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Three-dimensional topological field theory and symplectic algebraic geometry II

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Motivated by the path-integral analysis [6] of boundary conditions in a three-dimensional topological sigma model, we suggest a definition of the two-category $\ddot{\mathsf{L}}(X)$ associated with a holomorphic symplectic manifold X and study its properties. The simplest objects of $\ddot{\mathsf{L}}(X)$ are holomorphic lagrangian submanifolds $Y \subset X$. We pay special attention to the case when X is the total space of the cotangent bundle of a complex manifold U or a deformation thereof. In the latter case, the endomorphism category of the zero section is a monoidal category which is an A_∞ deformation of the two-periodic derived category of U.

1. Introduction

1.1. Sigma-models and categories

Let M be a real d-dimensional manifold and let $\mathcal{X} = (X, s)$ be a pair in which X is a real manifold and s is a geometric structure on X such as a complex structure or a symplectic structure. A d-dimensional topological sigma-model (TSM) with a *space–time* (also known as the world-sheet or the world-volume) M and a target-space $\mathcal{X}$ is a quantum field theory based on a path integral over the infinite-dimensional space of maps $M \to X$. The measure on the space of maps is determined by the structure s.

We are interested in two-dimensional and three-dimensional TSMs.

Path-integral-based arguments suggest that if manifolds $\mathcal{X}$ with a certain type of structure serve as target spaces for d-dimensional TSM then they form a d-category $\mathcal{C}$ with special features.[1] Let us briefly recall these features and illustrate them by two well-known examples.

The category $\mathcal{C}$ has a symmetric monoidal structure related to the cartesian product of manifolds:

$$\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \qquad (X_1, s_1) \times (X_2, s_2) = (X_1 \times X_2, s_1 \times s_2), \tag{1.1}$$

where $s_1 \times s_2$ is the natural structure on $X_1 \times X_2$. The monoidal structure has a unit element $\mathcal{X}_{\text{1-pt}} = (X_{\text{1-pt}}, s_{\text{1-pt}})$, where $X_{\text{1-pt}}$ is the manifold consisting of a single point and $s_{\text{1-pt}}$ is the corresponding trivial structure. This element has the property $\mathcal{X}_{\text{1-pt}} \times \mathcal{X} = \mathcal{X}$. For a structured manifold $\mathcal{X}$ we define a $(d-1)$-category of morphisms

$$\mathcal{C}_{\mathcal{X}} := \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_{\text{1-pt}}, \mathcal{X}).$$

This category is known in quantum field theory as the category of boundary conditions of the TSM related to $\mathcal{X}$.

The d-category $\mathcal{C}$ has a contravariant duality functor

$$\mathcal{C} \xrightarrow{\diamondsuit} \mathcal{C}, \qquad (X, s)^{\diamondsuit} = (X, s^{\diamondsuit}), \tag{1.2}$$

such that there is a canonical equivalence between $(d-1)$-categories of morphisms:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_1, \mathcal{X}_2) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_{\text{1-pt}}, \mathcal{X}_1^{\diamondsuit} \times \mathcal{X}_2) = \mathcal{C}_{\mathcal{X}_1^{\diamondsuit} \times \mathcal{X}_2}. \tag{1.3}$$

[1]The notion of a d-category for $d > 1$ is quite complicated, and in fact there is no generally accepted definition for $d > 2$. The difficulty lies in formulating coherence conditions on composition of k-morphisms, $1 \le k \le d$, which express a suitable version of associativity. Our treatment of d-categories for $d > 2$ will be heuristic because we will not check associativity. For $d = 2$ we will occasionally discuss associator two-morphisms, when they are not obvious, but we will not check whether they satisfy the pentagon identity. Our discussion will be even more informal when we invoke differential graded two-categories.

This equivalence implies that an object $\mathcal{O}_{12} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_1, \mathcal{X}_2)$ determines a functor between $(d-1)$-categories

$$\Phi\,[\mathcal{O}_{12}] : \mathcal{C}_{\mathcal{X}_1} \longrightarrow \mathcal{C}_{\mathcal{X}_2}, \tag{1.4}$$

which represents a composition of morphisms within $\mathcal{C}$. Moreover, a composition of morphisms of $\mathcal{C}$ corresponds to the composition of functors (1.4), so the structure of the d-category $\mathcal{C}$ is determined by the boundary condition categories $\mathcal{C}_{\mathcal{X}}$ and the functors (1.4).

Recall two examples of this general construction for $d = 2$, that is, when $\mathcal{C}$ is a two-category. The first example is related to the A-model. The structure s is a symplectic structure (that is, s is a symplectic form on X), the category of boundary conditions $\mathcal{C}_{\mathcal{X}}$ is the Fukaya–Floer category $\mathrm{F}(\mathcal{X})$, its simplest objects being lagrangian submanifolds of X, the action of the duality functor is $s^{\diamondsuit} = -s$, and the functor $\Phi\,[\mathcal{O}_{12}]$ is the lagrangian correspondence functor determined by a lagrangian submanifold $L_{12} \subset \mathcal{X}_1^{\diamondsuit} \times \mathcal{X}_2$.

The second example of the two-category $\mathcal{C}$ comes from the B-model: X is a Calabi–Yau manifold, s is its complex structure, the category of boundary conditions $\mathcal{C}_{\mathcal{X}}$ is the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(\mathcal{X})$, its simplest objects being complexes of holomorphic vector bundles on X, the duality functor acts trivially: $s^{\diamondsuit} = s$, and the functor $\Phi\,[\mathcal{O}_{12}]$ is the Fourier–Mukai transform corresponding to the object $\mathcal{O}_{12}$.

1.2. The three-category of holomorphic symplectic manifolds

For $d = 3$ a natural class of TSM comes from the Rozansky–Witten model [12]. In [6] we studied this TSM and its two-category of boundary conditions from the path-integral viewpoint. In this paper, we attempt to present a mathematical description of the three-category $\ddot{\mathsf{L}}$ formed by these theories and formulate conjectures about it.

Objects of $\ddot{\mathsf{L}}$ are holomorphic symplectic manifolds $\mathcal{X} = (X, \omega)$, where X is a complex manifold and $\omega \in \Omega^{2,0}$ is a holomorphic symplectic form: it is non-degenerate at every point of X and $d\omega = 0$. If the symplectic form is canonical, we abbreviate the notation (X, ω) down to X. The monoidal structure (1.1) comes from the product of manifolds and the sum of their symplectic structures: $(X_1, \omega_1) \otimes (X_2, \omega_2) = (X_1 \times X_2, \pi_1^*(\omega_1) + \pi_2^*(\omega_2),)$ where π_1 and π_2 are the projections of $X_1 \times X_2$ onto X_1 and X_2. The duality functor $\diamondsuit$ acts on objects by switching the sign of the symplectic form: $(X, \omega)^{\diamondsuit} = (X, -\omega)$.

The main purpose of the paper is to investigate the two-category $\mathcal{C}_{\mathcal{X}}$ associated to a holomorphic symplectic manifold $\mathcal{X} = (X, \omega)$. We denote it as $\ddot{\mathsf{L}}(X, \omega)$. The definition of $\ddot{\mathsf{L}}(X, \omega)$ for a general holomorphic symplectic manifold is rather complicated and requires a construction of a micro-local presheaf of two-categories on X. Therefore, we devote much of this paper to special (X, ω) and present an attempt at a general definition only in Section 6.

1.3. Algebraic approach

The first approach towards the description of the two-category $\ddot{\mathsf{L}}(X, \omega)$ is based on the fact that when X is a cotangent bundle of a complex manifold U, the category $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ can be described in terms of the properties of U. This description is algebraic in nature and there is no reference to $\mathrm{T}^\vee U$, so by looking at the definitions one would not see directly that symplectic structure is involved.

In Section 2 we study a "toy" two-category $\ddot{\mathsf{MF}}(\mathbf{x})$, $\mathbf{x} = x_1, \ldots, x_n$, which after a minor modification should be equivalent to the two-category $\ddot{\mathsf{L}}(\mathrm{T}^\vee \mathbb{C}^n)$ associated with a symplectic affine space, that is, with the cotangent bundle $\mathrm{T}^\vee \mathbb{C}^n$.

Recall that for a polynomial $W \in \mathbb{C}[\mathbf{x}]$, an object of the category of matrix factorizations $\mathsf{MF}(\mathbf{x}; W)$ is a free finite rank $\mathbb{Z}_2$-graded $\mathbb{C}[\mathbf{x}]$-module M with a degree-1 endomorphism (called a curved differential) D satisfying the condition $D^2 = W\, 1_M$. The polynomial W is called *a curving*. A curving of a tensor product of two matrix factorizations over $\mathbb{C}[\mathbf{x}]$ is the sum of their curvings.

The simplest objects of $\ddot{\mathsf{MF}}(\mathbf{x})$ are polynomials $W \in \mathbb{C}[\mathbf{x}]$. The category of morphisms between two polynomials W_1 and W_2 is the category of matrix factorizations of their difference

$$\mathrm{Hom}(W_1, W_2) = \mathsf{MF}(\mathbf{x}; W_2 - W_1), \tag{1.5}$$

and the composition of morphisms comes from the tensor product of matrix factorizations over $\mathbb{C}[\mathbf{x}]$.

In Section 3 we extend the algebraic construction of $\ddot{\mathsf{L}}(\mathrm{T}^\vee \mathbb{C}^n)$ to the cotangent bundle $\mathrm{T}^\vee U$ of a complex manifold U by defining algebraically a two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ which is supposed to be equivalent to $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$. Similar to the affine case, the simplest objects of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ are labeled by holomorphic functions W on U, and a category of morphisms between two such functions is the curved version of the derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(U)$:

$$\mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)}(W_1, W_2) = \mathsf{D}_{\mathbb{Z}_2}(U, W_2 - W_1). \tag{1.6}$$

This category is an analog of the category of matrix factorizations (1.5) when the algebra $\mathbb{C}[\mathbf{x}]$ is replaced by the differential graded Dolbeault algebra $(\Omega^{0,\bullet}(U), \bar{\partial})$. A perfect object of $\mathsf{D}_{\mathbb{Z}_2}(U, W)$ is a $\mathbb{Z}_2$-graded vector bundle $E \to U$ with a curved $(0,1)$ differential $\bar{\nabla}$ such that $\bar{\nabla}^2 = W\,1_E$, and the composition of morphisms of the type (1.6) comes from the tensor product of vector bundles.

1.4. Geometric approach: the case of a cotangent bundle

Path-integral analysis in [6] indicates that "in the classical approximation" the category $\ddot{\mathsf{L}}(X, \omega)$ should contain special "geometric" objects. These objects are holomorphic fibrations $\mathcal{Y} \to Y$, where $Y \subset X$ is a lagrangian submanifold. Generally, fibration objects $\mathcal{Y} \to Y$ have to be deformed because of quantum corrections, but this is unnecessary in two special cases. The first case is when $\mathcal{Y}$ is a one-point fibration, that is, the fiber is a point and the object is just the lagrangian submanifold Y itself. The second case is when X is isomorphic to a cotangent bundle: $X \cong \mathrm{T}^\vee U$.

In Section 4 we study morphisms between fibration objects of $\ddot{\mathsf{L}}(X, \omega)$ for $X = \mathrm{T}^\vee U$. Here is an overview of our conjectures for the simplest objects of $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ which are lagrangian submanifolds.

We say that two holomorphic submanifolds $Z_1, Z_2 \subset X$ have *a clean intersection* if any point $x \in Z_1 \cap Z_2$ has an open neighborhood $U_x \subset X$ which is isomorphic to a neighborhood of 0 in $\mathrm{T}_x X$, so that Z_1 and Z_2 are identified with $\mathrm{T}_x Z_1$ and $\mathrm{T}_x Z_2$. This condition implies that the intersection $Z_1 \cap Z_2$ is smooth.

Suppose that two lagrangian submanifolds $Y_1, Y_2 \subset X = \mathrm{T}^\vee U$ are Calabi–Yau, their intersection is clean, and the difference of dimensions $\dim Y_1 - \dim(Y_1 \cap Y_2)$ is even. Then the category of morphisms between them becomes fairly simple:

$$\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}(Y_1, Y_2) = \mathsf{D}_{\mathbb{Z}_2}(Y_1 \cap Y_2). \tag{1.7}$$

Moreover, if all intersections between three Calabi–Yau lagrangian submanifolds Y_1, Y_2 and Y_3 are clean, then the composition of morphisms $\mathcal{E}_{12} \in \mathrm{Hom}(Y_1, Y_2)$ and $\mathcal{E}_{23} \in \mathrm{Hom}(Y_2, Y_3)$ is a combination of pull-backs, tensor product and push-forward

$$\mathcal{E}_{23} \circ \mathcal{E}_{12} = (\iota_{13})_* \Big(\iota_{12}^*(\mathcal{E}_{12}) \otimes \iota_{23}^*(\mathcal{E}_{23}) \Big), \tag{1.8}$$

where ι_{ij} are injections

$$\begin{array}{ccccc} & & Y_1 \cap Y_2 \cap Y_3 & & \\ & \swarrow^{\iota_{12}} & \downarrow{\scriptstyle \iota_{13}} & \searrow^{\iota_{23}} & \\ Y_1 \cap Y_2 & & Y_1 \cap Y_3 & & Y_2 \cap Y_3. \end{array}$$

In particular, the endomorphism category of a lagrangian submanifold $Y \subset X$ is its two-periodic category:

$$\mathrm{End}_{\ddot{\mathsf{L}}(X,\omega)}(Y) = \mathsf{D}_{\mathbb{Z}_2}(Y), \tag{1.9}$$

and the composition of endomorphisms is just the tensor product:

$$\mathcal{E}_1 \circ \mathcal{E}_2 = \mathcal{E}_1 \otimes \mathcal{E}_2 \tag{1.10}$$

for any $\mathcal{E}_1, \mathcal{E}_2 \in \mathsf{D}_{\mathbb{Z}_2}(Y)$.

In Section 5 we study the geometric description of the category $\ddot{\mathsf{L}}(X,\omega)$ in case of a general holomorphic symplectic manifold (X,ω). We show that general features remain the same as for the cotangent bundle $X = \mathrm{T}^\vee U$, but almost everything is deformed. As we have mentioned, lagrangian submanifolds $Y \subset X$ remain objects of $\ddot{\mathsf{L}}(X,\omega)$, but the categories of morphisms (1.7) and the composition rules (1.8) are deformed. These A_∞ deformations are determined by the symplectic geometry of tubular neighborhoods of the lagrangian submanifolds involved, as summarized in Section 1.7.

1.5. Relation between algebraic and geometric approaches

In Section 4.3 we relate the algebraic and geometric descriptions of the two-category of a cotangent bundle. We describe the restriction of the equivalence functor

$$\ddot{\Phi}_{\cong} \colon \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U) \xrightarrow{\cong} \ddot{\mathsf{L}}(\mathrm{T}^\vee U) \tag{1.11}$$

(where the two-category $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ is a slight modification of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ defined in Section 3.5) to the objects of $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ whose images in $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ admit a geometric description as holomorphic fibrations.

Let us sketch this relation for $X = \mathrm{T}^\vee\mathbb{C}^n$. To a polynomial $W \in \mathbb{C}[\mathbf{x}]$, which is an object of the two-category $\ddot{\mathsf{MF}}(\mathbf{x})$, we associate the graph of its holomorphic differential

$$Y_W = \{(\mathbf{x}, \mathbf{p}) \in \mathrm{T}^\vee\mathbb{C}^n \,|\, \mathbf{p} = \boldsymbol{\partial} W\}. \tag{1.12}$$

Y_W is a lagrangian submanifold of X and it represents the object of $\ddot{\mathsf{L}}(\mathrm{T}^\vee\mathbb{C}^n)$ corresponding to W.

Let $\mathrm{Crit}(W)$ denote the critical locus of the polynomial W: $\mathrm{Crit}(W) = \{\mathbf{x} \in \mathbb{C}^n \,|\, \partial W(\mathbf{x}) = 0\}$. We say that the critical locus is clean, if $\mathrm{Crit}(W)$ is a smooth manifold and the Hessian of W is non-degenerate in the normal directions. A category $\mathsf{MF}(\mathbf{x}; W)$ "localizes" to $\mathrm{Crit}(W)$, and if $\mathrm{Crit}(W)$ is clean then we conjecture that $\mathsf{MF}(\mathbf{x}; W)$ is equivalent to $\mathsf{D}_{\mathbb{Z}_2}(\mathrm{Crit}(W))$ up to a certain categorical "shift" explained in Section 4.3.1. The intersection $Y_{W_1} \cap Y_{W_2} \subset \mathrm{T}^\vee\mathbb{C}^n$ projects onto $\mathrm{Crit}(W_2 - W_1) \subset \mathbb{C}^n$ and the projection establishes an isomorphism between them. The intersection of Y_{W_1} and Y_{W_2} is clean exactly when the difference $W_2 - W_1$ has a clean critical locus. Hence in this case the categories $\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}(W_1, W_2)$ and $\mathrm{Hom}_{\ddot{\mathsf{L}}(\mathrm{T}^\vee\mathbb{C}^n)}(Y_{W_1}, Y_{W_2})$ are equivalent (up to a shift).

1.6. The two-category of a deformed cotangent bundle

Path-integral analysis in paper [6] suggests that similarly to the derived category of coherent sheaves and in contrast to the Fukaya category, the two-category $\ddot{\mathsf{L}}(X, \omega)$ is local: the category of morphisms between two lagrangian submanifolds $\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}(Y_1, Y_2)$ is determined by a small tubular neighborhood of the intersection $Y_1 \cap Y_2$ and the composition of morphisms connecting Y_1, Y_2 and Y_3 is determined by a small tubular neighborhood of the triple intersection $Y_1 \cap Y_2 \cap Y_3$. Therefore, if we knew the properties of the two-category $\ddot{\mathsf{L}}$ of a tubular neighborhood of a lagrangian submanifold $Y \subset X$, we would know the morphisms involving Y and their compositions.

In real symplectic geometry, a small tubular neighborhood of a lagrangian submanifold is symplectomorphic to a small tubular neighborhood of the zero section of its cotangent bundle. This is no longer the case in holomorphic symplectic geometry: the holomorphic symplectic structure of the cotangent bundle $\mathrm{T}^\vee Y$ may have non-trivial deformations and a tubular neighborhood of $Y \subset X$ may be isomorphic to a tubular neighborhood of the zero section within this deformed bundle. Thus in order to apply the locality principle to the study of $\ddot{\mathsf{L}}(X, \omega)$, in Section 5 we explore the two-category $\ddot{\mathsf{L}}$ of a deformed cotangent bundle of a complex manifold U.

The best way for us to describe the two-category $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ is through its equivalence to the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$. Hence we describe the category $\ddot{\mathsf{L}}$ of the deformed cotangent bundle of U by constructing a deformation of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$. We assume that the deformation parameter $\varkappa$ of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ is the same parameter that describes the deformation of the holomorphic complex structure of $\mathrm{T}^\vee U$ and that the simplest objects of the deformed category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ are functions W on U, such that the graphs of their holomorphic differentials ∂W are lagrangian submanifolds of the deformed cotangent bundle $(\mathrm{T}^\vee U)_\varkappa$.

We find that the deformation parameter $\varkappa$ is an element of $\Omega^{0,1}\left(U,\mathrm{S}^\bullet\mathrm{T}U\right)$, $\deg_{\mathrm{S}^\bullet}\varkappa\geq 2$, satisfying the Maurer–Cartan equation $\bar\partial\varkappa+\frac{1}{2}\{\varkappa,\varkappa\}=0$, where $\{-,-\}$ is the Poisson–Schouten bracket on $\Omega^{0,1}(U,\mathrm{S}^\bullet\mathrm{T}U)$. The functions W parameterizing the simplest objects of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ must satisfy the equation $\bar\partial W=\varkappa(\partial W)$, where $\varkappa$ is regarded as a $(0,1)$ form on $\mathrm{T}^\vee U$ taking values in polynomial functions on the fibers of $\mathrm{T}^\vee U$. The category of morphisms between two objects W_1 and W_2 of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ turns out to be an A_∞-deformation of (1.6):

$$\mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(W_1,W_2)=\mathsf{D}_{\mathbb{Z}_2}(U;\lambda_{12}),\tag{1.13}$$

where the deformation parameter $\lambda_{12}=W_2-W_1+\cdots\in\Omega^{0,i}(U,\wedge^\bullet\mathrm{T}U)$ satisfies the Maurer–Cartan equation $\bar\partial\lambda_{12}+\frac{1}{2}\left[\lambda_{12},\lambda_{12}\right]=0$ and the bracket $[-,-]$ is the Schouten bracket. The composition of morphisms turns out to be a non-commutative deformation of the tensor product that described the composition of morphisms within $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$.

1.7. Geometric approach: the general case

The locality principle applied to formula (1.13) says that for a general holomorphic symplectic manifold (X,ω) the category of morphisms between lagrangian submanifolds Y_1 and Y_2 having a clean intersectionis given by a deformation of Equation (1.7):

$$\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}(Y_1,Y_2)=\mathsf{D}_{\mathbb{Z}_2}(Y_1\cap Y_2;\lambda_{\cap,12}),\tag{1.14}$$

where $\lambda_{\cap,12}$ is a deformation parameter of a special form: $\lambda_{\cap,12}=\lambda_{\cap,12,2}+\lambda_{\cap,12,3}+\cdots$ and $\lambda_{\cap,12,i}\in\Omega^{0,i}\left(U,\wedge^i\mathrm{T}(Y_1\cap Y_2)\right)$. Properties of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ suggest that if one of the exact sequences $\mathrm{T}Y_i\to\mathrm{T}X|_{Y_i}\to\mathrm{N}Y_i$, $i=1,2$ (for example, the one with $i=1$) splits and the other lagrangian submanifold

Y_2 can be presented as the graph of ∂W with $U = Y_1$ as described in Section 5.4.1, [2] then $\lambda_{\cap,12} = 0$, so that formula (1.7) holds true.

The non-split nature of the exact sequence $\mathrm{T}Y \to \mathrm{T}X|_Y \to \mathrm{N}Y$ is measured by a class $\check{\beta}_Y \in \mathrm{Ext}^1_Y(\mathscr{O}_Y, \mathrm{S}^2\mathrm{T}Y) \subset \mathrm{Ext}^1_Y(\mathrm{N}Y, \mathrm{T}Y)$, where $\mathscr{O}_Y$ is the structure sheaf of Y and we used the fact that Y is lagrangian, so $\mathrm{N}Y = \mathrm{T}^\vee Y$. If the exact sequence does not split or equivalently $\check{\beta}_Y \neq 0$, then the category of endomorphisms of Y is deformed:

$$\mathrm{End}_{\ddot{\mathsf{L}}(X,\omega)}(Y) = \mathsf{D}_{\mathbb{Z}_2}(Y; \lambda_Y),$$

where $\lambda_Y = \lambda_{Y,3} + \cdots$, $\lambda_{Y,i} \in \Omega^{0,i}\left(U, \wedge^i \mathrm{T}Y\right)$ and the leading term $\lambda_{Y,3}$ is quadratic in $\check{\beta}_Y$ and linear in the Atiyah class $\check{R}$ of the tangent bundle $\mathrm{T}Y$ (cf. Equation (5.46)).

To illustrate the deformation of the composition rule (1.8), consider the case when $Y_1 = Y_2 = Y_3 = Y$ and $\check{\beta}_Y \neq 0$. If the Atiyah class of the tangent bundle $\check{R}$ is zero, then, according to Section 5.6, the endomorphism category of Y remains undeformed as in Equation (1.9), but the composition rule (1.10) is deformed. For example, if $E_1, E_2 \in \mathsf{D}_{\mathbb{Z}_2}(Y)$ are two holomorphic vector bundles on Y, then their composition is the deformed tensor product

$$E_1 \circ E_2 = (E_1 \otimes E_2)_{\zeta_{12}}, \tag{1.15}$$

where ζ_{12} is a deformation parameter $\zeta_{12} = \zeta_{12,1} + \zeta_{12,2} + \cdots$, $\zeta_{12,i} \in \Omega^{0,2i+1}(\mathrm{End}(E_1 \otimes E_2))$, satisfying the Maurer–Cartan equation $\bar{\partial}\zeta_{12} + \frac{1}{2}[\zeta_{12}, \zeta_{12}] = 0$. The cohomology class of the leading component of ζ_{12} is proportional to $\check{\beta}_Y$ and to the Atiyah classes of the bundles E_1 and E_2:

$$\check{\zeta}_{12,3} = \check{\beta}_Y \llcorner (\check{F}_{E_1} \check{F}_{E_2}).$$

Note that $\check{\zeta}_{21,3} = -\check{\zeta}_{12,3}$, so the composition in the category $\mathrm{End}_{\ddot{\mathsf{L}}(X,\omega)}(Y)$ is non-commutative due to deformation (1.15).

If $\check{\beta}_Y = 0$, then $\zeta_{12} = 0$ and the composition rule (1.10) remains undeformed, however the associator isomorphism

$$\alpha_{123} \colon (E_1 \otimes E_2) \otimes E_3 \longrightarrow E_1 \otimes (E_2 \otimes E_3)$$

may be non-trivial.

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[2] We expect that such a presentation exists if the holomorphic bundle $\mathrm{T}Y_2|_{Y_1 \cap Y_2}/\mathrm{T}(Y_1 \cap Y_2)$ admits an $O(n, \mathbb{C})$ structure.

1.8. A presheaf definition of the two-category $\ddot{\mathsf{L}}(X,\omega)$

In Section 6 we sketch an approach to a rigorous definition of the two-category $\ddot{\mathsf{L}}(X,\omega)$ as the category of global sections of a certain presheaf $\ddot{\mathfrak{P}}(X,\omega)$ of two-categories defined on X.

Let $\mathbb{C}^n_{\mathbf{x}}$ be the affine space $\mathbb{C}^n$ with standard coordinates $\mathbf{x} = x_1,\dots,x_n$ and let $U_{\mathbf{x}} \subset \mathbb{C}^n_{\mathbf{x}}$ be an open subset inheriting the coordinates. A *symplectic rectangle* is a product of two such subsets $U_{\mathbf{x}} \times V_{\mathbf{y}}$; it has a natural holomorphic symplectic structure $\omega = \sum_{i=1}^n dy_i \wedge dx_i$. A *rectangular chart* is a symplectic embedding

$$f\colon U_{\mathbf{x}} \times V_{\mathbf{y}} \to X. \tag{1.16}$$

The images of these charts form an open cover of X.

Since the coordinates $\mathbf{x}$ on $U_{\mathbf{x}}$ provide a trivialization of the cotangent bundle $\mathrm{T}^{\vee}U$, there is a canonical symplectic embedding $U_{\mathbf{x}} \times V_{\mathbf{y}} \subset \mathrm{T}^{\vee}U$. To a symplectic rectangle $U_{\mathbf{x}} \times V_{\mathbf{y}}$ we associate a two-category $\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$ which is a "micro-localization" of the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$. The two-category $\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$ is a full subcategory of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ and its simplest objects are holomorphic functions W on U satisfying the condition that the associated lagrangian submanifolds (1.12) should lie within $U_{\mathbf{x}} \times V_{\mathbf{y}} \subset \mathrm{T}^{\vee}U$: for any point $u \in U_{\mathbf{x}}$, the differential $v = \partial_{\mathbf{x}}W \in \mathbb{C}^n_{\mathbf{y}}$ should belong to $V_{\mathbf{y}} \subset \mathbb{C}^n_{\mathbf{y}}$.

Thus to a rectangular chart (1.16) we associate the two-category $\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$. The structure of the micro-local presheaf $\ddot{\mathfrak{P}}(X,\omega)$ comes from two types of functors defined in Section 6.2: the restriction functor and the Legendre transform. A symplectic rectangle $U_{\mathbf{x}} \times V_{\mathbf{y}}$ has a lagrangian "q-fibration" formed by subspaces $u \times V_{\mathbf{y}}$, where $u \in U_{\mathbf{x}}$. The restriction functor $\ddot{\Phi}_{\mathrm{r},\varepsilon}\colon \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(U'_{\mathbf{x}'} \times V'_{\mathbf{y}'})$ is associated to a symplectic embedding $\varepsilon\colon U'_{\mathbf{x}'} \times V'_{\mathbf{y}'} \hookrightarrow U_{\mathbf{x}} \times V_{\mathbf{y}}$, which preserves the q-fibration. The Legendre transform is a special equivalence functor $\ddot{\Lambda}_{+}\colon \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(V_{\mathbf{y}} \times U_{-\mathbf{x}})$, which permutes the lagrangian fibrations $u \times V_{\mathbf{y}}$, $u \in U_{\mathbf{x}}$ and $U_{\mathbf{x}} \times v$, $v \in V_{\mathbf{y}}$ of the symplectic rectangle $U_{\mathbf{x}} \times V_{\mathbf{y}}$.

1.9. Derived categorical sheaves

In Section 7 we discuss the relationship between the RW model and the theory of derived categorical sheaves introduced by Toen and Vezzosi [14]. This relationship emerges when the target manifold X is the cotangent bundle of a complex manifold Y. In this special case, one can promote the $\mathbb{Z}_2$ grading of the RW model to a $\mathbb{Z}$-grading by declaring that natural fiber coordinates of the cotangent bundle sit in cohomological degree 2. Objects of the corresponding two-category of boundary conditions are naturally associated with sheaves of DG-categories over Y, i.e., with derived categorical sheaves. More precisely, objects of the kind mentioned above (complex fibrations over Y) correspond to rather special sheaves of DG-categories. However, we argue that more general sheaves of DG-categories, such as skyscraper sheaves, can also be related to boundary conditions in the RW model if we allow fibrations whose fibers are graded manifolds. Conjecturally, the two-category of boundary conditions in the $\mathbb{Z}$-graded RW model with target $\mathrm{T}^{\vee}Y$ is a full sub-two-category of the two-category of derived categorical sheaves over Y. We perform some simple checks of this conjecture.

2. The three-category of affine spaces

2.1. Two-periodic perfect derived category of a curved differential graded algebra

In this section we define the two-category of boundary conditions corresponding to the RW model whose target is a complex symplectic vector spaces. We also describe the three-category of all such RW models. Definitions of categories of morphisms between two objects of our two-categories follow the same general pattern that we are going to review in this subsection. We follow closely the exposition of Block [2], replacing $\mathbb{Z}$-grading with $\mathbb{Z}_2$-grading when needed.

