\frac{r g}{A} + \frac{1}{Re} \left[r^2 v \frac{d}{dr} \left(\frac{v}{r} \right) \right] + Q + E, \quad \text{I(6)}$$

where $g \equiv -\lambda dT/dr + (c_{p1} - c_{p2}) T \rho_1 u$, $C_p = c_{p2} + (c_{p1} - c_{p2}) \rho_1 / \rho$, λ is the coefficient of heat conduction, and c_{p1} and c_{p2} are the specific heats of the lower and higher molecular weight gases. Q is a heat source, which accounts for heat addition or loss due to external causes, and E is a constant of integration.

It is clear that solutions of these equations will in general be very complicated, because of the coupling between the diffusion and energy equations, and the non-linearity of both equations. In the general case where $c_{p2} \neq c_{p1}$, $Q \neq 0$, and heat conduction is important, numerical methods must be used. If only heat conduction is neglected, Eq. (6) reduces to an algebraic equation for T in terms of n_1/n and v , so that numerical integration should be simple in this case.

The present discussion will be limited to situations where $c_{p2} \approx c_{p1}$, $m_2 \approx m_1$. In this case, the coupling of Eqs. (3a) and (6) vanishes, and some analytical solutions can be found. Solutions for the tangential velocity and temperature have been given previously for this case^{(2),(3)} though not quite in the form given here.

Boundary Conditions:

Equation (3a) is of first order, so its solution will contain one arbitrary constant. Thus, only one boundary condition can be satisfied,

while the flow region in general has two boundaries. This apparent difficulty may be resolved by noting that the physical boundary conditions are different, depending on whether the fluid leaves or enters across the boundary.

Consider first the condition placed on ρ_i/ρ at a boundary where the fluid leaves the region in question, say through a porous wall. If the pore size is not small compared to the mean free path, the flow of each species through the wall will be proportional to its concentration at the surface. Thus, the condition for such a boundary is that,

$$\frac{\rho_i}{\rho}(r_b) = \frac{B}{A}. \quad \text{I(7)}$$

At a boundary where the fluid enters a given region, the relative mass flows of the two species are determined by external conditions, and are independent of the value of ρ_i/ρ just inside the boundary. A porous wall passing a mixture of a given composition is exactly equivalent to a wall penetrated by two sets of small jets, each passing a single species, in such amounts that the average composition is the same as for the porous wall flow. But the condition on mass flows is already included in Eq. (3a), in B/A . Thus, we see that the solution need satisfy no boundary condition at an inflow boundary. The value of ρ_i/ρ given by the solution, at the boundary, can be quite different from B/A .

Fundamentally, the difference between the outflow and inflow boundary conditions is that the entropy of the gas mixture would have to decrease in flowing through the boundary if Eq. (7) were not satisfied at an outflow boundary. It increases, by a mixing process, at an inflow boundary.

Solutions for Nearly Identical Species:

If the specific heats of the two species are nearly equal, heat transfer by diffusion becomes negligible, and the energy equation is not coupled to the diffusion equation. It can then be integrated, using Eq. (5). The result is,

$$c_p T = \frac{C^2}{2} \left[\frac{Pr Re}{Pr Re - 2(1+Re)} \right] r^{2(1+Re)} + CD \left[\frac{2 Pr}{Re - Pr Re} \right] r^{Re} - \frac{D^2}{2} \left[\frac{Pr Re (1+4/Re)}{Pr Re + 2} \right] r^{-2} \\ + F r^{Pr Re} + Pr Re r^{Pr Re} \int Q r^{-Pr Re} \frac{dr}{r} + G, \quad \text{I(8)}$$

where F and G are constants of integration, and $Pr = c_p \mu / \lambda$ is the Prandtl number.

