REMARKS ON THE TWO-MODE APPROXIMATION TO THE NONLINEAR BEHAVIOR OF PRESSURE OSCILLATIONS IN COMBUSTION CHAMBERS*

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ABSTRACT

This paper summarizes recent work on the approximation to nonlinear pressure oscillations using two modes. A large part of the effort has been concerned with the consequences of gasdynamic nonlinearities of second order in the fluctuations. It appears that the two-mode approximation is valid over a broad range of the linear parameters that govern the global qualitative behavior, particularly if the lower mode is the unstable mode. Exact solutions have been found for the existence and stability of limit cycles, allowing meaningful comparison with numerical solutions obtained with larger numbers of modes considered. The nonlinear analysis to second order does not contain "triggering" — nonlinear instability of a linearly stable system — because the nonlinear processes order terms involving the mean flow speed, or *DC* shifts in the amplitudes of oscillation do produce triggering.

1. INTRODUCTION

There are two strategies for investigating thoretically the problem of combustion instabilities: numerical analysis based the partial differential equations of conservation; or some sort of approximate analysis founded on reduced forms of the partial differential equations. A fully developed theory of the phenomena must involve both sorts of activities. In this paper we are concerned with a particularly useful form of approximate analysis constructed with a system of ordinary differential equations produced by spacial and time averaging of the partial differential equations.

It is well to remark first that numerical analysis of the partial differential equations is a fundamentally important part of the theoretical work. Because formal mathematical methods are several limited, numerical methods offer the only means of obtaining quantitative results for problems accurately modeling the physical circumstances existing in real problems. Work of this sort, which to-date has been limited to two-dimensional, axisymmetrical or, in most cases one-dimensional problems, constitutes essentially the application of computational fluid dynamics to internal flows. Even with the approximations inherent in modeling some of the processes (mainly combustion and turbulent flow), and the obvious characteristic that each numerical computation applies to a single well-defined problem, the results provide the only standard against which the accuracy of approximate solutions can be measured. Thus numerical analysis will in principle fulfill two needs: accurate simulations of complex internal flows, potentially a useful contribution in later stages of design and development; and checks of approximate results.

Approximate analysis likewise has two primary functions: to provide deeper understanding of the general behavior; and as a basis for less expensive computational methods usable in routine fashion and therefore available for preliminary design work and investigation of trends of behavior, such as parametric studies. Because numerical analysis provides results for a relatively small number of special cases, it is both expensive and time consuming to perceive reasons for the particular behavior found and to extract useful guidelines, "rules of thumb". Moreover, even with the recent advances in computational resources, numerical calculations of the sort required for internal flows are still expensive and restricted.

We have therefore concentrated on developing an approximate analysis to fill the requirements both for basic theoretical work and for convenient routine analysis of actual problems. Another important feature is that the formal results not only provide useful guidelines for design work but also serve as a framework for planning experiments and interpreting test results.

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The approximate analysis used here is based on a form of Galerkin's method and, in most of our work, time averaging as well. As a result, most of the calculations are based on systems of coupled nonlinear first order differential equations. This approach provides a convenient point of view for studying combustion instabilities and sets the mathematical developments squarely in the context of the contemporary theory of nonlinear dynamical systems.

Use of Galerkin's method to study combustion instabilities was first made by Zinn and Powell (1968, 1970) for liquid rockets and by Culick (1971, 1975, 1976) for solid rockets. The analysis described here is based largely on the last work, Culick (1976), in which the method of averaging was introduced. Because the derivation of the system of equations has been given in several places, we shall not repeat the process here, but rather concentrate on the specialization to two modes, for longitudinal oscillations.

Galerkin's method produces a system of second order ordinary differential equations representing the time evolution of an infinite set of coupled nonlinear oscillators. One oscillator is associated with each classical acoustic mode; the second order equation is for the amplitude of the mode, i.e. of the oscillator's motion. Time-averaging replaces the second order equation by two first order equations. Hence if N modes are considered, one has to solve $2N$ first order equations. The linear behavior of each mode is characterized by two parameters, the growth constant and the frequency shift, both due to the various perturbations of the classical acoustics prevailing in the absence of combustion and mean flow. Nonlinear gasdynamics introduces no new parameters, so if no other nonlinear processes are accounted for, both the linear and nonlinear behavior are determined by the values of the linear parameters, $2N$ for N modes.