A commutative curved differential graded algebra (CDGA) is a triple $\big(\mathcal{A}, \bar{\nabla}, W\big)$, where $\mathcal{A}$ is a $\mathbb{Z}$-graded associative commutative algebra $\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}^i$ with an associated $\mathbb{Z}_2$-grading

$$\mathcal{A} = \mathcal{A}^{\hat{0}} \oplus \mathcal{A}^{\hat{1}}, \quad \mathcal{A}^{\hat{0}} = \bigoplus_{i=0}^{\infty} \mathcal{A}^{2i}, \quad \mathcal{A}^{\hat{1}} = \bigoplus_{i=1}^{\infty} \mathcal{A}^{2i+1}, \tag{2.1}$$

$\bar{\nabla}$ is its differential of (possibly inhomogeneous) odd degree not less than 1:

$$\bar{\nabla}^2 = 0, \qquad \bar{\nabla}(\mathcal{A}^i) \subset \bigoplus_{j=0}^{\infty} \mathcal{A}^{i+2j+1},$$

and a curving W is a $\bar{\nabla}$-closed element of $\mathcal{A}$ of even $\mathbb{Z}_2$-degree: $W \in \mathcal{A}^{\hat{0}}$, $\bar{\nabla} W = 0$.

We adopt the notations $|-|_{\mathbb{Z}}$ and $|-|$ for $\mathbb{Z}$-degree and $\mathbb{Z}_2$-degree, respectively, and we denote the elements of $\mathbb{Z}_2$ as $\hat{0}$ and $\hat{1}$.

A $\mathbb{Z}_2$-graded differential module ($\mathbb{Z}_2$-GDM) over a CDGA $\big(\mathcal{A},\bar{\nabla},W\big)$ is a pair $\mathcal{M}=\big(M,\bar{\nabla}_M\big)$, where M is a $\mathbb{Z}_2$-graded module over $\mathcal{A}$, while $\bar{\nabla}_M$ is its curved differential: $\bar{\nabla}_M$ is a $\mathbb{C}$-linear map $M\xrightarrow{\bar{\nabla}_M}M$, $|\bar{\nabla}_M|=\hat{1}$, satisfying the Leibniz identity

$$\bar{\nabla}_M(am)=(\bar{\nabla}a)\,m+(-1)^{|a|}\,a\,(\bar{\nabla}_M m),\qquad\forall a\in\mathcal{A},\quad\forall m\in M,$$

and having the curving W:

$$\bar{\nabla}_M\circ\bar{\nabla}_M=W\,\mathtt{1}_M,$$

where $\mathtt{1}_M$ is the identity endomorphism of M. The module M can be rolled out into a two-periodic twisted complex

$$\cdots\xrightarrow{\bar{\nabla}_{M^{\hat{1}}}}M^{\hat{0}}\xrightarrow{\bar{\nabla}_{M^{\hat{0}}}}M^{\hat{1}}\xrightarrow{\bar{\nabla}_{M^{\hat{1}}}}M^{\hat{0}}\xrightarrow{\bar{\nabla}_{M^{\hat{0}}}}\cdots$$
$$\bar{\nabla}_{M^{\hat{0}}}\circ\bar{\nabla}_{M^{\hat{1}}}=W\mathtt{1}_{M^{\hat{1}}},\qquad\bar{\nabla}_{M^{\hat{1}}}\circ\bar{\nabla}_{M^{\hat{0}}}=W\mathtt{1}_{M^{\hat{0}}},$$

hence the name of the category.

Suppose that two $\mathbb{Z}_2$-graded modules (or vector spaces) M_1 and M_2 have endomorphisms A_1 and A_2 of a similar nature. Then for a linear map $f\colon M_1\to M_2$ we use the commutator notation for the following expression:

$$[A,f]=A_2f-(-1)^{|A||f|}fA_1. \tag{2.2}$$

For two $\mathbb{Z}_2$-GDMs $\mathcal{M}_1$ and $\mathcal{M}_2$ over a CDGA $\big(\mathcal{A},\bar{\nabla},W\big)$, the space of homomorphisms $\operatorname{Hom}_{\mathcal{A}}(M_1,M_2)$ has a differential d:

$$df=[\bar{\nabla}_M,f],\qquad f\in\operatorname{Hom}_{\mathcal{A}}(M_1,M_2).$$

Thus $\mathbb{Z}_2$-GDMs are objects of a DG-category.

We define the tensor product of a $\mathbb{Z}_2$-GDM $\mathcal{M}_1$ over a CDGA $\big(\mathcal{A},\bar{\nabla},W_1\big)$ and a $\mathbb{Z}_2$-GDM $\mathcal{M}_2$ over a CDGA $\big(\mathcal{A},\bar{\nabla},W_2\big)$ as a $\mathbb{Z}_2$-GDM over a CDGA $\big(\mathcal{A},\bar{\nabla},W_1+W_2\big)$ by the formula

$$\mathcal{M}_1\otimes_{\mathcal{A}}\mathcal{M}_2=\big(M_1\otimes_{\mathcal{A}}M_2,\bar{\nabla}_{M_1}\otimes\mathtt{1}_M+(-1)^{|-|}\otimes\bar{\nabla}_{M_2}\big). \tag{2.3}$$

Also we define the dual $\mathbb{Z}_2$-GDM as

$$\big(M,\bar{\nabla}_M\big)^\vee=\big(M^\vee,\bar{\nabla}_M^\vee\big). \tag{2.4}$$

Note that $\big(M,\bar{\nabla}_M\big)^\vee$ is a $\mathbb{Z}_2$-GDM over the CDGA $\big(\mathcal{A},\bar{\nabla},-W\big)$.

A $\mathbb{Z}_2$-GDM $\mathcal{P} = (P, \bar{\nabla}_P)$ is called *perfect* if the $\mathcal{A}$-module P has the form

$$P = \check{P} \otimes_{\mathcal{A}^0} \mathcal{A}, \tag{2.5}$$

where $\check{P}$ is a projective $\mathbb{Z}_2$-graded module over $\mathcal{A}^0$. The *two-periodic perfect derived category* $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}, \bar{\nabla}, W)$ of a CDGA $(\mathcal{A}, \bar{\nabla}, W)$ is defined as a graded category whose objects are perfect $\mathbb{Z}_2$-GDMs, and morphisms are defined by

$$\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2) = \mathrm{H}^{\bullet}_{d}\Big(\mathrm{Hom}_{\mathcal{A}}(P_1, P_2)\Big). \tag{2.6}$$

One may enhance the category $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}, \bar{\nabla}, W)$ by adding new "admissible" objects which are declared isomorphic to the existing perfect objects according to the following rule. A $\mathbb{Z}_2$-GDM $\mathcal{M}$ is called *admissible*, if there exists a perfect $\mathbb{Z}_2$-GDM $\mathcal{P}$ such that for any perfect $\mathbb{Z}_2$-GDM $\mathcal{P}'$ there is an isomorphism

$$\mathrm{Hom}(\mathcal{P}', \mathcal{M}) = \mathrm{Hom}(\mathcal{P}', \mathcal{P}) \tag{2.7}$$

and for any other admissible $\mathbb{Z}_2$-GDM $\mathcal{M}'$ we define

$$\mathrm{Ext}(\mathcal{M}, \mathcal{M}') := \mathrm{Hom}(\mathcal{P}, \mathcal{M}').$$

2.2. Categories of matrix factorizations

2.2.1. Definition of the category. A category of matrix factorizations is a particular case of a two-periodic perfect derived category defined in Section 2.1. For a finite set of commuting variables

$$\mathbf{x} = x_1, \ldots, x_n \tag{2.8}$$

consider the algebra of polynomial functions $\mathcal{A} = \mathcal{A}^0 = \mathbb{C}[\mathbf{x}]$ regarded as $\mathbb{Z}$-graded CDGA placed in zero degree, with zero differential $\bar{\nabla}_{\mathcal{A}} = 0$, and the curving being a polynomial $W \in \mathbb{C}[\mathbf{x}]$. Then the category of matrix factorization $\mathsf{MF}(\mathbf{x}; W)$ is the corresponding two-periodic perfect derived category:

$$\mathsf{MF}(\mathbf{x}; W) := \mathsf{D}_{\mathbb{Z}_2}\big(\mathbb{C}[\mathbf{x}], 0, W\big). \tag{2.9}$$

According to the general definition, an object of $\mathsf{MF}(\mathbf{x}; W)$ is a pair $\mathcal{M} = (M, D_M)$, where M is a free $\mathbb{Z}_2$-graded $\mathbb{C}[\mathbf{x}]$-module, while D_M is its curved differential:

$$D_M \in \mathrm{End}_{\mathbb{C}[\mathbf{x}]}(M), \qquad |D_M| = \hat{1}, \qquad D_M^2 = W\, 1_M.$$

Morphisms between objects are defined by Equation (2.6). The tensor product (2.3) gives a functor

$$\mathsf{MF}(\mathbf{x}; W_1) \times \mathsf{MF}(\mathbf{x}; W_2) \xrightarrow{\otimes_{\mathbb{C}[\mathbf{x}]}} \mathsf{MF}(\mathbf{x}; W_1 + W_2)\,. \tag{2.10}$$

Let $\mathbf{y} = y_1, \ldots, y_k$ be another list of variables, generally of a different length. For $W_1 \in \mathbb{C}[\mathbf{x}]$ and $W_2 \in \mathbb{C}[\mathbf{y}]$, a matrix factorization $\mathcal{M}_{12} \in \mathsf{MF}(\mathbf{x}, \mathbf{y}; W_2 - W_1)$ determines a functor

$$\mathsf{MF}(\mathbf{x}; W_1) \xrightarrow{\Phi[\mathcal{M}_{12}]} \mathsf{MF}(\mathbf{y}; W_2) \tag{2.11}$$

which acts by taking a tensor product with $\mathcal{M}_{12}$: for a matrix factorization $\mathcal{M} \in \mathsf{MF}(\mathbf{x}; W_1)$,

$$\Phi\left[\mathcal{M}_{12}\right](M) = \mathcal{M} \otimes_{\mathbb{C}[\mathbf{x}]} \mathcal{M}_{12} \in \mathsf{MF}(\mathbf{y}; W_2). \tag{2.12}$$

Note that since W_1 cancels from the curving of the tensor product (2.12), we can forget its $\mathbb{C}[\mathbf{x}]$-module structure, thus turning it into an object of $\mathsf{MF}(\mathbf{y}; W_2)$.

All categories $\mathsf{MF}(\mathbf{x}; W)$ can be unified into a single two-category $\ddot{\mathsf{MF}}$ of Landau–Ginzburg B-models with affine target spaces along the lines explained in Section 1.1. This two-category should be thought of as a two-category of boundary conditions for the RW model whose target is a point. An object of $\ddot{\mathsf{MF}}$ is a pair $(\mathbf{x}\,; W)$, $W \in \mathbb{C}[\mathbf{x}]$ or, equivalently, a category of matrix factorizations $\mathsf{MF}(\mathbf{x}; W)$. Morphisms between two objects also form a category of matrix factorization

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}}\left((\mathbf{x}\,; W_1)\,, (\mathbf{y}\,; W_2)\right) = \mathsf{MF}(\mathbf{x}, \mathbf{y}; W_2 - W_1), \tag{2.13}$$

and the composition of morphisms corresponds to the composition of functors (2.11): for two morphisms $\mathcal{M}_{12} \in \mathrm{Hom}((\mathbf{x}\,; W_1)\,, (\mathbf{y}\,; W_2))$ and $\mathcal{M}_{23} \in \mathrm{Hom}((\mathbf{y}\,; W_2)\,, (\mathbf{z}\,; W_3))$ we define

$$\mathcal{M}_{23} \circ \mathcal{M}_{12} = \mathcal{M}_{23} \otimes_{\mathbb{C}[\mathbf{y}]} \mathcal{M}_{12}.$$

2.2.2. Koszul matrix factorizations A Koszul complex corresponding to a list of polynomials $\mathbf{p} = p_1, \ldots, p_k \in \mathbb{C}[\mathbf{x}]$ is the tensor product of complexes

$$K(\mathbf{p}) = \bigotimes_{i=1}^{k} \Big(\mathbb{C}[\mathbf{x}]_{\hat{0}} \xrightarrow{p_i} \mathbb{C}[\mathbf{x}]_{\hat{1}} \Big), \tag{2.14}$$

where $\mathbb{C}[\mathbf{x}]_i$ denotes a rank-1 $\mathbb{C}[\mathbf{x}]$-module of $\mathbb{Z}_2$-degree i. A Koszul matrix factorization is defined similarly: for two sequences of polynomials $\mathbf{p}, \mathbf{q} \in \mathbb{C}[\mathbf{x}]$ of equal length k, a Koszul matrix factorization $K(\mathbf{p};\mathbf{q})$ is a tensor product of rank-$(1,1)$ matrix factorizations

$$K(\mathbf{p};\mathbf{q}) = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ \cdots & \cdots \\ p_k & q_k \end{pmatrix} = \bigotimes_{i=1}^{k} \Big(\mathbb{C}[\mathbf{x}]_{\hat{0}} \underset{q_i}{\overset{p_i}{\rightleftarrows}} \mathbb{C}[\mathbf{x}]_{\hat{1}} \Big). \tag{2.15}$$

Obviously, $K(\mathbf{p};\mathbf{q}) \in \mathsf{MF}(\mathbf{x}; \mathbf{p}\cdot\mathbf{q})$, where we use the notation

$$\mathbf{p}\cdot\mathbf{q} = \sum_{i=1}^{k} p_i q_i.$$

Suppose that an ideal $(\mathbf{p}) \subset \mathbb{C}[\mathbf{x}]$ is generated by a regular sequence $\mathbf{p}$ and $W \in (\mathbf{p})$. Then the polynomial W has a presentation $W = \mathbf{p}\cdot\mathbf{q}$, where $\mathbf{q} \in \mathbb{C}[\mathbf{x}]$. It is easy to check that for fixed W, the isomorphism class of the matrix factorization (2.15) does not depend on the choice of the polynomials $\mathbf{q}$, so we can use an abbreviated notation

$$K_W(\mathbf{p}) = K(\mathbf{p};\mathbf{q}) \tag{2.16}$$

for the matrix factorization $K(\mathbf{p};\mathbf{q})$ of Equation (2.15).

The identity endofunctor of a matrix factorization category $\mathsf{MF}(\mathbf{x};W)$ can be presented in the form (2.11) with the help of a Koszul matrix factorization. For a list of variables $\mathbf{x}$, consider another list $\mathbf{x}'$ of the same length. The difference $W(\mathbf{x}') - W(\mathbf{x})$ belongs to the ideal $(\mathbf{x}' - \mathbf{x}) \subset \mathbb{C}[\mathbf{x}, \mathbf{x}']$, and the corresponding Koszul matrix factorization

$$1_{\mathbf{x};W} := K_{W(\mathbf{x}')-W(\mathbf{x})}(\mathbf{x}' - \mathbf{x})$$

determines the functor (2.11)

$$\mathsf{MF}(\mathbf{x};W) \xrightarrow{\Phi[1_{\mathbf{x};W}]} \mathsf{MF}(\mathbf{x}';W)\ ,$$

which becomes the identity functor, if we identify the categories $\mathsf{MF}(\mathbf{x};W)$ and $\mathsf{MF}(\mathbf{x}';W)$ by identifying the variables $\mathbf{x}$ and $\mathbf{x}'$.

2.2.3. Knorrer periodicity and the translation two-functor. If the list of variables $\mathbf{x}$ is empty and the curving W is zero, then the corresponding category $\mathsf{MF}(\varnothing;0)$ is equivalent to the category of $\mathbb{Z}_2$-graded vector spaces. The latter has only two indecomposable objects: the one-dimensional vector space $\mathbb{C}$ in degree 0 and its translation $\mathbb{C}[\hat{1}]$.

The category $\mathsf{MF}(y_1,y_2;y_1^2+y_2^2)$ also has only two indecomposable objects: the Koszul matrix factorization

$$\mathcal{M}_{y_1,y_2} := K_{y_1^2+y_2^2}(y_1-\sqrt{-1}\,y_2)$$

and its translation $\mathcal{M}_{y_1,y_2}[\hat{1}] = K_{y_1^2+y_2^2}(y_1+\sqrt{-1}\,y_2)$.

The Knorrer periodicity theorem states that there is an equivalence of categories

$$\mathsf{MF}(\varnothing;0) \cong \mathsf{MF}(y_1,y_2;y_1^2+y_2^2),$$

established by the functor

$$\mathsf{MF}(\varnothing;0) \xrightarrow{\Phi[\mathcal{M}_{y_1,y_2}]} \mathsf{MF}_{y_1,y_2;y_1^2+y_2^2}\ .$$

More generally, for any curving polynomial $W\in\mathbb{C}[\mathbf{x}]$ there is an equivalence of categories

$$\mathsf{MF}(\mathbf{x},y_1,y_2;W(\mathbf{x})+y_1^2+y_2^2) \cong \mathsf{MF}(\mathbf{x};W),$$

established by the functor

$$\mathsf{MF}(\mathbf{x};W) \xrightarrow{\Phi[1_{\mathbf{x};W}\otimes\mathcal{M}_{y_1,y_2}]} \mathsf{MF}(\mathbf{x}',y_1,y_2;W(\mathbf{x}')+y_1^2+y_2^2)\ . \tag{2.17}$$

The translation two-functor $[1]_2\colon \ddot{\mathsf{MF}}\to\ddot{\mathsf{MF}}$ acts on a matrix factorization category $\mathsf{MF}(\mathbf{x};W)$ by adding an extra variable to the list $\mathbf{x}$ and adding its square to the curving polynomial W:

$$\mathsf{MF}(\mathbf{x};W)[1]_2 := \mathsf{MF}(\mathbf{x},y;W(\mathbf{x})+y^2). \tag{2.18}$$

The action of $[1]_2$ on categories of morphisms is provided by the Knorrer periodicity functors (2.17). Knorrer periodicity also implies that the square of the translation two-functor is equivalent to the identity two-functor:

$$[2]_2 := [1]_2 \circ [1]_2 \simeq 1_{\ddot{\mathsf{MF}}}.$$

Definition (2.18) of the translation two-functor $[1]_2$ can be adapted to the general setting of the two-periodic perfect derived category of a CDGA (see Section 2.1). Consider a special CDGA $\mathcal{A}_{y^2} = (\mathbb{C}[y], 0, y^2)$, so that according to Equation (2.9) $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}_{y^2}) = \mathsf{MF}(y;y^2)$. For a CDGA $\mathcal{A}$ we define the translation of its two-periodic perfect derived category as

$$\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[1]_2 = \mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}\otimes\mathcal{A}_{y^2}). \tag{2.19}$$

Knorrer periodicity establishes an equivalence of categories

$$\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[2]_2 \cong \mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}).$$

For a list of variables $\mathbf{y} = y_1,\dots,y_n$ define a CDGA $\mathcal{A}_{\mathbf{y}^2} = (\mathbb{C}[\mathbf{y}], 0, \mathbf{y}^2)$, so that by our definition $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[n]_2 = \mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}\otimes\mathcal{A}_{\mathbf{y}^2})$. The category $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}\otimes\mathcal{A}_{\mathbf{y}^2})$ has an equivalent "intrinsic" description in terms of objects and morphisms of $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})$. Namely, consider a category $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[n]_2'$, whose objects are pairs $(\mathcal{P}, \mathbf{f}_{\mathcal{P}})$, where $\mathcal{P} = \big(P, \bar{\nabla}_P\big)$ is a perfect module of $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})$, while

$$\mathbf{f}_{\mathcal{P}} = f_{\mathcal{P},1},\dots f_{\mathcal{P},n} \in \mathrm{Ext}^{\hat{1}}(\mathcal{P},\mathcal{P})$$

satisfies the property

$$\{f_{\mathcal{P},i}, f_{\mathcal{P},j}\} = \delta_{ij}\, 1_{\mathcal{P}}.$$

Morphisms between two objects $(\mathcal{P}, \mathbf{f}_{\mathcal{P}})$ and $(\mathcal{P}', \mathbf{f}_{\mathcal{P}'})$ are $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})$-morphisms $g \in \mathrm{Ext}^{\bullet}(\mathcal{P},\mathcal{P}')$ which intertwine the lists $\mathbf{f}_{\mathcal{P}}$ and $\mathbf{f}_{\mathcal{P}'}$: $g\,\mathbf{f}_{\mathcal{P}} = \mathbf{f}_{\mathcal{P}'}g$. An equivalence functor between the categories $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[n]_2'$ and $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}\otimes\mathcal{A}_{\mathbf{y}^2})$ maps an object $(\mathcal{P}, \mathbf{f}_{\mathcal{P}})$ of $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A})[n]_2'$ into an object $\big(P\otimes\mathbb{C}[\mathbf{y}], \bar{\nabla}_P + \mathbf{y}\cdot\mathbf{f}_{\mathcal{P}}\big)$ of $\mathsf{D}_{\mathbb{Z}_2}(\mathcal{A}\otimes\mathcal{A}_{\mathbf{y}^2})$.

2.3. The two-category of relative curved differential graded polynomial algebras

Fix a finite set of variables $\mathbf{x}$ of length n. The two-category of relative curved differential graded (CDG) polynomial algebras $\ddot{\mathsf{MF}}(\mathbf{x})$ is a result of "fibering" the two-category $\ddot{\mathsf{MF}}$ over the algebra $\mathbb{C}[\mathbf{x}]$. One should regard this two-category as a two-category of boundary conditions for the RW model whose target is $\mathrm{T}^\vee\mathbb{C}^n$. (These are not the most general boundary conditions: more general ones will be described in the next section.) Objects of $\ddot{\mathsf{MF}}(\mathbf{x})$ are pairs $(\mathbf{y}\,;W)$, where $\mathbf{y} = y_1,\ldots,y_k$ is a list of "extra" variables of arbitrary length and the curving W is an element of the algebra $\mathbb{C}[\mathbf{x},\mathbf{y}]$ over the ring $\mathbb{C}[\mathbf{x}]$. The category of morphisms between two objects is defined as the category of matrix factorization of the difference of curvings:

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}\left((\mathbf{y}\,;W_1)\,,(\mathbf{z}\,;W_2)\right) := \mathsf{MF}(\mathbf{x},\mathbf{y},\mathbf{z};W_2(\mathbf{x},\mathbf{z}) - W_1(\mathbf{x},\mathbf{y})) \tag{2.20}$$

(cf. Equation (2.13)). The composition of morphisms between objects is given by the tensor product functor (2.10): for $\mathcal{M}_{12} \in \mathrm{Hom}((\mathbf{y}\,;W_1)\,,(\mathbf{z}\,;W_2))$ and $\mathcal{M}_{23} \in \mathrm{Hom}((\mathbf{z}\,;W_2)\,,(\mathbf{u}\,;W_3))$ we define

$$\mathcal{M}_{23}\circ\mathcal{M}_{12} = \mathcal{M}_{23}\otimes_{\mathbb{C}[\mathbf{x},\mathbf{z}]}\mathcal{M}_{12} \in \mathsf{MF}(\mathbf{x},\mathbf{y},\mathbf{u};W_3 - W_1)$$

(cf. Equation (2.12)).

The simplest objects of $\ddot{\mathsf{MF}}(\mathbf{x})$ are the ones without extra variables, and we denote them as $W := (\varnothing\,;W)$, where $W \in \mathbb{C}[\mathbf{x}]$. The category of morphisms between such objects is

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}(W_1,W_2) = \mathsf{MF}(\mathbf{x};W_2 - W_1).$$

An object $(\mathbf{z}\,;W_{12}) \in \ddot{\mathsf{MF}}(\mathbf{x},\mathbf{y})$ determines a correspondence two-functor

$$\ddot{\mathsf{MF}}(\mathbf{x}) \xrightarrow{\ddot{\Phi}[\mathbf{z}\,;W]} \ddot{\mathsf{MF}}(\mathbf{y})\,, \tag{2.21}$$

which turns an object $(\mathbf{u}\,;W) \in \ddot{\mathsf{MF}}(\mathbf{x})$ into an object

$$\ddot{\Phi}\,[\mathbf{z}\,;W]\,(\mathbf{u}\,;W) = (\mathbf{x},\mathbf{u},\mathbf{z}\,;W + W_{12})\,. \tag{2.22}$$

The action of the correspondence two-functor (2.21) on categories of morphisms between objects is defined with the help of Koszul matrix factorizations. For objects $(\mathbf{u}\,;W_1)\,,(\mathbf{w}\,;W_2) \in \ddot{\mathsf{MF}}(\mathbf{x})$ we have to define the functor

$$\begin{aligned}\mathrm{Hom}((\mathbf{u}\,;W_1)\,,(\mathbf{w}\,;W_2)) \xrightarrow{\Phi_{12}[\mathbf{z}\,;W]} \\ \mathrm{Hom}(\ddot{\Phi}\,[\mathbf{z}\,;W]\,(\mathbf{u}\,;W_1)\,,\ddot{\Phi}\,[\mathbf{z}\,;W]\,(\mathbf{w}\,;W_2)),\end{aligned}$$

or more explicitly, according to Equations (2.20) and (2.22),

$$\begin{aligned}\mathsf{MF}(\mathbf{x},\mathbf{u},\mathbf{w};W_2(\mathbf{x},\mathbf{w})-W_1(\mathbf{x},\mathbf{u})) \xrightarrow{\Phi_{12}[\mathbf{z}\,;W]} \\ \mathsf{MF}(\mathbf{y},\mathbf{x}',\mathbf{u}',\mathbf{z}',\mathbf{x}'',\mathbf{w}'',\mathbf{z}'';W_2^{\mathrm{tot}}),\end{aligned} \tag{2.23}$$

where

$$W_2^{\mathrm{tot}} = W_2(\mathbf{x}'',\mathbf{w}'') + W_{12}(\mathbf{x}'',\mathbf{y},\mathbf{z}'') - W_1(\mathbf{x}',\mathbf{u}') - W_{12}(\mathbf{x}',\mathbf{y},\mathbf{z}')$$

and for some lists of variables we used primed and double-primed lists of the same length in order to change the names of variables. Functor (2.23) can be written in the form (2.12)

$$\Phi_{12}\,[\mathbf{z}\,;W] = \Phi\,[\mathcal{M}_{12}]$$

for a certain matrix factorization $\mathcal{M}_{12} \in \mathsf{MF}(R^{\mathrm{tot}};W_{12}^{\mathrm{tot}})$, where

$$R^{\mathrm{tot}} = \mathbb{C}[\mathbf{x},\mathbf{u},\mathbf{w},\mathbf{y},\mathbf{x}',\mathbf{u}',\mathbf{z}',\mathbf{x}'',\mathbf{w}'',\mathbf{z}'']$$

and

$$\begin{aligned}W_{12}^{\mathrm{tot}} = \Big(W_2(\mathbf{x}'',\mathbf{w}'') - W_2(\mathbf{x},\mathbf{w})\Big) - \Big(W_1(\mathbf{x}',\mathbf{u}') - W_1(\mathbf{x},\mathbf{u})\Big) \\ + \Big(W_{12}(\mathbf{x}'',\mathbf{y},\mathbf{z}'') - W_{12}(\mathbf{x}',\mathbf{y},\mathbf{z}')\Big).\end{aligned}$$

The full formula for the matrix factorization $\mathcal{M}_{12}$ is rather bulky, so we describe it indirectly in terms of the Koszul matrix factorization (2.16). Denote

$$\mathbf{p} = (\mathbf{x}''-\mathbf{x},\mathbf{w}''-\mathbf{w},\mathbf{x}'-\mathbf{x},\mathbf{u}'-\mathbf{u},\mathbf{z}''-\mathbf{z}').$$

Then $W_{12}^{\mathrm{tot}} \in (\mathbf{p})$, and we set

$$\mathcal{M}_{12} = K_{W^{\mathrm{tot}}}(\mathbf{p}).$$

When the lists $\mathbf{x}$ and $\mathbf{y}$ have equal number of variables, there exist two important equivalence two-functors of type (2.21). The first one is the two-functor

$$\ddot{\mathbb{1}}_{\mathbf{a}} = \ddot{\Phi}\,[\mathbf{a};\mathbf{a}\cdot(\mathbf{y}-\mathbf{x})]\,. \tag{2.24}$$

If we identify $\ddot{\mathsf{MF}}(\mathbf{x})$ with $\ddot{\mathsf{MF}}(\mathbf{y})$ by identifying the variables $\mathbf{x}$ and $\mathbf{y}$, then $\ddot{\mathbb{1}}_{\mathbf{a}}$ becomes equivalent to the identity two-functor. We leave the details of the proof to the reader, while illustrating this statement by the following example: after the identification of $\mathbf{x}$ and $\mathbf{y}$, the action of $\ddot{\mathbb{1}}_{\mathbf{a}}$ on an object $W \in \ddot{\mathsf{MF}}(\mathbf{x})$ becomes $\ddot{\mathbb{1}}_{\mathbf{a}}(W) = \left(\mathbf{x}', \mathbf{a}\,; \widetilde{W}\right)$, where $\widetilde{W}(\mathbf{x}, \mathbf{x}', \mathbf{a}) = W(\mathbf{x}') + \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}')$, and the isomorphism between W and $\ddot{\mathbb{1}}_{\mathbf{a}}(W)$ is established by the Koszul matrix factorization $K_{\widetilde{W}-W}(\mathbf{x}' - \mathbf{x})$.

The second equivalence two-functor is the "Legendre transform" and it has two versions:

$$\ddot{\Lambda}_+ = \ddot{\Phi}\left[\varnothing; \mathbf{x} \cdot \mathbf{y}\right], \qquad \ddot{\Lambda}_- = \ddot{\Phi}\left[\varnothing; -\mathbf{x} \cdot \mathbf{y}\right]. \tag{2.25}$$

Again, we leave the verification of their equivalence nature to the reader. Note that the equivalence of morphism categories $\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}(W_1, W_2)$ and $\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{y})}\left(\ddot{\Lambda}_\pm(W_1), \ddot{\Lambda}_\pm(W_2)\right)$ is a corollary of the Knorrer periodicity.

The composition of two Legendre transforms with opposite signs is equivalent to the identity two-functor:

$$\ddot{\Lambda}_+ \circ \ddot{\Lambda}_- \simeq \ddot{\Lambda}_- \circ \ddot{\Lambda}_+ \simeq \ddot{\mathbb{1}}.$$

The translation two-functor is an equivalence two-functor $[1]_2\colon \ddot{\mathsf{MF}}(\mathbf{x}) \to \ddot{\mathsf{MF}}(\mathbf{x})$, which acts on an object $(\mathbf{y}\,; W)$ by adding a new variable a to the extra variable list and adding its square to the polynomial W:

$$(\mathbf{y}\,; W)\,[1]_2 := \left(\mathbf{y}, a\,; W + a^2\right). \tag{2.26}$$

An equivalence between the morphism categories $\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}((\mathbf{y}\,; W_1)\,, (\mathbf{z}\,; W_2))$ and $\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}\,(\mathbf{y}\,; W_1)\,[1]_2, (\mathbf{z}\,; W_2)\,[1]_2$ is established by the Knorrer periodicity functor in view of an obvious equivalence of categories

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}\left((\mathbf{y}\,; W_1)\,[1]_2, (\mathbf{z}\,; W_2)\right) = \mathrm{Hom}_{\ddot{\mathsf{MF}}(\mathbf{x})}\left((\mathbf{y}\,; W_1)\,, (\mathbf{z}\,; W_2)\right)[1]_2. \tag{2.27}$$

The two-translation by 2 is isomorphic to the identity endofunctor: $[2]_2 \cong 1_{\ddot{\mathsf{MF}}(\mathbf{x})}$.

