ANALYSIS OF PRESSURE OSCILLATIONS IN RAMJETS; REVIEW OF RESEARCH: 1983–1988 *

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ABSTRACT

This paper is a summary of work carried out during the past five years as part of an ONR/NAVAIR Research Initiative. The effort has been devoted to developments and applications of an approximate analysis intended to provide understanding of observed behavior and guidelines for design and development. Although most of the work has been carried out for longitudinal modes, quite extensive results have been obtained for transverse modes in a circular chamber, for both second and third order nonlinear gasdynamics. Especially important are the useful conclusions based on the simplified two-mode approximation. Preliminary results obtained for the influences of stochastic sources suggest that some of the observed behavior of the amplitudes of oscillations may be attributed to random inputs, such as flow separation and turbulence. Some important aspects of the problem of existence and stability of limit cycles in linearly stable systems ("triggering") remain unclear.

1. INTRODUCTION

This program has been devoted to analytical work concerned with answering the question: what causes unsteady motions in ramjet engines to grow and reach the limiting amplitudes observed in practice? The effort therefore encompasses both the causes of mechanisms of instabilities; and the linear and nonlinear processes influencing the time evolution of the motions. We are primarily concerned with application of an approximate analysis, in contrast to numerical analysis carried out to produce solutions of "complete" equations of motion. Subjects covered here include results for linear and nonlinear analysis of acoustics with vortex shedding entropy waves; application of the method of averaging to problems having arbitrary distributions of modal frequencies; explicit formulation of Rayleigh's criterion; and the excitation of nonlinear acoustic waves by random sources.

The approximate analysis, within which almost all of the work has been accomplished in this program, offers several important advantages. First, because it begins by replacing the partial differential conservation equations by total differential equations, the expense of obtaining specific results is greatly reduced. Second, the formulation is generally applicable to combustion instabilities in any combustion chamber. A recent review [Culick (1988)] has been given of instabilities in the three main types of liquid-fueled propulsion systems — liquid rockets, gas turbines (mainly thrust augmentors) and ramjet engines. One purpose was to suggest how the various phenomena can be accommodated within the framework of the approximate analysis used here. In particular, one may treat any known mechanism; the problem of analyzing a special case comes down to modeling the important processes. Thus it becomes possible to assess quantitatively the relative influences of the energy gains and losses associated with the physical processes accounted for.

Third, for theoretical purposes, the formulation is convenient because any combustion instability is represented as a system of coupled nonlinear oscillators. This form permits easy interpretation of the behavior and is accessible to analysis by contemporary methods of nonlinear dynamical systems [Paparizos and Culick (1988a, 1988b)]. A corollary is that for practical purposes, numerical computations can be routinely performed with assurance that the truth of the results can be established, within, of course, the physical approximations used to establish the initial system of equations. Insufficient work has been done to determine broadly the accuracy of the method. Early work some years ago, and a limited amount of recent work [Culick and Yang (1989)] have shown that the accuracy is very good indeed for both linear and nonlinear behavior, at least when the amplitudes of pressure oscillation are less than 10%.

Finally, the structure of the formulation, in which an arbitrary motion is expressed as a synthesis of normal modes, lends itself directly to the general problem of active control. The recent developments of high-capacity, light-weight and inexpensive high-speed computers makes conceivable the active control of combustion instabilities in full-scale propulsion systems. A few laboratory tests at Cambridge University and Ecole Centrale have shown some success controlling oscillations in small devices. Application to large systems is far off and will no doubt require many sensors and control inputs. Development of practical systems will require

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a sweeping combination of theoretical and experimental work in combustion instabilities and control systems. Representation of the physical system in the form used here constitutes the most natural framework for applying modern control theory. Yang et al (1988) have given an initial discussion of this subject.

Details of the formulation of the analysis and results of this program have been given in many papers prepared during the past five years. The discussion here will therefore be confined to a brief summary.

2. APPROXIMATE ANALYSIS

In general, the presence of a liquid phase must be accounted for. The conservation equations for the two phases can be combined in the following convenient form [Culick and Yang (1989); Culick (1988)]:

$$\frac{\partial \rho}{\partial t} + \vec{u}_g \cdot \nabla \rho = \mathcal{W} \tag{2.1}$$

$$\rho \frac{\partial \vec{u}_g}{\partial t} + \rho \vec{u}_g \cdot \nabla \vec{u}_g = -\nabla p + \vec{\mathcal{F}} \tag{2.2}$$

$$\frac{\partial p}{\partial t} + \bar{\gamma} p \nabla \cdot \vec{u}_g = -\vec{u}_g \cdot \nabla p + \mathcal{P} \tag{2.3}$$

where $\vec{u}_g$ is the gas velocity, but ρ is the mass-averaged density of the two-phase mixture. The formulas for the source terms are given in the references cited and need not be repeated here.

We shall drop the subscript on the velocity, $\vec{u}_g \rightarrow \vec{u}$, and all of the following discussion will be phrased as if we are dealing with a single gas. That simplifies matters without excluding any essentials. In our work, we have accounted explicity for the liquid phase in only one specific problem reported by Yang and Culick (1984a).