Numerical solutions to the set of equations is inexpensive and easy to achieve. The point here is to investigate the formal behavior and for that purpose it is necessary to truncate the system to two modes, i.e. four equations. It happens that if gasdynamics is the only nonlinear process included, and only second order nonlinearities are allowed, the four equations can be reduced to three by a simple transformation. That simplification is the basis for the theoretical work described here.

2. FORMULATION OF THE TWO-MODE APPROXIMATION

After the dependent variables are expressed as sums of mean and fluctuating parts, a nonlinear wave equation can be formed for the pressure, with its boundary condition:

$$\nabla^2 p' - \frac{1}{\bar{a}^2} - \frac{\partial^2 p'}{\partial t^2} = h \tag{2.1}$$

$$\hat{n} \cdot \nabla p' = -f \tag{2.2}$$

The pressure and velocity fluctuations are then expanded in the normal modes $\psi_n(\bar{r})$ with time-varying amplitudes $\eta_n(t)$:

$$p' = \bar{p} \sum \eta_n(t) \psi_n(\bar{r}) \tag{2.3}$$

$$\vec{u}\,' = \sum \frac{1}{\gamma k_n{}^2} \dot{\eta}_n(t) \nabla \psi_n(\bar{r}) \tag{2.4}$$

Now multiply (2.1) by ψ_m, substitute (2.3) for p' and integrate over volume. After use of the boundary condition (2.2), some manipulations give the equation for η_n:

$$\frac{d^2 \eta_n}{dt^2} + \omega_n^2 \eta_n = F_n \tag{2.5}$$

where the force is the spacial average of the functions k and f:

$$F_n = -\frac{\bar{a}^2}{\bar{p} E_n^2} \left\{ \int h \psi_n dV + \oiint f \psi_n dS \right\} \tag{2.6}$$

For many practical situations, the amplitudes are oscillatory with slowly varying amplitudes and phases. Thus we assume that a good approximation for η_n is

$$\eta_n(t) = r_n(t) \cos\big(\omega_n t - \phi_n(t)\big) = A_n(t) \sin \omega_n t + B_n(t) \cos \omega_n t \tag{2.7}$$

This equation really serves to define the function $A_n(t)$ and $B_n(t)$. Time averaging eventually leads to the equations for A_n and B_n:

$$
\begin{aligned}
\frac{dA_n}{dt} &= \frac{1}{\omega_n \tau} \int_t^{t+\tau} F_n \cos \omega_n t' dt' \\
\frac{dB_n}{dt} &= \frac{-1}{\omega_n \tau} \int_t^{t+\tau} F_n \sin \omega_n t' dt'
\end{aligned}
\tag{2.8a,b}
$$

where τ is the interval of averaging.

The case of longitudinal modes, for which $\omega_n = n\omega_1$, is special. The integrands in (2.8)a,b are then periodic and the range of integration can be shifted from $(t, t+\tau)$ to $(0, \tau)$. Then if only gasdynamic nonlinearities to second order are accounted for, the equations can be written

$$
\begin{aligned}
\frac{dA_n}{dt} &= \alpha_n A_n + \theta_n B_n + \frac{n\beta}{2} \sum_{i=1}^{n-1} (A_i A_{n-i} - B_i B_{n-i}) \\
&\quad - n\beta \sum_{i=1}^{\infty} (A_{n+i} A_i + B_{n+i} B_i) \\
\frac{dB_n}{dt} &= \alpha_n B_n - \theta_n A_n + \frac{n\beta}{2} \sum_{i=1}^{n-1} (A_i B_{n-i} + B_i A_{n-i}) \\
&\quad + n\beta \sum_{i=1}^{\infty} (A_{n+i} B_i - B_{n+i} A_i)
\end{aligned}
\tag{2.9a,b}
$$