2.4. The three-category $\dddot{\mathsf{MF}}$ of polynomial algebras

The two-categories described in the previous subsection can be combined into a single three-category. This three-category should be thought of as the three-category of RW models whose target spaces have the form $\mathrm{T}^\vee\mathbb{C}^n$ for some non-negative integer n.

An object of the three-category $\dddot{\mathsf{MF}}$ is a list of variables $\mathbf{x}$. The two-category of morphisms between two objects $\mathbf{x},\mathbf{y}\in\dddot{\mathsf{MF}}$ is the correspondence two-category $\ddot{\mathsf{MF}}(\mathbf{x},\mathbf{y})$:

$$\mathrm{Hom}(\mathbf{x},\mathbf{y})=\ddot{\mathsf{MF}}(\mathbf{x},\mathbf{y}).$$

Each correspondence determines a 2-functor (2.21), and the composition of correspondences as morphisms of $\dddot{\mathsf{MF}}$ is defined to agree with the composition of the corresponding two-functors. Namely, the composition of two correspondences $(\mathbf{u}\,;W_{12})\in\ddot{\mathsf{MF}}(\mathbf{x},\mathbf{y})$ and $(\mathbf{w}\,;W_{23})\in\ddot{\mathsf{MF}}(\mathbf{y},\mathbf{z})$ is the correspondence

$$(\mathbf{w}\,;W_{23})\circ(\mathbf{u}\,;W_{12})=(\mathbf{u},\mathbf{w},\mathbf{y}\,;W_{12}+W_{23})\in\ddot{\mathsf{MF}}(\mathbf{x},\mathbf{z}).$$

The identity endomorphism of an object $\mathbf{x}$ can be represented by the correspondence

$$1_{\mathbf{x}}\simeq\big(\mathbf{a}\,;\mathbf{a}\cdot(\mathbf{x}'-\mathbf{x})\big)\in\ddot{\mathsf{MF}}(\mathbf{x},\mathbf{x}')=\mathrm{End}(\mathbf{x}),\tag{2.28}$$

the lists $\mathbf{x}'$ and $\mathbf{a}$ having the same length as $\mathbf{x}$ (cf. Equation (2.24)).

The three-category $\dddot{\mathsf{MF}}$ has a symmetric monoidal structure corresponding to Equation (1.1). The product of objects corresponds to the concatenation of lists

$$\mathbf{x}\times\mathbf{y}=\mathbf{x},\mathbf{y}$$

and the unit element is the empty list $\varnothing$. The duality endofunctor $\diamondsuit$ of Equation (1.2) acts on $\dddot{\mathsf{MF}}$ as the identity.

3. The three-category of complex manifolds

3.1. The two-periodic derived category of a curved complex manifold

The category of matrix factorization (2.9) is based on a polynomial algebra $\mathbb{C}[\mathbf{x}]$. One can define a similar "analytic" category $\mathsf{MFA}(\mathbf{x};W)$ based on the algebra of holomorphic functions on $\mathbb{C}^n$. The two-periodic derived category of a curved complex manifold $\mathsf{D}_{\mathbb{Z}_2}(U,W)$, which is the two-periodic perfect derived category of its curved Dolbeault algebra, generalizes the category

$\mathsf{MFA}(\mathbf{x};W)$ from $\mathbb{C}^n$ to a general complex manifold U, so that if $U=\mathbb{C}^n$, then there is an equivalence of categories

$$\mathsf{D}_{\mathbb{Z}_2}(\mathbb{C}^n,W)\simeq \mathsf{MFA}(\mathbf{x};W).$$

From the physical point of view $\mathsf{D}_{\mathbb{Z}_2}(U,W)$ is the category of boundary conditions for the B-model with target U deformed by a curving W. In the special case when W is a holomorphic function on U, this theory is the topological Landau–Ginzburg model with target U and the superpotential W.

Following the general construction of Section 2.1, let us define the curved Dolbeault algebra of a complex manifold U. The $\mathbb{Z}$-graded CDGA $\mathcal{A}$ of Equation (2.1) is the algebra of $(0,\bullet)$-forms $\Omega^{0,\bullet}(U)$, the differential $\bar\nabla$ is $\bar\partial$, and the curving W is a $\bar\partial$-closed even form:

$$\bar\nabla=\bar\partial,\qquad W\in\Omega^{0,\hat 0}(U),\qquad \bar\partial W=0. \tag{3.1}$$

Let E be a smooth (not necessarily holomorphic) $\mathbb{Z}_2$-graded vector bundle over U, and let $\Omega^{0,\bullet}(E)$ be the space of $(0,\bullet)$-forms with values in E:

$$\Omega^{0,\bullet}(E)=\Gamma\big(E\otimes\wedge^\bullet\bar{\mathrm{T}}^\vee X\big)=\Gamma(E)\otimes_{\Omega^{0,0}(U)}\Omega^{0,\bullet}(U),$$

where $\Gamma(E)$ is the space of sections of E. The space $\Omega^{0,\bullet}(E)$ is a $\mathbb{Z}_2$-graded module over $\Omega^{0,\bullet}(U)$ of the form (2.5). A *curved quasi-holomorphic vector bundle* is a pair $(E,\bar\nabla_E)$, where $\bar\nabla_E$ is a curved $(0,\hat 1)$-differential acting on $\Omega^{0,\bullet}(E)$, that is, $\bar\nabla_E$ is a $\mathbb{C}$-linear operator

$$\bar\nabla_E\in\mathrm{End}_{\mathbb{C}}(\Omega^{0,\bullet}(E)),$$

which satisfies the following properties:

$$|\bar\nabla_E|=\hat 1, \tag{3.2}$$

$$\bar\nabla_E\,(\alpha\wedge\sigma)=(\bar\partial\alpha)\wedge\sigma+(-1)^{|\alpha|}\alpha\wedge(\bar\nabla_E\,\sigma), \tag{3.3}$$

$$\bar\nabla_E^2\,\sigma=W\wedge\sigma, \tag{3.4}$$

where $\alpha\in\Omega^{0,\bullet}(U)$ and $\sigma\in\Omega^{0,\bullet}(E)$.

We call the pair $(E,\bar\nabla_E)$ quasi-holomorphic, because even if $W=0$, the bundle E is not necessarily holomorphic: if we split the differential $\bar\nabla_E$

according to the Dolbeault degree:

$$\bar{\nabla}_E = \sum_{i=0}^{\dim U} \bar{\nabla}_{E,i}, \qquad \bar{\nabla}_{E,i}\colon \Omega^{0,\bullet}(E) \longrightarrow \Omega^{0,\bullet+i}(E),$$

then $\bar{\nabla}^2_{E,1} = -\{\bar{\nabla}_{E,0}, \bar{\nabla}_{E,2}\}$ rather than $\bar{\nabla}^2_{E,1} = 0$, so in general $\bar{\nabla}_{E,1}$ does not determine a holomorphic structure on E.

If $(E, \bar{\nabla}_E)$ is a curved quasi-holomorphic vector bundle, then the pair

$$\mathcal{E} = \big(\Omega^{0,\bullet}(E), \bar{\nabla}_E\big) \tag{3.5}$$

is a perfect $\mathbb{Z}_2$-GDM over the CDGA $(\Omega^{0,\bullet}(U), \bar{\partial}, W)$. In fact, all perfect $\mathbb{Z}_2$-GDMs over this CDGA originate in this way from vector bundles with curved differentials.

The pair (U, W) will be called a *curved complex manifold*. We define its two-periodic derived category $\mathsf{D}_{\mathbb{Z}_2}(U, W)$ as the two-periodic perfect derived category of its curved Dolbeault algebra:

$$\mathsf{D}_{\mathbb{Z}_2}(U, W) = \mathsf{D}_{\mathbb{Z}_2}(\Omega^{0,\bullet}(U), \bar{\partial}, W), \tag{3.6}$$

its perfect objects being the pairs (3.5) and morphisms defined according to the general formula (2.6). The monoidal structure and the action of the duality functor $^\vee$ also follow the general definitions (2.3) and (2.4).

For a perfect object (3.5) we use an abbreviated notation

$$\mathcal{E} = (E, \bar{\nabla}_E). \tag{3.7}$$

If $W = 0$, then we will abbreviate the notation (3.6) down to

$$\mathsf{D}_{\mathbb{Z}_2}(U) := \mathsf{D}_{\mathbb{Z}_2}(U, 0). \tag{3.8}$$

The latter category contains a full subcategory which is equivalent to the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(U)$:

$$\mathsf{D}^{\mathrm{b}}(U) \hookrightarrow \mathsf{D}_{\mathbb{Z}_2}(U). \tag{3.9}$$

An object of $\mathsf{D}^{\mathrm{b}}(U)$, represented by a chain complex of holomorphic vector bundles

$$E^0 \xrightarrow{\sigma_0} E^1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{k-1}} E^k\ , \tag{3.10}$$

corresponds to an object $(E, \bar{\partial} + \sigma)$ of $\mathsf{D}_{\mathbb{Z}_2}(U)$, where E is the total $\mathbb{Z}_2$-graded vector bundle

$$E = \bigoplus_{i-\text{even}} E^i \oplus \bigoplus_{i-\text{odd}} E^i,$$

while $\bar{\partial}$ is the $(0, \hat{1})$ differential for holomorphic vector bundles and σ is the combined differential of the complex (3.10): $\sigma = \sum_{i=1}^{k} \sigma_i$.

The category $\mathsf{D}_{\mathbb{Z}_2}(U, W)$ admits a certain "Dolbeault filtration". Consider two perfect $\mathbb{Z}_2$-GDMs, which share the same vector bundle E: $\mathcal{E} = (E, \bar{\nabla}_E)$ and $\mathcal{E}' = (E, \bar{\nabla}'_E)$. We say that the objects $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic up to order k, if the difference between their connections is of higher degree as an element of $\Omega^{0,\bullet}(E)$:

$$\bar{\nabla}'_E - \bar{\nabla}_E = O_k, \qquad O_k \in \bigoplus_{i>k} \Omega^{0,i}(E).$$

The tensor product of perfect $\mathbb{Z}_2$-GDMs over $\Omega^{0,\bullet}(U)$ corresponds to the tensor product of vector bundles:

$$\mathcal{E}_1 \otimes \mathcal{E}_2 = \big(E_1 \otimes E_2, \bar{\nabla}_{E_1} + \bar{\nabla}_{E_2}\big). \tag{3.11}$$

Since

$$(\bar{\nabla}_{E_1} + \bar{\nabla}_{E_2})^2 = (W_1 + W_2)\, 1_{\mathcal{E}_1 \otimes \mathcal{E}_2},$$

the tensor product (3.11) gives rise to a functor

$$\mathsf{D}_{\mathbb{Z}_2}(U, W_1) \times \mathsf{D}_{\mathbb{Z}_2}(U, W_2) \xrightarrow{\otimes} \mathsf{D}_{\mathbb{Z}_2}(U, W_1 + W_2).$$

For a holomorphic map $F \colon U' \longrightarrow U$ and for a $(0, \bullet)$-form W of (3.1) let $F^*(W)$ denote its pull-back to U'. We introduce a 'derived' pull-back functor F^* and a push-forward functor F_*:

$$\mathsf{D}_{\mathbb{Z}_2}\Big(U', F^*(W)\Big) \underset{F^*}{\overset{F_*}{\rightleftarrows}} \mathsf{D}_{\mathbb{Z}_2}(U, W)\ .$$

The pull-back functor F^* acts on perfect objects (3.5) by pulling back quasi-holomorphic vector bundles. The definition of the push-forward functor F_*

is a bit tricky. There is a pull-back homomorphism of CDGAs

$$(\Omega^{0,\bullet}(U'), \bar{\partial}, F^*(W)) \longleftarrow (\Omega^{0,\bullet}(U), \bar{\partial}, W),$$

which turns a perfect $\mathbb{Z}_2$-GDM $\mathcal{E}'$ over the CDGA $(\Omega^{0,\bullet}(U'), \bar{\partial}, F^*(W))$ into a $\mathbb{Z}_2$-GDM over the CDGA $(\Omega^{0,\bullet}(U), \bar{\partial}, W)$. We conjecture that the latter $\mathbb{Z}_2$-GDM is admissible (cf. Equation (2.7)) and use it as the definition of $F_*(\mathcal{E}')$.

3.2. The two-category of curved complex manifolds

The two-category of curved complex manifolds $\ddot{\mathsf{D}}_{\mathbb{Z}_2}$ is a generalization of the two-category of analytic matrix factorizations $\ddot{\mathsf{MF}}\mathsf{A}$. From the physical viewpoint, it is the two-category of curved B-models, or equivalently the two-category of boundary conditions for the RW model whose target is a point. An object of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}$ is a curved complex manifold (U, W). A morphism between two curved complex manifolds (U_1, W_1) and (U_2, W_2) is a perfect (or an admissible) $\mathbb{Z}_2$-GDM

$$\mathcal{F} \in \mathsf{D}_{\mathbb{Z}_2}(U_1 \times U_2,\ \pi_2^*(W_2) - \pi_1^*(W_1)),$$

where π_1 and π_2 are projections

$$\begin{array}{ccc} & U_1 \times U_2 & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ U_1 & & U_2 \end{array} \qquad (3.12)$$

Such an object $\mathcal{F}$ determines a Fourier–Mukai transform functor

$$\mathsf{D}_{\mathbb{Z}_2}(U_1, W_1) \xrightarrow{\Phi[\mathcal{F}]} \mathsf{D}_{\mathbb{Z}_2}(U_2, W_2),$$

with the standard action on objects $\mathcal{E} \in \mathsf{D}_{\mathbb{Z}_2}(U_1, W_1)$:

$$\Phi\,[\mathcal{F}]\,(\mathcal{E}) = \pi_{2,*}\Big(\mathcal{F} \otimes \pi_1^*(\mathcal{E})\Big). \qquad (3.13)$$

We define the composition of morphisms $\mathcal{F}$ as the composition of the corresponding functors, so for $\mathcal{F}_{12} \in \mathrm{Hom}((U_1, W_1), (U_2, W_2))$ and $\mathcal{F}_{23} \in$

$\mathrm{Hom}((U_2, W_2), (U_3, W_3))$ the composition is

$$\mathcal{F}_{23} \circ \mathcal{F}_{12} = \pi_{13,*}(\pi_{12}^*(\mathcal{F}_{12}) \otimes \pi_{23}^*(\mathcal{F}_{23})), \tag{3.14}$$

where the maps π_{ij} are the projections

$$\begin{array}{ccccc} & & U_1 \times U_2 \times U_3 & & \\ & \swarrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_{23}} & \searrow^{\pi_{13}} & \\ U_1 \times U_2 & & U_2 \times U_3 & & U_1 \times U_3 \end{array} \tag{3.15}$$

3.3. The two-category of curved fibrations

Let us fix a complex manifold U. A holomorphic fibration over U is a smooth fiber bundle

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{U} \\ & & \downarrow{\scriptstyle p} \\ & & U \end{array} \tag{3.16}$$

where $\mathcal{U}$ is a complex manifold and the projection p is holomorphic. Bundle (3.16) does not have to be holomorphic, so the complex structure of the fiber V may be different over different points of the base U.

Let $\times_U$ denote the fibered product of two fibrations with the same base U. It has natural projections

$$\begin{array}{ccc} & \mathcal{U}_1 \times_U \mathcal{U}_2 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{U}_1 & & \mathcal{U}_2 \end{array} \tag{3.17}$$

The two-category of curved fibrations $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ is obtained by "fibering" the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}$ over U, the fibers V being the analogs of the manifolds U appearing in Section 3.2. This category is also a generalization of the analytic two-category $\mathsf{M\ddot{F}A}(\mathbf{x})$:

$$\ddot{\mathsf{D}}_{\mathbb{Z}_2}(\mathbb{C}^n) = \mathsf{M\ddot{F}A}(\mathbf{x}).$$

From the physical viewpoint, $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ is the two-category of boundary conditions for the RW model with target $\mathsf{T}U$.

An object of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ is a curved fibration $(\mathcal{U}, W)$ where, according to definition (3.1), $W \in \Omega^{0,\hat{0}}(\mathcal{U})$ and $\bar{\partial} W = 0$. The category of morphisms between two objects is the two-periodic derived category of their curved fibered product:

(3.18)
$$\mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)}\big((\mathcal{U}_1, W_1), (\mathcal{U}_2, W_2)\big) = \mathsf{D}_{\mathbb{Z}_2}\big(\mathcal{U}_1 \times_U \mathcal{U}_2,\ \pi_2^*(W_2) - \pi_1^*(W_1)\big).$$

An object $\mathcal{F}$ of category (3.18) generates a Fourier–Mukai transform functor

$$\mathsf{D}_{\mathbb{Z}_2}(\mathcal{U}_1, W_1) \xrightarrow{\Phi[\mathcal{F}]} \mathsf{D}_{\mathbb{Z}_2}(\mathcal{U}_2, W_2)$$

which acts on an object $\mathcal{E}_1 \in \mathsf{D}_{\mathbb{Z}_2}(\mathcal{U}_1, W_1)$ according to formula (3.13). The composition of morphisms $\mathcal{F}_{12} \in \mathrm{Hom}((\mathcal{U}_1, W_1), (\mathcal{U}_2, W_2))$ and $\mathcal{F}_{23} \in \mathrm{Hom}((\mathcal{U}_2, W_2), (\mathcal{U}_3, W_3))$ is defined again by formula (3.14), where the maps π_{ij} are projections

$$\begin{array}{ccccc} & & \mathcal{U}_1 \times_U \mathcal{U}_2 \times_U \mathcal{U}_3 & & \\ & \swarrow^{\pi_{12}} & \downarrow^{\pi_{23}} & \searrow^{\pi_{13}} & \\ \mathcal{U}_1 \times_U \mathcal{U}_2 & & \mathcal{U}_2 \times_U \mathcal{U}_3 & & \mathcal{U}_1 \times_U \mathcal{U}_3 \end{array}$$

so that this composition agrees with the composition of Fourier–Mukai functors.

The simplest objects of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ are curved one-point fibrations $(\mathcal{T}_U, W)$, where

$$\mathcal{T}_U = \begin{array}{ccc} \{\text{1-point}\} & \longrightarrow & U \\ & & \downarrow \\ & & U \end{array} \tag{3.19}$$

is a one-point fibration, that is, a fibration whose fiber is a single point, and W is a holomorphic function on U. We denote the objects $(\mathcal{T}_U, W)$ simply as W. According to the general definition (3.18), the category of morphisms between two curved one-point fibrations is the curved two-periodic derived category

$$\mathrm{Hom}(W_1, W_2) = \mathsf{D}_{\mathbb{Z}_2}(U, W_2 - W_1), \tag{3.20}$$

and, in particular, the endomorphisms of a curved one-point fibration W form the two-periodic derived category of U:

$$\mathrm{End}(W) = \mathsf{D}_{\mathbb{Z}_2}(U) \tag{3.21}$$

containing the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(X)$ as a subcategory. The composition of endomorphisms in $\mathrm{End}(W)$ corresponds to the tensor product (3.11), in other words, to the standard monoidal structure on $\mathsf{D}_{\mathbb{Z}_2}(U)$.

The two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ has a "pseudo-monoidal" structure. For two objects $(\mathcal{U}_1, W_1)$ and $(\mathcal{U}_2, W_2)$ we define

$$(\mathcal{U}_1, W_1)\tilde{\times}_U(\mathcal{U}_2, W_2) = (\mathcal{U}_1 \times_U \mathcal{U}_2, \pi_1^*(W_1) + \pi_1^*(W_2)).$$

The main property of the pseudo-monoidal structure is that for the object $(\mathcal{T}_U, 0)$ consisting of the one-point fibration over U and $W = 0$, the following isomorphism holds:

$$\mathrm{Hom}((\mathcal{U}_1, W_1)\tilde{\times}_U(\mathcal{U}_2, -W_2), (\mathcal{T}_U, 0)) \cong \mathrm{Hom}((\mathcal{U}_1, W_1), (\mathcal{U}_2, W_2)).$$

The two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ has a translation endo-two-functor similar to (2.26), which acts on a 0-object $(\mathcal{U}, W)$ by adding a space $\mathbb{C}_a$ (which is $\mathbb{C}$ with the standard coordinate a) to fibers of $\mathcal{U}$ and adding a^2 to the curving W:

$$(\mathcal{U}, W)[1]_2 = (\mathcal{U} \times \mathbb{C}_a, W + a^2).$$

For a holomorphic map $F\colon U' \longrightarrow U$ there exists a pull-back two-functor

$$\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U') \xleftarrow{\ddot{F}^*} \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$$

which acts on an object $(\mathcal{U}, W) \in \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ by pulling back the fibration $\mathcal{U}$ and the Dolbeault cohomology class W:

$$\ddot{F}^*(\mathcal{U}, W) = (F^*(\mathcal{U}), F^*(W))$$

If U' is a holomorphic fibration over U and the map F is its projection, then there is also a push-forward two-functor

$$\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U') \xrightarrow{\ddot{F}_*} \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U).$$

Indeed, for an object $(\mathcal{U}', W') \in \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U')$, a fibration $\mathcal{U}' \xrightarrow{p'} U'$ can be pushed forward to a fibration $F_*(\mathcal{U}')$ over U: $\mathcal{U}' \xrightarrow{F\circ p'} U$, so we can keep the form $W' \in \Omega^{0,\hat{0}}(\mathcal{U}')$ and define

$$\ddot{F}_*(\mathcal{U}', W') = (F_*(\mathcal{U}'), W').$$

3.4. The three-category of complex manifolds

The two-categories $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ can be assembled into a three-category $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$. It should be thought of as the three-category of RW models with target-spaces of the form $\mathrm{T}^\vee U$, for all complex manifolds U. Objects of $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$ are complex manifolds U (or, equivalently, categories $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$). The category of morphisms between two objects is

$$\mathrm{Hom}(U_1, U_2) = \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_1 \times U_2). \tag{3.22}$$

Objects of category (3.22) are called *correspondences*. A correspondence $(\mathcal{U}_{12}, W_{12}) \in \mathrm{Hom}(U_1, U_2)$ determines a two-functor

$$\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_1) \xrightarrow{\ddot{\Phi}[\mathcal{U}_{12}, W_{12}]} \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_2) \tag{3.23}$$

acting on an object $(\mathcal{U}_1, W_1) \in \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_1)$ according to the formula

$$\ddot{\Phi}\,[\mathcal{U}_{12}, W_{12}]\,(\mathcal{U}_1, W_1) = \pi_{2,*}((\mathcal{U}_{12}, W_{12})\ \tilde{\times}_{(U_1\times U_2)}\ \pi_1^*(\mathcal{U}_1, W_1)),$$

where π_1 and π_2 are the projections (3.12). The composition of correspondences $(\mathcal{U}_{12}, W_{12}) \in \mathrm{Hom}(U_1, U_2)$ and $(\mathcal{U}_{23}, W_{23}) \in \mathrm{Hom}(U_2, U_3)$ as morphisms in the three-category $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$ is defined to agree with the composition of their functors:

$$\mathcal{U}_{23} \circ \mathcal{U}_{12} = \pi_{13,*}\left(\pi_{12}^*(\mathcal{U}_{12}, W_{12})\ \tilde{\times}_{(U_1\times U_2\times U_3)}\ \pi_{23}^*(\mathcal{U}_{23}, W_{23})\right),$$

where the maps π_{ij} are the projections (3.15).

The three-category $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$ has a symmetric monoidal structure corresponding to that of Equation (1.1). The product of objects corresponds to the product of the underlying complex manifolds $U_1 \times U_2$ and the unit object is the complex manifold $U_{\text{1-pt}}$ consisting of a single point. The duality endofunctor $\diamondsuit$ of Equation (1.2) acts on $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$ as the identity.

3.5. Augmented categories

The curving W enters the path-integral formulation of the RW model with boundaries only through it derivative ∂W. This suggests that one should define the two-category of boundary conditions in such a way that W is defined only up to addition of a locally constant function. Below we describe such a modification of the two-category $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$. It is also necessary for a geometric interpretation of $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ in terms of the cotangent bundle $\mathrm{T}^\vee U$, as we will see in the next section.

For an element $W \in \Omega^{0,\hat{0}}(\mathcal{U})$, such that $\bar{\partial}W = 0$, define an augmented category $\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U,W)$ as a formal union over all locally constant functions W_{lc}:

$$\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U,W) = \bigcup_{\partial W_{\mathrm{lc}}=0} \mathsf{D}_{\mathbb{Z}_2}(U, W + W_{\mathrm{lc}}).$$

We define the two-category $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ in exactly the same way as $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$, except that in the definition of morphisms (3.18) we replace the two-periodic category $\mathsf{D}_{\mathbb{Z}_2}$ with its augmented version $\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}$:

(3.24)
$$\mathrm{Hom}_{\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)}\big((\mathcal{U}_1, W_1), (\mathcal{U}_2, W_2)\big) = \mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}\big(\mathcal{U}_1 \times_U \mathcal{U}_2,\ \pi_2^*(W_2) - \pi_1^*(W_1)\big).$$

Two objects $(\mathcal{U}, W_1)$ and $(\mathcal{U}, W_2)$ are isomorphic within $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ if the difference $W_2 - W_1$ is locally constant on $\mathcal{U}$, that is, if $\partial W_1 = \partial W_2$.

The augmented three-category $\dddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}$ is defined in the same way as $\dddot{\mathsf{D}}_{\mathbb{Z}_2}$, except that we replace the categories $\ddot{\mathsf{D}}_{\mathbb{Z}_2}$ appearing in its definition with augmented categories $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}$.

The augmented matrix factorization category is defined as the formal union of categories

$$\mathsf{MF}^{\mathrm{a}}_{\mathbf{x};W} = \bigcup_{C\in\mathbb{C}} \mathsf{MF}(\mathbf{x}; W + C),$$

and the augmented categories $\ddot{\mathsf{MF}}^{\mathrm{a}}(\mathbf{x})$ and $\dddot{\mathsf{MF}}^{\mathrm{a}}$ are defined similar to $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ and $\dddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}$.

4. The two-category $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$: a geometric description and a relation to $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$

4.1. A geometric description of the two-category $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$

Let (X, ω) be a holomorphic symplectic manifold. The RW model associates to (X, ω) a two-category of boundary conditions $\ddot{\mathsf{L}}(X, \omega)$. Path-integral arguments suggest that a certain part of $\ddot{\mathsf{L}}(X, \omega)$ can be described in geometric terms. In this subsection, we consider the geometric description when (X, ω) is the cotangent bundle of a complex manifold U: $X = \mathrm{T}^\vee U$. The description uses only the holomorphic symplectic structure of X, and in our definitions we never refer to the cotangent bundle structure. Conjecturally, this property should hold also for the whole two-category $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$, in the sense that it should be acted upon by the group of symplectic automorphisms of $\mathrm{T}^\vee U$. This is far from obvious from the algebraic definition of $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ as the two-category $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ given in the previous section.

4.1.1. $O(n,\mathbb{C})$ bundles and matrix factorizations. A holomorphic $O(n,\mathbb{C})$ vector bundle B over a complex manifold U determines a "B-twisted" version $\mathsf{D}_{\mathbb{Z}_2}(U,W)[B]_{\mathrm{tw}}$ of the category $\mathsf{D}_{\mathbb{Z}_2}(U,W)$. The $O(n,\mathbb{C})$ structure determines, up to a non-zero constant factor, a holomorphic function W_{q} on the total space of B, which is quadratic along the fibers. The B-twisting replaces U with the total space of the bundle B and adds W_{q} to the curving:

$$\mathsf{D}_{\mathbb{Z}_2}(U,W)[B]_{\mathrm{tw}} = \mathsf{D}_{\mathbb{Z}_2}(B, W + W_{\mathrm{q}}). \tag{4.1}$$

It is easy to see that the composition of two-translations corresponds to the sum of vector bundles: $[B_1]_{\mathrm{tw}}[B_2]_{\mathrm{tw}} = [B_1 \oplus B_2]_{\mathrm{tw}}$, and if the bundle B is trivial, then $\mathsf{D}_{\mathbb{Z}_2}(U,W)[B]_{\mathrm{tw}} = \mathsf{D}_{\mathbb{Z}_2}(U,W)[\operatorname{rank} B]_2$, where $[1]_2$ is the translation two-functor (2.19).

A line bundle $L \to U$ has an $O(1,\mathbb{C})$ structure if and only if it is self-dual, that is, if it is isomorphic to its dual: $L \cong L^\vee$. The top exterior power $\bigwedge^{\mathrm{top}} B$ of an $O(n,\mathbb{C})$ bundle B is self-dual, and there is a canonical equivalence of categories[3]

$$\mathsf{D}_{\mathbb{Z}_2}(U,W)[B]_{\mathrm{tw}} = \mathsf{D}_{\mathbb{Z}_2}(U,W)[\textstyle\bigwedge^{\mathrm{top}} B]_{\mathrm{tw}}[\operatorname{rank} B - 1]_2. \tag{4.2}$$

Similar to the untwisted two-translation discussed in Section 2.2.3, category (4.1) has an alternative "intrinsic" description in terms of objects of $\mathsf{D}_{\mathbb{Z}_2}(U,W)$. Consider the case when B is a line bundle L. Let $\mathcal{L} = (\Omega^{0,\bullet}(L), \bar\partial)$ be the perfect $\mathbb{Z}_2$-GDM corresponding to L. The self-duality of L determines an isomorphism $f_{\mathrm{b},L}$ between $\mathcal{L} \otimes \mathcal{L}$ and the structure sheaf $\mathscr{O}_U$:

$$f_{\mathrm{b},L} \in \mathrm{Ext}^{\hat{0}}\,(\mathcal{L} \otimes \mathcal{L}, \mathscr{O}_U).$$

[3]We thank M. Kontsevich for pointing this out.