The diffusion equation is simplified considerably, since if $m_2 \approx m_1$, $\rho_i/\rho \approx n_i/n$. Further, the Schmidt number is near unity, being $5/6$ for hard-sphere molecules, and about $3/4$ for real gases. Using the latter value, we find

$$\left(\frac{B}{A} - \frac{n_1}{n} \right) = - \frac{4}{3 Re} \left\{ r \frac{d(n_1/n)}{dr} + \frac{n_1}{n} \left(1 - \frac{n_1}{n} \right) S(\gamma M^2) \right\}. \quad \text{I(3b)}$$

Solution for $n_1/n \ll 1$:

For $n_1/n \ll 1$, Eq. (3b) is linear. Using the outflow boundary condition, Eq. (7), its solution may be written as,

$$\frac{n_1}{n} = \frac{B}{A} \left\{ -e^{-g(r)} \frac{3Re}{4} \int_{r_b}^r e^{g(r)} \frac{dr}{r} + e^{g(r_b) - g(r)} \right\}, \quad \text{I(9)}$$

where r_b is the radius of the outflow boundary, and

$$g(r) = \int [\delta(\gamma M^2) - 3Re/4] \frac{dr}{r}. \quad \text{I(10)}$$

Thus, the solution in any region, i. e., any range of r with fixed Re , depends only on the nature of $g(r)$ in that region.

In some simple cases, the integration in Eq. (9) can be carried out quite easily. The simplest case is that for which $v = v_b$, a constant, and $T = T_b$, a constant. Then $\delta(\gamma M^2) = \gamma M_b^2 (m_2 - m_1 / \overline{m}) \equiv \epsilon$, and the solution becomes,

$$\frac{n_1}{n} = \frac{B}{A} \left\{ \left(\frac{r}{r_b} \right)^{3Re/4 - \epsilon} + \frac{3Re}{4} \left[\frac{1 - \left(\frac{r}{r_b} \right)^{3Re/4 - \epsilon}}{3Re/4 - \epsilon} \right] \right\}. \quad \text{I(11)}$$

The logarithm of the function in brackets, i. e., $\log(n_1 A / n B)$ is shown in Fig. (3) as a function of r/r_b for several values of ϵ and Re . In view of the requirement that the outflow boundary be at r_b , the solution is meaningful only for $Re < 0$ when $r/r_b > 1$, and for $Re > 0$ when $r/r_b < 1$, so it is shown only for these combinations.

An interesting characteristic of the solutions is that for $|3Re/4|$ fairly large, the value of $\log(n_1 A / n B)$ behaves asymptotically as r/r_b deviates from unity. It is easy to show that for small $|4\epsilon/3Re|$,

$$\log \left(\frac{n_1 A}{n B} \right) \sim \frac{4\epsilon}{3Re} \left[1 - \left(\frac{r}{r_b} \right)^{3Re/4} \right]. \quad \text{I(11a)}$$

Since $Re < 0$ for $r/r_b > 1$, and $Re > 0$ for $r/r_b < 1$, the bracketed factor approaches unity very rapidly as r/r_b deviates from unity, if $|3Re/4|$ is large.

Though it is not apparent from the plot of Fig. (3), it can be seen from Eq. (11a) that for small ϵ , $\log(n_1 A / n B)$ changes only in sign if the sign of Re is reversed while r/r_b is changed to r_b/r .

A second case in which the solution can be obtained fairly simply is that for $T = T_b$, a constant, and $v r = \Gamma_b$, a second constant. This is an exact solution of the energy and momentum equations for $|Re|$ approaching zero and physical boundaries whose velocities are inversely proportional to their radii. In this case $\delta(\gamma M^2) = \gamma M_b^2 (m_2 - m_1 / \overline{m}) (\Gamma_b / r)^2 \equiv \epsilon (r_b / r)^2$. Under the present assumptions, $m_2 \approx m_1$, and $\epsilon \ll 1$, so that the solution may be expressed as a power series in ϵ . We find,

$$\frac{n_i}{n} = \frac{B}{A} \left(\frac{r}{r_b}\right)^{3Re/4} e^{\frac{\epsilon}{2} \left(\frac{r}{r_b}\right)^2} \left\{ \frac{3Re}{4} \sum_{\nu=0}^{\infty} \frac{(-\epsilon/2)^{\nu}}{(2\nu+3Re/4)\nu!} \left[\left(\frac{r}{r_b}\right)^{-(2\nu+3Re/4)} - 1 \right] + e^{-\epsilon/2} \right\} \quad \text{I(12)}$$