These are slightly re-arranged but equivalent forms of the corresponding equations given in Culick (1976) and used in many works since. The quantity β shown explictly here can be absorbed as a time scale, so it is not a true parameter:

$$
\beta = \left(\frac{\gamma+1}{8\gamma}\right)\omega_1 \tag{2.10}
$$

For the case of two modes, the form governing equations can be transformed to the following set of three [Paparizos and Culick (1988a)]:

$$
\begin{aligned}
\frac{dy_1}{dt} &= (\alpha_1 + y_2) y_1 \\
\frac{dy_2}{dt} &= \alpha_2 y_2 + |\theta_2 - 2\theta_1| y_3 + 2{y_3}^2 - {y_1}^2 \\
\frac{dy_3}{dt} &= -|\theta_2 - 2\theta_1| y_2 + \alpha_2 y_3 - 2 y_2 y_3
\end{aligned}
\tag{2.11a,b,c}
$$

where

$$
\begin{aligned}
y_1 &= \beta r_1 \\
y_2 &= \beta r_2 \sin(\phi_2 - 2\phi_1) \\
y_3 &= \frac{|\theta_2 - 2\theta_1|}{(\theta_2 - 2\theta_1)} \beta r_2 \cos(\phi_2 - 2\phi_1)
\end{aligned}
\tag{2.12a,b,c}
$$

where $r_n = ({A_n}^2 + {\beta_n}^2)^{\frac{1}{2}}$ and $\tan\phi_n = A_n/B_n$. These equations can be solved exactly and therefore provide complete information about the behavior of a two-mode system, when only second order gasdynamical nonlinearities are considered. The results are significant not only for their own sake as an exact solution for a special case, but also because they form a convenient basis from which the analysis can be extended to cover other perturbations, including third order terms and stochastic sources, for example.

3. EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS (LIMIT CYCLES)

Besides reducing the number of equations by one (possible because the zero phase of the oscillations in the limit cycle can be freely chosen), the transformation (2.12)a,b,c also maps periodic solutions of the equations for (A_1, A_2, B_1, B_2) into fixed points of equations (2.11)a,b,c. That is, limit cycles of the original system are determined in the variable (y_1, y_2, y_3) by setting $dy_n/dt = 0$. With $\dot{y}_n = 0$, equations (2.11)a,b,c have two solutions for the fixed points; the trivial one for which $y_i{}^p = 0$, and

$$y_1{}^p = \left[-\alpha_1\alpha_2\left(1+\left(\frac{\theta_2-2\theta_1}{\alpha_2+2\alpha_1}\right)^2\right)\right]^{\frac{1}{2}}$$

$$y_2{}^p = -\alpha_1 \qquad (3.1)a,b,c$$

$$y_3{}^p = -\frac{\alpha_1(\theta_2-2\theta_1)}{\alpha_2+2\alpha_1}$$

It follows from (3.1)a that the condition for existence (y_1 must be real) is

$$\alpha_1\alpha_2 < 0 \qquad (3.2)$$

This reflects the physical attribute that in the limit cycle, because the energy in the oscillation is constant, one mode must be unstable ($\alpha > 0$) and the other unstable ($\alpha < 0$). Note that the case $\alpha_1\alpha_2 > 0$ implies that both modes are either unstable or both are stable. The first condition implies that the energy of the system increases without limit, an unstable motion; the second condition characterizes the trivial "limit cycle," the state of no motion.

Local stability of the limit cycle is obtained by investigating small motions about the motion represented by the fixed point (3.1)a,b,c. The Routh-Hurwitz criteria for stability give the three conditions:

$$\alpha_1\left(1+\frac{\alpha_2}{\alpha_1}\right) < 0$$

$$\alpha_1\left(2+\frac{\alpha_2}{\alpha_1}\right) < 0 \qquad (3.3)a,b,c$$

$$\alpha_1\left(\frac{\alpha_2}{\alpha_1}-x_1\right)\left(\frac{\alpha_2}{\alpha_1}-x_2\right) > 0$$

where

$$x_{1,2} = \frac{(\beta_0-1)\pm\sqrt{(\beta_0-1)(3\beta_0+1)}}{1+\beta_0} \qquad (3.4)$$

and

$$\beta_0 = \left(\frac{\theta_2-2\theta_1}{\alpha_2+2\alpha_1}\right)^2 \qquad (3.5)$$

The condition (3.2) for existence shows that there are two cases to consider: ($\alpha_1 > 0$, $\alpha_2 < 0$), for which the first mode is unstable; and ($\alpha_1 < 0$, $\alpha_2 > 0$), when the second mode is unstable. Figure 1 shows the ranges of the parameters in which the oscillations for these two cases are stable. Note that the coordinates are β_0 and α_2/α_1; in contrast to the results found by Awad and Culick (1985), here we have found the conditions covering all values of the linear parameters.