An object of $\mathsf{D}_{\mathbb{Z}_2}(U,W)[L]'_{\mathrm{tw}}$ is a pair $(\mathcal{E}, f_{\mathcal{E}})$, where $\mathcal{E} = (E, \bar{\nabla}_E)$ is a perfect $\mathbb{Z}_2$-GDM (3.5) and the extension $f_{\mathcal{E}} \in \mathrm{Ext}^{\hat{1}}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L})$ satisfies the property

$$f_{\mathrm{b},L} \circ f_{\mathcal{E}} \circ f_{\mathcal{E}} = 1_{\mathcal{E}}.$$

Morphisms between two objects $(\mathcal{E}, f_{\mathcal{E}}), (\mathcal{E}', f_{\mathcal{E}'}) \in \mathsf{D}_{\mathbb{Z}_2}(U,W)[L]_{\mathrm{tw}}$ are morphisms $g \in \mathrm{Ext}^{\bullet}(\mathcal{E}, \mathcal{E}')$ which intertwine the extensions: $g f_{\mathcal{E}} = f_{\mathcal{E}'} g$. We conjecture that the categories $\mathsf{D}_{\mathbb{Z}_2}(U,W)[L]_{\mathrm{tw}}$ and $\mathsf{D}_{\mathbb{Z}_2}(U,W)[L]'_{\mathrm{tw}}$ are equivalent.

4.1.2. A holomorphic fibration with a lagrangian base as a geometric object. Let K_U denote the canonical line bundle of a complex manifold U: $K_U := \bigwedge^{\mathrm{top}} \mathrm{T}^{\vee} U$. Let $Y \subset X$ be a lagrangian submanifold, that is, Y is a holomorphic submanifold of X, such that $\dim_{\mathbb{C}} Y = \frac{1}{2} \dim_{\mathbb{C}} X$ and $\omega|_Y = 0$. We are going to consider 'geometric' objects of $\ddot{\mathsf{L}}(X,\omega)$ which are pairs $(\mathcal{Y}, L_{\mathcal{Y}})$, where $\mathcal{Y}$ is a fibration

$$\begin{array}{ccc} Z \longrightarrow & \mathcal{Y} & \\ & \downarrow{\scriptstyle p_{\mathcal{Y}}} & \\ & Y & \subset X \end{array} \tag{4.3}$$

with a lagrangian base Y, and $L_{\mathcal{Y}}$ is a holomorphic line bundle on $\mathcal{Y}$, whose square is the pull-back of the canonical bundle of Y: $L_{\mathcal{Y}}^{\otimes 2} = p_{\mathcal{Y}}^* K_Y$.

A particularly simple type of a holomorphic fibration (4.3) is a one-point fibration

$$\mathcal{T}_Y = \begin{array}{cc} Y & \\ \downarrow & \\ Y & \subset X \end{array} \tag{4.4}$$

Unless there is a danger of confusion, we denote such a fibration simply as Y. The pairs (Y, L_Y), where $L_Y \to Y$ is a line bundle such that $Y^{\otimes 2} = K_Y$, are the simplest objects of the type $(\mathcal{Y}, L_{\mathcal{Y}})$.

4.1.3. Morphisms between geometric objects. We say that two holomorphic submanifolds $Y_1, Y_2 \subset X$ have a *clean intersection*, if any point $x \in Y_1 \cap Y_2$ has an open neighborhood U_x which is isomorphic to a neighborhood of the origin of the tangent space $\mathrm{T}_x X$ in such a way that Y_1 and Y_2 correspond to their tangent spaces $\mathrm{T}_x Y_1, \mathrm{T}_x Y_2 \subset \mathrm{T}_x X$. This condition guarantees that the intersection $Y_{12} := Y_1 \cap Y_2$ is a disjoint union of holomorphic submanifolds of X.

Define the X-product of two fibrations $\mathcal{Y}_1$ and $\mathcal{Y}_2$ as a fibration over the intersection of their bases:

$$\mathcal{Y}_1 \times_X \mathcal{Y}_2 := \mathcal{Y}_1|_{Y_{12}} \times_{Y_{12}} \mathcal{Y}_2|_{Y_{12}}, \qquad \mathcal{Y}_1 \times_X \mathcal{Y}_2 \xrightarrow{p_{12}} Y_{12}.$$

There are obvious projections

$$\begin{array}{ccc} & \mathcal{Y}_1 \times_X \mathcal{Y}_2 & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathcal{Y}_1|_{Y_{12}} & & \mathcal{Y}_2|_{Y_{12}} \end{array}$$

Suppose that the lagrangian bases $Y_1, Y_2 \subset X$ of the fibrations of two objects $(\mathcal{Y}_1, L_{\mathcal{Y}_1})$, $(\mathcal{Y}_2, L_{\mathcal{Y}_2})$ have a clean intersection, so the intersection Y_{12} is a complex manifold. The line bundle

$$L_{12} := \pi_1^*(L_{\mathcal{Y}_1}|_{Y_{12}}) \otimes \pi_2^*(L_{\mathcal{Y}_2}|_{Y_{12}}) \otimes p_{12}^*(K_{Y_{12}}^{-1}) \tag{4.5}$$

is self-dual. Indeed, for $i = 1, 2$ we have $p_{12} = p_{\mathcal{Y}_i} \circ \pi_i$, and since $L_{\mathcal{Y}_i}^2 = p_{\mathcal{Y}_i}^* K_{Y_i}$, the square of L_{12} can be presented as the pull-back of the product of canonical bundles:

$$L_{12}^2 = p_{12}^* \left(K_{Y_1}|_{Y_{12}} \otimes K_{Y_2}|_{Y_{12}} \otimes K_{Y_{12}}^{-2} \right). \tag{4.6}$$

Now consider the quotient bundles

$$Q_i = \mathrm{T}Y_i|_{Y_{12}} / \mathrm{T}Y_{12}, \qquad i = 1, 2. \tag{4.7}$$

The holomorphic symplectic form ω produces a non-degenerate pairing between Q_1 and Q_2, so their top exterior powers are dual to each other and, as a result, the tensor product $\bigwedge^{\mathrm{top}} Q_1 \otimes \bigwedge^{\mathrm{top}} Q_2$ is a trivial line bundle. At the same time, $K_{Y_i}|_{Y_{12}} = \bigwedge^{\mathrm{top}} Q_i^\vee \otimes K_{Y_{12}}$, so $K_{Y_1}|_{Y_{12}} \otimes K_{Y_2}|_{Y_{12}} = K_{Y_{12}}^2$ and the tensor square (4.6) is trivial, that is, L_{12} is self-dual.

Having established the self-duality of L_{12}, we propose that the category of morphisms between the objects, whose lagrangian bases have a clean intersection, is the shifted two-periodic derived category of the X-product $\mathcal{Y}_1 \times_X \mathcal{Y}_2$:

$$\begin{aligned} \mathrm{Hom}_{\ddot{\mathsf{L}}(X)}((\mathcal{Y}_1, L_{\mathcal{Y}_1}), (\mathcal{Y}_2, L_{\mathcal{Y}_2})) = \mathsf{D}_{\mathbb{Z}_2}(\mathcal{Y}_1 \times_X \mathcal{Y}_2)\, [L_{12}]_{\mathrm{tw}} \\ \times \left[\tfrac{1}{2} \dim X - \dim Y_{12} - 1\right]_2. \end{aligned} \tag{4.8}$$

Roughly speaking, the category of morphisms $\mathrm{Hom}_{\ddot{\mathsf{L}}(X)}\left((\mathcal{Y}_1, L_{\mathcal{Y}_1}), (\mathcal{Y}_2, L_{\mathcal{Y}_2})\right)$ is the two-periodic derived category of coherent sheaves on the product $\mathcal{Y}_1 \times_X \mathcal{Y}_2$. The origin of the shifts in the r.h.s. of Equation (4.8) will become clear when we compare Equations (4.8) and (3.18).

In the special case when $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are one-point fibrations (4.3) with the same base $Y_1 = Y_2 = Y$ and accompanying line bundles are the same, formula (4.8) becomes

$$\mathrm{End}_{\ddot{\mathsf{L}}(X)}(Y, L_{\mathcal{Y}}) = \mathsf{D}_{\mathbb{Z}_2}(Y). \tag{4.9}$$

4.1.4. Composition of morphisms. We describe the geometric composition of morphisms under the simplifying assumption that the lagrangian bases Y_i of the fibrations $\mathcal{Y}_i$, $i = 1, 2, 3$ are Calabi–Yau and the accompanying line bundles $L_{\mathcal{Y}_i}$ are trivial. This implies that a clean intersection of two lagrangian submanifolds $Y_{ij} = Y_i \cap Y_j$ is "semi-Calabi–Yau", that is, $K_{Y_{ij}}^{\otimes 2}$ is trivial. Let us make a stronger assumption that Y_{ij} is Calabi–Yau. Then the general formula (4.8) simplifies

$$\mathrm{Hom}(\mathcal{Y}_i, \mathcal{Y}_j) = \mathsf{D}_{\mathbb{Z}_2}\left(\mathcal{Y}_i \times_X \mathcal{Y}_j\right) \tag{4.10}$$

and we suggest that the composition of morphisms is a combination of derived pull-backs, push-forwards and the tensor product: for two morphisms $\mathcal{E}_{12} \in \mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2)$ and $\mathcal{E}_{23} \in \mathrm{Hom}(\mathcal{Y}_2, \mathcal{Y}_3)$ their composition is

$$\mathcal{E}_{23} \circ \mathcal{E}_{12} = (\pi_{13})_* \left(\pi_{12}^*(\mathcal{E}_{12}) \otimes \pi_{23}^*(\mathcal{E}_{23})\right), \tag{4.11}$$

where π_{ij} are the embedding projections

$$\begin{array}{ccccc} & & \mathcal{Y}_1 \times_X \mathcal{Y}_2 \times_X \mathcal{Y}_3 & & \\ & \swarrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_{13}} & \searrow^{\pi_{23}} & \\ \mathcal{Y}_1 \times_X \mathcal{Y}_2 & & \mathcal{Y}_1 \times_X \mathcal{Y}_3 & & \mathcal{Y}_2 \times_X \mathcal{Y}_3 \end{array}$$

In the special case when all $\mathcal{Y}_i$ are one-point fibrations (4.3) with the same base $Y_1 = Y_2 = Y_3 = Y$, their categories of morphisms are given by Equation (4.9) and the composition rule (4.11) reduces to the tensor product within $\mathsf{D}_{\mathbb{Z}_2}(Y)$: for $\mathcal{E}, \mathcal{E}' \in \mathsf{D}_{\mathbb{Z}_2}(Y)$

$$\mathcal{E} \circ \mathcal{E}' = \mathcal{E} \otimes \mathcal{E}'. \tag{4.12}$$

4.2. Holomorphic lagrangian correspondences and the three-category of holomorphic symplectic manifolds

In this subsection we describe part of the three-category of RW models in geometric terms. Throughout this subsection we will ignore the line bundles $L_{\mathcal{Y}}$ in the definition of the objects $(\mathcal{Y}, L_{\mathcal{Y}})$ of $\ddot{\mathsf{L}}(X,\omega)$. To be more precise, we may assume that all complex manifolds appearing here are Calabi–Yau and all these bundles are trivial. Moreover, we assume that all intersections are clean.

The statements of this subsection apply when (X,ω) are cotangent bundles: $X = \mathrm{T}^{\vee}U$, but in the case of one-point fibrations the statements apply to general holomorphic symplectic manifolds (X,ω).

4.2.1. Lagrangian correspondence two-functors. Let us forget for a moment that the complex manifold X has a holomorphic symplectic structure and that the base Y of a fibration $\mathcal{Y}$ must be lagrangian. Then we may define pull-back and push-forward functors associated with a holomorphic map $F\colon X \longrightarrow X'$. A pull-back of a fibration $\mathcal{Y}' \to Y' \subset X'$ is a fibration $F^*(\mathcal{Y}') \to F^{-1}(Y')$ constructed by pulling back $\mathcal{Y}'$ by the restriction $F|_{F^{-1}(Y')}$. In order to define a push-forward of a fibration $\mathcal{Y} \to Y \subset X$ we assume that the restricted map $F|_Y\colon Y \to F(Y)$ represents a holomorphic fibration. Then we define the fibration $F_*(\mathcal{Y})$ as $\mathcal{Y} \to F(Y)$, whose projection is the composition of projections $\mathcal{Y} \to Y \to F(Y)$.

A holomorphic fibration $\mathcal{Y}_{12} \in \ddot{\mathsf{L}}\left((X_1, -\omega_1) \times (X_2, \omega_2)\right)$ determines a lagrangian correspondence two-functor

$$\ddot{\mathsf{L}}(X_1,\omega_1) \xrightarrow{\ddot{\Phi}[\mathcal{Y}_{12}]} \ddot{\mathsf{L}}(X_2,\omega_2) \tag{4.13}$$

defined by the formula

$$\ddot{\Phi}\,[\mathcal{Y}_{12}] = (\pi_2)_*(\mathcal{Y}_{12} \times_{(X_1\times X_2)} \pi_1^*), \tag{4.14}$$

where π_1 and π_2 are projections onto the factors of the product $X_1 \times X_2$:

$$\begin{array}{ccccc} & & X_1 \times X_2 & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ X_1 & & & & X_2 \end{array} \tag{4.15}$$

The action of the two-functor (4.14) on the bases of holomorphic fibrations is described by a simple set-theoretic formula: if $\mathcal{Y}_2 = \Phi\,[\mathcal{Y}_{12}]\,(\mathcal{Y}_1)$, then

$$Y_2 = \pi_2\left(Y_{12} \cap \pi_1^{-1}(Y_1)\right), \tag{4.16}$$

where Y_1 and Y_2 are the bases of the corresponding fibrations.

Although the operations π_1^* and $(\pi_2)_*$ do not correspond to well-defined two-functors for two-categories $\ddot{\mathsf{L}}$, their composition (4.14) is well-defined, because if the base Y_1 of the fibration $\mathcal{Y}_1$ is lagrangian then so is the base Y_2 of its image determined by Equation (4.16).

The one-point fibration

$$\mathcal{T}_{\Delta_X} = \begin{array}{c} \Delta_X \\ \downarrow \\ \Delta_X \end{array} \subset X \times X \tag{4.17}$$

over the diagonal $\Delta_X \subset (X, -\omega) \times (X, \omega)$ determines the identity endo-2-functor $\ddot{\Phi}\,[\mathcal{T}_{\Delta_X}]$ of the category $\ddot{\mathsf{L}}(X, \omega)$.

4.2.2. A geometric description of the three-category $\dddot{\mathsf{L}}$. As we mentioned in Section 1.2, objects of the three-category $\dddot{\mathsf{L}}$ are holomorphic symplectic manifolds (X, ω). The duality functor $\diamondsuit$ switches the sign of the symplectic form: $(X, \omega)^{\diamondsuit} = (X, -\omega)$, and we define the two-category of morphisms between two objects in accordance with the general formula (1.3):

$$\mathrm{Hom}_{\dddot{\mathsf{L}}}\left((X_1, \omega_1), (X_2, \omega_2)\right) = \ddot{\mathsf{L}}\left((X_1, -\omega_1) \times (X_2, \omega_2)\right).$$

The composition of morphisms represented by holomorphic fibrations with lagrangian bases is defined in such a way that it would agree with the composition of their correspondence functors (4.14): the composition of two morphisms

$$\mathcal{Y}_{12} \in \mathrm{Hom}((X_1, \omega_1), (X_2, \omega_2))\,, \qquad \mathcal{Y}_{23} \in \mathrm{Hom}((X_2, \omega_2), (X_3, \omega_3))$$

is

$$\mathcal{Y}_{23} \circ \mathcal{Y}_{12} = (\pi_{13})_*\left(\pi_{12}^*(\mathcal{Y}_{12}) \times_{X_1 \times X_2 \times X_3} \pi_{23}^*(\mathcal{Y}_{23})\right),$$

where π_{ij} are the projections

$$\begin{array}{ccccc} & & X_1 \times X_2 \times X_3 & & \\ & \swarrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_{13}} & \searrow^{\pi_{23}} & \\ X_1 \times X_2 & & X_1 \times X_3 & & X_2 \times X_3 \end{array}$$

The identity endomorphism $\mathbb{1}_{(X,\omega)} \in \mathrm{End}(X,\omega)$ is the one-point fibration (4.17):

$$\mathbb{1}_{(X,\omega)} = \mathcal{T}_{\Delta_X}. \tag{4.18}$$

As we mentioned in Section 1.1, the three-category $\ddot{\mathsf{L}}$ has a symmetric monoidal structure:

$$(X_1,\omega_1)\times(X_2,\omega_2) = (X_1\times X_2, \pi_1^*(\omega_1)+\pi_2^*(\omega_2)),$$

where π_1 and π_2 are projections of diagram (4.15). The unit object is the holomorphic symplectic manifold $X_{1\text{-pt}}$ consisting of a single point.

4.3. A relation between the geometric and the algebraic descriptions

We have already outlined the equivalence of categories (1.11) in Section 1.5. Here we provide a more detailed description of the equivalence two-functor $\ddot{\Phi}_{\cong}$.

4.3.1. Localization. Let us split the curving W, which determines the category $\mathsf{D}_{\mathbb{Z}_2}(U,W)$, into zero degree and positive degree parts:

$$W = W_0 + W_+,\qquad W_0\in\Omega^{0,0}(U),\qquad W_+\in\bigoplus_{i\geq 1}\Omega^{0,i}(U). \tag{4.19}$$

We say that the set of critical points of W_0

$$U_{W_0} = \{x\in U\mid dW_0(x)=0\}$$

is *clean* if it is a smooth holomorphic submanifold of U and the quadratic form induced by the Hessian of W_0 (that is, by the quadratic form of its second derivatives) on the normal bundle $\mathrm{N}U_{W_0}$ is non-degenerate. The non-degenerate Hessian gives rise to an $O(n,\mathbb{C})$ structure on $\mathrm{N}U_{W_0}$, and we conjecture the following equivalence of categories:

$$\mathsf{D}_{\mathbb{Z}_2}(U,W) = \mathsf{D}_{\mathbb{Z}_2}(U_{W_0}, W|_{U_{W_0}})[\mathrm{N}U_{W_0}]_{\mathrm{tw}}. \tag{4.20}$$

In other words, we expect that the category $\mathsf{D}_{\mathbb{Z}_2}(U,W)$ localizes to a small tubular neighborhood of U_{W_0} and that if the dominant terms in the expansion of W_0 in the normal directions to U_{W_0} are quadratic then lower degree terms do not matter.

Since W_0 is locally constant on its critical set, relation (4.20) implies the following category equivalence:

$$\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U,W) = \mathsf{D}_{\mathbb{Z}_2}(U_{W_0}, W_+|_{U_{W_0}})[\mathrm{N}U_{W_0}]_{\mathrm{tw}}. \tag{4.21}$$

Note that we did not have to augment the r.h.s. category, because the connected parts of the critical set U_{W_0} will contribute to it only when the constant value of W on them is zero.

In view of Equation (4.2), equivalence (4.21) can be simplified:

$$\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U,W) = \mathsf{D}_{\mathbb{Z}_2}(U_{W_0}, W_+|_{U_{W_0}})[\textstyle\bigwedge^{\mathrm{top}}\mathrm{N}U_{W_0}]_{\mathrm{tw}}[\operatorname{rank}\mathrm{N}U_{W_0} - 1]_2. \tag{4.22}$$

Finally, if $W_+ = 0$, that is, if W is just a holomorphic function on U, then the equivalence takes the form

$$\mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U,W) = \mathsf{D}_{\mathbb{Z}_2}\,(U_W)\,[\textstyle\bigwedge^{\mathrm{top}}\mathrm{N}U_W]_{\mathrm{tw}}[\operatorname{rank}\mathrm{N}U_W - 1]_2. \tag{4.23}$$

4.3.2. The relation between objects. Choose a line bundle $L_0 \to U$ such that $L_0^{\otimes 2} = K_U$. For simplicity we will consider only curved fibrations $(\mathcal{U}, W) \in \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$, for which $W_+ = 0$ (see Equation (4.19)), that is, W is just a holomorphic function on $\mathcal{U}$. Recall that $\mathcal{U}$ is a fibration (3.16): $p\colon \mathcal{U} \to U$. For a point $u \in \mathcal{U}$ let $V_u \subset \mathcal{U}$ be the fiber to which u belongs: $V_u = p^{-1}(p(u))$. Let $\mathcal{U}_W$ be the set of "fiber-critical" points of W: $\mathcal{U}_W = \{u \in \mathcal{U} \mid \partial W|_{\mathrm{T}_u V_u} = 0\}$. In other words, there is an exact sequence $\mathrm{T}^\vee_{p(u)} U \xrightarrow{a} \mathrm{T}^\vee_u \mathcal{U} \xrightarrow{b} \mathrm{T}^\vee_u V_u$ and $u \in \mathcal{U}_W$ if $b\bigl(\partial W(u)\bigr) = 0$. The latter condition means that $\partial W(u)$ is in the image of a. Hence there is a map $f\colon \mathcal{U}_W \to \mathrm{T}^\vee U$ such that $f(u) = a^{-1}\bigl(\partial W(u)\bigr) \in \mathrm{T}^\vee_{p(u)}$. The *support* of the object $(\mathcal{U}, W)$ is defined to be the image of f, and we denote it suggestively as $Y_{(\mathcal{U},W)}$:

$$Y_{(\mathcal{U},W)} = \operatorname{supp}(\mathcal{U}, W) := f(\mathcal{U}_W) \subset \mathrm{T}^\vee U. \tag{4.24}$$

Generally, $Y_{(\mathcal{U},W)}$ is an isotropic submanifold with respect to the symplectic structure $\mathrm{T}^\vee U$. Assume that for all $x \in U$, the critical locus of the function $W|_{p^{-1}(x)}$ is clean and that $\mathcal{U}_W$ is a complex manifold. Then $Y_{(\mathcal{U},W)} \subset \mathrm{T}^\vee U$ is a lagrangian submanifold and the map $f\colon \mathcal{U}_W \to Y_{(\mathcal{U},W)}$ is a fibration. Let B_x denote the normal bundle to the critical set of W restricted to the fiber $p^{-1}(x)$. This bundle has an $O(n,\mathbb{C})$ structure given by the Hessian of $W|_{p^{-1}(x)}$, and all these bundles together form a holomorphic $O(n,\mathbb{C})$ bundle B over $\mathcal{U}_W$. Thus we define the action of the equivalence two-functor (1.11) on the curved fibration $(\mathcal{U}, W)$ as follows:

$$\ddot{\Phi}_{\cong}(\mathcal{U}, W) = (\mathcal{U}_W, (p^* L_0)|_{\mathcal{U}_W} \otimes \textstyle\bigwedge^{\mathrm{top}} B)[\operatorname{rank} B - 1]_2,$$

that is, the pair in the r.h.s. of this equation is the object of $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ corresponding to the object $(\mathcal{U}, W) \in \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$.

The geometric object corresponding to the pair $(\mathcal{U}, W)$ is particularly simple, if $\mathcal{U}$ is a one-point fibration over U. Then W is just a holomorphic function on U and the object of $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ corresponding to W is the pair $(Y_W, g^* L_0)$, where Y_W is the graph of ∂W:

$$Y_W = \{p \in \mathrm{T}^\vee_x U \mid x \in U,\ p = \partial W|_x\} \tag{4.25}$$

and $g\colon Y_W \to U$ is the restriction of the projection $\mathrm{T}^\vee U \to U$ to Y_W (it establishes the isomorphism between Y_W and U as complex manifolds).

4.3.3. The relation between categories of morphisms. We will compare the categories of morphisms within two-categories $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ and $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)$ only for the simplest objects. Let W_1 and W_2 be holomorphic functions on U such that their difference $W_{12} = W_2 - W_1$ has a clean set of critical points. This is equivalent to saying that Y_{W_1} and Y_{W_2} have a clean intersection.

According to definition (3.20) and equivalence (4.23), the category of morphisms within $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ is

$$\begin{aligned}\operatorname{Hom}_{\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)}(W_1, W_2) &= \mathsf{D}^{\mathrm{a}}_{\mathbb{Z}_2}(U, W_{12}) \\ &= \mathsf{D}_{\mathbb{Z}_2}(U_{W_{12}})\,[\mathrm{N}U_{W_{12}}]_{\mathrm{tw}}[\operatorname{rank} \mathrm{N}U_{W_{12}} - 1]_2.\end{aligned} \tag{4.26}$$

At the same time, according to Equation (4.8),

$$\begin{aligned}&\operatorname{Hom}_{\ddot{\mathsf{L}}(\mathrm{T}^\vee U)}((Y_{W_1}, g_1^* L_0), (Y_{W_2}, g_2^* L_0)) \\ &\quad= \mathsf{D}_{\mathbb{Z}_2}(Y_{12})[L_{12}]_{\mathrm{tw}}[\dim U - \dim Y_{12} - 1]_2,\end{aligned} \tag{4.27}$$

where $Y_{12} := Y_{W_1} \cap Y_{W_2}$, the maps $g_i\colon Y_{W_i} \to U$ are the restrictions of the projection $\mathrm{T}^\vee U \to U$ and

$$L_{12} = (g_1^* L_0)|_{Y_{12}} \otimes (g_2^* L_0)|_{Y_{12}} \otimes K^{-1}_{Y_{12}}. \tag{4.28}$$

The maps g_1 and g_2, as well as another projection restriction $g_{12}\colon Y_{12} \to U_{W_{12}}$, establish isomorphisms between the corresponding complex manifolds.

Therefore, there is an equivalence of categories

$$\mathsf{D}_{\mathbb{Z}_2}(Y_{12})[L_{12}]_{\mathrm{tw}} = \mathsf{D}_{\mathbb{Z}_2}(U_{W_{12}})[g_{12,*}L_{12}]_{\mathrm{tw}},$$

and according to Equation (4.28), the push-forward of the line bundle L_{12} is

$$g_{12,*}L_{12} = L_0^2 \otimes K_{Y_{12}}^{-1} = K_U \otimes K_{Y_{12}}^{-1} = \bigwedge^{\mathrm{top}} \mathrm{N}U_{W_{12}}.$$

Since

$$\operatorname{rank} \mathrm{N}U_{W_{12}} = \dim U - \dim U_{W_{12}}, \qquad U_{W_{12}} = Y_{12},$$

we established the equivalence of categories (4.26) and (4.27) provided by the two-functor $\ddot{\Phi}_{\cong}$.

4.3.4. The relation between two-functors. Let ω be the canonical symplectic form of the cotangent bundle $\mathrm{T}^\vee U$. The symplectomorphism $\tau : (\mathrm{T}^\vee U, \omega) \to (\mathrm{T}^\vee U, -\omega) = (\mathrm{T}^\vee U)^\diamondsuit$ reverses the cotangent vectors: for $p \in \mathrm{T}_q^\vee U$ we define $\tau(p) := -p$. For two cotangent bundles we define the symplectomorphism $\tau_1 \colon (\mathrm{T}^\vee U_1, \omega_1) \times (\mathrm{T}^\vee U_2, \omega_2) \to (\mathrm{T}^\vee U_1, -\omega_1) \times (\mathrm{T}^\vee U_2, \omega_2)$ which reverses the orientation of the first cotangent vector: $\tau_1 := \tau \times 1$.

The symplectomorphism τ acts as a two-functor $\ddot{\tau} \colon \ddot{\mathsf{L}}(\mathrm{T}^\vee U) \to \ddot{\mathsf{L}}(\mathrm{T}^\vee U)^\diamondsuit$ by transforming the bases of holomorphic fibrations $\mathcal{Y} \to Y \subset \mathrm{T}^\vee U$. Similarly, τ_1 acts as a two-functor $\ddot{\tau}_1 \colon \ddot{\mathsf{L}}(\mathrm{T}^\vee U_1 \times \mathrm{T}^\vee U_2) \to \ddot{\mathsf{L}}\left((\mathrm{T}^\vee U_1)^\diamondsuit \times \mathrm{T}^\vee U_2\right)$. Now consider a composition of two-functors

$$\ddot{\tau}_1 \circ \ddot{\Phi}_{\cong,12} \colon \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(\mathrm{T}^\vee U_1 \times \mathrm{T}^\vee U_2) \longrightarrow \ddot{\mathsf{L}}\Big((\mathrm{T}^\vee U_1)^\diamondsuit \times \mathrm{T}^\vee U_2\Big).$$

We leave it for the reader to check that the two-functors (3.23) and (4.13) coming from the objects related by (1.9) are intertwined by two-functors $\ddot{\Phi}_{\cong}$, that is, the diagram

$$\begin{array}{ccc}
\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_1) & \xrightarrow{\ \ddot{\Phi}[\mathcal{U}_{12}, W_{12}]\ } & \ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_2) \\
\Big\downarrow{\scriptstyle \ddot{\Phi}_{\cong,1}} & & \Big\downarrow{\scriptstyle \ddot{\Phi}_{\cong,2}} \\
\ddot{\mathsf{L}}(\mathrm{T}^\vee U_1) & \xrightarrow{\ \ddot{\Phi}[\mathcal{Y}_{12}]\ } & \ddot{\mathsf{L}}(\mathrm{T}^\vee U_2)
\end{array}$$

is commutative, if $\mathcal{Y}_{12} = \ddot{\tau}_1 \circ \ddot{\Phi}_{\cong,12}(\mathcal{U}, W)$. The easiest part of this commutativity is the verification that the support of a curved fibration from $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U_1)$

is transformed as in Equation (4.16):

$$Y_{\ddot{\Phi}[\mathcal{U}_{12},W_{12}](\mathcal{U}_1,W_1)} = \pi_2\left(Y_{(\mathcal{U}_{12},W_{12})} \cap \pi_1^{-1}\left(Y_{(\mathcal{U}_1,W_1)}\right)\right), \tag{4.29}$$

where π_1 and π_2 are the projections

$$\begin{array}{ccc} & \mathrm{T}^\vee U_1 \times \mathrm{T}^\vee U_2 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathrm{T}^\vee U_1 & & \mathrm{T}^\vee U_2 \end{array}$$

5. The two-category of a deformed cotangent bundle

5.1. Outline

The geometric description of the two-category $\ddot{\mathsf{L}}(X,\omega)$ provided in Section 4.1 is completely correct only when (X,ω) is a cotangent bundle: $X = \mathrm{T}^\vee U$. However, path-integral considerations [6] suggest that the main feature of that description holds true for a general holomorphic symplectic manifold (X,ω): a pair (Y, L_Y), where $Y \subset X$ is a lagrangian submanifold and the line bundle L_Y is a square root of the canonical bundle of Y, always represents an object in $\ddot{\mathsf{L}}(X,\omega)$.