Again, $\log(n_i A/nB)$ is plotted in Fig. (4) as a function of r/r_b for several values of $3Re/4$ and ϵ . It should be noted that the scales for the positive range of $\log(n_i A/nB)$ are 10 times as large as those for the negative range.

The principal qualitative difference between the results of Fig. (4) and those of Fig. (3) is the occurrence of minima of $\log(n_i A/nB)$ for $3Re/4 < 0$ in Fig. (4), and not in Fig. (3). This difference may be understood quite readily from Eq. (3b). If a stationary point of n_i/n exists, then the quantity in braces is stationary in r at that point. Thus if $\delta(\gamma M^2)$ is decreasing at the point, as for the second example, $d(n_i/n)/dr$ must increase and a minimum results. In the first example, since $\delta(\gamma M^2)$ is constant, the only stationary point is at infinity.

More generally, if $\delta(\gamma M^2) = \epsilon (r/r_b)^{2\nu}$, as it is for $\Gamma = T_b$, $v/w_b = (r/r_b)^{\nu}$, then minima can occur for $\epsilon > 0, \nu < 0$, and maxima for $\epsilon < 0, \nu < 0$, if $3Re/4 < 0$. For $3Re/4 > 0$, maxima will occur for $\epsilon > 0$ and $\nu > 0$, while minima will occur for $\epsilon < 0$ and $\nu > 0$. Thus, stationary points may occur in an inflow region if $\delta(\gamma M^2)$ decreases with increasing r , and in an outflow region if $\delta(\gamma M^2)$ increases with increasing r .

Application to Gas Separation

The above solutions for nearly identical gases have an interesting application to the separation of gases, particularly isotopes. The assumptions made in the analysis, e. g., that $C_{p2} \approx C_{p1}$, are very precisely valid in this case.

In order that a gas composed of a mixture of lighter and heavier molecules be enriched in one of the species, the flow must be divided into two parts, one enriched and the other depleted. This might be done by an arrangement such as is sketched in Fig. (2). The original gas mixture is introduced with a tangential velocity v_f , at a radius r_f , and the fluid flows outward and inward from r_f , leaving through porous cylinders of radii r_o and r_i .

The portions of flow outside and inside of r_f form two distinct regions, in each of which the solutions given above are applicable. The radius r_f is an inflow boundary for each of the two regions, while the porous cylinders are outflow boundaries. Although there is no connection between n_i/n and B/A at r_f , since it is an inflow boundary, n_i/n must be continuous at r_f . Thus, if $n_i A/nB$ denotes the function in braces in Eq. (9), we must have

$$\frac{B_o A_o}{B_i A_i} = \frac{n_i A_o (r_f)}{n B_o (r_o)} / \frac{n_i A_i (r_f)}{n B_i (r_i)}, \quad \text{I(13)}$$

the two functions on the right being given by Figs. (2) and (3) for the first two examples. The ratio $B_i A_o / B_o A_i$ is a measure of the amount of separation achieved.

From the physical viewpoint, the factors which can be specified are the relative concentrations of the two species in the feed, the two radius ratios, r_f / r_o and r_f / r_i , and the two mass flows, A_i and A_o . The flows of one species then distribute according to Eq. (13).