It is important to recognize that the two cases ($\alpha_1 > 0$, $\alpha_2 < 0$) and ($\alpha_1 < 0$, $\alpha_2 > 0$) exhibit substantially different physical behavior because the gasdynamic nonlinearities tend naturally to cause energy to flow upward in the spectrum.

<u>$\alpha_1 > 0$, $\alpha_2 < 0$</u>

For this case, the inequality (3.3)b gives

$$\alpha_2 + 2\alpha_1 < 0 \qquad (3.6)$$

Irrespective of the values of θ_1, θ_2, this condition must be met if a disturbance is to evolve into a stable periodic limiting motion. Figure 2 shows trajectories in the (y_1, y_2, y_3) plane, each trajectory characterized by different initial conditions. Note that for all initial conditions, the motion is the same in the limit cycle, represented by the point P whose location is a function of α_1, α_2, θ_1 and θ_2. As the values of the linear parameters are changed, P traces a line in the space, called the "center manifold" in the theory of dynamical systems. The equation for the manifold is found by setting $\dot{y}_2 = \dot{y}_3 = 0$ in (2.11)b,c. From (2.11)c, one has

$$y_3 = \frac{\theta_2-2\theta_1}{\alpha_2-2y_2} \qquad (3.7)$$

and substitution in (2.11)b leads to a third order polynomial $y_2(y_1)$. Hence the curve is readily computed. On

One may then show that a unique stable solution exists for $y_2 > \alpha_2/2$. Figure 3 shows the projection of the center manifold in the (y_2, y_3) plane. (Note that $\alpha_2/2 < 0$ for the conditions assumed here).

$\underline{\alpha_1 > 0\,,\, \alpha_2 < 0}$

When the second mode is unstable, the behavior is more complicated, a consequence of the fact that in the limit cycle, energy must flow downward to the first mode, opposing the preferred transfer associated with the gasdynamics. The evolution of the motion to the limit cycle is much more erratic, as shown by the time histories of the amplitudes in Figure 3.

4. REMARKS ON THE INFLUENCES OF HIGHER MODES

Because of the obvious convenience of approximating an actual problem with two modes, it is important to gain some understanding of the circumstances under which one should expect the approximation to be accurate. According to earlier observations, we should anticipate difficulties when the higher of the two modes is unstable and dissipation of energy is forced to occur entirely from the first mode.

Figure 4 is a comparison of the time-dependent amplitudes of the two modes computed for several cases in which higher modes are included. The linear parameters were chosen from an example treated by Culick (1976) and Culick and Levine (1974); the modes for $n \geq 3$ are stable. The accuracy of the approximation is clearly shown. However, the conclusion is not entirely general, for if one or more of the higher modes are only weakly stable, they may be driven to higher amplitudes, and the two-mode approximation will lose accuracy.

A contrary case for which the second mode is unstable and all others are stable is shown in Figure 5. While the two-mode approximation predicts no stable limit cycle (the amplitude of the second mode grows without limit) including higher modes does produce a stable motion for long times.

Further discussion of the consequences of truncating the modal expansion is given by Paparizos and Culick (1988a). This aspect of the general problem is not fully understood. For example, application of the two-mode approximation to adjacent modes within the spectrum (e.g. modes 5 and 6, say) is a possibility not yet explored.

5. INFLUENCES OF THIRD ORDER ACOUSTICS: TRIGGERING

Experience gained during the past 10–15 years suggests that much of the nonlinear behavior of many combustion instabilities in real systems can be quite well accommodated by second order acoustics. That is the case providing the amplitude is not too high, certainly up to about 10% and possibly higher — the limit is not satisfactorily set. However, the important problem of nonlinear instability or "triggering" is not contained in nonlinear gasdynamics to second order. That definite statement rests on the particular structure of the gasdynamic nonlinearities. Other processes may give second order nonlinearities capable of describing triggering, but none are known at this time and we shall not consider that possibility.