All dependent variables are now written as sums of mean and fluctuating values and terms retained to the desired order. Eventually a nonlinear wave equation for the pressure fluctuation can be formed, with its boundary condition:

$$\nabla^2 p' - \frac{1}{\bar{a}^2} \frac{\partial^2 p'}{\partial t^2} = h \tag{2.4}$$

$$\hat{n} \cdot \nabla p' = -f \tag{2.5}$$

We now use a form of the method of least residuals, essentially Galerkin's method. This approach was first applied to combustion instabilities in liquid rockets by Zinn and Powell (1968, 1970). Independently, essentially the same idea was worked out for solid propellant rockets by Culick (1971, 1975, 1976), the basis for the discussion here.

For most problems of practical interest, the motions may be approximated quite well as combinations of a small number of classical acoustic modes, denoted here $\psi_n(r)$. Thus it is reasonable to expand the field in the normal modes with time varying amplitudes:

$$p' = \bar{p} \Sigma \eta_n(t) \psi_n(\vec{r}) \tag{2.6}$$

$$\vec{u}' = \Sigma \frac{1}{\gamma {k_n}^2} \dot{\eta}_n(t) \nabla \psi_n(\vec{r}) \tag{2.7}$$

The classical modes satisfy the equations for problems without sources:

$$\nabla^2 \psi_n + {k_n}^2 \psi_n = 0 \tag{2.8}$$

$$\hat{n} \cdot \nabla \psi_n = 0 \tag{2.9}$$

In some situations, it may be useful to use an inhomogeneous boundary condition for ψ_n. It is a practical necessity to ensure that the ψ_n are orthogonal,

$$\int \psi_n \psi_m dV = {E_n}^2 \delta_{nm} \tag{2.10}$$

Substitution and some manipulations with equations (2.4) and (2.5) lead to the equation for η_n:

$$\frac{d^2 \eta_n}{dt^2} + {\omega_n}^2 \eta_n = F_n \tag{2.11}$$

where

$$F_n = -\frac{\bar{a}^2}{\bar{p}E_n{}^2}\left\{\int h\psi_n dV + \oiint f\psi_n ds\right\} \tag{2.12}$$

These two equations are the basis for the approximate nonlinear analysis.

A major part of the effort in analyzing specific problems divides into two parts: construction of appropriate contributions to the functions h and f; and solution to the linear problems associated with the physical situation being studied. Then nonlinear problems can be treated. Thus, for example, here we shall discuss the linear behavior of acoustics with entropy waves before nonlinear behavior is analyzed.

3. APPLICATION OF THE METHOD OF AVERAGING

The observed behavior of interest here generally consists of oscillations having slowly varying amplitudes and phases. Thus it is sensible to write the amplitudes $\eta_n(t)$ in the form

$$\eta_n(t) = r_n(t)\sin\big(\omega_n t + \phi_n(t)\big) = A_n(t)\sin\omega_n t + B_n(t)\cos\omega_n t \tag{3.1}$$

Then the "velocity" of the oscillator is

$$\dot{\eta}_n = \omega_n\big[A_n\cos\omega_n t - B_n\sin\omega_n t\big] + \big[\dot{A}_n\sin\omega_n t + \dot{B}_n\cos\omega_n t\big]$$

Because a single function $\eta_n(t)$ has been replaced by two functions, $A_n(t)$ and $B_n(t)$, we are free to place a restriction. Namely, we require that the second set of terms vanish always,

$$\dot{A}_n\sin\omega_n t + \dot{B}_n\cos\omega_n t = 0 \tag{3.2}$$

so the velocity has the same form as that for an oscillator having fixed energy:

$$\dot{\eta}_n = \omega_n\big[A_n\cos\omega_n t - B_n\sin\omega_n t\big] \tag{3.3}$$

Substitution in (2.11) and some arithmetic leads to the equations for A_n and B_n, still with no approximations:

$$\begin{aligned}\frac{dA_n}{dt} &= \epsilon\frac{1}{\omega_n}F_n\cos\omega_n t\\ \frac{dB_n}{dt} &= -\epsilon\frac{1}{\omega_n}F_n\sin\omega_n t\end{aligned} \tag{3.4}a,b$$

The quantity ϵ has been introduced as a measure of the "smallness" of F_n : $\epsilon \to 0$ corresponds to vanishingly small perturbations or sources, represented in F_N.

When $\epsilon = 0$, the amplitudes and phases of η_n are constant; when $\epsilon \neq 0$ but is small, the amplitudes and phases vary slowly. It is convenient to introduce two time scales: $\tau_n = 2\pi/\omega_n$ the period of the oscillation, and the long time scale τ_n/ϵ characterizing slow variations. Correspondingly, we define the "fast" and "slow" dimensionless time variables,

$$t_f = \omega_n t \quad ; \quad t_s = \epsilon\omega_n t \tag{3.5}$$

Hence equations (2.4)a,b may be written

$$\begin{aligned}\frac{dA_n}{dt_s} &= \frac{1}{\omega_n{}^2}F_n\\ \frac{dB_n}{dt_s} &= -\frac{1}{\omega_n{}^2}F_n\end{aligned} \tag{3.6}a,b$$