Path-integral arguments [6] also suggest that the two-category $\ddot{\mathsf{L}}(X,\omega)$ is local. From the physics perspective, locality means that there are no instanton corrections to the path integral, that is, there are no three-dimensional analogs of A-model holomorphic disks which lie at the heart of the Floer homology and the Fukaya category. From the mathematical perspective, locality means that the category of morphisms between two objects of $\ddot{\mathsf{L}}(X,\omega)$ is determined by the structure of X in the formal neighborhood of the intersection of their supports, that is, $\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}((Y_1, L_{Y_1}), (Y_2, L_{Y_2}))$ is determined by the formal neighborhood of $Y_1 \cap Y_2 \subset X$ (when X is a cotangent bundle and the intersection of supports is clean, the category of morphisms is determined just by the intersection itself, as suggested by Equation (4.8)). The composition of morphisms is determined by the structure of X near the triple intersection of supports (cf. Equation (4.11)).

Let $Y \subset X$ be a lagrangian submanifold of a general holomorphic symplectic manifold (X,ω). Locality of $\ddot{\mathsf{L}}(X,\omega)$ means that if we knew the structure of the two-category $\ddot{\mathsf{L}}$ of the formal neighborhood of Y, we would know exactly all categories of morphisms involving the objects (Y, L_Y) as well as the compositions of such morphisms.

In real symplectic geometry, a sufficiently small tubular neighborhood of a lagrangian submanifold Y is symplectomorphic to a tubular neighborhood of the zero section of the cotangent bundle $\mathrm{T}^{\vee}Y$. However, in holomorphic case this is no longer true: there may be non-trivial deformations of the holomorphic symplectic structure of (the formal neighborhood of the zero section) of the cotangent bundle $\mathrm{T}^{\vee}Y$ and the formal neighborhood of $Y \subset X$ may be isomorphic to a deformed formal neighborhood of the zero section of $\mathrm{T}^{\vee}Y$. Therefore, in order to gain information about the morphisms involving the object (Y, L_Y) of $\ddot{\mathsf{L}}(X,\omega)$ we have to explore the two-category corresponding to a deformed cotangent bundle $(\mathrm{T}^{\vee}Y)_{\varkappa}$, where $\varkappa$ is a deformation parameter of the holomorphic symplectic structure of $\mathrm{T}^{\vee}Y$.

To understand the two-category $(\mathrm{T}^{\vee}Y)_{\varkappa}$ we follow the algebraic approach: for a complex manifold U, which plays the role of Y, we construct a deformation $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ of the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$. The construction of this deformation is based on two path-integral-motivated assumptions: the deformation parameter $\varkappa$ of the two-category is the same as the deformation parameter of the holomorphic symplectic structure of $\mathrm{T}^{\vee}U$ and the simplest objects of the deformed category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ (that is, the objects corresponding to one-point fibrations and described by holomorphic functions W on U in the category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$) should be related to the lagrangian submanifolds of the deformed cotangent bundle $(\mathrm{T}^{\vee}U)_{\varkappa}$ in the same way as in the undeformed case discussed in Section 4.3.

5.2. Differential Lie–Gerstenhaber algebras and the Maurer–Cartan equation

Let us review some well-known facts about algebras governing the deformations of objects and categories appearing in this paper.

5.2.1. General definitions. A differential Lie–Gerstenhaber algebra $\mathfrak{L}$ is a $\mathbb{Z}_2$-graded vector space endowed with a differential D and a compatible graded Lie bracket $[\cdot,\cdot]_{\mathrm{LG}}$ which may be even or odd. Let $d_{[\,]}$ be the $\mathbb{Z}_2$-degree of the Lie bracket. If $d_{[\,]} = \hat{0}$ then $\mathfrak{L}$ is called a differential Lie algebra. If in addition $\mathfrak{L}$ has a supercommutative associative product compatible both with D and the Lie bracket, then $\mathfrak{L}$ is called a differential Poisson algebra. If $d_{[\,]} = \hat{1}$ and $\mathfrak{L}$ has a supercommutative associative product compatible both with D and $[\cdot,\cdot]_{\mathrm{LG}}$, then $\mathfrak{L}$ is called a differential Gerstenhaber algebra.

The graded Lie bracket of $\mathfrak{L}$ descends to its D-homology $\mathrm{H}_D(\mathfrak{L})$.

If an element $\alpha \in \mathfrak{L}$ of $\mathbb{Z}_2$-degree $d_{[\,]} + 1$ satisfies the Maurer–Cartan equation

$$D\alpha + \tfrac{1}{2}\,[\alpha,\alpha]_{\mathrm{LG}} = 0, \tag{5.1}$$

then the operator

$$D_\alpha = D + [\alpha,\cdot]_{\mathrm{LG}} \tag{5.2}$$

is also a differential and it determines a deformed differential Lie–Gerstenhaber algebra $\mathfrak{L}_\alpha$. If two Maurer–Cartan elements are related by a "gauge transformation" ϕ, whose infinitesimal form is

$$\delta\alpha = D_\alpha\phi, \qquad \phi \in \mathfrak{L}, \quad \deg_{\mathbb{Z}_2}\phi = d_{[\,]}, \tag{5.3}$$

then the corresponding deformed algebras are isomorphic.

If a Maurer–Cartan element is presented as a formal power series in a parameter ϵ:

$$\alpha_\epsilon = \sum_{i=1}^{\infty} \alpha_{|i}\,\epsilon^i,$$

then the leading coefficient $\alpha_{|1}$ is D-closed and, due to the gauge symmetry, the corresponding deformation of $\mathfrak{L}$ is determined by its class $\breve{\alpha}_{|1} \in \mathrm{H}_D(\mathfrak{L})$. The ϵ^2 part of the Maurer–Cartan equation (5.1) says that

$$D\alpha_{|2} + \tfrac{1}{2}\,[\alpha_{|1},\alpha_{|1}]_{\mathrm{LG}} = 0,$$

so $\breve{\alpha}_{|1}$ satisfies the condition

$$[\breve{\alpha}_{|1},\breve{\alpha}_{|1}]_{\mathrm{LG}} = 0. \tag{5.4}$$

The importance of differential Lie–Gerstenhaber algebras stems from the fact that they appear as deformation complexes of objects in categories and Hochschild complexes of categories and two-categories. Equivalence classes of Maurer–Cartan elements parameterize deformations of those objects, categories and two-categories. We need three particular examples: the differential Lie algebra $\mathscr{C}(E,\bar{\nabla}_E)$ which governs deformations of a curved quasi-holomorphic vector bundle $(E,\bar{\nabla}_E)$, the differential Gerstenhaber algebra $\mathscr{G}(U)$ governing A_∞ deformations of the two-periodic derived category $\mathsf{D}_{\mathbb{Z}_2}(U)$ of a complex manifold U, and the differential Poisson algebra $\mathscr{P}(X,\omega)$ which, according to path-integral considerations [6], governs deformations of the two-category $\ddot{\mathsf{L}}(X,\omega)$.

5.2.2. The differential Lie algebra of a curved quasi-holomorphic vector bundle. Let the pair $(E, \bar{\nabla}_E)$, where $\bar{\nabla}_E^2 = W 1_E$, be a curved quasi-holomorphic vector bundle defined in Section 3.1. The corresponding differential Lie algebra of relative connections $\mathscr{C}(E, \bar{\nabla}_E)$ is the algebra of Dolbeault forms $\Omega^{0,\bullet}\big(\operatorname{End} E\big)$ (with total grading), the differential is the covariant Dolbeault differential $\bar{\nabla}_E$, and the Lie bracket is the supercommutator: $[\zeta_1, \zeta_2] := \zeta_1\zeta_2 - (-1)^{|\zeta_1||\zeta_2|}\zeta_2\zeta_1$.

A Maurer–Cartan element ζ determines a deformed quasi-holomorphic vector bundle

$$(E, \bar{\nabla}_E)_\zeta := (E, \bar{\nabla}_E + \zeta)$$

(in the r.h.s. of this formula ζ denotes an odd bundle map $\zeta\colon \Omega^{0,\bullet}(E) \to \Omega^{0,\bullet}(E)$).

5.2.3. The differential Gerstenhaber algebra of a complex manifold. The well-known differential Gerstenhaber algebra $\mathscr{G}(U)$ of a complex manifold U is the algebra $\Omega^{0,\bullet}(U, \wedge^\bullet \mathrm{T}U)$, the $\mathbb{Z}_2$-grading coming from the total degree of forms and wedge-powers, with the differential $\bar{\partial}$ and the Schouten–Nijenhuis bracket $[\cdot,\cdot]$. In fact, $\mathscr{G}(U)$ has $\mathbb{Z}$-grading, and its degree 2 Maurer–Cartan elements parameterize deformations of the derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(U)$. More generally, even Maurer–Cartan elements of $\mathscr{G}(U)$ parameterize A_∞ deformations of $\mathsf{D}_{\mathbb{Z}_2}(U)$.

A holomorphic function $W \in \Omega^{0,0}(U)$, $\bar{\partial} W = 0$ obviously satisfies the Maurer–Cartan equation and hence determines a deformation of $\mathscr{G}(U)$ which we denote as $\mathscr{G}_W(U)$. It has a new differential $\bar{\partial}_{W} = \bar{\partial} + [W, \cdot]$ (cf. Equation (5.2)). The corresponding deformation of $\mathsf{D}_{\mathbb{Z}_2}(U)$ is the curved category $\mathsf{D}_{\mathbb{Z}_2}(U, W)$.

Define a *relative* grading on $\mathscr{G}(U)$ as

$$\deg_{\mathrm{rel}} \Omega^{0,n}(U, \wedge^m \mathrm{T}U) = n - m,$$

then, obviously,

$$\deg_{\mathrm{rel}} \bar{\partial} = \deg_{\mathrm{rel}}[\cdot,\cdot] = 1.$$

We say that a Maurer–Cartan element λ is *relatively non-negative* if $\deg_{\mathrm{rel}} \lambda \geq 0$ and *relatively balanced* if $\deg_{\mathrm{rel}} \lambda = 0$.

5.2.4. The differential Poisson algebra of a holomorphic symplectic manifold. The differential Poisson algebra $\mathscr{P}(X, \omega)$ of a holomorphic symplectic manifold (X, ω) is defined as the algebra $\Omega^{0,\bullet}(X)$ of $(0, \bullet)$-forms on X with the differential $\bar{\partial}$ and with the Poisson bracket coming from ω. If $X = \mathrm{T}^\vee U$, where U is a complex manifold, then we may consider a simpler version of this algebra, which we denote as $\mathscr{P}_{\mathrm{T}^\vee}(U)$: it is the algebra of $\mathrm{S}^\bullet\mathrm{T}U$-valued $(0,\bullet)$ forms $\Omega^{0,\bullet}(U,\mathrm{S}^\bullet\mathrm{T}U)$, where $\mathrm{S}^\bullet$ is the symmetric algebra, and its differential is $\bar\partial$. There is a natural injection

$$\Omega^{0,\bullet}(U,\mathrm{S}^\bullet\mathrm{T}U)\hookrightarrow\Omega^{0,\bullet}(\mathrm{T}^\vee U), \tag{5.5}$$

which turns an element of $\Omega^{0,\bullet}(U,\mathrm{S}^\bullet\mathrm{T}U)$ into a $(0,\bullet)$ differential form on $\mathrm{T}^\vee U$ having a polynomial dependence on fiber coordinates and restricting to zero on all fibers. The bracket of $\mathscr{P}_{\mathrm{T}^\vee}(U)$ is a well-defined restriction of the Poisson bracket of $\Omega^{0,\bullet}(\mathrm{T}^\vee U)$, so injection (5.5) becomes an injection of differential Poisson algebras

$$\mathscr{P}_{\mathrm{T}^\vee}(U)\hookrightarrow\mathscr{P}(\mathrm{T}^\vee U). \tag{5.6}$$

5.2.5. A_∞-algebra and its modules. Following Keller's review [7] let us recall the main definitions. An A_∞-algebra is a $\mathbb{Z}_2$-graded vector space $\mathscr{A}$ endowed with a series of n-linear maps (n-multiplications) $\mathbf{a}=a_0,a_1,\ldots$,

$$a_n\colon\mathscr{A}^{\otimes n}\to\mathscr{A},\qquad \deg_{\mathbb{Z}_2}a_n=n,\qquad n=0,1,2,\ldots,$$

satisfying the relations

$$\sum_{\substack{m_1,m_2,n\geq 0\\ m_1+m_2+n=N}}(-1)^{m_1+nm_2}\,a_{m_1+m_2+1}(1^{\otimes m_1}\otimes a_n\otimes 1^{\otimes m_2})=0 \tag{5.7}$$

for all $N\geq 0$. Among other things, these relations indicate that if $a_0=0$, then a_1 is a differential ($a_1^2=0$) and it satisfies the usual Leibnitz rule with respect to the multiplication a_2.

If $a_0\neq 0$ then the A_∞-algebra $\mathscr{A}$ is called *weak* or *curved.*

A module over an A_∞-algebra $\mathscr{A}$ is a vector space M endowed with a series of n-linear maps (n-actions) $\mathbf{a}^M=a_1^M,a_2^M,\ldots$

$$a_n^M\colon M\otimes\mathscr{A}^{\otimes(n-1)}\to\mathscr{A},\qquad \deg_{\mathbb{Z}_2}a_n^M=n,\qquad n=1,2,\ldots,$$

satisfying the relations

$$\sum_{\substack{m_2,n\geq 0\\ m_2+n=N}}(-1)^{nm_2}\,a^M_{m_2+1}(a^M_n\otimes 1^{\otimes m_2}) + \sum_{\substack{m_1\geq 1\\ m_2,n\geq 0\\ m_1+m_2+n=N}}(-1)^{m_1+nm_2}\,a^M_{m_1+m_2+1}(1^{\otimes m_1}\otimes a_n\otimes 1^{\otimes m_2})=0 \tag{5.8}$$

for all $N\geq 0$.

For two $\mathscr{A}$-modules M_1 and M_2, the vector space $\widetilde{\operatorname{Hom}}(M_1, M_2)$ is formed by sequences $\mathbf{f} = (f_1, f_2, \ldots)$, f_n being n-linear maps

$$f_n\colon M_1 \otimes \mathscr{A}^{\otimes(n-1)} \to M_2.$$

Define a differential d acting on $\widetilde{\operatorname{Hom}}(M_1, M_2)$ by the following formula for each term in $d\mathbf{f}$:

$$(5.9) \qquad (d\mathbf{f})_N = \sum_{\substack{m_2, n \geq 0 \\ m_2+n=N}} (-1)^{nm_2}\, a^{M_2}_{m_2+1}(f_n \otimes 1^{\otimes m_2}) - \sum_{\substack{m_1 \geq 1 \\ m_2, n \geq 0 \\ m_1+m_2+n=N}} (-1)^{m_1+nm_2}\, f_{m_1+m_2+1}(1^{\otimes m_1} \otimes a_n \otimes 1^{\otimes m_2}) - \sum_{\substack{m_2, n \geq 0 \\ m_2+n=N}} (-1)^{nm_2}\, f_{m_2+1}(a^{M_1}_n \otimes 1^{\otimes m_2}).$$

5.2.6. A filtered Dolbeault A_∞-algebra and the perfect homotopy category of its modules. Now we adapt the general definitions to the Dolbeault setting.

First of all, note that the Dolbeault algebra $(\Omega^{0,\bullet}(U), \bar\partial)$ associated with a complex manifold U has a canonical sequence of n-multiplications $\mathbf{a}^{(0)}$ which turn it into an A_∞-algebra: $a^{(0)}_1 = \bar\partial$, $a^{(0)}_2$ is the wedge-product, and $a^{(0)}_n = 0$ for $n \neq 1, 2$.

We define *a filtered Dolbeault A_∞-algebra* (FDA_∞) on a complex manifold U as the space $\Omega^{0,\bullet}(U)$ endowed with n-multiplications $\mathbf{a}$ satisfying relation (5.7) and the following restriction on Dolbeault degrees of their deviation from the canonical n-multiplications:

$$(5.10) \qquad \deg_{\mathrm{Dlb}}(a_n - a^{(0)}_n) \geq n, \qquad \forall n \geq 0.$$

If $\lambda \in \Omega^{0,\bullet}(U, \wedge^\bullet \mathrm{T}U)$ is a relatively non-negative Maurer–Cartan element, then the corresponding deformation $\left(\Omega^{0,\bullet}(U), \bar\partial, \lambda\right)$ of the Dolbeault algebra is a filtered Dolbeault A_∞-algebra, the relation between the components of λ and n-multiplications being fairly complicated. If the deformation parameter λ is relatively balanced, that is,

$$\lambda = \sum_{n=0}^{\infty} \lambda_n, \qquad \lambda_n \in \Omega^{0,n}(U, \wedge^n \mathrm{T}U),$$

then the dominant part of the deviations $a_n - a_n^{(0)}$ is determined by the formula

$$(5.11) \qquad (a_n - a_n^{(0)})(\alpha_1, \ldots, \alpha_n) = \lambda_n \llcorner (\partial\alpha_1, \ldots, \partial\alpha_n) + \cdots ,$$

where $\alpha_1, \ldots, \alpha_n \in \Omega^{0,\bullet}(U)$ and the correction terms have higher Dolbeault degree than the first term in the r.h.s. of this equation.

A simple example of a relatively balanced deformation of the Dolbeault algebra $(\Omega^{0,\bullet}(U), \bar{\partial})$ is $\lambda = W$, where W is a holomorphic function on U. Then formula (5.11) has no correction terms and the deformation results in the curved Dolbeault algebra $(\Omega^{0,\bullet}(U), \bar{\partial}, W)$ discussed already in Section 3.1.

Consider again the Dolbeault algebra $(\Omega^{0,\bullet}(U), \bar{\partial})$ as a filtered Dolbeault A_∞-algebra with n-multiplications $\mathbf{a}^{(0)}$. Its perfect $\mathbb{Z}_2$-GDM (3.5) with $\bar{\nabla}_E^2 = 0$ has a canonical structure of an A_∞-module if we endow it with n-actions $\mathbf{a}^{E,(0)}$ such that $a_1^{E,(0)} = \bar{\nabla}_E$, $a_2^{E,(0)}$ is the standard multiplication and $a_n^{E,(0)} = 0$ for $n > 2$. We define a perfect module over a filtered Dolbeault A_∞-algebra as the vector space $\Omega^{0,\bullet}(E)$ endowed with a sequence of n-actions $\mathbf{a}^E$ satisfying relations (5.8) and the restriction

$$\deg_{\mathrm{Dlb}}(a_n^E - a_n^{E,(0)}) \geq n, \qquad \forall n \geq 1.$$

Finally, we define the two-periodic perfect derived category of a filtered Dolbeault A_∞-algebra $(\Omega^{0,\bullet}(U); \mathbf{a})$: its objects are perfect A_∞-modules $(\Omega^{0,\bullet}(E); \mathbf{a}^E)$ and morphisms between two modules are homologies of differential (5.9).

If $\lambda \in \Omega^{0,\bullet}(U, \wedge^\bullet \mathrm{T}U)$ is a relatively non-negative Maurer–Cartan element, then $\mathsf{D}_{\mathbb{Z}_2}(U, \lambda)$ denotes the two-periodic perfect derived category corresponding to the deformed A_∞-algebra $\left(\Omega^{0,\bullet}(U), \bar{\partial}, \lambda\right)$. In other words, $\mathsf{D}_{\mathbb{Z}_2}(U, \lambda)$ is the result of deforming $\mathsf{D}_{\mathbb{Z}_2}(U)$ with λ. In particular, if $\lambda = W$, where W is a holomorphic function on U, then $\mathsf{D}_{\mathbb{Z}_2}(U, W)$ is the category (3.6).

5.3. Deformation of a holomorphic symplectic structure

5.3.1. The general case. Let (X, ω) be a holomorphic symplectic manifold. Deformations of its holomorphic symplectic structure which preserve the de Rham cohomology class of ω are parameterized up to gauge equivalence by Maurer–Cartan elements of the differential Poisson algebra

$\mathscr{P}(X, \omega)$ defined in Section 5.2.4. Namely, if an element

$$\varkappa \in \Omega^{0,1}(X) \subset \mathscr{G}(X)$$

satisfies the Maurer–Cartan equation

$$\bar{\partial}\varkappa + \tfrac{1}{2}\{\varkappa, \varkappa\} = 0, \tag{5.12}$$

where $\{\ ,\ \}$ is the Poisson bracket corresponding to the symplectic form ω, then the corresponding deformation of the complex structure of X is described by the Beltrami differential

$$\mu = \omega^{-1}(\partial\varkappa), \tag{5.13}$$

that is, the $(0,1)$ part of the deformed Dolbeault differential is

$$\bar{\partial}' = \bar{\partial} + \omega^{-1}(\partial\varkappa) \llcorner \partial, \tag{5.14}$$

while the symplectic form ω is replaced by

$$\omega' = \omega + d\varkappa,$$

so that it remains of type $(2,0)$ relative to the new complex structure. In formula (5.14) we defined $\omega^{-1} \colon \Gamma(\mathrm{T}^{\vee}X) \to \Gamma(\mathrm{T}X)$ as the inverse of $\iota_{-}\,\omega$.

If the deformation of (X, ω) is perturbative, that is, if $\varkappa$ is a formal power series

$$\varkappa_{\epsilon} = \sum_{i=1}^{\infty} \varkappa_{|i}\, \epsilon^{i}, \tag{5.15}$$

then relation (5.12) says that the leading coefficient $\varkappa_{|1}$ must be $\bar{\partial}$-closed, and its gauge equivalence class is determined by its Dolbeault cohomology class $\check{\varkappa}_{|1} \in \mathrm{H}^{1}_{\bar{\partial}}(X)$, while the relation

$$\bar{\partial}\varkappa_{|2} + \tfrac{1}{2}\{\varkappa_{|1}, \varkappa_{|1}\} = 0$$

implies the 'integrability condition' for $\check{\varkappa}_{|1}$:

$$\{\check{\varkappa}_{|1}, \check{\varkappa}_{|1}\} = 0.$$

5.3.2. Deformation of the holomorphic symplectic structure of a cotangent bundle. If $X = \mathrm{T}^\vee U$, then we restrict ourselves to Maurer–Cartan elements $\varkappa$ belonging to the subalgebra $\mathscr{P}_{\mathrm{T}^\vee}(U) \hookrightarrow \mathscr{P}(\mathrm{T}^\vee U)$ defined in Section 5.2.4. Moreover, we consider only the deformations which do not deform the complex structure of the zero section $U_0 \subset \mathrm{T}^\vee U$, so we impose the condition $\mu|_{U_0} = 0$ on the Beltrami differential (5.13). This condition is satisfied if $\varkappa$ is at least quadratic as a function of holomorphic coordinates on fibers of $\mathrm{T}^\vee U$:

$$\varkappa = \varkappa_2 + \varkappa_3 + \cdots, \qquad \varkappa_i \in \Omega^{0,1}\big(U, \mathrm{S}^i\mathrm{T}U\big). \tag{5.16}$$

The first two terms in this sum play a particularly important role in what follows and we give them special names:

$$\varkappa_2 = \beta, \qquad \varkappa_3 = \gamma. \tag{5.17}$$

According to Equation (5.12), they satisfy the equations

$$\bar\partial\beta = 0, \qquad \bar\partial\gamma + \tfrac{1}{2}\{\beta,\beta\} = 0. \tag{5.18}$$

Thus β is $\bar\partial$-closed, and its gauge equivalence class is determined by the Dolbeault cohomology class that it represents:

$$\check\beta \in \mathrm{H}^1_{\bar\partial}(U, \mathrm{S}^2\mathrm{T}U), \qquad \{\check\beta, \check\beta\} = 0.$$

The class $\check\beta$ has a simple geometric interpretation. For a holomorphic submanifold $Y \subset X$ of a complex manifold X, the exact sequence of vector bundles on Y

$$\mathrm{T}Y \longrightarrow \mathrm{T}X|_Y \longrightarrow \mathrm{N}Y \tag{5.19}$$

determines an extension class $\tilde\beta_Y \in \mathrm{Ext}^1(\mathrm{N}Y, \mathrm{T}Y)$.[4] If Y is a lagrangian submanifold of a holomorphic symplectic manifold X then the symplectic form ω establishes an isomorphism $\mathrm{N}Y \simeq \mathrm{T}^\vee Y$, so $\tilde\beta_Y \in \mathrm{Ext}^1(\mathrm{T}^\vee Y, \mathrm{T}Y)$. The zero-section $U \subset \mathrm{T}^\vee U$ is a lagrangian submanifold and its exact sequence (5.19) splits, so in this case $\tilde\beta_U = 0$. However, if we consider the zero-section of the deformed bundle $(\mathrm{T}^\vee U)_\varkappa$ then sequence (5.19) does not

[4]If X is Kähler, then the class $\tilde\beta_Y$ may be represented by the anti-holomorphic part of the second fundamental form of Y contracted with the Kähler metric and with its inverse in order to turn two anti-holomorphic indices on the second fundamental form into the holomorphic ones.

have to split. The injection $\mathrm{H}^1_{\bar\partial}(U, \mathrm{S}^2\mathrm{T}U) \hookrightarrow \mathrm{Ext}^1(\mathrm{T}^\vee U, \mathrm{T}U)$ turns $\check\beta$ into an extension class within $\mathrm{Ext}^1(\mathrm{T}^\vee U, \mathrm{T}U)$ and, in fact,

$$\check\beta = \tilde\beta_U. \tag{5.20}$$

In other words, the leading coefficient $\check\beta$ in expansion (5.16) of $\varkappa$ reflects the fact that the sequence (5.19) for the zero-section of the deformed cotangent bundle $(\mathrm{T}^\vee U)_\varkappa$ does not split.

Injection (5.5) turns an element $\varkappa \in \Omega^{0,\bullet}(U, \mathrm{S}^\bullet\mathrm{T}U)$ into a $\bar{\mathrm{T}}^\vee U$-valued function (or, rather, a formal power series) on the total space of $\mathrm{T}^\vee U$. We denote this function by the same letter $\varkappa$. The evaluation of $\varkappa$ on a section of $\mathrm{T}^\vee U$ gives a map

$$\varkappa\colon \Gamma(\mathrm{T}^\vee U) \to \Omega^{0,1}(U). \tag{5.21}$$

The restriction of the $(1,0)$ part of the differential $\partial\varkappa$ of an element $\varkappa \in \Omega^{0,\bullet}(\mathrm{T}^\vee U)$ to the fibers of $\mathrm{T}^\vee U$ determines a vertical holomorphic differential map

$$\partial_{\mathrm{vrt}}\varkappa\colon \Gamma(\mathrm{T}^\vee U) \to \Omega^{0,1}(U, \mathrm{T}U). \tag{5.22}$$

Recall that if W is a function on U, then $Y_W \subset \mathrm{T}^\vee U$ denotes the graph of ∂W defined by Equation (4.25). If W is holomorphic, then Y_W is a lagrangian submanifold. Let $(\mathrm{T}^\vee U)_\varkappa$ denote the total space of the cotangent bundle $\mathrm{T}^\vee U$ whose holomorphic symplectic structure is deformed by the Maurer–Cartan element (5.16). The deformed cotangent bundle $(\mathrm{T}^\vee U)_\varkappa$ is canonically diffeomorphic to $\mathrm{T}^\vee U$, so for an arbitrary function W, the graph Y_W is still a submanifold in $(\mathrm{T}^\vee U)_\varkappa$, but this time Y_W is lagrangian if W satisfies the equation

$$\bar\partial W = \varkappa(\partial W), \tag{5.23}$$

where $\varkappa(\cdot)$ is the map (5.21). If we consider a perturbative deformation (5.15), then the generating function becomes a formal power series

$$W_\epsilon = W_0 + \sum_{i=1}^{\infty} W_i\,\epsilon^i. \tag{5.24}$$

The leading term W_0 is a holomorphic function describing a lagrangian submanifold $Y \subset X$ and it has a special property

$$\check\varkappa_{|1}(\partial W_0) = 0,$$

which guarantees that Y can be deformed to the first order in ϵ, while the first-order perturbation W_1 satisfies the equation

$$\bar{\partial} W_1 - \varkappa_{|1}(\partial W_0) = 0.$$

The complex structure of the lagrangian submanifold Y_W determined by the function W satisfying condition (5.23) can be described by saying that the bundle projection of $\mathrm{T}^\vee U$ establishes an isomorphism between Y_W and the base U, whose complex structure is deformed by the Beltrami differential

$$\mu_W = -\partial_{\mathrm{vrt}} \varkappa(\partial W), \tag{5.25}$$

where $\partial_{\mathrm{vrt}} \varkappa$ is the map (5.22).

5.4. Deformation of the two-category of curved one-point fibrations: deformation of the category of morphisms

5.4.1. Objects of the deformed category. Following the outline of Section 5.1, we conjecture that the Maurer–Cartan element $\varkappa$ parameterizing the deformations of the holomorphic symplectic structure of $\mathrm{T}^\vee U$, parameterizes also the deformations of the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$. We are going to discuss the structure of the deformed category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \varkappa)$, but we will limit ourselves to simplest objects in it, which are the analogs of curved one-point fibration objects $(\mathcal{T}_U, W)$ denoted simply as W.

Recall that in the undeformed case the function W labeling an object of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ is holomorphic, so that the graph of its holomorphic differential (4.25) is a lagrangian submanifold of $\mathrm{T}^\vee U$. We conjecture that in the case of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \varkappa)$, a similar object is (parameterized by) a function W on U, which satisfies Equation (5.23), because then the graph of its holomorphic differential Y_W defined by the same Equation (4.25) is again a lagrangian submanifold of the deformed cotangent bundle $(\mathrm{T}^\vee U)_\varkappa$, and this is in line with our conjecture that lagrangian submanifolds represent the objects of $\ddot{\mathsf{L}}(X, \omega)$ not only when (X, ω) is an undeformed cotangent bundle, but also when it is a general holomorphic symplectic manifold.

5.4.2. The universal Maurer–Cartan element. Consider the tensor algebra over $\mathbb{C}$ of *Dolbeault tensor fields*

$$\mathcal{T}(U) := \bigoplus_{k,l=1}^{\infty} \mathcal{T}_l^k(U), \qquad \mathcal{T}_l^k(U) := \Omega^{0,\bullet}(\mathrm{T}^k U \otimes \mathrm{T}^{\vee,l} U).$$

Fix a ∂-connection $\nabla\colon \mathcal{T}^\bullet_\bullet(U) \to \mathcal{T}^\bullet_{\bullet+1}(U)$. For a tensor field $\tau \in \mathcal{T}(U)$, let $\boldsymbol{\nabla}\tau$ denote a sequence of multiple covariant derivatives: $\boldsymbol{\nabla}\tau := \tau, \nabla\tau, \nabla^2\tau, \ldots$. For tensor fields $\tau_1, \ldots, \tau_k$, let $\mathcal{T}_{\boldsymbol{\nabla}}[\tau_1, \ldots, \tau_k]$ denote a subalgebra of $\mathcal{T}(U)$ generated by all tensor fields $\boldsymbol{\nabla}\tau_1, \ldots, \boldsymbol{\nabla}\tau_k$ and by all possible contractions within their tensor products.