As an example, consider a mixture of the two isotopic forms of uranium hexafluoride, i. e., $U^{235}F_6$ and $U^{238}F_6$. At the sublimation temperature of about 600°R, a tangential velocity of 1200 feet per second gives $\epsilon = 0.15$. For the case of $\nu = \nu_b$, $T = T_b$, if $Re_i = Re_o = 4$, and $r_f / r_o = r_i / r_f = 0.6$, the ratio of the weight fraction of light gas in the inner bleed to that in the feed is 1.04. While this is not a large factor, it should be noted that the corresponding factor for the gaseous diffusion process is only 1.0043. The mass flow rate is about 0.6×10^{-3} lb/sec per foot of tube length.

II. DIFFUSION OF ELECTRONS IN A VORTEX

The second problem which will be considered is that of the diffusion, as a result of a strong pressure gradient, of the electrons in an ionized gas. The same model will be adopted for the flow as was used in the preceding problem, except that the boundary conditions are altered by the fact that the electrons can penetrate a solid conducting boundary, while the ions cannot. It is assumed that there are no externally impressed magnetic fields, that the flow is azimuthally symmetric, and that it has only radial and tangential components. Viscous effects will be neglected for the sake of simplicity.

Whereas in the previous problem the selective effect of the radial pressure gradient manifested itself in a physical separation of the lighter and heavier molecules, it will appear in the present problem as a force between the electrons and ionized particles, which leads to a radial electric field. Physical separation of the electrons and positive particles is very small as long as the Debye length is small compared to other characteristic lengths of the problem, such as the radius of curvature of the streamlines.

General Formulation

Two new variables, the electric field, \underline{E} , and the magnetic field, \underline{B} , must be added to those of the previous problem in order to complete the description of an ionized gas. The additional relations required for specification of these variables are provided by Maxwell's equations,

$$\nabla \cdot \underline{E} = \frac{1}{\epsilon_o} (n_1 e_1 + n_2 e_2) \quad ; \quad \nabla \times \underline{E} = 0 \quad \text{II(1)}$$

$$\nabla \times \underline{B} = \mu_o [\underline{j} + \underline{u} (n_1 e_1 + n_2 e_2)] \quad ; \quad \nabla \cdot \underline{B} = 0 \quad \text{II(2)}$$

Here n_1 and n_2 are the concentrations, and e_1 and e_2 the charges, of

electrons and ionized molecules respectively. \underline{u} is the mass averaged velocity of the fluid and \underline{j} is the electric current. MKS units have been assumed here and will be used throughout this section.

The principal coupling between Maxwell's equations and the fluid mechanical equations occurs through the current, \underline{j} . It is generally true that $n_e e + n_i e_i$ may be neglected in Eq. (2), though not in Eq. (1), and this procedure will be followed here.

The connection between the current and the fluid mechanical variables is given by a "generalized Ohm's Law." Such expressions have been derived by Schuler⁽⁴⁾ and revised by Finkelburg and Maecher⁽⁵⁾. The expressions given by these authors can be reduced to the following form, for the present case,

$$\underline{j} = \sigma \left[\underline{E} + \underline{u} \times \underline{B} - \frac{1}{n_e e_i} \nabla p_i + \frac{1}{n_e e_i} \underline{j} \times \underline{B} \right] \quad \text{II(3)}$$

In making this reduction, the ion velocity has been replaced by \underline{u} , the fluid velocity, since the mobility of the ions is small compared to that of the electrons. A simple derivation of this relation is also given in Ref. (6).

It is the third term on the right which gives rise to the effects to be considered here. The order of magnitude of this term can be seen by writing it as, $\nabla p_i / n_e e_i = kT \nabla p_i / e_i p_i$. If, for example, $T = 5,000^\circ\text{K}$ and $\nabla p_i / p_i = 100 \text{ M}^{-1}$, the field due to the electron pressure gradient is about 50 volts M^{-1} . A point worthy of note is that the magnitude of this effect does not depend on the extent of ionization of gas, which enters only in the conductivity, σ .