Now the approximation is that A_n and B_n are functions *only* of τ_s and these equations are averaged over some interval τ in the fast variable:

$$\begin{aligned}\frac{1}{\tau}\int_{t_f}^{t_f+\tau}\frac{dA_n}{dt_s}d\tau_f{}' \cong \frac{dA_n}{dt_s} &= \frac{1}{\omega_n{}^2}\frac{1}{\tau}\int_{t_f}^{t_f+\tau}F_n\cos t_f' dt_f'\\ \frac{dB_n}{dt_s} &= -\frac{1}{\omega_n{}^2}\frac{1}{\tau}\int_{t_f}^{t_f+\tau}F_n\sin t_f' dt_f'\end{aligned} \tag{3.7}a,b$$

The function F_n contains three types of terms: those which vary rapidly — i.e. depend harmonically on the fast variable t_f; factors which are functions of t_s, and hence vary slowly; and some oscillatory terms having periods which are less obviously short or long. The slowly varying terms (usually products of the A_n and B_n) are regarded as constant over the interval of averaging and hence may be taken outside the integral, but still treated as function of t_s. The rapidly varying functions must be retained under the integral.

The third kind of term is handled less unambiguously. These arise, for example, from products of harmonics in $\omega_n t$, leading to sum and difference terms. Difference terms, say $\sin(\omega_m - \omega_n)t$ may be slowly or rapidly varying depending on whether $\omega_m - \omega_n$ is large or sufficiently smaller than ω_n. Thus, there is no rigorous rule for treating such terms, and it is really a matter of experience to determine what constitutes slowly varying.

Some elementary special cases have been treated to gain some idea of the range of validity of the method. For linear oscillations, with one or two modes accounted for, the method of averaging reproduces exact solutions to first order in the parameter ϵ characterizing the perturbations. For nonlinear oscillations, the case of two modes, with nonlinear arising from second order gasdynamics, both the method of averaging and the method of multiple time scales have been applied. Again good agreement has been obtained to first order in ϵ.

More generally, the question of validity is less easy to answer when many modes are accounted for. Some work has been to determine the conditions under which the method exhibits unacceptable errors but the results are incomplete [Yang and Culick (1986)]. For example, the amplitude of the tenth harmonic, say, may suffer a small fractional change during a single period of its motion, but the change during one period of the fundamental mode may be substantial. On the other hand, contributions from high frequency components may have relatively small effects on the global behavior of the system, although this is not necessarily true. Hence it is unlikely that firm universal rules can be established.

4. TRANSVERSE MODES

Concern with the method of averaging first arose in connection with problems involving transverse modes. More generally, the issues addressed in the preceding section must be faced in any problem for which the natural frequencies are not those for purely longitudinal modes, i.e. those for which $\omega_n = n\omega_1$, is *not* true. Hence the results described in the preceding section have quite general applicability, but they are best illustrated, and checked, by treating the case of transverse modes in a cylindrical chamber.

Compared with problems of longitudinal modes, which to date are the only cases treated with the approximate analysis, transverse modes immediately offer additional complications for two reasons. First, there are two types of modes, radial and tangential. And second, the tangential may be either standing or spinning. It is true that longitudinal modes may be either standing or travelling, but it happens that there are qualitative differences between travelling longitudinal modes and spinning tangential modes. Some of these were noted in an early paper by Maslen and Moore (1956) who showed that there is much less tendency for nonlinear steepening of spinning tangential modes.

To begin study of the transverse modes, we restrict the analysis to three modes: the first two tangential modes and the first radial mode, characterized by the following wavenumbers κ_n and mode shapes ψ_n:

	1st Tangential	1st Radial	2nd Tangential
Wavenumber	$\kappa_1 = 1.8412$	$\kappa_2 = 3.8317$	$\kappa_3 = 3.0542$
Mode Shapes	$\psi_1 = \cos\theta J_1(\kappa, r)$ $\psi_4 = \sin\theta J, (\kappa, r)$	$\psi_2 = J_0(\kappa_2 r)$	$\psi_3 = \cos 2\theta J_2(\kappa_3 r)$ $\psi_5 = \sin 2\theta J_2(\kappa_3 r)$

For *standing* transverse oscillations, we consider ψ_1, ψ_2, and ψ_3. From the analysis described in Sections 1 and 2, we find the following equations for the A_n and B_n:

$$\begin{aligned}\frac{dA_1}{dt} &= \alpha_1 A_1 + \theta_1 B_1 + a_1\{(A_1A_2 + B_1, B_2)\cos\Omega_1 t + (A_1B_2 - A_2B_1)\sin\Omega_1 t\}\\ &\quad + a_2\{(A_1A_3 + B_1B_3)\cos\Omega_2 t + (A_1B_3 - A_3B_1)\sin\Omega_2 t\}\\ \frac{dB_1}{dt} &= \alpha_1 B_1 - \theta_1 A_1 - a_1\{(A_1A_2 + B_1B_2)\sin\Omega_1 t - (A_1B_2 - A_2B_1)\cos\Omega_1 t\}\\ &\quad - a_2\{(A_1A_3 + B_1B_3)\sin\Omega_2 t - (A_1B_3 - A_3B_1)\cos\Omega_2 t\}\end{aligned} \qquad (4.1)a,b$$

$$\frac{dA_2}{dt} = a_2A_2 + \theta_2B_2 + b_1\{A_1{}^2 - B_1{}^2) \cos\Omega_1 t - 2A_1B_1 \sin\Omega_1 t\}$$
$$\frac{dB_2}{dt} = \alpha_2B_2 - \theta_2A_2 + b_1\{(A_1{}^2 - B_1{}^2) \sin\Omega_1 t + 2A_1B_1 \cos\Omega_1 t\} \qquad (4.2)a,b$$