The equations to third order have been given by Yang and Culick (1987). Transformation to the variables (y_1, y_2, y_3) gives the equations

$$\begin{aligned}
\frac{dy_1}{dt} &= (\alpha_1 + y_2)y_1 \\
\frac{dy_2}{dt} &= \alpha_2 y_2 + (\theta_2 - 2\theta_1)y_3 + 2y_3{}^2 - y_1{}^2 + \frac{1}{4\beta\gamma}\, y_3(y_2{}^2 + y_3{}^2 - y_1{}^2) \\
\frac{dy_3}{dt} &= -(\theta_2 - 2\theta_1)y_2 + \alpha_2 y_3 - 2y_2y_3 - \frac{1}{4\beta\gamma}\, y_2(y_2{}^2 + y_3{}^2 - y_1{}^2)
\end{aligned} \qquad (5.1)a,b,c$$

Once again fixed points of these equations correspond to periodic solutions or limit cycles of the original equations for the oscillators. The amplitudes in the nontrivial limit cycles, the solutions to (5.1)a,b,c with $\dot{y}_1 = \dot{y}_2 = \dot{y}_3 = 0$, are

$$\begin{aligned}
-\alpha_1(y_1{}^p)^2 &= \alpha_2\left[(y_2{}^p)^2 + (y_3{}^p)^2\right] \\
y_2{}^p &= -\alpha_1 \\
y_3{}^p &= -\frac{\alpha_2 + 2\alpha_1}{\delta^{\text{th}}} \pm \frac{1}{\delta^{\text{th}}}\sqrt{(\alpha_2 + 2\alpha_1)^2 - 4\delta^{\text{th}}\alpha_1(\theta_2 - 2\theta_1 + \delta^{\text{th}}\alpha_1)}
\end{aligned} \qquad (5.2)a,b,c$$

where

$$\delta^{\text{th}} = \frac{\alpha_1 + \alpha_2}{4\beta\gamma} \qquad (5.3)$$

Equations (5.2)a,b are the same as those found for second order acoustics, but (5.2)c shows that there may be two nontrivial limit cycles. These can be displayed explicitly by writing the formula for α_n/ω_n and δ^{th} small; which must be true for time-averaging to work:

$$
\begin{aligned}
y_3^{p,1} &\approx -\frac{\alpha_1(\theta_2 - 2\theta_1 + \delta^{th})}{\alpha_2 + 2\alpha_2} \\
y_3^{p,2} &\approx -\frac{\alpha_2 + 2\alpha_1}{\delta^{th}}
\end{aligned}
\tag{5.4)a,b}
$$

The first, $y_3^{p,1}$ is a perturbation of (3.1)c obtained for second order acoustics while the second is new.

Unfortunately, it is not possible to derive explicit conditions for stability of these limit cycles. Thus, the stability has been studied by using the idea of attractive manifolds. The results show that the behavior is qualitatively the same as that for second order acoustics over almost the entire range of linear parameters; that conclusion is consistent with that reached by Yang, Kim and Culick (1987, 1988a). However, the present results show that in a narrow range near the stability boundary ($\alpha_2 + 2\alpha_1 = 0$), the possibility for triggering to unstable limit cycles may exist.

6. THE INFLUENCE OF A MEAN PRESSURE SHIFT

We have also considered the influences of 'drift' contributions to the amplitudes; calculations show that none appears in η_1 and that in η_2 is proportional to the squared amplitude of η_1 and so is of second order. Thus the amplitudes have the form

$$
\begin{aligned}
\eta_1(t) &= A_1(t)\sin\omega_1 t + B_1(t)\cos\omega_1 t + \text{third order terms} \\
\eta_2(t) &= A_2(t)\sin\omega_2 t + B_2(t)\cos\omega_2 t + \frac{1}{4\gamma}(A_1^2 + B_1^2) \\
&\qquad + \text{third order terms}
\end{aligned}
\tag{6.1)a,b}
$$

This modification amounts to accounting for a slowly varying drift of the mean pressure, represented as a synthesis of the acoustic modes:

$$
\bar{p} = \bar{p}_0\left[1 + \sum_{i=1}^{\infty} D_i\psi_i(\vec{r})\right] \tag{6.2}
$$