Neglecting viscous effects and currents due to charge separation, the momentum equation is

$$\rho \frac{D\underline{u}}{Dt} = \underline{j} \times \underline{B} - \nabla p \quad \text{II(4)}$$

where ρ and p are the total density and pressure.

Equations (1) and (2) add so to the complexity of the diffusion problem that no general solution seems available. As a first step, we may consider the case of zero currents, which may then be extended to apply for small currents, by a perturbation technique.

Solution for Zero Currents

If there are no currents, and radial velocities are small compared to tangential velocities, the momentum equations for the tangential and radial directions become $v r = \Gamma$, and $d\rho/dr = \rho v^2 / r$, where Γ is the constant circulation. Thus, $(kT/e_i p)(dp/dr) = \bar{m} \Gamma^2 / e_i r^3$, where \bar{m} is the mean molecular mass. Finally, if we assume that the fraction of the gas which is ionized is a constant, then $(1/p)(dp/dr) = (1/p)(dp/dr)$, and the radial component of Eq. (3) gives, for $\underline{j} = 0$, and $\underline{B} = 0$,

$$E_r = \frac{\bar{m}}{e_i} \frac{\Gamma^2}{r^3} \quad \text{II(5)}$$

Integrating, we find for the radial potential $V_i - V_o$, between electrodes at radii r_i and r_o ,

$$V_i - V_o = \frac{\bar{m} \Gamma^2}{2 \epsilon_1} \left(\frac{1}{r_i^2} - \frac{1}{r_o^2} \right) = \frac{\gamma M_o^2}{2} \left(\frac{k T_o}{\epsilon_1} \right) \left(\frac{r_o^2}{r_i^2} - 1 \right), \quad \text{II(6)}$$

where M_o is the tangential Mach number at r_o .

The extent of charge separation can be computed from Eq. (1). The net volumetric charge is $n_1 e_1 + n_2 e_2 = 3 \epsilon_o \bar{m} \Gamma^2 / e_1 r^5$.

In the absence of all current flow, the above solution gives a complete description of the flow field, and of the electromagnetic fields. The effect of small currents will be computed in the next section.

Solution for Small Currents and Magnetic Fields

It has been assumed that there are no externally impressed magnetic fields. Since the term $\underline{j} \times \underline{B}$ is of second order in small quantities, Eq. (4) is not modified for small currents, and the pressure and velocity fields are unchanged, to a first approximation.

Similarly, this term will be neglected in Eq. (3), however $\underline{u} \times \underline{B}$ is of first order importance. Thus, we write $\underline{j}^{(1)} = \sigma (\underline{E}^{(1)} + \underline{u}^{(0)} \times \underline{B}^{(1)})$, where the superscripts denote the orders of the various quantities. Eliminating $\underline{j}^{(1)}$ by means of Eq. (2), and writing out the radial, azimuthal and axial components, we find

$$j_r^{(1)} = -\frac{1}{\mu_o} \frac{\partial B_\theta^{(1)}}{\partial z} = \sigma (E_r^{(1)} + v^{(0)} B_z^{(1)}), \quad \text{II(7)}$$

$$j_\theta^{(1)} = \frac{1}{\mu_o} \left(\frac{\partial B_r^{(1)}}{\partial z} - \frac{\partial B_z^{(1)}}{\partial r} \right) = -\sigma u^{(0)} B_z^{(1)}, \quad \text{II(8)}$$

$$j_z^{(1)} = \frac{1}{\mu_o} \frac{\partial}{\partial r} (r B_\theta^{(1)}) = \sigma (E_z^{(1)} + u^{(0)} B_\theta^{(1)} - v^{(0)} B_r^{(1)}). \quad \text{II(9)}$$

The second of Eqs. (2) gives,

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r^{(1)}) + \frac{\partial B_z^{(1)}}{\partial z} = 0 \quad \text{II(10)}$$

while the second of Eqs. (1) becomes,

$$\frac{\partial E_r^{(1)}}{\partial z} - \frac{\partial E_z^{(1)}}{\partial r} = 0 \quad \text{II(11)}$$