$$\frac{dA_3}{dt} = \alpha_3A_3 + \theta_3B_3 + b_2\{(A_1{}^2 - B_1{}^2) \cos\Omega_2 t - 2A_1B_1 \sin\Omega_2 t\}$$
$$\frac{dB_3}{dt} = \alpha_3B_3 - \theta_3A_3 + b_3\{(A_1{}^2 - B_1{}^2) \sin\Omega_2 t + 2A_1B_1 \cos\Omega_2 t\} \qquad (4.3)a,b$$

where

$$\Omega_1 = 2\omega_1 - \omega_2$$
$$\Omega_2 = 2\omega_1 - \omega_3 \qquad (4.4)a,b$$

Note that $\omega_n = \bar{a}k_n$, so $2\omega_1 - \omega_2 = \bar{a}(2k_1 - k_2)$ and $2\omega_1 - \omega_3 = \bar{a}(2k_1 - k_3)$ are less than one-third as large as ω_1 and are regarded as "small": i.e. $\cos\Omega_1 t \cos\Omega_2 t$ etc. have been treated as "slowly varying functions." The presence of these time dependent factors is crucial. The equations for longitudinal modes do *not* contain such terms: that is, they are autonomous equations, the right hand sides being independent of time.

The coefficients a_1, a_2, b_1, b_2 are independent of frequency and are known but not given here, they arise from the integrals on the definition of F_n. Conditions for existence and stability of limit cycles for these equations can be determined with calculations similar to those reported by Awad and Culick (1985). For longitudinal modes, a stable limit cycle exists in which the A_n, B_n are constant, so η_2 oscillates at frequency ω_2, twice the frequency of η_1, when only two modes are considered.

A similar result is found here when only the first tangential and first radial modes are treated, except that the functions A_n, B_n oscillate: $A_1 = r_1 \cos(\nu_1 t + \varsigma_1)$ etc. But eventually one finds that the amplitudes are given by

$$\eta_1(t) = r_1 \sin[\omega_1 + \nu_1)t + \varsigma_1]$$
$$\eta_2(t) = r_2 \sin[2(\omega_1 + \nu_1)t + \varsigma_2] \qquad (4.5)a,b$$

where ν_1, ς_1, and ς_2 are constants depending on α_1, θ_1, α_2, θ_2. Thus the amplitude of the first radial mode oscillates at exactly twice the frequency of the first tangential. This result was found in numerical calculations done previously at Georgia Tech.

The case with three modes has also been treated. It is a curious result that in the limit cycle the second tangential mode (i.e. η_3) oscillates at the same frequency as the first tangential. We have no explanation for this "frequency locking."

To treat a spinning mode, we add the equations for ψ_4 and ψ_5. The first tangential mode then contains both parts, ψ_1 and ψ_2 containing $\cos\theta$ and $\sin\theta$ respectively. By summing these with appropriate phases in time, a spinning wave can be constructed, varying as $\cos(\omega_1 t + \phi_1)$. Similar remarks apply to the second tangential mode.

Two cases have been treated, each for two modes. The first is for the first tangential and first radial modes. Once again, the conditions for a stable limit cycle can be determined. The frequencies and phases are the same for the spinning and standing modes in the limit cycle but the magnitude r_1 of the first tangential mode is reduced by $1/\sqrt{2}$ for the spinning wave. These formal results have been confirmed by numerical integration of the differential equations. An interesting theoretical result is that if linear coupling between the modes is absent (i.e. $\alpha_n = \theta_n = \Omega$) then the limit cycle is *always* a standing wave, irrespective of the initial conditions.

The second case is for the first and second tangential modes. It is not possible to deduce simple conclusions for this case. There are eight algebraic equations for the four magnitudes and four phases required, and no simple solutions have been found. It appears that standing, spinning, and possibly some mixed forms, are all possible. Details of the work on transverse modes have been reported by Yang and Culick (1986, 1988).

5. VORTEX SHEDDING AND CONVECTIVE WAVES AS MECHANISMS OF COMBUSTION INSTABILITIES

Research during the past eight years has established that vortex shedding and convective waves are two dominant causes of combustion instabilities in ramjet combustors. Discussions of these phenomena may be found in many recent papers covering both experimental work and interpretation of the results. Accommodation of those mechanisms within the framework described here has been examined by Culick (1988a, 1988b).

For some time there has been considerable interest in the possible importance of "entropy waves" in ramjet combustors. The chief concern has been the possibility that fluctuations of entropy convected by the mean flow into a choked exhaust nozzle may produce pressure fluctuations which propagate upstream. Subsequent interactions with the combustion zone at the dump plane may then generate entropy fluctuations, closing the cycle.

Entropy fluctuations are local inhomogeneities of temperature or density caused by irreversible processes. In addition to combustion, flow separation and oscillations of shock waves may be significant sources. In this work, to investigate some of the elementary aspects of these processes, and in particular to analyze the stability of pressure oscillations, we examine an idealized model of a dump combustor.