Let $\varkappa_i$ denote an element of $\Omega^{0,1}\big(U, \mathrm{S}^i\mathrm{T}U\big)$. The Poisson–Schouten bracket $\{\varkappa_i, \varkappa_j\}$ of two such elements is an element of $\mathcal{T}_{\boldsymbol{\nabla}}[\varkappa_i, \varkappa_j]$ and it is universal in the sense that the coefficients of its expression in terms of the appropriate contractions of $\varkappa_i \otimes \nabla\varkappa_j$ and $\varkappa_j \otimes \nabla\varkappa_i$ are universal constants. The same holds true for elements $\lambda_{ij} \in \Omega^{0,i}(U, \wedge^j\mathrm{T}U)$ and their Schouten–Nijenhuis bracket $[-,-]$.

For a Maurer–Cartan element $\varkappa \in \Omega^{0,1}(U, \mathrm{S}^\bullet\mathrm{T}U)$ satisfying Equation (5.12) and for two functions W_1, W_2 on U satisfying Equation (5.23), denote

$$\mathcal{T}_{\boldsymbol{\nabla},12} := \mathcal{T}_{\boldsymbol{\nabla}}[\partial W_1, \partial W_2, R, \varkappa_2, \varkappa_3, \ldots], \tag{5.26}$$

where $\varkappa_i$ are the components (5.16), while $R := [\bar\partial, \nabla] \in \Omega^{0,1}\big(\mathrm{S}^2\mathrm{T}^\vee U \otimes \mathrm{T}U\big)$ is the curvature of the tangent bundle $\mathrm{T}U$ corresponding to the connection ∇. The Dolbeault differential $\bar\partial$ acts universally on the elements of $\mathcal{T}_{\boldsymbol{\nabla},12}$. Indeed, its action on the components of $\varkappa$ is prescribed by the Maurer–Cartan equation (5.12), its action on ∂W_i is prescribed by Equation (5.23), $\bar\partial R = 0$, and a permutation of $\bar\partial$ and ∇ generates the curvature tensor R.

Conjecture 5.1. *For a Maurer–Cartan element $\varkappa \in \Omega^{0,1}(U, \mathrm{S}^\bullet\mathrm{T}U)$ satisfying Equation (5.12) and for two functions W_1, W_2 on U satisfying Equation (5.23) there exists a universal relatively balanced Maurer–Cartan element*

$$\lambda_{12} = \sum_{i=0}^{\infty} \lambda_{12,i}, \qquad \lambda_{12,i} \in \Omega^{0,i}(U, \wedge^i\mathrm{T}U), \tag{5.27}$$

$$\bar\partial\lambda_{12} + \tfrac{1}{2}\,[\lambda_{12}, \lambda_{12}] = 0 \tag{5.28}$$

such that

$$\lambda_{12,0} = W_{12} := W_2 - W_1 \tag{5.29}$$

and $\lambda_{12,i} \in \mathcal{T}_{\boldsymbol{\nabla},12}$ for $i \geq 1$, where $\varkappa_i$ are the components (5.16)*. The universality of λ_{12} means that the coefficients in the expression of its components $\lambda_{12,i}$ in terms of the tensor fields and their derivatives are constants that do* *not depend on U. The universal element λ_{12} is unique up to gauge equivalence* (5.2)*, and different choices of ∇ also lead to gauge equivalent elements λ_{12}.*

Simply put, if two functions W_1, W_2 satisfy Equation (5.12) then their difference is not necessarily holomorphic and hence it cannot serve as a deformation parameter of the category $\mathsf{D}_{\mathbb{Z}_2}(U)$. However, we conjecture that there is a special unique correction to W_{12} which turns it into a Maurer–Cartan element suitable for deforming $\mathsf{D}_{\mathbb{Z}_2}(U)$. Hence we conjecture that the category of morphisms between the objects of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)$ represented by W_1 and W_2 is the deformed category $\mathsf{D}_{\mathbb{Z}_2}(U)$:

$$\mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(W_1,W_2)=\mathsf{D}_{\mathbb{Z}_2}(U;\lambda_{12}), \tag{5.30}$$

where λ_{12} is the unique universal deformation parameter of Conjecture 5.1. Note that for a fixed manifold U, the sum in Equation (5.27) is effectively finite, the highest value of i being the complex dimension of U.

5.4.3. Perturbative construction of the universal Maurer–Cartan element. We construct the universal expression for λ_{12} perturbatively in the Dolbeault degree $\deg_{\mathrm{Dlb}}$ defined in an obvious way:

$$\deg_{\mathrm{Dlb}} W_i=0,\qquad \deg_{\mathrm{Dlb}} R=1,\qquad \deg_{\mathrm{Dlb}} \varkappa_{|i}=1,\qquad \deg_{\mathrm{Dlb}} \nabla=0. \tag{5.31}$$

We substitute expansion (5.27) into the Maurer–Cartan equation (5.28) and find the equation determining $\lambda_{12,n}$ in terms of $\lambda_{12,i}$ with $i<n$:

$$[W_{12},\lambda_{12,n}]=-\bar{\partial}\lambda_{12,n-1}-\frac{1}{2}\sum_{i=1}^{n-1}[\lambda_{12,i},\lambda_{12,n-i}]. \tag{5.32}$$

We will find expressions for the first three corrections in expansion (5.27). We introduce a special notation for them:

$$\lambda_{12,1}=\mu,\qquad \lambda_{12,2}=\nu,\qquad \lambda_{12,3}=\xi$$

(the indices 12 at μ, ν and ξ are dropped temporarily in this subsection). We are particularly interested in the third correction, because, as we will see shortly, this is the dominant (that is, the lowest Dolbeault degree) term in λ_{12} when $W_1=W_2=0$, that is, from the $\ddot{\mathsf{L}}(\mathrm{T}^\vee U)_\varkappa$ perspective it describes the leading deformation of the category of endomorphisms of the zero-section of $\mathrm{T}^\vee U$.

According to Equation (5.32), the first three corrections are determined by the equations

$$[W_{12}, \mu] = -\bar{\partial} W_{12}, \tag{5.33}$$

$$[W_{12}, \nu] = -\bar{\partial} \mu - \tfrac{1}{2}\,[\mu, \mu], \tag{5.34}$$

$$[W_{12}, \xi] = -\bar{\partial} \nu - [\mu, \nu]. \tag{5.35}$$

First of all, we find an exact universal solution for μ. Equation (5.33) can be rewritten simply as

$$\bar{\partial} W_{12} + \mu \llcorner \partial W_{12} = 0. \tag{5.36}$$

In order to solve it, we have to introduce divided difference notations. Let V and V' be vector spaces. An element $\alpha \in \mathrm{S}^{\bullet} V \otimes V'$ determines a polynomial function (or a formal power series) $\alpha\colon V^{\vee} \to V'$. The first divided difference of α is a symmetric map $\partial_{\mathrm{s}} \alpha\colon V^{\vee} \times V^{\vee} \to V \otimes V'$ defined by the property

$$\partial_{\mathrm{s}} \alpha(v_1, v_2) \llcorner (v_2 - v_1) = \alpha(v_2) - \alpha(v_1), \qquad \forall\, v_1, v_2 \in V.$$

The second divided difference of α is a totally symmetric map $\partial_{\mathrm{s}}^2 \alpha\colon V \times V \times V \to \mathrm{S}^2 V \otimes V'$ defined by the property

$$\partial_{\mathrm{s}}^2 \alpha(v_1, v_2, v_3) \llcorner (v_3 - v_2) = \partial_{\mathrm{s}} \alpha(v_1, v_3) - \partial_{\mathrm{s}} \alpha(v_1, v_2), \qquad \forall\, v_1, v_2, v_3 \in V.$$

In application to an element $\varkappa \in \Omega^{0,1}\,(U,\ \mathrm{S}^{\bullet}\mathrm{T}U)$ divided differences produce symmetric maps

$$\partial_{\mathrm{s}} \varkappa\colon \Gamma(\mathrm{T}^{\vee} U)^{\times 2} \to \Omega^{0,1}\,(U, \mathrm{T}U)\,, \qquad \partial_{\mathrm{s}}^2 \varkappa\colon \Gamma(\mathrm{T}^{\vee} U)^{\times 3} \to \Omega^{0,1}\,\big(U, \mathrm{S}^2\mathrm{T}U\big).$$

According to condition (5.23),

$$\bar{\partial} W_{12} = \varkappa(\partial W_2) - \varkappa(\partial W_1) = \partial_{\mathrm{s}} \varkappa(\partial W_1, \partial W_2) \llcorner \partial W_{12}.$$

Hence the universal solution to Equation (5.36) is the divided difference of $\varkappa$ evaluated on the differentials ∂W_1 and ∂W_2:

$$\mu = -\partial_{\mathrm{s}} \varkappa(\partial W_1, \partial W_2). \tag{5.37}$$

We will find the universal expressions for ν_{12} and ξ_{12} only approximately, up to certain powers of W_1 and W_2:

$$\mu = \mu_{\approx} + O(W^3), \quad \nu = \nu_{\approx} + O(W^2), \quad \xi = \xi_{\approx} + O(W), \tag{5.38}$$

where $O(W^n)$ denotes an expression which is at least of combined degree n in W_1 and W_2. According to Equation (5.37),

$$\mu_{\approx} = -\partial_{\mathrm{s}}\beta(\partial W_1, \partial W_2) - \partial_{\mathrm{s}}\gamma(\partial W_1, \partial W_2),$$

where β and γ are defined by Equation (5.17). By substituting Equation (5.38) into Equations (5.34) and (5.35) we find

$$[W_{12}, \nu_{\approx}] = -\bar{\partial}\mu_{\approx} - \tfrac{1}{2}\,[\mu_{\approx}, \mu_{\approx}], \tag{5.39}$$

$$[W_{12}, \xi_{\approx}] = -\bar{\partial}\nu_{\approx}, \tag{5.40}$$

which determine the universal expressions for $\nu_{\approx}$ and $\xi_{\approx}$.

In order to simplify the calculations required to derive the universal formula for $\nu_{\approx}$ and $\xi_{\approx}$, we present formula (5.37) in a different form. Consider the relation between the Lie bracket on a manifold, and the Poisson bracket on the total space of its cotangent bundle. For a function W on a complex manifold U, let $\widetilde{W}$ denote its pull-back to the total space of the cotangent bundle $\mathrm{T}^{\vee}U$. For an element $\mu \in \Omega^{0,\bullet}(U, \mathrm{T}U)$ let $\tilde{\mu}$ denote the corresponding $(0,\bullet)$-form on the total space of $\mathrm{T}^{\vee}U$ which is linear along the fibers. In our previous notations, $\mu = \partial_{\mathrm{vrt}}\tilde{\mu}$. The Lie bracket on U and the Poisson bracket on $\mathrm{T}^{\vee}U$ are related as follows:

$$\widetilde{[\mu, W]} = -\{\tilde{\mu}, \widetilde{W}\}, \qquad \widetilde{[\mu, \mu']} = -\{\tilde{\mu}, \tilde{\mu}'\}.$$

For a function W on U, let $\widehat{W} = \{\widetilde{W}, \cdot\}$ be a linear operator acting on functions on the total space of $\mathrm{T}^{\vee}U$. The operators $\widehat{W}$ commute with each other: for two functions W_1 and W_2

$$[\widehat{W_1}, \widehat{W_2}] = \{\widehat{W_1, W_2}\} = 0.$$

For $\varkappa \in \Omega^{0,\bullet}(U, \mathrm{S}^{\bullet}\mathrm{T}U)$ let $\varkappa|_i$ denote its component in $\Omega^{0,\bullet}(U, \mathrm{S}^{i}\mathrm{T}U)$. It is easy to see that in our new notations the r.h.s. of Equations (5.23) and (5.37) can be presented as

$$\bar{\partial}W = \varkappa(\partial W) = \left(e^{\widehat{W}}\varkappa\right)\Big|_0, \tag{5.41}$$

$$\mu_{12} = -\partial_{\mathrm{s}}\varkappa(\partial W_1, \partial W_2) = -\partial_{\mathrm{vrt}}\left(\frac{e^{\widehat{W_2}} - e^{\widehat{W_1}}}{\widehat{W_2} - \widehat{W_1}}\,\varkappa\right)\Bigg|_1 \tag{5.42}$$

(in the r.h.s. of these formulas $\varkappa$ is considered to be a $\bar{\mathrm{T}}^\vee U$-valued function on the total space of $\mathrm{T}^\vee U$). According to the first formula,

$$\bar\partial W = \beta(\bar\partial W) + O(W^3) = \tfrac{1}{2}\,\widehat{W}^2\beta + O(W^3). \tag{5.43}$$

According to the second formula,

$$\mu_\approx = -\partial_{\mathrm{vrt}}\Big(\tfrac{1}{2}\,(\widehat{W}_1 + \widehat{W}_2)\beta + \tfrac{1}{6}\,(\widehat{W}_1^2 + \widehat{W}_1\widehat{W}_2 + \widehat{W}_2^2)\,\gamma\Big).$$

Both sides of Equation (5.39) are elements of $\Omega^{0,2}(U,\mathrm{T}U)$, so applying $\tilde{}$ to them (that is, turning them into $(0,2)$-forms on the total space of $\mathrm{T}^\vee U$, which are linear along fibers), we find

$$\begin{aligned}
\widetilde{[W_{12},\nu_\approx]} &= -\bar\partial\tilde\mu_\approx + \tfrac{1}{2}\,\{\tilde\mu_\approx,\tilde\mu_\approx\} + O(W^3)\\
&= \tfrac{1}{2}\,\{(\bar\partial W_1 + \bar\partial W_2),\beta\} + \tfrac{1}{6}\,(\widehat{W}_1^2 + \widehat{W}_1\widehat{W}_2 + \widehat{W}_2^2)\,\bar\partial\gamma\\
&\qquad + \tfrac{1}{8}\,\{(\widehat{W}_1 + \widehat{W}_2)\beta,(\widehat{W}_1 + \widehat{W}_2)\beta\} + O(W^3)\\
&= -\tfrac{1}{8}\,\{\widehat{W}_{12}\beta,\widehat{W}_{12}\beta\} + \tfrac{1}{24}\,\widehat{W}_{12}^2\{\beta,\beta\}.
\end{aligned} \tag{5.44}$$

We used formulas (5.18) and (5.43) as well as the Jacobi identity for the Poisson bracket in order to derive the last line in this equation.

In order to solve Equation (5.44) for $\nu_\approx$, we express its r.h.s. in terms of a torsionless covariant $(1,0)$-differential on the tangent bundle $\mathrm{T}U$:

$$\nabla\colon \Gamma(\mathrm{T}U) \to \Omega^{1,0}(U,\mathrm{T}U).$$

We use index notations: let x^I, $I = 1,\ldots,\dim_{\mathbb{C}} U$ be local holomorphic coordinates on U. The corresponding frames in $\mathrm{T}U$ and $\mathrm{T}^\vee U$ are formed by ∂_I and dx^I. In our formulas, we assume summation over repeated indices appearing on opposite levels (this corresponds to applying contraction to tensor products). Anti-holomorphic indices are hidden. Thus in our notations

(5.45)
$$\begin{gathered}
\beta = \beta^{IJ}\,\partial_I\partial_J, \qquad \beta^{IJ} = \beta^{JI}, \qquad \nu = \nu^{IJ}\,\partial_I\wedge\partial_J, \qquad \nu^{IJ} = -\nu^{JI},\\
\{W,\beta\} = 2\beta^{IJ}\,(\partial_I W)\,\partial_J, \qquad \{\beta,\beta\} = -4\beta^{IL}\,(\nabla_L\beta^{JK})\,\partial_I\partial_J\partial_K,\\
[W,\nu] = 2\nu^{IJ}(\partial_J W)\,\partial_I.
\end{gathered}$$

A straightforward computation shows that if we lift the tilde in Equation (5.44) by applying ∂_{vrt} to both sides, then its r.h.s. can be rewritten as

$$[W_{12}, \nu_{\approx}] = \big(\beta^{JK}\beta^{IL}\,(\partial_J W_{12})\,(\nabla_K \partial_L W_{12}) - \tfrac{1}{3}\,\big(\beta^{IL}\,(\nabla_L \beta^{JK}) \\ -\ \beta^{JL}\,(\nabla_L \beta^{IK})\big)\,(\partial_J W_{12})(\partial_K W_{12})\big)\,\partial_I.$$

Comparing this expression with the last equation of (5.45) we find a formula for the leading term in ν:

$$\nu_{\approx} = \tfrac{1}{2}\ \big(\beta^{JK}\beta^{IL}\,(\nabla_K \partial_L W_{12}) - \tfrac{2}{3}\,\beta^{IL}\,(\nabla_L \beta^{JK})(\partial_K W_{12})\big)\,\partial_I \wedge \partial_J.$$

Finally, we solve Equation (5.40) for $\xi_{\approx}$. In order to compute its r.h.s., we introduce the Riemann curvature tensor $R \in \Omega^{0,1}\big(U, \mathrm{S}^2\mathrm{T}^\vee U \otimes \mathrm{T}U\big)$ of the connection ∇ by the formula $R = [\bar\partial, \nabla]$. In index notations

$$R = R^I_{JK}\,dx^J dx^K\,\partial_I, \qquad R^I_{JK} = R^I_{KJ}.$$

Then a straightforward computation shows that

$$\bar\partial\nu_{\approx} = \tfrac{1}{3}\,\big(\beta^{IL}\beta^{JM}R^K_{LM} + \beta^{JL}\beta^{KM}R^I_{LM} + \beta^{KL}\beta^{IM}R^J_{LM}\big)\,(\partial_K W_{12})\,\partial_I \wedge \partial_J.$$

Since in index notations

$$\xi = \xi^{IJK}\,\partial_I \wedge \partial_J \wedge \partial_K, \qquad [W, \xi] = 3\,\xi^{IJK}(\partial_K W)\,\partial_I \wedge \partial_J,$$

we find that

$$\xi_{\approx} = \tfrac{1}{3}\,\beta^{IL}\beta^{JM}R^K_{LM}\ \partial_I \wedge \partial_J \wedge \partial_K. \tag{5.46}$$

5.4.4. Semi-classical grading. The algebra $\mathcal{T}_{\boldsymbol{\nabla},12}$ of Equation (5.26) has an important *semi-classical* grading defined as follows:

$$\deg_{\mathrm{sc}} W_i = -2, \qquad \deg_{\mathrm{sc}} R = 0, \qquad \deg_{\mathrm{sc}} \varkappa_i = i - 3, \qquad \deg_{\mathrm{sc}} \nabla = 1. \tag{5.47}$$

The defining relation $R = [\bar\partial, \nabla]$ implies $\deg_{\mathrm{sc}} \bar\partial = -1$. If we set

$$\deg_{\mathrm{sc}} \lambda_{12} = -2, \tag{5.48}$$

then all three defining relations of our deformation construction:

$$\bar\partial\varkappa + \tfrac{1}{2}\,\{\varkappa, \varkappa\} = 0, \qquad \bar\partial W = \varkappa(\partial W), \qquad \bar\partial\lambda_{12} + \tfrac{1}{2}\,[\lambda_{12}, \lambda_{12}] = 0$$

respect the semi-classical grading. Hence the universal Maurer–Cartan element, determined recursively by Equation (5.32), must have the semi-classical degree -2: its zeroth term $\lambda_0 = W_{12}$ has degree -2 in virtue of $\deg_{\mathrm{sc}} W_{12} = -2$ and the degree of higher terms is expressed through Equation (5.32) in terms of the degrees of the lower terms.

Conjecture 5.1 together with relation (5.48) has three easy corollaries.

Consider a new grading on the algebra $\mathcal{T}_{\boldsymbol{\nabla},12}$:

$$(5.49)\quad \deg_{\partial W} \boldsymbol{\nabla}\varkappa_i = \deg_{\partial W} \boldsymbol{\nabla} R = 0, \qquad \deg_{\partial W} \nabla^k \partial W = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Corollary 5.2. *If $\beta = 0$, then $\deg_{\partial W} \lambda_i \geq 2$ for $i \geq 1$.*

Corollary 5.3. *If $\beta = 0$ and $W_1 = W_2 = 0$, then $\lambda = 0$.*

Let $\check{R} \in \mathrm{Ext}^1(\mathrm{S}^2\mathrm{T}U, \mathrm{T}U)$ denote the Atiyah class of the tangent bundle $\mathrm{T}U$ (it is the class represented by the curvature tensor R).

Corollary 5.4. *If $\check{R} = 0$ and $W_1 = W_2 = 0$, then $\lambda = 0$.*

Observe that among all generators $\boldsymbol{\nabla}\varkappa_i$, $\boldsymbol{\nabla} R$ and $\boldsymbol{\nabla}\partial W_i$ of $\mathcal{T}_{\boldsymbol{\nabla},12}$ only β and ∂W_i have negative semi-classical degrees: $\deg_{\mathrm{sc}} \beta = \deg_{\mathrm{sc}} \partial W_i = -1$. At the same time, $\deg_{\mathrm{sc}} \lambda = -2$, so if $\beta = 0$, then each term in the expression of λ_i, $i \geq 1$ must have at least two powers of ∂W. This proves Corollary 5.2. If $W_1 = W_2 = 0$, then, obviously, $\lambda_i = 0$ for $i \geq 1$ and, at the same time, $\lambda_0 = W_{12} = 0$. This proves Corollary 5.3.

In order to prove Corollary 5.4, we introduce two more gradings on the algebra $\mathcal{T}_{\boldsymbol{\nabla},12}$. The first grading reflects the difference between the total numbers of upper and lower indices in a tensor field:

$$\deg_{\mathrm{bal}} \nabla = -1, \quad \deg_{\mathrm{bal}} \varkappa_i = i - 1, \quad \deg_{\mathrm{bal}} \partial W_i = -1, \quad \deg_{\mathrm{bal}} R = -2.$$

Since the universal deformation parameter λ is relatively balanced, its degree must be zero: $\deg_{\mathrm{bal}} \lambda = 0$.

The second degree is the sum: $\deg_{\mathrm{tot}} = \deg_{\mathrm{sc}} + \deg_{\mathrm{bal}}$, so

$$\deg_{\mathrm{tot}} \nabla = 0, \quad \deg_{\mathrm{tot}} \varkappa_i = 2i - 4, \quad \deg_{\mathrm{tot}} \partial W_i = -2, \quad \deg_{\mathrm{tot}} R = -2.$$

Since $\deg_{\mathrm{sc}} \lambda = -2$ and $\deg_{\mathrm{bal}} \lambda = 0$, then $\deg_{\mathrm{tot}} \lambda = -2$. However, $\deg_{\mathrm{tot}} \varkappa_i \geq 0$ for $i \geq 2$, so if we set $W_1 = W_2 = 0$ and $R = 0$, then all remaining tensor fields in $\mathcal{T}_{\boldsymbol{\nabla},12}$ must have non-negative degree, so λ must be zero. This proves Corollary 5.4.

5.5. Deformation of the two-category of curved one-point fibrations: deformation of the composition of morphisms.

Describing the deformation of the composition functor requires the following steps: first, for a pair of objects W_1, W_2 we have to describe the filtered Dolbeault A_∞-algebra $\big(\Omega^{0,\bullet}(U), \bar{\partial}, \lambda_{12}\big)$ corresponding to the Maurer–Cartan element λ_{12} found in Section 5.4, by working out the expressions for its n-multiplications. Second, we have to describe the objects and morphisms of the two-periodic perfect category of $\big(\Omega^{0,\bullet}(U), \bar{\partial}, \lambda_{12}\big)$. After that we can define the composition between objects of $\mathrm{Hom}(W_1, W_2)$ and $\mathrm{Hom}(W_2, W_3)$.

5.5.1. A first-order deformation of the holomorphic symplectic structure. The procedure is simplified if we stay within the realm of categories $\mathsf{D}_{\mathbb{Z}_2}(U, W_2 - W_1)$ which are very similar to derived categories of coherent sheaves. We achieve this by considering only infinitesimal deformations of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$ by $\varkappa$. In other words, we introduce the algebra $\mathbb{C}[\epsilon]/(\epsilon^2)$ and consider a deformation of the holomorphic symplectic structure of $\mathrm{T}^\vee U$ by an element $\epsilon\varkappa$, where $\varkappa$ is of the form (5.16). The quadratic term in the Maurer–Cartan equation (5.12) drops out, hence $\varkappa$ must be holomorphic:

$$\bar{\partial}\varkappa = 0.$$

A function W describing an object of the deformed category has the form

$$W = W_{|0} + \epsilon W_{|1}.$$

It must satisfy Equation (5.23), which reduces to two relations

$$\bar{\partial} W_{|0} = 0, \qquad \bar{\partial} W_{|1} = \varkappa(\partial W_{|0}). \tag{5.50}$$

Let us introduce two shortcut notations related to functions W_i satisfying conditions (5.50):

(5.51)
$$\partial_{\mathrm{s}}\varkappa_{ij} := \partial_{\mathrm{s}}\varkappa(\partial W_{i|0}, \partial W_{j|0}), \qquad \partial^2_{\mathrm{s}}\varkappa_{ijk} := \partial^2_{\mathrm{s}}\varkappa(\partial W_{i|0}, \partial W_{j|0}, \partial W_{k|0}).$$

For two functions W_1, W_2 satisfying conditions (5.50) and thus defining objects of the deformed category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \epsilon\varkappa)$, the corresponding Maurer–Cartan element has the form

$$\lambda_{12} = W_{12|0} + \epsilon(W_{12|1} + \mu_{12}), \tag{5.52}$$

where, according to Equation (5.37),

$$\mu_{12} = -\partial_{\mathrm{s}}\varkappa_{12} = -\partial_{\mathrm{s}}\beta(\partial W_{1|0} + \partial W_{2|0}) + O(W^2).$$

This "pseudo-Beltrami" differential satisfies the equations

$$\bar\partial\mu_{12} = 0, \qquad \bar\partial W_{12|1} + \mu_{12} \llcorner \partial W_{12|0} = 0,$$

but generally does not satisfy the integrability condition $[\mu_{12}, \mu_{12}] = 0$.

5.5.2. Deformation of the two-periodic category of a curved complex manifold. According to Equation (5.52), the category of morphisms between two objects W_1, W_2 of the deformed two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \epsilon\varkappa)$ is

(5.53)
$$\mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\epsilon\varkappa)}(W_1, W_2) = \mathsf{D}_{\mathbb{Z}_2}\Big(U, W_{12|0} + \epsilon(W_{12|1} + \mu_{12})\Big) = \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12}),$$

where $U_{\epsilon\mu_{12}}$ denotes the complex manifold U whose complex structure is deformed by the Beltrami differential $\epsilon\mu_{12}$. Hence the category (5.53) is defined along the lines of Section 3.1. However, before we go into specifics, let us recall the definition of the curved Atiyah class.

Recall that a perfect object (3.5) of a two-periodic category $\mathsf{D}_{\mathbb{Z}_2}(U, W)$ defined in Section 3.1 is determined by a pair $\mathcal{E} = (E, \bar\nabla_E)$, where E is a curved quasi-holomorphic vector bundle over a complex manifold U, while $\bar\nabla_E$ is its curved differential satisfying properties (3.2)–(3.4). Let us endow E also with a (possibly curved) $(1, 0)$ covariant differential

$$\Omega^{i,\bullet}(E) \xrightarrow{\nabla_E} \Omega^{i+1,\bullet}(E), \qquad |\nabla_E| = \hat{0},$$

which satisfies the analog of Equation (3.3):

$$\nabla_E\,(\alpha \wedge \sigma) = (\partial\alpha) \wedge \sigma + (-1)^{\deg_{\mathbb{Z}_2}\alpha} \alpha \wedge (\nabla_E\,\sigma). \tag{5.54}$$

The choice of ∇_E is not unique, and the difference of two differentials ∇_E, ∇'_E satisfying (5.54) is a differential form

$$\nabla'_E = \nabla_E + a, \qquad a \in \Omega^{1,\bullet}(\mathrm{End}\,E). \tag{5.55}$$

In other words, all possible differentials ∇_E form an affine space based on a vector space $\Omega^{1,\hat{0}}(\mathrm{End}\,E)$. The commutator of $(1, 0)$ and $(0, \hat{1})$ differentials is the $(1,\hat{1})$-curvature

$$[\bar{\nabla}_E, \nabla_E] = F_E \in \Omega^{1,\bullet}(\operatorname{End} E),$$

which satisfies the curved Bianchi identity

$$\bar{\nabla}_E F_E = -(\partial W)\, 1_E. \tag{5.56}$$

The curvature F_E is determined by the object $\mathcal{E}$ (that is, by $\bar{\nabla}_E$) up to a $\bar{\nabla}_E$-exact element: if we replace ∇_E by ∇'_E of Equation (5.55), then F_E is replaced by $F'_E = F_E + \bar{\nabla}_E a$. Hence $\mathcal{E}$ determines the *curved Atiyah class*

$$\check{F}_E \in \Omega^{1,\bullet}(\operatorname{End} E)\Big/\bar{\nabla}_E\left(\Omega^{1,\bullet}(\operatorname{End} E)\right).$$

If $W = 0$, then the Bianchi identity (5.56) implies that

$$\check{F}_E \in \mathrm{H}_{\bar{\nabla}_E}(\operatorname{End} E) = \operatorname{Ext}(\mathcal{E}, \mathcal{E}).$$

A perfect object of the deformed category $\mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12})$ is a pair

$$\mathcal{E} = (E, \bar{\nabla}_E), \qquad \bar{\nabla}_E = \bar{\nabla}_{E|0} + \epsilon\bar{\nabla}_{E|1},$$

(cf. Equation (3.7)), such that the pair $\mathcal{E}_{|0} = (E, \bar{\nabla}_{E|0})$ is an object of the undeformed category $\mathsf{D}_{\mathbb{Z}_2}(U, W_{12})$, while

$$\bar{\nabla}_{E|1} = \mu_{12} \llcorner \nabla_E + b, \qquad b \in \Omega^{0,\bullet}(\operatorname{End} E) \tag{5.57}$$

and b satisfies the condition

$$\bar{\nabla}_{E|0}\, b = W_{12|1}\, 1_E - \mu_{12} \llcorner F_E. \tag{5.58}$$

Here F_E is the $(1,1)$-curvature of $\mathcal{E}_{|0}$, so $\bar{\nabla}_E$ satisfies condition (3.4): $\bar{\nabla}_E^2 = W_{12} 1_E$. A change (5.55) in the choice of ∇_E is compensated by the corresponding replacement of b by $b' = b - \mu_{12} \llcorner a$.