We ask whether there exists a solution, for small \underline{j} and \underline{B} , such that $E_r^{(1)} = E_r^{(1)}(r)$, and the current flow is purely radial. If such is to be the case then the dependence of Eq. (7) must vanish. This requires that $B_\theta^{(1)} = z f(r)$, and $B_z^{(1)} = B_z^{(1)}(r)$. Then from Eq. (10), $B_r^{(1)} = D/r$, D a constant. Further, from Eq. (11), since $E_r^{(1)} = E_r^{(1)}(r)$, $E_z^{(1)} = E_z^{(1)}(z)$. Now Eq. (9) becomes,

$$j_z^{(1)} = \frac{1}{\mu_o r} \frac{\partial}{\partial r} (r z f(r)) = \sigma [E_z^{(1)}(z) + u^{(0)} z f(r) - D v^{(0)} / r].$$

It seems clear that the only consistent solution is obtained for $E_z^{(1)}(z) = 0$, $D = 0$, in which case we have,

$$\frac{d}{dr}(rf) - \mu_0 \sigma u^{(0)} r f = 0. \quad \text{II(12)}$$

Since $u^{(0)}$ is known, we can compute $B_\theta^{(1)} = z f(r)$. Further, since $B_r^{(1)} = 0$, Eq. (8) indicates that apart from a constant, $B_z^{(1)} = r f(r)$. In terms of $f(r)$, the currents are thus, $j_r^{(1)} = -f(r)/\mu_0$, $j_\theta^{(1)} = -\sigma u^{(0)} r f(r)$, $j_z^{(1)} = \sigma u^{(0)} z f(r)$.

The solution of Eq. (12) is

$$f = \frac{F}{r} \exp\left[\mu_0 \sigma \int_{r_0}^r u^{(0)} dr\right], \quad \text{II(13)}$$

where F is a constant. Thus, the condition which must be satisfied in order that the current flow be radial, is clearly that $\mu_0 \sigma \int_{r_0}^r u^{(0)} dr \ll 1$. In this case, $f = F/r$, and $j_r^{(1)} = -F/\mu_0 r$. If the current at r_0 , the outside electrode, is $j_r^{(1)}(r_0)$, then we have $r j_r^{(1)} = r_0 j_r^{(1)}(r_0)$ and $F = -\mu_0 r_0 j_r^{(1)}(r_0)$, whence $j_z^{(1)} = -\mu_0 \sigma u^{(0)} r_0 j_r^{(1)}(r_0) z/r$, and $j_\theta^{(1)} = \mu_0 \sigma u^{(0)} r_0 j_r^{(1)}(r_0) c$, so that in the above limit both $j_z^{(1)}$ and $j_\theta^{(1)}$ tend to zero.

Now $E_r^{(1)}$ may be computed from Eq. (7), which gives,

$$E_r^{(1)} = r_0 j_r^{(1)}(r_0) \left[\frac{1}{\sigma} + \mu_0 \Gamma \right] \frac{1}{r}. \quad \text{II(14)}$$

Integrating, we find the first order correction to the potential difference to be

$$(V_0 - V_i)^{(1)} = (J/2\pi) \left[\frac{1}{\sigma} + \mu_0 \Gamma \right] \log(r_0/r_i), \quad \text{II(15)}$$

where $J = 2\pi r_0 j_r^{(1)}(r_0)$ is the total current per unit length.

The Possibility of Power Generation by Vortices

From Eq. (6), it follows that for $\gamma = 1.667$, a temperature of 2000°K , a Mach number of unity, and a radius ratio of 3, the potential difference for zero current flow is 1.15 volts. In this range of temperature, the conductivity of gases is small enough that the term $\mu_0 \Gamma$ in Eq. (15) is completely dominated by $1/\sigma$. For cesium at 2000°K and one atmosphere, for example, σ is about 10 mho/M . Thus, a cesium vortex under these conditions would have a voltage-current characteristic with an initial slope of about $-.018 \text{ volts/(amp/M)}$. The potential difference would be one volt at a power of 6.4 watts/M . Although this is a very small power, the performance would improve rapidly with increases in temperature, because of the increases in both σ and kT/e_i .