The inlet shock system is represented by a single normal shock, with a uniform inlet to the combustor. All combustion processes are collapsed to a single plane flame, the flow downstream being uniform to the exhaust nozzle. Linear acoustic and entropy waves are coupled at the inlet shock, the flame, and the nozzle. Analysis is carried out for steady waves; results are then found for stability and mode shapes of the various normal modes of the system.

Results obtained by Culick and Rogers (1983) are used to represent the unsteady behavior of the normal shock. The response of the normal flame is based on the work of Chu (1953). Interactions of entropy waves with an exhaust nozzle have been most thoroughly developed in the work by Marble (1973) and Marble and Candel (1977, 1978); similar considerations have been incorporated in the present work.

Much of the work has been concerned with fitting the pieces of the problem together to form an elementary model of a dump combustor in which both acoustic waves and entropy fluctuations are accounted for. The three main intentions have been to investigate the stability of the system; to study the influences of entropy waves on the classical acoustic modes; and to determine the possible existence of new modes.

A wide range of special cases have been treated, thoroughly documented in the thesis by Humphrey (1987). The formal analysis comes down to matching one-dimensional waves, the chief parameters being associated with the boundary conditions and the characteristics of the flame. Resonant frequencies are found as solutions to a transcendental equation.

The oscillating shock wave is a source of entropy waves incident upon the flame. Interactions then cause a conversion of combustion energy to acoustic energy and hence are sources of driving which depends upon both frequency and location of the flame. Both the nozzle and the shock wave absorb acoustic energy because they transmit portions of incident waves. Hence, existence of an unstable mode requires that the flame provide sufficient conversion of combustion energy to acoustic energy to compensate the losses in the inlet and the nozzle.

An isolated plane flame is intrinsically stable in a diverging duct, unstable in a converging duct, and neutrally stable in a uniform channel. An entropy wave incident on a flame in a uniform channel will cause the flame to oscillate about its mean position, emitting pressure both up and downstream. Calculations have been done for an artificial case in which the upstream acoustic wave produces the entropy wave at some upstream location but is not itself reflected. The flame is then found to be stable only at a discrete frequency and a particular value of the amplitude of the entropy fluctuation.

Special cases have been computed for a flame with a downstream chocked nozzle and a perfectly reflecting upstream inlet, but with incident entropy waves. The results show unstable modes in the low frequency range, providing the entropy fluctuations are sufficiently large.

Computations have been carried out to find neutrally stable steady oscillations for the complete system. In addition to modes which reduce to classical acoustic modes in the absence of entropy, there are modes which require entropy fluctuations for their existence. These are reasonably called entropy modes, but involve pressure waves as well. There is usually one entropy mode having frequency less than that of the fundamental acoustic mode. The frequencies of the higher entropy modes are approximately integer multiples of the frequency for the lowest entropy mode. The fact that the frequencies of coupled entropy/acoustic modes are measurably different from the classical acoustic frequencies may afford a means for determining experimentally when convective waves are an important mechanism. We should note that similar calculations can be carried out for convected vorticity waves.

Some results have been obtained for the influence of entropy fluctuations on nonlinear acoustics. The purposes are to incorporate the results described in the preceding section in the approximate analysis developed in reference 1, and to include nonlinear entropy/acoustic interactions. At present this effort is far from complete.

The physical problem treated is that formulated above, for a plane flame in a dump combustor with a single normal shock wave and choked exit nozzle. Unfortunately, the approximate analysis described in Section 2 becomes more complicated because the boundary conditions strictly produce a set of basic functions ψ_n which are not orthogonal. An orthogonal set can be formed using the Gramm-Schmidt procedure, but the formalism then becomes considerably more cumbersome. Those difficulties have been avoided by using idealized boundary conditions giving an orthogonal set directly. Entropy waves produced at the upstream boundary are assumed to be proportional to the incident pressure waves and hence can be expanded in the same series of basic functions but with amplitudes proportional to the η_n.

With the approximations used in these calculations, the gasdynamic nonlinearities still dominate. Hence the results are modified, from earlier analyses for purely acoustic waves, primarily by the influences of the linear coefficients which depend strongly on the presence of entropy waves. The presence of the linear entropy modes have been accounted for.

Although a simple model has been proposed, vortex shedding and combustion has not been analyzed within the framework used in this program. Details of these results for entropy waves have been reported by Humphrey and Culick (1986, 1987a, 1987b).

6. EXPLICIT FORMULATION OF RAYLEIGH'S CRITERION

Rayleigh formulated a concise statement of the qualitative conditions under which unsteady heat addition will drive acoustic waves. Probably no other principle has been so widely invoked in studies of combustion instabilities. Sometimes the idea has been incorrectly extended or applied. Also, recent work intended to confirm the criterion by using measurements of light emission as an indicator of heat addition has raised questions concerning its explicit form. In particular, while the criterion apparently refers to the unsteady heat addition Q' per unit volume and time, the wave equation (2.4) has its rate of change, $\partial Q'/\partial t$ as a source term. Moreover, it has not been clear how the criterion should be applied quantitatively to nonlinear oscillations that do not have simple sinusoidal variations in time.