The space of morphisms $\operatorname{Hom}_{\mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12})}(\mathcal{E}_1, \mathcal{E}_2)$ between two perfect objects is defined by means of an obvious deformation of the general formula (2.6). A morphism between two objects $\sigma \in \operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is represented by a $\bar{\nabla}_E$-closed sum

$$\begin{aligned} \sigma = \sigma_{|0} + \epsilon\sigma_{|1}, \qquad & \sigma_{|0}, \sigma_{|1} \in \Omega^{0,\bullet}(E_2 \otimes E_1^*), \\ \bar{\nabla}_{E|0}\,\sigma_{|0} = 0, \qquad & \bar{\nabla}_{E|0}\,\sigma_{|1} = -\bar{\nabla}_{E|1}\,\sigma_{|0} \end{aligned} \tag{5.59}$$

up to $\bar{\nabla}_E$-exact elements. Note that the dominant component $\sigma_{|0}$ defines a morphism between the undeformed objects $\mathcal{E}_{1|0}$ and $\mathcal{E}_{2|0}$.

5.5.3. Deformation of the composition of morphisms. The composition of morphisms between three objects W_1, W_2 and W_3 of the deformed two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \epsilon\varkappa)$ is a bi-functor

$$\mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12}) \times \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{23}}, W_{23}) \longrightarrow \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{13}}, W_{13}). \tag{5.60}$$

The composition of two morphisms $\mathcal{E}_{12} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12})$ and $\mathcal{E}_{23} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{23}}, W_{23})$ is the appropriately deformed tensor product:

$$\mathcal{E}_{23} \circ \mathcal{E}_{12} = (E_{23}, \bar{\nabla}_{E_{23}}) \circ (E_{12}, \bar{\nabla}_{E_{12}}) = (E_{23} \otimes E_{12}, \bar{\nabla}_{E_{12}\otimes E_{23}} + \epsilon\zeta_{12,23}), \tag{5.61}$$

where the deformation term is

$$\begin{aligned} \zeta_{12,23} &= \partial_{\mathrm{s}}^2 \varkappa_{123} \llcorner \big((\partial W_{12|0})\nabla_{E_{23}} - (\partial W_{23|0})\nabla_{E_{12}} + F_{E_{12}} F_{E_{23}}\big) \\ &= \beta \llcorner (F_{E_{12}} F_{E_{23}}) + O(W), \end{aligned} \tag{5.62}$$

and $\partial_{\mathrm{s}}^2 \varkappa_{123}$ is a shortcut notation defined by Equation (5.51).

The first two terms in the r.h.s. of this equation are related to the fact that each of three categories in (5.60) has its own deforming "pseudo-Beltrami" differential. Hence we had to add correction terms to $\bar{\nabla}_{E_{13}}$ so that it would satisfy the condition (5.54) or, more precisely, condition (5.57), that is, that the difference

$$\bar{\nabla}_{E_{13}} - \epsilon\mu_{13} \llcorner \nabla_{E_{12}\otimes E_{23}}$$

must be just an odd element of $\Omega^{0,\bullet}(\operatorname{End} E_{13})$. The third correction term in Equation (5.62) is required to comply with condition (5.58).

The composition of morphisms $\mathcal{E}_{23} \circ \mathcal{E}_{12}$ defined by Equation (5.61) is independent (up to an isomorphism) of the choice of $(1, 0)$ differentials

$\nabla_{E_{12}}$ and $\nabla_{E_{23}}$. Indeed, if we replace $\nabla_{E_{12}}$ with $\nabla'_{E_{12}} = \nabla_{E_{12}} + a$ as in Equation (5.55), then $\zeta_{12,23}$ is replaced by

$$\begin{aligned}\zeta'_{12,23} &= \zeta_{12,23} - \partial_{\mathrm{s}}^2 \varkappa_{123} \llcorner \left((\partial W_{23|0})a - (\bar{\nabla}_{E_{12}|0}\, a) F_{E_{23}}\right)\\ &= \zeta_{12,23} + \bar{\nabla}_{E_{12}|0}\, \left(\partial_{\mathrm{s}}^2 \varkappa_{123} \llcorner (a\, F_{E_{23}})\right),\end{aligned}$$

so it changes by a $\bar{\nabla}_{E_{13}|0}$-exact term.

The bi-functorial nature of the map (5.60) means that an object $\mathcal{E}_{23} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{23}}, W_{23})$ determines a functor

$$\mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12}) \xrightarrow{\Phi[\mathcal{E}_{23}]} \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{13}}, W_{13})\ .$$

Its action on objects is defined by composition (5.61). Consider now its action on morphisms. For $\sigma_{12} \in \mathrm{Hom}(\mathcal{E}_{12}, \mathcal{E}'_{12})$, where $\mathcal{E}_{12}, \mathcal{E}'_{12} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{12}}, W_{12})$, we define

$$\Phi\,[\mathcal{E}_{23}]\,(\sigma_{12}) = \sigma_{12} \otimes 1_{23} + \epsilon\,(\partial_{\mathrm{s}}^2 \varkappa_{123}) \llcorner \left(\nabla_{E_{12}} \sigma_{12\,|\,0} \otimes F_{E_{23}}\right).$$

The correction term is required to satisfy relation (5.59). Similarly, an object $\mathcal{E}_{12} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\mu_{12}}, W_{12|\epsilon})$ determines a functor

$$\mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{23}}, W_{23}) \xrightarrow{\Phi[\mathcal{E}_{12}]} \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{13}}, W_{13})\ ,$$

which maps a morphism $\sigma_{23} \in \mathrm{Hom}(\mathcal{E}_{23}, \mathcal{E}'_{23})$, where $\mathcal{E}_{23}, \mathcal{E}'_{23} \in \mathsf{D}_{\mathbb{Z}_2}(U_{\epsilon\mu_{23}}, W_{23})$ into a morphism

$$\Phi\,[\mathcal{E}_{12}]\,(\sigma_{23}) = 1_{12} \otimes \sigma_{23} - \epsilon\,(\partial_{\mathrm{s}}^2 \varkappa_{123}) \llcorner \left(F_{E_{12}} \otimes \nabla_{E_{23}} \sigma_{23\,|\,0}\right).$$

The images of morphisms $\sigma_{12\,|\,\epsilon}$ and $\sigma_{23\,|\,\epsilon}$ commute in the following sense:

$$\Phi\left[\mathcal{E}'_{12}\right](\sigma_{23}) \circ \Phi\left[\mathcal{E}_{23}\right](\sigma_{12}) - \Phi\left[\mathcal{E}'_{23}\right](\sigma_{12}) \circ \Phi\left[\mathcal{E}_{12}\right](\sigma_{23}) = 0$$

in $\mathrm{Hom}(\mathcal{E}_{23} \circ \mathcal{E}_{12}, \mathcal{E}'_{23} \circ \mathcal{E}'_{12})$, because

$$\begin{aligned}&\Phi\left[\mathcal{E}'_{12}\right](\sigma_{23}) \circ \Phi\left[\mathcal{E}_{23}\right](\sigma_{12}) - \Phi\left[\mathcal{E}'_{23}\right](\sigma_{12}) \circ \Phi\left[\mathcal{E}_{12}\right](\sigma_{23})\\ &\quad = \epsilon\, \bar{\nabla}_{E_{13}|0} \left((\partial_{\mathrm{s}}^2 \varkappa_{123}) \llcorner \left(\nabla_{E_{12}} \sigma_{12\,|\,0} \otimes \nabla_{E_{23}} \sigma_{23\,|\,0}\right)\right).\end{aligned}$$

In the special case $W_1 = W_2 = W_3 = 0$, formula (5.62) says the following. Let 0 denote the object of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \epsilon\varkappa)$ corresponding to the trivial fibration over U with $W = 0$. The endomorphism category $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\epsilon\varkappa)}(0)$ is a monoidal category which is equivalent to $\mathsf{D}_{\mathbb{Z}_2}(U)$ as a category, but with a monoidal structure given by the deformed tensor product

$$(E, \bar{\nabla}_E) \circ (E', \bar{\nabla}_{E'}) = \big(E \otimes E', \bar{\nabla}_{E\otimes E'} + \epsilon\beta \llcorner (F_E F_{E'})\big), \tag{5.63}$$

5.5.4. Deformation of the monoidal structure beyond the first order. Now we return to the two-category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U, \varkappa)$ for a general Maurer–Cartan element $\varkappa$ and consider the category $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U)$ of endomorphisms of the one-point fibration with $W = 0$ denoted here as 0_U. This category has a monoidal structure corresponding to the composition of endomorphisms. According to the general formula (5.30), the endomorphism category itself is a deformation of the category $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$: $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U) = \mathsf{D}_{\mathbb{Z}_2}(U, \lambda)$, where $\lambda = \lambda_3 + \lambda_4 + \cdots$ and $\lambda_i \in \Omega^{0,i}\left(U, \wedge^i \mathrm{T}U\right)$, while its monoidal structure is a deformation of the monoidal structure of $\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U)$, the latter being the tensor product (3.11).

Let us assume that the Atiyah class $\breve{R}$ of the tangent bundle $\mathrm{T}U$ is zero. Then, according to Corollary 5.4, the deformation parameter is zero: $\lambda = 0$, so we have an equivalence of categories

$$\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U) \simeq \mathsf{D}_{\mathbb{Z}_2}(U). \tag{5.64}$$

However, as the study of the first-order perturbation in Section 5.5.3 demonstrated, the monoidal structure of $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U)$ is still a non-trivial deformation of the tensor product monoidal structure of $\mathsf{D}_{\mathbb{Z}_2}(U)$. The relatively simple nature of category (5.64) allows us to discuss the properties of this deformation without invoking A_∞-algebras and their modules.

A deformation of the monoidal structure of the category $\mathsf{D}_{\mathbb{Z}_2}(U)$ is described by two sets of data. First, for every pair of quasi-holomorphic vector bundles $(E_1, \bar{\nabla}_{E_1})$, $(E_2, \bar{\nabla}_{E_2})$ there is a Maurer–Cartan element $\zeta_{12} \in \Omega^{0,\bullet}\left(\mathrm{End}(E_1 \otimes E_2)\right)$,

$$\bar{\partial}\zeta_{12} + \tfrac{1}{2}\left[\zeta_{12}, \zeta_{12}\right] = 0, \tag{5.65}$$

which determines the deformed monoidal bifunctor of the composition within the endomorphism category $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U)$:

$$(E_1, \bar{\nabla}_{E_1}) \circ (E_2, \bar{\nabla}_{E_2}) = (E_1 \otimes E_2, \bar{\nabla}_{E_1\otimes E_2} + \zeta_{12}). \tag{5.66}$$

Second, for every triple of quasi-holomorphic vector bundles $(E_1, \bar{\nabla}_{E_1})$, $(E_2, \bar{\nabla}_{E_2})$, $(E_3, \bar{\nabla}_{E_3})$ there is an associator $\alpha_{123} \in \Omega^{0,\bullet}(\mathrm{End}(E_1 \otimes E_2 \otimes E_3))$ which establishes the associativity isomorphism

$$\alpha_{123} \colon \big((E_1, \bar{\nabla}_{E_1}) \circ (E_2, \bar{\nabla}_{E_2})\big) \circ (E_3, \bar{\nabla}_{E_3}) \xrightarrow{\cong} (E_1, \bar{\nabla}_{E_1}) \circ \big((E_2, \bar{\nabla}_{E_2}) \circ (E_3, \bar{\nabla}_{E_3})\big).$$

If both sides of the associativity isomorphism have the presentation

$$\begin{aligned} \big((E_1, \bar{\nabla}_{E_1}) \circ (E_2, \bar{\nabla}_{E_2})\big) \circ (E_3, \bar{\nabla}_{E_3}) &= (E_1 \otimes E_2 \otimes E_3, \bar{\nabla}_{123} + \zeta_{123}), \\ (E_1, \bar{\nabla}_{E_1}) \circ \big((E_2, \bar{\nabla}_{E_2}) \circ (E_3, \bar{\nabla}_{E_3})\big) &= (E_1 \otimes E_2 \otimes E_3, \bar{\nabla}_{123} + \zeta'_{123}), \end{aligned} \tag{5.67}$$

where $\bar{\nabla}_{123} := \bar{\nabla}_{E_1 \otimes E_2 \otimes E_3}$, then α_{123} is an invertible element satisfying the equation

$$\bar{\nabla}_{123} \alpha_{123} + \zeta'_{123}\, \alpha_{123} - \alpha_{123}\, \zeta_{123} = 0. \tag{5.68}$$

We conjecture that there exist unique universal formulas for the element ζ_{12} and for the associator α_{123} related to the deformation of the tensor product monoidal structure of $\mathsf{D}_{\mathbb{Z}_2}(U)$ into the monoidal structure of the endomorphism category $\mathrm{End}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U,\varkappa)}(0_U)$. These universal formulas express ζ_{12} and α_{123} in terms of the deformation parameter $\varkappa$, $(1,1)$-curvatures F_{E_i} and their holomorphic covariant derivatives:

(5.69)

$$\zeta_{12} \in \mathcal{T}_{\boldsymbol{\nabla}}[F_{E_1}, F_{E_2}, \varkappa_2, \varkappa_3, \ldots], \qquad \alpha_{123} \in \mathcal{T}_{\boldsymbol{\nabla}}[F_{E_1}, F_{E_2}, F_{E_3}, \varkappa_2, \varkappa_3, \ldots].$$

We propose to derive the universal formulas perturbatively. Define the Dolbeault degree by Equation (5.31) and by the additional formula $\deg_{\mathrm{Dlb}} F_{E_i} = 1$ (note that generally $F_{E_i} \in \Omega^{0,\bullet}(\mathrm{End}\, E_i)$ and its Dolbeault degree coincides with j of $\Omega^{0,j}$ only when E_i is a holomorphic vector bundle with the operator $\bar{\nabla}_{E_i}$ not containing forms of degree other than 1). We present the deformation parameter ζ_{12} and the associator α_{123} as the sums

$$\zeta_{12} = \sum_{i=1}^{\infty} \zeta_{12,i}, \qquad \alpha_{123} = 1_{E_1 \otimes E_2 \otimes E_3} + \sum_{i=2}^{\infty} \alpha_{123,i},$$

where

$$\deg_{\mathrm{Dlb}} \zeta_{12,i} = 2i + 1, \qquad \deg_{\mathrm{Dlb}} \alpha_{123,i} = 2i$$

(the reason for assuming that the Dolbeault degree of ζ_{12} is even and the Dolbeault degree of α_{123} is odd will become clear shortly). The Maurer–Cartan equation (5.65) splits

$$\bar{\partial}\zeta_{12,n} + \sum_{i=1}^{n-1} \zeta_{12,i}\, \zeta_{12,n-i} = 0, \tag{5.70}$$

and the associativity equation (5.68) splits

$$\bar{\partial}\alpha_{123,n} + \zeta'_{123,n} - \zeta_{123,n} + \sum_{i=2}^{n-1}(\zeta'_{123,n-i}\, \alpha_{123,i} - \alpha_{123,i}\, \zeta_{123,n-i}) = 0. \tag{5.71}$$

The action of the Dolbeault differential $\bar{\partial}$ on the elements of the algebras (5.69) follows from its action on the elementary tensor fields prescribed by the Maurer–Cartan equation (5.12) and by the Bianchi identity $\bar{\nabla}_{E_i} F_{E_i} = 0$,[5] and from the defining equation of the curvature tensor $F_{E_i} = [\bar{\nabla}_{E_i}, \nabla_{E_i}]$.

We introduce the notation $\zeta[-,-]$ to emphasize the dependence of the universal deformation parameter ζ_{12} on curvatures: $\zeta_{12} = \zeta[F_{E_1}, F_{E_2}]$. The parameters ζ_{123} and ζ'_{123} of Equation (5.67) can be expressed in terms of $\zeta[-,-]$:

$$\begin{aligned}
\zeta_{123} &= \zeta[F_{E_1}, F_{E_2}] + \zeta[F_{E_1} + F_{E_2} + \nabla_{E_1\otimes E_2}\zeta[F_{E_1}, F_{E_2}], F_{E_3}],\\
\zeta'_{123} &= \zeta[F_{E_2}, F_{E_3}] + \zeta[F_{E_1}, F_{E_2} + F_{E_3} + \nabla_{E_1\otimes E_2}\zeta[F_{E_2}, F_{E_3}]].
\end{aligned}$$

These formulas allow us to present the difference $\zeta'_{123,n} - \zeta_{123,n}$ appearing in Equation (5.71) in the form

$$\zeta'_{123,n} - \zeta_{123,n} = \delta_{123}\zeta_n + \tilde{\zeta}_{123,n}, \tag{5.72}$$

where

$$\zeta_{123,n} = \zeta_n[F_{E_2}, F_{E_3}] + \zeta_n[F_{E_1}, F_{E_2} + F_{E_3}] - \zeta_n[F_{E_1}, F_{E_2}] - \zeta_n[F_{E_1} + F_{E_2}, F_{E_3}]$$

and the expression $\tilde{\zeta}_{123,n}$ contains the deformation parameter components ζ_i only with $i < n$.

[5] We assume for simplicity that the curvatures of the ∂-connections ∇_{E_i} are zero.

After the substitution (5.72), the associativity equation (5.71) becomes

$$(5.73)\quad \bar{\partial}\alpha_{123,n} + \delta_{123}\zeta_n + \tilde{\zeta}_{123,n} + \sum_{i=2}^{n-1}(\zeta'_{123,n-i}\,\alpha_{123,i} - \alpha_{123,i}\,\zeta_{123,n-i}) = 0.$$

We conjecture that the Maurer–Cartan equation (5.70) together with the associativity equation (5.73) can be solved perturbatively over the Dolbeault degree, thus producing the unique universal solutions ζ_{12} and α_{123} if, following Equation (5.63), we set

$$(5.74)\qquad \zeta_{12,1} = \beta \llcorner (F_{E_1}F_{E_2}) = \beta^{IJ}F_{E_1,I}F_{E_2,J},$$

where we used the index notations explained at the end of Section 5.4.3, as well as the notation $F_E = F_{E,I}\,dx^I$ for the $(1,1)$-curvature tensor components. The parity of Dolbeault degrees of $\zeta_{12,i}$ and $\alpha_{123,i}$ is dictated by these equations.

It is easy to verify that expression (5.74) satisfies Equations (5.70) and (5.73) for $n = 1$. We leave it for the reader to verify that the following expressions

$$(5.75)\qquad \begin{aligned}\zeta_{12,2} &= \tfrac{1}{3}\,\beta^{JL}(\nabla_L\beta^{IK})F_{E_1,I}(F_{E_1,J} - F_{E_2,J})F_{E_3,K}\\ &\quad + \tfrac{1}{2}\,\beta^{IJ}\beta^{KL}\,((\nabla_I F_{E_1,K})\,F_{E_2,J}F_{E_2,L} + (\nabla_I F_{E_2,K})\,F_{E_1,J}F_{E_1,L}),\end{aligned}$$

$$(5.76)\qquad \alpha_{123,2} = \tfrac{2}{3}\,\gamma^{IJK}F_{E_1,I}F_{E_2,J}F_{E_3,K}.$$

satisfy these equations for $n = 2$.

We extend the semi-classical grading of Section 5.4.4 to the algebras (5.69) by setting $\deg_{\mathrm{sc}} F_{E_i} = 0$. Solving the system of equations (5.70) and (5.71) recursively over n with the initial condition (5.74) determines the semi-classical degrees of the deformation parameter ζ_{12} and of the associator α_{123}:

$$\deg_{\mathrm{sc}}\zeta_{12} = -1, \qquad \deg_{\mathrm{sc}}\alpha_{123} = 0.$$

Let us consider what happens if we set $\beta = 0$ in the universal formulas for ζ_{12} and α_{123}. Since β is the only generator of the algebras (5.69) with negative semi-classical grading, then $\deg_{\mathrm{sc}}\zeta_{12} = -1$ implies

$$\zeta_{12}|_{\beta=0} = 0,$$

that is, the composition part (5.66) of the monoidal structure remains undeformed. However, Equation (5.76) indicates that if $\gamma \neq 0$, then

$\alpha_{123}|_{\beta=0} \neq \mathbb{1}$. This means that if for a complex manifold U there exists a non-trivial Maurer–Cartan element $\varkappa \in \Omega^{0,\bullet}(U, \mathrm{S}^\bullet \mathrm{T}U)$ such that $\beta \equiv \varkappa_2 = 0$ while $\gamma \equiv \varkappa_3 \neq 0$, then the tensor product monoidal structure of the category $\mathsf{D}_{\mathbb{Z}_2}(U)$ has a non-trivial associator $\alpha_{123} \neq \mathbb{1}$ in addition to the standard one. This situation is realized, for example, when U is a holomorphic symplectic manifold X [4]. The element $\varkappa$ in this case describes the formal neighborhood of the diagonal in $X \times X$. It follows that the $\mathbb{Z}_2$-graded derived category of any holomorphic symplectic manifold admits a non-trivial monoidal structure with a deformed associator. This provides an underlying reason for the results of Roberts and Willerton [11].

If $\beta = 0$, then all remaining generators of the algebras (5.69) have non-negative semi-classical degrees. Among them, only F_{E_i} and γ have zero degrees, and all others, including holomorphic derivatives, have positive degrees. Since $\deg_{\mathrm{sc}} \alpha_{123} = 0$, this means that α_{123} belongs to the algebra generated by γ and F_{E_i}:

$$\alpha_{123}|_{\beta=0} \in \mathcal{T}[\gamma, F_{E_1}, F_{E_2}, F_{E_3}].$$

In fact, we conjecture that if $\beta = 0$, then the associator is a pure exponential:

$$\alpha_{123}|_{\beta=0} = \exp\left(\tfrac{2}{3}\,\gamma^{IJK} F_{E_1,I} F_{E_2,J} F_{E_3,K}\right).$$

5.6. A geometric description of the two-category $\ddot{\mathsf{L}}(X, \omega)$

Following the outline of Section 5.1, we apply the results of the previous subsection to formulate conjectures about a geometric description of the category $\ddot{\mathsf{L}}(X, \omega)$, where (X, ω) is a general holomorphic symplectic manifold. Our goal is to explain how the statements of Section 4.1 referring to the case of $X = \mathrm{T}^\vee U$, should be modified for a general (X, ω).

The pairs (Y, L_Y), where $Y \subset X$ is a lagrangian submanifold and $L_Y \to Y$ is a line bundle such that $L_Y^{\otimes 2} = K_Y$, are still objects of the two-category $\ddot{\mathsf{L}}(X, \omega)$. We conjecture that the analogs of holomorphic fibration objects $(\mathcal{Y}, L_{\mathcal{Y}})$ also appear, but this time $\mathcal{Y} \to Y$ is not a holomorphic fibration, as in the case of $X = \mathrm{T}^\vee U$, but rather a special “non-holomorphic” deformation of a holomorphic fibration. The reason for this deformation is similar to the non-holomorphicity of the functions W which solve Equation (5.23), but we will not explore this subject further.

Suppose that two lagrangian submanifolds $Y_1, Y_2 \subset X$ have a clean intersection. We conjecture that the category of morphisms between them is the deformed and shifted two-periodic category of their intersection:

$$\begin{aligned}(5.77)\qquad &\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}\left((Y_1, L_{Y_1}), (Y_2, L_{Y_2})\right)\\ &\quad = \mathsf{D}_{\mathbb{Z}_2}(Y_{12}; \lambda_{\cap,12})[L_{12}]_{\mathrm{tw}}[\tfrac{1}{2}\dim X - \dim Y_{12} - 1]_2,\end{aligned}$$

where $Y_{12} := Y_1 \cap Y_2$, the line bundle $L_{12} \to Y_{12}$ is defined by Equation (4.5) adapted to the case of one-point fibrations:

$$L_{12} := L_{Y_1}|_{Y_{12}} \otimes L_{Y_2}|_{Y_{12}} \otimes K^{-1}_{Y_{12}}$$

and $\lambda_{\cap,12} \in \Omega^{0,\bullet}(\bigwedge^{\bullet}\mathrm{T}Y_{12})$ is a special Maurer–Cartan element which determines the A_∞-deformation of the category $\mathsf{D}_{\mathbb{Z}_2}(Y_{12})$.

Based on the results of the previous subsection, we make the following conjectures about $\lambda_{\cap,12}$:

(1) The Maurer–Cartan element $\lambda_{\cap,12}$ is relatively balanced and $\deg \lambda_{\cap,12} \geq 2$:

$$\lambda_{\cap,12} = \sum_{i=2}^{\infty} \lambda_{\cap,12,i}, \qquad \lambda_{\cap,12,i} \in \Omega^{0,i}(U, \wedge^i Y_{12}).$$

(2) If at least one of the classes $\tilde{\beta}_{Y_1}$, $\tilde{\beta}_{Y_2}$ determined by the exact sequences (5.19) is zero and the other lagrangian submanifold has a presentation of Section 5.4.1 as the graph of a differential ∂W, where W is a function on the first lagrangian surface, then $\lambda_{12} = 0$.

(3) If $Y_1 = Y_2 = Y$, then $\lambda_{\cap,12,2} = 0$ and $\lambda_{\cap,12,3}$ is given by formula (5.46), where β represents $\tilde{\beta}_Y$ and R is the curvature of the tangent bundle $\mathrm{T}Y$.

(4) If $Y_1 = Y_2 = Y$ and the Atiyah class $\check{R}$ of the tangent bundle $\mathrm{T}Y$ is zero, then $\lambda_{\cap,12} = 0$.

In order to derive these conjectures from Equation (5.32), we consider a tubular neighborhood of Y_1 (or Y_2) as a tubular neighborhood of the zero section of a deformed cotangent bundle $(\mathrm{T}^\vee Y_1)_\varkappa$ with an appropriate deformation parameter $\varkappa$. Then the object Y_1 corresponds to the zero section and hence it is represented by the holomorphic function $W_1 = 0$. We assume that within $(\mathrm{T}^\vee Y_1)_\varkappa$ the second object Y_2 is of the form Y_W for an appropriate function W on Y_1. Generally, this is not true, but we expect that conjecture 1 holds true independently of whether such a presentation exists, while Conjectures 3 and 4 correspond to the case $W = 0$. Finally, we assume that the line bundle L_{Y_2} is the pull-back of the deformation of L_{Y_1} under the projection of Y_2 onto the zero-section of $\mathrm{T}^\vee Y_1$ (recall that the complex structure of the projection of Y_2 onto the base Y_1 of the cotangent bundle has a complex structure corresponding to the Beltrami differential (5.25), hence the bundle L_{Y_1} has to be deformed in order to be holomorphic with respect to it). Under these assumptions

(5.78)
$$\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}\left((Y_1, L_{Y_1}), (Y_2, L_{Y_2})\right) = \mathrm{Hom}_{\ddot{\mathsf{D}}_{\mathbb{Z}_2}(Y_1,\varkappa)}(0, W) = \mathsf{D}_{\mathbb{Z}_2}(Y_1; \lambda_{12})$$

(cf. Equation (5.30)), where λ_{12} is the deformation parameter determined by Equation (5.32), in which we set $W_1 = 0$, $W_2 = W$ and, consequently, $W_{12} = W$. Hence λ_{12} has the expansion (5.27): $\lambda_{12} = \sum_{i=0}^{\infty} \lambda_{12,i}$, $\lambda_{12,i} \in \Omega^{0,i}(U, \wedge^i \mathrm{T}Y_1)$. Here $\lambda_{12,0} = W$, while all other terms $\lambda_{12,i}$ depend on W by being polynomials in ∂W and its covariant holomorphic differentials $\nabla^k \partial W$, $k \geq 1$.

Since we assumed that Y_1 and Y_2 have a clean intersection, it follows that W has a clean critical locus $\mathrm{Crit}(W)$ which is isomorphic to the intersection Y_{12}. We conjecture that the category $\mathsf{D}_{\mathbb{Z}_2}(Y_1; \lambda_{12})$ localizes to $\mathrm{Crit}(W)$:

$$\mathsf{D}_{\mathbb{Z}_2}(Y_1; \lambda_{12}) = \mathsf{D}_{\mathbb{Z}_2}(Y_{12}; \lambda_{\cap,12})[L_{12}]_{\mathrm{tw}}[\tfrac{1}{2}\dim X - \dim Y_{12} - 1]_2,$$

and the deformation parameter $\lambda_{\cap,12}$ is determined somehow by the restriction $\lambda_{12}|_{\mathrm{Crit}(W)}$. We do not understand this relation precisely, but we can still make conjectures about $\lambda_{\cap,12}$ based on the properties of $\lambda_{12}|_{\mathrm{Crit}(W)}$.

Consider the degree $\deg_{\partial W}$ defined by Equation (5.49). $\partial W = 0$, hence if $\deg_{\partial W} \lambda_{12,i} \geq 1$, then $\lambda_{12,i}|_{\mathrm{Crit}(W)} = 0$. Then explicit formula (5.37) for $\lambda_{12,1} = \mu$ implies that $\lambda_1|_{\mathrm{Crit}(W)} = 0$. Since W is locally constant at $\mathrm{Crit}(W)$, we may also assume that $\lambda_{12,0} = 0$. Thus our first conjecture is that $\lambda_{\cap,12,0} = \lambda_{\cap,12,1} = 0$. We also conjecture that $\lambda_{\cap,12}$ is relatively balanced, because the same is true for λ_{12}.

Formula (5.20) states that $\check{\beta} = \tilde{\beta}_{Y_1}$, so if $\tilde{\beta}_{Y_1} = 0$, then $\check{\beta} = 0$ and we can use the gauge transformation of the Maurer–Cartan element $\varkappa$ in order to set $\beta = 0$. Now Corollary 5.2 says that $\deg_{\partial W} \lambda_i \geq 2$ for $i \geq 2$, so $\lambda_{12,i}|_{\mathrm{Crit}(W)} = 0$ for all i. Hence we conjecture that if $\tilde{\beta}_{Y_1} = 0$ and Y_2 has a presentation as the graph of ∂W, then $\lambda_{\cap,12} = 0$.