III. CONCLUDING REMARKS

Vortices with high tangential and low radial Mach numbers are of considerable technological interest, as well as the simplest flows in which to investigate the mechanism of diffusion due to strong pressure gradients normal to the principal flow direction.

As far as the realization of these flows is concerned, a major obstacle is presented by the combination of low radial Mach number and high tangential Mach number, which results in very strong viscous retardation if the vortex is bounded by a stationary wall. Rotation of the wall is a possible solution for gases of high molecular weight, but is not attractive when the speed of sound in the gas is high.

Two problems, not treated here, which may be of interest are the influence of pressure diffusion on the heat transfer in a dissociating gas, and the influence of heat addition, by chemical or other means, on the diffusion. The heat transfer in a reacting gas between two parallel walls at different temperatures has been treated by Hirschfelder ⁽⁷⁾. Conductive heat transfer is augmented by the diffusion of the dissociation products to the walls, and their surface recombination. It is to be expected that a pressure gradient would either increase or decrease this heat transfer, depending on whether it enhances or retards the diffusion. In the vortex, the heat transfer between concentric cylinders would be increased if the outer cylinder were hotter than the inner cylinder, and decreased if it were colder.

Heat addition, through its effect on the gas temperature would provide a means for establishing a quite different variation of the tangential Mach number with radius than is found for any adiabatic flow. The resulting concentration profiles would then be qualitatively different from those found in the present study.

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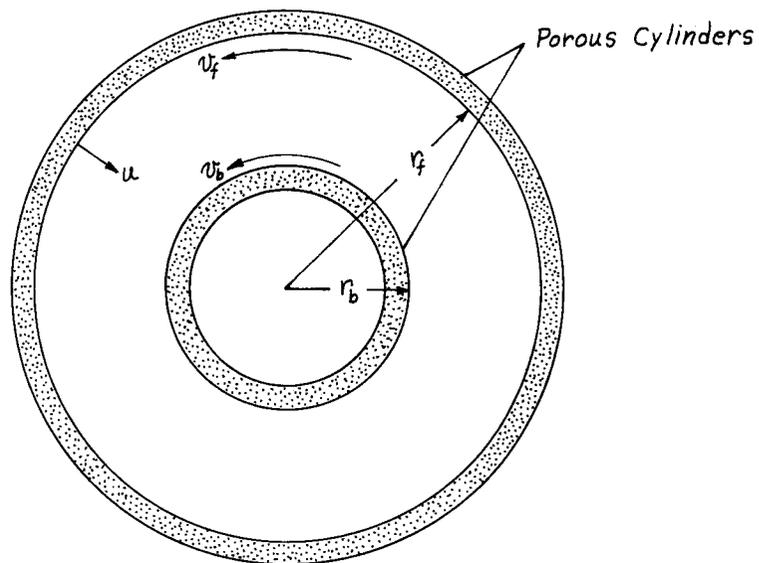


Fig. 1. Model for a One-Region Vortex Flow.