Those considerations prompted the calculations leading to a note on the criterion, Culick (1987). It was shown there that the approximate analysis could be used to establish the desired formula valid for each mode. Define $\mathcal{E}_n$, the energy of the oscillator associated with the nth acoustic mode,

$$\mathcal{E}_n = \frac{1}{2}(\dot{\eta}_n^2 + {\omega_n}^2{\eta_n}^2) \tag{6.1}$$

Note that this is the energy calculated for the oscillator whose motion is governed by equation (2.11); the units of $\mathcal{E}_n$ are therfore sec^{-1}. The idea now is to calculate the change of energy, $\Delta\mathcal{E}_n$, in one cycle τ_n of oscillation. This is non-zero due to the force F_n which does work at the rate $F_n\dot{\eta}_n$; consequently, $\Delta\mathcal{E}_n$ is

$$\Delta\mathcal{E}_n = \int_t^{t+\tau_n} F_n\dot{\eta}_n dt' \tag{6.2}$$

If only heat addition is accounted for, this relation becomes

$$\Delta\mathcal{E}_n = (\bar{\gamma}-1)\frac{{\omega_n}^2}{\bar{p}^2{E_n}^2}\int dV \int_t^{t+\tau_n} p_n Q' dt' \tag{6.3}$$

where $p_n' = \bar{p}\eta_n(t)\psi_n(\bar{r})$ is the time-dependent pressure in the nth mode. This formula is the explicit formulation of Rayleigh's criterion, valid for nonlinear as well as linear oscillations.

The criterion has been generalized by Culick (1988a) to account for all sources of energy gain and loss. One should then expect that $\Delta\mathcal{E}_n$ is then related to the growth constant α of a linear motion for α is proportional to the fractional rate of change of energy in the mode. That is indeed the case, and the simple relation holds,

$$\Delta\mathcal{E}_n = 2\pi\omega_n\alpha \tag{6.4}$$

7. THE TWO-MODE APPROXIMATION TO NONLINEAR COMBUSTION INSTABILITIES

The approximate analysis produces initially an infinite set of equations, one second order or two first order equations for each classical acoustic mode. Even for the simplest cases, the infinite set cannot be solved in closed form. However, if only two modes are considered, an exact solution can be found for the limit cycles. In itself that result is less important than the fact that it provides a convenient starting point for learning how the hierarchy of modes might be reasonably replaced by two modes only. This is also a basis for examining the consequences of including special or higher order nonlinear processes.

When two modes are considered, the method of averaging produces four coupled nonlinear equations. For longitudinal modes the general forms of the equations for $A_n(t)$ and $B_n(t)$ are defined in equation (3.1) are

$$\frac{dA_n}{dt} = \alpha_n A_n + \theta_n B_n + \frac{n\beta}{2}\sum_{i=1}^{n-1}(A_iA_{n-i} - B_iB_{n-i}) - n\beta\sum_{i=1}^{\infty}(A_{n+i}A_i + B_{n+i}B_i) \tag{7.1}$$

$$\frac{dB_n}{dt} = \alpha_n B_n - \theta_n A_n + \frac{n\beta}{2}\sum_{i=1}^{n-1}(A_iB_{n-i} + B_iA_{n-i}) + n\beta\sum_{i=1}^{\infty}(A_{n+i}B_i - B_{n+i}A_i) \tag{7.2}$$

where $\beta = \omega_1(\bar{\gamma}+1)/8\bar{\gamma}$. Set $n = 1, 2$ in succession gives the four equations. By transformation worked out by Paparizos and Culick (1988a), the four equations can be reduced to three:

$$\begin{aligned}
\frac{dy_1}{dt} &= (\alpha_1 + y_2)y_1 \\
\frac{dy_2}{dt} &= \alpha_2 y_2 + |\theta_2 - 2\theta_1|y_3 + 2{y_3}^2 - {y_1}^2 \\
\frac{dy_3}{dt} &= -|\theta_2 - 2\theta_1|y_2 + \alpha_2 y_3 - 2y_2y_3
\end{aligned} \tag{7.3)a,b,c \text{(}$$

where

$$\begin{aligned}
y_1 &= \beta r_1 \\
y_2 &= \beta r_2 \sin(\phi_2 - 2\phi_1) \\
y_3 &= \frac{|\theta_2 - 2\theta_1|}{\theta_2 - 2\theta_1}\beta r_2 \cos(\phi_2 - 2\phi_1)
\end{aligned} \tag{7.4)a,b,c \text{(}$$

and ${r_n}^2 = {A_n}^2 + {B_n}^2$. The reduction of the number of equations by one is due to the arbitrary definition of the phase of the oscillations. Here it is fixed by setting the phase equal to zero for the component y_1 in the new variables $(y_1,\, y_2)$.

The system (7.3)a,b,c allows exact solution and has the great advantage that because there are three dependent variables, the behavior can be displayed in three-dimensional space. Especially important is the existence of stable periodic solutions of the original equations for $(\eta_1(t),\ \eta_2(t))$, which here arise as fixed points in the space $(y_1,\, y_2,\, y_3)$: $dy_n/dt = 0$. Some of the results were previously obtained by Award and Culick (1985), but those have been extended to cover arbitrary values of the parameters $(\theta_1,\, \theta_2)$ that represent frequency shifts in the limit cycle. Figure 1 shows the regions of parameters in which stable limit cycles exist. Note that the two cases are shown: $(\alpha_1 > 0,\, \alpha_2 < 0)$ and $(\alpha_1 < 0,\, \alpha_2 > 0)$ representing the circumstances for which the first mode is stable and unstable respectively. If one mode is unstable, the other must be stable in order that the energy of the system remain constant in the limit cycle.