The analysis then shows that $D_1 = 0$ for longitudinal modes. After time-averaging and transformation of variables we find the eequations

$$
\begin{aligned}
\frac{dy_1}{dt} &= (\alpha_1 + y_2)y_1 \\
\frac{dy_2}{dt} &= \alpha_2 y_2 + (\theta_2 - 2\theta_1)y_3 + 2y_3^2 - y_1^2 + \frac{\delta^d}{\alpha_2}y_1^2 y_3 \\
\frac{dy_3}{dt} &= -(\theta_2 - 2\theta_1)y_2 + \alpha_2 y_3 - 2y_2 y_3 - \frac{\delta^d}{\alpha_2}y_1^2 y_2
\end{aligned}
\tag{6.3)a,b,c}
$$

Limited numerical calculations have shown that this approximation yields results that compare very well with the solutions to the second order oscillator equations (i.e. without time averaging) up to pressure amplitudes as large as 50% of the mean pressure.

The drift terms do not alter the conservation of energy: the rate of energy lost from one mode to the environment equals the rate of gain to the unstable mode. Consequently, this contribution does not provide a mechanism for triggering in the case when both modes are stable, as for the third order terms discussed in Section 5. However, the drift adds terms $-(\delta^d/\alpha_2)y_1^2 y_2$ to equation (5.1)c, where

$$
\delta^d = \frac{5(\gamma - 1)\alpha_2}{\gamma(\gamma + 1)\beta} \tag{6.4}
$$

Thus $\delta^d/\delta^{th} = 40(\gamma - 1)/(\gamma + 1) > 4$, so these third order are numerically more significant than those arising directly with third order gasdynamic nonlinearities.

7. THE INFLUENCE OF MEAN VELOCITY ON TRIGGERING

Initially in the approximate analysis described here, we deal with an expansion in two small parameters, one (denoted μ) measuring the Mach number of the mean flow field, and the other (ϵ) measuring the amplitude of the oscillations. The ordering of terms in the equations is such that classical linear acoustics is of order ϵ, those defining linear stability are of order $\mu\epsilon$, second order acoustics contributions are of order ϵ^2 and so forth. Here we consider for the first time contributions that depend on the mean flow and are nonlinear in the fluctuations. We assume a mean flow field varying linearly along the axis of the chamber, $\bar{u} = \bar{a}M_N(z/L)$ where M_N is the Mach number at the entrance to the nozzle. Retaining terms of order $\mu\epsilon^2$, we eventually find the equations for (y_1, y_2, y_3):

$$\frac{dy_1}{dt} = (\alpha_1 + y_2)y_1 - 7\mu y_1 y_3$$

$$\frac{dy_2}{dt} = \alpha_2 y_2 + (\theta_2 - 2\theta_1)y_3 + 2{y_3}^2 - {y_1}^2 + 14\mu y_2 y_3 \qquad (7.1)a,b,c$$

$$\frac{dy_3}{dt} + -(\theta_2 - 2\theta_1)y_2 + \alpha_2 y_3 - 2y_2y_3 + \mu {y_1}^2 - 14\mu {y_2}^2$$

where

$$\mu = \frac{M_N}{4\pi(\gamma+1)}$$

For small ν, the solutions for the limit cycle are

$$({y_1}^p)^2 = -\frac{\alpha_z}{\alpha_1}\left[({y_2}^p)^2 + ({y_3}^p)^2\right]\left(1 + 6\frac{\mu}{\alpha_1}{y_3}^p\right)$$

$${y_2}^p = -\alpha_1 + 7\mu {y_3}^p$$

$${y_3}^{p,1} = -\frac{\alpha_1(\theta_2 - \theta_1)}{\alpha_2 + 2\alpha_1} + \mu\frac{\alpha_1(14\alpha_1 + \alpha_2)}{\alpha_2 + 2\alpha_1} \qquad (7.2)a,b,c,d$$

$${y_3}^{p,2} = \frac{\alpha_2 + 2\alpha_1}{12\mu}$$

The solution (7.2)a,b,c represents a perturbation of the result found for second order acoustics. The second possibility, with ${y_3}^{p,2}$ given by (7.2)d is new and particularly interesting because it contains triggering. However, as in all cases found above, triggering leads to an unstable motion, not a stable limit cycle.

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