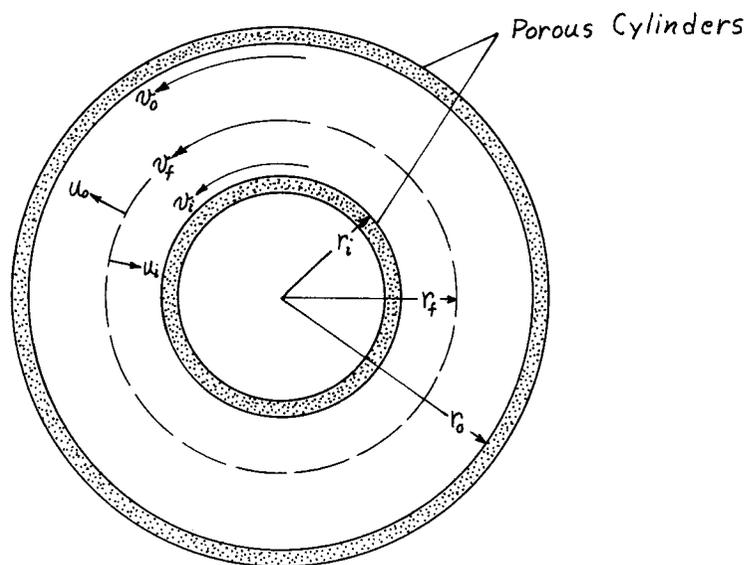


Fig. 2. Model of Two-Region Vortex Flow, for Gas Separation.

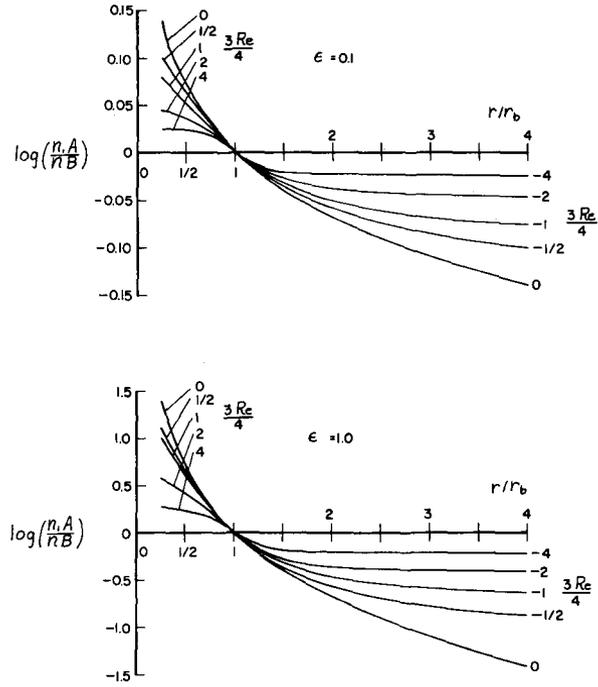


Fig. 3. Solutions of the Diffusion Equation for $v = v_b, T = T_b$.

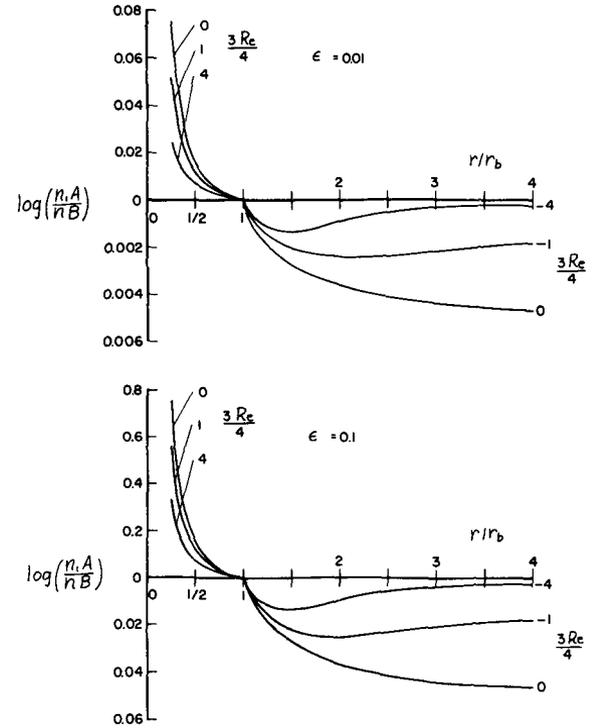


Fig. 4. Solutions of the Diffusion Equation for $vr = \Gamma_b, T = T_b$.