For the common case of an unstable first mode, the general global behavior can be displayed in the simple form shown in Figure 2. Independently of the initial conditions, all solutions to a single fixed point P whose location on a curved line — technically called a one-dimensional "center manifold" in nonlinear dynamical systems theory — is fixed by the values of the parameters $(\alpha_n\ \theta_n)$ characterizing the system.

This formulation permits systematic analysis of many of detailed features on nonlinear behavior not readily understood from purely numerical solutions to the equations (7.1) and (7.2). Numerical solutions have confirmed the accuracy of truncation to two modes if the first mode is unstable. Because the governing processes (nonlinear gasdynamics) preferentially cause flow of energy from lower modes to higher modes, truncation of the system to two modes does not produce reliable results if the upper mode is unstable.

8. 'TRIGGERING' OF COMBUSTION INSTABILITIES

If a system is linear stable but unstable to a sufficiently large initial disturbance, it is said to be nonlinearly unstable; the phenomenon has come to be called 'triggering'. Usually the distinction is not made whether the subsequent instability does or does not evolve to a stable limit cycle.

Nonlinear instabilities have been found mainly in liquid rockets and solid rockets. The use of small explosive charges to assess the stability of motors has long been standard practice, a test intended to simulate the effect of chunks of material passing through the nozzle, or, in liquid rockets, rapid burning of accumulated liquids. Rapid changes of fuel flow can produce similar effects in a ramjet engine. Thus analysis of triggering is directed to an important practical problem. The subject also raises particularly interesting theoretical questions that have yet to be answered.

One trivial reason for triggering — but possibly true in many cases — is dependence of the growth constant α on the amplitude of the initial disturbance. If all α_n are negative for small amplitudes, an arbitrary disturbance is stable. But if one or more α_n should become positive when the initial amplitude is sufficiently large, then the motion is unstable. Such a possibility could arise if the parameters defining the system — i.e. the processes contributing to the α_n — are drastically affected by large disturbances.

Here, however, we are concerned with determining the conditions under which triggering may occur when the linear parameters (α_n, θ_n) are strictly constant. Thus the results must depend entirely on the form and structure of the nonlinear terms in the governing equations (2.4), (2.5), (2.11), and especially (3.7)a,b which have been used in all the analyses carried out in this program.

Much work was done on triggering in the 1960's and early 1970's because of its importance in the development of liquid rockets. All of the results were based on numerical solutions to the one-dimensional partial differential conservation equations or to the second order equations (2.11) for the amplitudes. The purpose of our work has been to obtain explicit formal results if possible, but in any event to clarify the general conditions under which triggering will occur.

On the basis of the equation for a single degree of freedom system one would expect that triggering requires nonlinear terms of at least third order, and will likely occur if third order terms are included. In most of our work, we have used equations containing only second order nonlinearities, such as those shown in equations (7.1), (7.2) and (7.3)a,b,c. Therefore we have investigated solutions to the equations obtained when the gasdynamic nonlinearities are carried out to third order [Yang, Kim, and Culick (1987, 1988a,b,c)]. It appears that the equations governing all modes (longitudinal or transverse) do not contain triggering to stable limit cycles. The conclusion is based on numerical solutions since it has not been possible to find analytical results. Later results [Paparizos and Culick (1988b)] show that there may be a narrow range of parameters in which a nonlinear instability may occur, in the neighborhood of the boundary of the region in which stable limit cycles are possible. However, in those cases the fundamental mode of the system is unstable: small disturbances evolve to a unique stable limit cycle and large disturbances produce a growing instability.

That conclusion is contrary to that reached by Flandro (1986) who devised a model for nonlinear motions to third order and demonstrated the existence of triggering to stable limit cycles. However, he assumed that the time dependence is the same for all modes, during the transient growth and in the limit cycle, producing a single third order equation for the energy in an oscillation. That assumption is false except possibly in some very special cases; his conclusion therefore seems not to be generally valid.

Our conclusions may be consistent with some earlier works apparently showing triggering under some conditions with the third order equations. In some of those cases it seems that the phenomenon found may have been an unstable limit cycle. However, the numerical results may reflect the same behavior noted above [Paparizos and Culick (1988b)]. It is difficult to give a close comparison between the numerical and analytical results.

Perhaps more significantly, nonlinear processes other than purely gasdynamic broaden greatly the conditions under which triggering will occur. In particular, nonlinear interactions with the mean flow field (second order in the acoustic fluctuations and first order in the mean flow speed) do produce triggering [Paparizos and Culick (1988b)]. It is also true that nonlinear combustion processes of the proper form can provide conditions for triggering, a circumstance that amounts to allowing the constant α_n to be a function of the amplitude.

The problem of triggering still requires further attention. Not only are the results not yet in a form useful for applications, but the differing conclusions reached in the various works on the subject have not been satisfactorily reconciled. It's an important theoretical and practical problem that should be pursued.

9. NONLINEAR COMBUSTION INSTABILITIES WITH RANDOM SOURCES OF NOISE

Any combustion chamber contains stochastic or random sources associated with turbulent flow, regions of separated flow and turbulent combustion. The most elementary observations confirm that the noise emitted by a reacting flow is enormously greater than that radiated by a non-reacting flow. We are not presently concerned with the details of the sources, or with the particular origins of noise. Whatever may be the causes, they must always be represented by randomly varying sources of mass, momentum or energy in the conservation equations (2.1)–(2.3). Hence they will contribute to h and f in (2.4) and (2.5) and to F_n in (2.11). We therefore assume, in the spirit of the approximate analysis, that the physical processes can somehow be modeled — not necessarily an easy task to give realistic results — and become grouped as random times varying terms in the F_n.

If we simply add such terms to the second order equations (2.11) for the amplitudes, simple truncation of the modal expansion gives, for the case of two modes, the two equations:

$$\frac{d^2\eta_1}{dt^2} + \omega_1{}^2\eta_1 = 2\alpha_1\frac{d\eta_1}{dt} + 2\omega_1\theta_1\eta_1 - F_{11}\frac{d\eta_1}{dt}\frac{d\eta_2}{dt} - F_{12}\eta_1\eta_2 - b_1(t)$$
$$\frac{d^2\eta_2}{dt^2} + \omega_2{}^2\eta_2 = 2\alpha_2\frac{d\eta_2}{dt} + 2\omega_2\theta_2\eta_2 - F_{21}\left(\frac{d\eta_1}{dt}\right)^2 - F_{22}\eta_1{}^2 - b_2(t) \tag{9.1)a,b$$

where F_{11}, F_{12}, F_{21}, F_{22} are constant, depending only on γ and ω_1; $b_1(t)$ and $b_2(t)$ represent the random excitations.

Equations (9.1)a,b have been sölved by Yang and Culick (1988) using a method due to Rice (1945) and Caughey (1959). The idea is first to write the solutions as sums of periodic and random parts:

$$\eta_n = \eta_{np} + \eta_{nR} \tag{9.2}$$

The periodic part is defined as the statistical mean of η_n, because by assumption the statistical mean of the random source is zero:

$$E\{\eta_n\} = E\{\eta_{np}\} = \eta_{np}$$

where $E\{\}$ means statistical mean, or, equivalently, the ensemble average. After substitution of (9.2) in (9.1)a,b taking the statistical means and with a bit of manipulation, the equations for the periodic and randomly varying amplitudes can be found. For the first mode, these are

$$\begin{aligned}\frac{d^2\eta_{1p}}{dt^2} + \omega_1{}^2\eta_{1p} &= 2\alpha_1\frac{d\eta_{1p}}{dt} + 2\omega_1\theta_1\eta_{1p} - F_{11}\left[\frac{d\eta_{1p}}{dt}\frac{d\eta_{2p}}{dt} + E\left\{\frac{d\eta_{1R}}{dt}\frac{d\eta_{2R}}{dt}\right\}\right]\\
&\quad - F_{12}\left[\eta_{1p}\eta_{2p} + E\left\{\eta_{1R}\eta_{2R}\right\}\right]\\
\frac{d^2\eta_{1R}}{dt^2} + \omega_1{}^2\eta_{1R} &= 2\omega_1\frac{d\eta_{1R}}{dt} + 2\omega_1\theta_1\eta_{1R}\\
&\quad - F_{11}\left[\frac{d\eta_{1p}}{dt}\frac{d\eta_{2R}}{dt} + \frac{d\eta_{1R}}{dt}\frac{d\eta_{2R}}{dt} - E\left\{\frac{d\eta_{1R}}{dt}\frac{d\eta_{2R}}{dt}\right\}\right]\\
&\quad - F_{12}\left[\eta_{1P}\eta_{2R} + \eta_{1R}\eta_{2p} + \eta_{1R}\eta_{2R} - E\left\{\eta_{1R}\eta_{2R}\right\}\right]\\
&\quad - b_1(t)\end{aligned} \tag{9.3)a,b$$

Solutions have been obtained with the assumption that the random part of the amplitude, η_{nR} is much smaller than the periodic part. One important consequence is that the nonlinear terms involving the randomly varying amplitudes can be ignored in the equations governing the random amplitudes; the justification for this assumption has been given by Rice (1945) and Caughey (1959).

With those assumptions, it becomes possible first to determine the periodic amplitudes — which contain a DC contribution depending on the correlation function of the randomly varying amplitude (at this point unknown). Then the random part of the amplitude can be obtained as the solution to equation (9.3)b and its counterpart for η_{2R}. Owing to coupling between the periodic and random amplitudes, explicit solutions cannot be obtained. Thus results have beem computed numerical by iteration.

The results obtained so far with this method are not extensive. For the case when the first mode is unstable, and the second mode stable, it appears that the periodic oscillations cause a transfer of energy from the random part of the first mode to that of the second.

It is particularly interesting that random sources produce a DC shift of the periodic amplitude (perhaps not surprising now that it has been established). Some results computed with the two mode approximation [Paparizos and Culick (1988b)] show that if a DC shift is allowed, the nonlinear equations to second order admit triggering. Although the necessary confirming analysis has not been completed, the suggestion is that triggering of nonlinear oscillations may be rendered more likely if random sources are present.

A quite different analysis of the influences of stochastic sources in being pursued and will be reported at a later date in a paper yet to be prepared. Some of the preliminary results will be reported in this meeting [Paparizos and Culick (1988c)].

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