If $\beta = 0$ and Y_2 is presented as the graph of ∂W for a function W satisfying Equation (5.23), then the normal bundle $Q_1 := \mathrm{T}Y_1|_{Y_{12}}/\mathrm{T}(Y_{12})$ appearing in Equation (4.7) admits an $O(n, \mathbb{C})$ structure. Indeed, $Y_{12} = \mathrm{Crit}(W)$, so $\partial W|_{Y_{12}} = 0$ and there is a well-defined Hessian $\partial^2 W \in \Gamma(\mathrm{S}^2 Q_1^\vee)$. This

Hessian is non-degenerate, because we assumed that Y_1 and Y_2 have a clean intersection. Equation (5.23) implies that generally it satisfies the equation

$$\bar{\partial}(\partial^2 W) = \beta \llcorner \big((\partial^2 W)\,(\partial^2 W)\big),$$

but since we assumed that $\beta = 0$ we find that the Hessian is holomorphic: $\bar{\partial}(\partial^2 W) = 0$. The holomorphic non-degenerate Hessian provides the $O(n,\mathbb{C})$ structure for the bundle Q_1. In fact, we suspect that the converse is also true: if $\beta = 0$ and the bundle Q_1 has an $O(n,\mathbb{C})$ structure then the lagrangian submanifold Y_2 has a presentation as the graph of ∂W at least in a tubular neighborhood of $Y_1 \cap Y_2$.

If $Y_1 = Y_2 = Y$, then $Y_{12} = Y$ and $W = 0$, so Equation (5.78) says that $\lambda_{\cap,12} = \lambda_{12}$. Hence $\deg \lambda_{\cap,12} \geq 3$ and $\lambda_{\cap,12,3}$ is given by formula (5.46). Also, if $\tilde{\beta}_{Y_1} = 0$, then $\lambda_{\cap,12} = 0$ follows directly from Corollary 5.2 without any further conjectures regarding the localization properties of the deformed category $\mathsf{D}_{\mathbb{Z}_2}(Y_1; \lambda_{12})$.

Finally, if $Y_1 = Y_2 = Y$ and the Atiyah class $\check{R}$ of $\mathrm{T}Y$ is zero, then Corollary 5.4 says that $\lambda_{12} = 0$. Since in this case $\lambda_{\cap,12} = \lambda_{12}$, then $\lambda_{\cap,12} = 0$.

We cannot say much about the deformation of composition (4.10) except that when all $\mathcal{Y}_i$ are one-point fibrations with the same base $Y = Y_1 = Y_2 = Y_3$, and the Atiyah class $\check{R}$ of $\mathrm{T}Y$ is zero, then the deformation of the composition rule (4.12) is described by the formulas of subsection 5.5.4 in which we replace U with Y.

6. Micro-local definition of the two-category $\ddot{\mathsf{L}}(X, \omega)$

6.1. Symplectic rectangles

Let $U_{\mathbf{x}}$ denote an n-dimensional Stein complex manifold U equipped with holomorphic coordinate functions $\mathbf{x} = x_1, \ldots x_n$. The functions $\mathbf{x}$ determine an embedding $U_{\mathbf{x}} \hookrightarrow \mathbb{C}^n_{\mathbf{x}}$, where $\mathbb{C}^n_{\mathbf{x}}$ is the affine space $\mathbb{C}^n$ equipped with the standard coordinates $\mathbf{x}$. In other words, $U_{\mathbf{x}}$ is just an open subspace of $\mathbb{C}^n_{\mathbf{x}}$ with inherited coordinates.

A *symplectic rectangle* is a product $U_{\mathbf{x}} \times V_{\mathbf{y}}$ with the holomorphic symplectic structure determined by the two-form $\omega = \sum_{i=1}^n dy_i \wedge dx_i$. The identity map establishes an isomorphism between $U_{\mathbf{x}} \times V_{\mathbf{y}}$ and $U_{-\mathbf{x}} \times V_{-\mathbf{y}}$. The permutation map $\sigma\colon U \times V \to V \times U$ establishes the isomorphisms $U_{\mathbf{x}} \times V_{\mathbf{y}} \to V_{-\mathbf{y}} \times U_{\mathbf{x}}$ and $U_{\mathbf{x}} \times V_{\mathbf{y}} \to V_{\mathbf{y}} \times U_{-\mathbf{x}}$.

A symplectic rectangle $U_{\mathbf{x}} \times V_{\mathbf{y}}$ has a pair of transversal lagrangian fibrations: a q-fibration $u \times V_{\mathbf{y}}$ for $u \in U$ and a p-fibration $U_{\mathbf{x}} \times v$ for $v \in V$. A *q-embedding*

$$\varepsilon\colon U'_{\mathbf{x}'}\times V'_{\mathbf{y}'}\hookrightarrow_q U_{\mathbf{x}}\times V_{\mathbf{y}} \tag{6.1}$$

is a symplectic embedding such that there exists an embedding $\varepsilon_q\colon U'\hookrightarrow U$ for which the diagram

$$\begin{array}{ccc} U'\times V' & \xrightarrow[q]{\varepsilon} & U\times V \\ \downarrow & & \downarrow \\ U' & \xrightarrow{\varepsilon_q} & U \end{array}$$

is commutative. In other words, a q-embedding must preserve the q-fibration. A composition of q-embeddings

$$U_{3,\mathbf{x}_3}\times V_{3,\mathbf{y}_3}\xrightarrow[q]{\varepsilon_{23}}U_{2,\mathbf{x}_2}\times V_{2,\mathbf{y}_2}\xrightarrow[q]{\varepsilon_{12}}U_{1,\mathbf{x}_1}\times V_{1,\mathbf{y}_1} \tag{6.2}$$

is a q-embedding.

The cotangent bundle $\mathrm{T}^\vee U_{\mathbf{x}}$ has a canonical structure of a symplectic rectangle, because the holomorphic differentials $\partial\mathbf{x}$ form a frame of the cotangent bundle $\mathrm{T}^\vee U_{\mathbf{x}}$ thus providing an isomorphism

$$\mathrm{T}^\vee U_{\mathbf{x}}\xrightarrow{\cong}U_{\mathbf{x}}\times\mathbb{C}^n_{\mathbf{y}}. \tag{6.3}$$

Moreover, an embedding $V_{\mathbf{y}}\hookrightarrow\mathbb{C}^n_{\mathbf{y}}$ generates an embedding $U_{\mathbf{x}}\times V_{\mathbf{y}}\hookrightarrow \mathrm{T}^\vee U_{\mathbf{x}}$, which preserves the symplectic structure as well as both lagrangian fibrations.

6.2. Two-categories of symplectic rectangles and their functors

We define the two-category $\ddot{\mathsf{L}}(U_{\mathbf{x}}\times V_{\mathbf{y}})$ as a full subcategory of $\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$. A curved fibration $(\mathcal{U},W)\in\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U)$ is an object of $\ddot{\mathsf{L}}(U_{\mathbf{x}}\times V_{\mathbf{y}})$ if its support (4.24) lies within $U_{\mathbf{x}}\times V_{\mathbf{y}}$ as embedded into $\mathrm{T}^\vee U_{\mathbf{x}}$:

$$Y_{(\mathcal{U},W)}\subset U_{\mathbf{x}}\times V_{\mathbf{y}}\subset\mathrm{T}^\vee U_{\mathbf{x}}.$$

Isomorphism (6.3) implies the equivalence of categories

$$\ddot{\mathsf{L}}(U_{\mathbf{x}}\times\mathbb{C}^n_{\mathbf{y}})\simeq\ddot{\mathsf{D}}_{\mathbb{Z}_2}(U). \tag{6.4}$$

A curved fibration $(\mathcal{U}_{12},W_{12})\in\ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U_1\times U_2)$ determines the two-functor $\ddot{\Phi}\,[\mathcal{U}_{12},W_{12}]$ of Equation (3.23). Formula (4.29) describing the transformation of the support of a curved fibration under the action of $\ddot{\Phi}\,[\mathcal{U}_{12},W_{12}]$

implies that if the support of $(\mathcal{U}_{12}, W_{12})$ fits within the product of symplectic rectangles:

$$Y_{(\mathcal{U}_{12},W_{12})} \subset (U_{1,\mathbf{x}_1} \times V_{1,\mathbf{y}_1}) \times (U_{2,\mathbf{x}_2} \times V_{2,\mathbf{y}_2}), \tag{6.5}$$

then the two-functor $\ddot{\Phi}\,[\mathcal{U}_{12}, W_{12}]$ restricts to the two-functor

$$\ddot{\Phi}\,[\mathcal{U}_{12}, W_{12}] : \ddot{\mathsf{L}}(U_{1,\mathbf{x}_1} \times V_{1,\mathbf{y}_1}) \longrightarrow \ddot{\mathsf{L}}(U_{2,\mathbf{x}_2} \times V_{2,\mathbf{y}_2}). \tag{6.6}$$

A particular example of the two-functor (6.6) is the analog of Legendre transforms (2.25). This time the Legendre transforms are the two-functors

$$\ddot{\Lambda}_+ : \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(V_{\mathbf{y}} \times U_{-\mathbf{x}}), \qquad \ddot{\Lambda}_- : \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(V_{-\mathbf{y}} \times U_{\mathbf{x}}) \tag{6.7}$$

determined through Equation (3.23) by the one-point fibration and the curving $\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{n} x_i y_i$: $\ddot{\Lambda}_\pm := \ddot{\Phi}\,[\pm\mathbf{x} \cdot \mathbf{y}]$. It is easy to see that the curvings $\pm\mathbf{x} \cdot \mathbf{y}$ satisfy condition (6.5) and, moreover, the Legendre two-functors essentially do not change the supports of objects: for a curved fibration $(\mathcal{U}, W) \in \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$

$$Y_{\ddot{\Lambda}_\pm(\mathcal{U},W)} = \sigma\left(Y_{(\mathcal{U},W)}\right).$$

We conjecture that the composition of Legendre two-functors yields the identity two-functor: $\ddot{\Lambda}_+ \circ \ddot{\Lambda}_- \simeq \ddot{\Lambda}_- \circ \ddot{\Lambda}_+ \simeq 1_{\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})}$, so the Legendre two-functors themselves establish equivalences of two-categories in Equation (6.7).

An important class of two-functors related to two-categories $\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$ are restrictions. Suppose that U' is a submanifold of U of the same dimension. Then there is the restriction two-functor $\ddot{\Phi}_{\mathrm{r}} : \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U) \to \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U')$, which acts on curved fibrations and their morphisms by restricting them from U to U'. The two-functor $\ddot{\Phi}_{\mathrm{r}}$ can be restricted to the subcategory:

$$\ddot{\Phi}_{\mathrm{r}} : \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U'). \tag{6.8}$$

If the subset $U' \subset U$ inherits the coordinates $\mathbf{x}$ then the image of the two-functor (6.8) lies within $\ddot{\mathsf{L}}(U'_{\mathbf{x}} \times V_{\mathbf{y}})$:

$$\ddot{\Phi}_{\mathrm{r}} : \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(U'_{\mathbf{x}} \times V_{\mathbf{y}}). \tag{6.9}$$

For a q-embedding (6.1) we define the restriction two-functor

$$\ddot{\Phi}_{\mathrm{r},\varepsilon} : \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) \longrightarrow \ddot{\mathsf{L}}(U'_{\mathbf{x}'} \times V'_{\mathbf{y}'})$$

as the composition of five two-functors:

$$\begin{array}{ccccccc}
 & & & \overset{\ddot{\Phi}_{\mathrm{r},\varepsilon}}{\curvearrowright} & & & \\
\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}}) & \xrightarrow{\ddot{\Phi}_{\mathrm{r},1}} & \ddot{\mathsf{D}}^{\mathrm{a}}_{\mathbb{Z}_2}(U') & \xrightarrow{\mathtt{1}} & \ddot{\mathsf{L}}(U'_{\mathbf{x}'} \times \mathbb{C}_{\mathbf{y}'}) & & \ddot{\mathsf{L}}(U'_{\mathbf{x}'} \times V'_{\mathbf{y}'}) \\
 & & & & \downarrow{\scriptstyle \ddot{\Lambda}_+} & & \uparrow{\scriptstyle \ddot{\Lambda}_-} \\
 & & & & \ddot{\mathsf{L}}(\mathbb{C}_{\mathbf{y}'} \times U'_{-\mathbf{x}'}) & \xrightarrow{\ddot{\Phi}_{\mathrm{r},2}} & \ddot{\mathsf{L}}(V'_{\mathbf{y}'} \times U'_{-\mathbf{x}'})
\end{array}$$

In this diagram the restriction two-functor $\ddot{\Phi}_{\mathrm{r},1}$ is of the type (6.8), the restriction two-functor $\ddot{\Phi}_{\mathrm{r},2}$ is of the type (6.9) and the equivalence $\mathtt{1}$ is of the type (6.4).

We conjecture that the restriction two-functor of the composition of q-embeddings is isomorphic to the composition of individual restrictions, that is, for a chain of q-embeddings (6.2)

$$\ddot{\Phi}_{\mathrm{r},\varepsilon_{12}\circ\varepsilon_{23}} \simeq \ddot{\Phi}_{\mathrm{r},\varepsilon_{12}} \circ \ddot{\Phi}_{\mathrm{r},\varepsilon_{23}}. \tag{6.10}$$

6.3. A presheaf of two-categories

A *rectangular chart* in a holomorphic symplectic manifold (X, ω) is a symplectic map

$$f \colon U_{\mathbf{x}} \times V_{\mathbf{y}} \to X \tag{6.11}$$

To every rectangular chart we associate the two-category $\ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$. Relation (6.10) suggests that these chart two-categories form a presheaf $\ddot{\mathfrak{P}}(X, \omega)$. An object O of the category $\ddot{\mathsf{L}}(X, \omega)$ is defined to be a global section of this presheaf: to every rectangular chart (6.11) we associate an object $O_f \in \ddot{\mathsf{L}}(U_{\mathbf{x}} \times V_{\mathbf{y}})$ with two conditions: for any commutative triangle

$$\begin{array}{ccc}
U_{\mathbf{x}} \times V_{\mathbf{y}} & \xrightarrow{\sigma} & V_{\mathbf{y}} \times U_{-\mathbf{x}} \\
 & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\
 & X &
\end{array}$$

there is a relation $O_{f'} \cong \ddot{\Lambda}_+ O_f$, and for any commutative triangle

$$\begin{array}{ccc} U'_{\mathbf{x}'} \times V'_{\mathbf{y}'} & \xhookrightarrow[q]{\varepsilon} & U_{\mathbf{x}} \times V_{\mathbf{y}} \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle f} & \\ & X & \end{array}$$

there should be a relation $O_{f'} \cong \ddot{\Phi}_{\mathrm{r},\varepsilon} O_f$.

Two global sections $O_1, O_2 \in \ddot{\mathsf{L}}(X, \omega)$ determine a presheaf of categories $\mathcal{H}om(O_1, O_2)$: to every rectangular chart (6.11) we associate the category $\mathrm{Hom}(O_{1,f_1}, O_{2,f_2})$ and we define $\mathrm{Hom}_{\ddot{\mathsf{L}}(X,\omega)}(O_1, O_2)$ as the category of global sections of this presheaf.

7. Categorified algebraic geometry and the RW model

7.1. RW model of a graded cotangent bundle

In the case when $X = \mathrm{T}^\vee Y$ one can promote the RW model from a $\mathbb{Z}_2$-graded TQFT to a $\mathbb{Z}$-graded one, as explained in [6]. To this end, one assigns cohomological degree 2 to linear coordinates on the fiber of the projection $\mathrm{T}^\vee Y \to Y$. From the physical viewpoint, the degree is the weight with respect to a $U(1)$ ghost number symmetry. We will call the resulting graded manifold $\mathrm{T}^\vee Y[2]$. The sheaf of holomorphic functions on $\mathrm{T}^\vee Y[2]$ is a quasicoherent sheaf of graded algebras on Y:

$$\mathscr{O}_X = \bigoplus_p \mathrm{Sym}^p \mathrm{T}Y.$$

The RW model with the target $\mathrm{T}^\vee Y[2]$ has $U(1)$ ghost number symmetry, and it is natural to consider boundary conditions and topological defects which preserve this symmetry. This gives a $\mathbb{Z}$-graded version of the model.

The two-category of boundary conditions supported on Y has a distinguished object: the zero section of $\mathrm{T}^\vee Y[2]$. It is easy to see that this boundary condition is invariant with respect to the $U(1)$ ghost number symmetry. The corresponding endomorphism category is $\mathsf{D}^{\mathrm{b}}(Y)$, the bounded derived category of coherent sheaves on Y. From the physical viewpoint, it arises as the homotopy category of a DG-category $\mathfrak{D}(Y)$. Objects of $\mathfrak{D}(Y)$ are perfect DG-modules over the $\mathbb{Z}$-graded Dolbeault DG-algebra $(\Omega^{0,\bullet}(Y), \bar{\partial})$, with morphisms being the usual morphisms of DG-modules. $\mathsf{D}^{\mathrm{b}}(Y)$ is a symmetric monoidal DG-category; as discussed in [6], the monoidal structure is the standard one (this is easy to see on the classical level, but it takes some work to show that there are no quantum corrections). The algebra of boundary local operators for the distinguished boundary condition (i.e., the endomorphism algebra of the unit object in the endomorphism category) is isomorphic to $H^*(\mathscr{O}_Y)$.

In the $\mathbb{Z}$-graded case, infinitesimal deformations of a boundary condition correspond to degree-2 elements in the algebra of local boundary operators. Thus infinitesimal deformations of the distinguished boundary condition are parameterized by $H^2(\mathscr{O}_Y)$. If Y is compact and Kähler, such deformations are unobstructed. Indeed, we can choose a harmonic representative B of a class in $H^2(\mathscr{O}_Y)$, and then the deformation of the boundary action is simply

$$\int_{\partial M} \phi^* B,$$

where ϕ is a map from the space–time M to the target X. Since the form B is closed, such deformation is obviously BRST-invariant and does not affect BRST-transformations of any fields. We will call such a deformation a B-field deformation, by analogy with the 2d sigma models.

Let (Y, B) denote the distinguished boundary condition deformed by B. The category of morphisms from (Y, B_1) to (Y, B_2) is the bounded derived category of twisted coherent sheaves on Y, where the twist is given by the class of $B_2 - B_1$. We will denote this category $\mathsf{D}(Y, B_2 - B_1)$. The composition of morphisms is the obvious one (tensor product of twisted coherent sheaves). Physically, $\mathsf{D}^{\mathrm{b}}(Y, B)$ arises as the homotopy category of a certain DG-category which we denote $\mathfrak{D}(Y, B)$. It is the category of perfect CDG-modules over the CDGA $(\Omega^{0,\bullet}(Y), \bar{\partial}, B)$.

More complicated boundary conditions can be obtained by considering complex fibrations $\mathcal{Y} \to Y$ equipped with a B-field $B \in H^2(\mathscr{O}_{\mathcal{Y}})$. The category of morphisms from $(\mathcal{Y}_1, B_1)$ to $(\mathcal{Y}_2, B_2)$ is the bounded derived category of twisted coherent sheaves on $\mathcal{Y}_1 \times_Y \mathcal{Y}_2$ with the twist given by $\pi_2^* B_2 - \pi_1^* B_1$, where π_s is the projection from $\mathcal{Y}_1 \times_Y \mathcal{Y}_2$ to $\mathcal{Y}_s$, $s = 1, 2$.

To understand the resulting two-category better, note that an object of $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_1 \times_Y \mathcal{Y}_2, B_2 - B_1)$ defines a functor from $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_1, B_1)$ to $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_2, B_2)$. Composition of morphisms in the two-category of boundary conditions is simply the composition of functors. Moreover, this functor intertwines the natural action of $\mathsf{D}^{\mathrm{b}}(Y)$, regarded as a monoidal category, on $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_1, B_1)$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_2, B_2)$. That is, if we regard the categories $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_s, B_s)$, $s = 1, 2$ as modules over the monoidal category $\mathsf{D}^{\mathrm{b}}(Y)$, then this functor defines a morphism in the two-category of modules.

Note that for a $\mathbb{C}$-linear (or DG) monoidal category $\mathcal{C}$ there are two very different notions of a module: a module over $\mathcal{C}$ regarded simply as a $\mathbb{C}$-linear (or DG) category, and a module over $\mathcal{C}$ regarded as a monoidal $\mathbb{C}$-linear (or monoidal DG) category. The former is a functor from $\mathcal{C}$ to the category of complex vector spaces $\mathrm{Vect}_{\mathbb{C}}$ (or the category of differential graded complex vector spaces); the latter is a $\mathbb{C}$-linear (or DG) category which is acted upon by $\mathcal{C}$. To avoid confusion, we will call the latter notion a two-module over $\mathcal{C}$. This terminology is not standard,[6] but natural, if we think about a monoidal category as a two-algebra, i.e., a categorification of an algebra. Two-modules over a monoidal category $\mathcal{C}$ form a two-category.

One could hope that any morphism in the two-category of two-modules is represented by an object of $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_1 \times_Y \mathcal{Y}_2, B_2 - B_1)$. Then the two-category of boundary conditions in the RW model would be a full sub-two-category of the two-category of two-modules. This statement is incorrect as formulated, however, apparently it does become correct if we replace the derived category of (twisted) coherent sheaves with its enhancement. Recall that an enhancement of a triangulated category $\mathcal{C}$ is a DG-category $\mathfrak{C}$ whose homotopy category $H^0(\mathfrak{C})$ is triangulated and an equivalence of $H^0(\mathfrak{C})$ and $\mathcal{C}$. From the physical viewpoint, a natural enhancement of $\mathsf{D}^{\mathrm{b}}(Y)$ is the DG-category $\mathfrak{D}(Y)$. Similarly, a natural enhancement of $\mathsf{D}^{\mathrm{b}}(Y, B)$ is the DG-category $\mathfrak{D}(Y, B)$ of perfect CDG-modules over the CDGA $(\Omega^{0,\bullet}(Y), \bar{\partial}, B)$. The category $\mathfrak{D}(Y)$ is a monoidal DG-category which acts by DG-functors on the DG-category $\mathfrak{D}(\mathcal{Y}, B)$. Any object of $\mathfrak{D}(\mathcal{Y}_1 \times_Y \mathcal{Y}_2, B_2 - B_1)$ determines a DG-functor from $\mathfrak{D}(\mathcal{Y}_1, B_1)$ to $\mathfrak{D}(\mathcal{Y}_2, B_2)$ which intertwines the action of $\mathfrak{D}(Y)$. The improved version of the conjecture is that any such DG-functor is represented by an object of $\mathfrak{D}(Y)$.

In [13] this conjecture was proved for Y being a point. In [1] the proof was extended to the case when Y is a more general scheme.

The conclusion is that the two-category of boundary conditions in the $\mathbb{Z}$-graded version of the RW model with target $\mathrm{T}^{\vee}Y[2]$ is the homotopy category of a full sub-two-category in the two-category of two-modules over the monoidal DG-category $\mathfrak{D}(Y)$.

7.2. Derived categorical sheaves

Complex fibrations over Y play a role in the RW model similar to that played by holomorphic vector bundles in the B-model with target Y. But it is well-known that more general coherent sheaves also arise as B-branes, and it is natural to ask if boundary conditions in the RW model can be similarly generalized.

[6]The more standard name for a two-module is a module category.

It is convenient to take a more algebraic viewpoint and replace complex fibrations over Y with families of algebras or DG-algebras over Y. Likely this entails no essential loss of generality. For example, it is known that for any sufficiently nice (quasi-compact and quasi-separated) scheme Z the derived category of complexes of sheaves on Z with quasicoherent cohomology is equivalent to the derived category of modules over some DG-algebra with bounded cohomology [3]. Thus we will replace the fibration $\mathcal{Y}$ with a sheaf of DG-algebras over Y. One may conjecture that any sheaf of DG-algebras over Y can be interpreted as a boundary condition in the RW model.

To test this conjecture, we need to have a reasonable definition of the category of morphisms between sheaves of DG-algebras. A natural definition has been sketched by Toen and Vezzosi [14]. They work with more general objects called derived categorical sheaves over Y. A derived categorical sheaf is a sheaf of DG-categories over Y. This means that to any affine open subscheme $\operatorname{Spec} A = U \subset Y$ one attaches a DG-category $\mathfrak{C}(U)$ over A, to any inclusion of affine open subschemes $U' \subset U$ one attaches a morphism of DG-categories $r_{U'U} : \mathfrak{C}(U) \to \mathfrak{C}(U')$, and to any inclusion of affine open subschemes $U'' \subset U' \subset U$ one attaches an invertible two-morphism from $r_{U''U'} \circ r_{U'U}$ to $r_{U''U}$. These data must satisfy a number of conditions which are spelled out, for example, in [8]. A sheaf of DG-algebras can be thought of as a special case of this, with the DG-category $\mathfrak{C}(U)$ having a single object for any U.

The category of morphisms from the derived categorical sheaf T_1 to the derived categorical sheaf T_2 is defined as follows. First of all, one can define the derived tensor product of two derived categorical sheaves which is again a sheaf of DG-categories. In [14] it is denoted $T_1 \otimes^{\mathbb{L}} T_2$. The category of morphisms from T_1 to T_2 is defined to be the derived category of modules over the sheaf of DG-categories $\mathsf{T}_1^{op} \otimes^{\mathbb{L}} T_2$, where T_1^{op} is the opposite of T_1. In this way one gets a two-category of derived categorical sheaves over Y. There are versions of this definition which depend on which modules precisely one considers.

The simplest object in this two-category is the structure sheaf $\mathscr{O}_Y$ regarded as a sheaf of DG-algebras with zero differential. It corresponds to the distinguished boundary condition in the two-category of boundary conditions for the RW model with target $\mathrm{T}^\vee Y[2]$. Its endomorphism category is $\mathsf{D}^{\mathrm{b}}(Y)$; this agrees with the endomorphism category of the distinguished boundary condition in the RW model. Given any other derived categorical sheaf T over Y, the category of morphisms from the distinguished object to T is a two-module over the monoidal category $\mathsf{D}^{\mathrm{b}}(Y)$. Thus the two-category of derived categorical sheaves over Y is embedded into the two-category of two-modules over $\mathsf{D}^{\mathrm{b}}(Y)$.

A simple but interesting example of a derived categorical sheaf is a skyscraper sheaf, i.e., a sheaf of DG-categories such that $\mathfrak{C}(U)$ is quasi-equivalent to the trivial category if U does not contain a point $p \in Y$, and is quasi-equivalent to a fixed DG-category $\mathfrak{C}_0$ otherwise. We may call this a skyscraper sheaf with stalk $\mathfrak{C}_0$. We now explain how to construct the corresponding boundary condition in the RW model by allowing the fibration $\mathcal{Y}$ over Y to carry a non-trivial curving $W \in H^0(\mathscr{O}_{\mathcal{Y}})$. Such boundary conditions should be regarded as 3d analogues of 0-branes. For simplicity, we will assume that the DG-category $\mathfrak{C}_0$ is simply a DG-algebra $\mathcal{A}$, and is moreover of a geometric origin, i.e., its derived category of modules $\mathsf{D}(\mathcal{A})$ is equivalent to the derived category of coherent sheaves on some complex manifold V.

First we note that in order for a curving W to preserve the ghost number symmetry, we have to allow the fiber $\mathcal{Y}$ to be a graded manifold with a non-trivial $\mathbb{C}^*$ action. Then the space $H^0(\mathscr{O}_Z)$ is also graded, and W must sit in its degree-2 component. We will call such W a superpotential. A graded fibration $\mathcal{Y} \to Y$ equipped with a superpotential W of degree 2 defines a boundary condition for the RW model.

The category of morphisms from the distinguished boundary condition to the boundary condition $(\mathcal{Y}, W)$ is $H^\bullet(\mathfrak{D}(\mathcal{Y}, W)) = \mathsf{D}^{\mathrm{b}}(\mathcal{Y}, W)$. Note that this category is equivalent to a trivial one if W has no critical points [5, 9]. This is a local statement: given an open set $U \subset Y$ we may consider the restriction $(\mathcal{Y}_U, W_U)$ of $(\mathcal{Y}, W)$ to U and the category $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_U, W_U)$; this category is trivial if $\mathcal{Y}_U$ does not contain critical points of W. Therefore a natural candidate for an analogue of a skyscraper sheaf is a pair $(\mathcal{Y}, W)$ such that all critical points of W are contained in the fiber over a point $p \in Y$.

To be concrete, let us consider the case when Y is the n-dimensional affine space $\mathbb{A}_n$ with coordinates $y^1, \ldots, y^n$. We will describe a boundary condition in the RW model with target $\mathrm{T}^\vee Y[2]$ which corresponds to a skyscraper sheaf over Y with the stalk at $y = 0$ being a DG-algebra $\mathcal{A}$ of a geometric origin. Let $\mathcal{Y} = \mathbb{A}_n[2] \times Y \times V$, where $\mathbb{A}_n[2]$ denotes the affine space with linear coordinates $a_1, \ldots, a_n$ of cohomological degree 2. The graded manifold $\mathcal{Y}$ is a trivial fibration over Y. The superpotential will be

$$W = \sum_i y^i a_i,$$

We may think of y^i and a_i as coordinates on $\mathrm{T}^\vee Y[2]$.

The category of morphisms from the distinguished boundary condition to $(\mathcal{Y}, W)$ is $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}, W)$. By Knörrer periodicity, it is equivalent to $\mathsf{D}^{\mathrm{b}}(V)$. Furthermore, W has a single critical point $y = a = 0$, so the category $\mathsf{D}^{\mathrm{b}}(\mathcal{Y}_U, W_U)$ is equivalent to a trivial one if U does not contain the point $y = 0$.

We propose that this boundary condition corresponds to the skyscraper sheaf with the stalk $\mathcal{A}$ at $y = 0$. By definition, the category of morphisms from the derived categorical sheaf $\mathscr{O}_Y$ to this skyscraper is the category $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathsf{D}^{\mathrm{b}}(V)$, which agrees with the RW model.

To check this proposal further, let us compare the endomorphism category of $(\mathcal{Y}, W)$ regarded as a boundary condition in the RW model and the endomorphism category of the skyscraper sheaf. The former is the category $\mathsf{D}^{\mathrm{b}}(\mathbb{A}_n[2] \times \mathbb{A}_n[2] \times V \times V \times Y, \tilde{W})$. The superpotential $\tilde{W}$ is given by

$$\tilde{W} = y^i(a_i - \tilde{a}_i),$$

where $\tilde{a}_i$ denote the coordinates on the second copy of $\mathbb{A}_n[2]$. By Knörrer periodicity, this category is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathbb{A}_n[2] \times V \times V)$.

To compute the endomorphism category of a skyscraper sheaf, we first need to compute its derived tensor product with itself. Since the base is an affine space $\mathbb{A}_n$, we can think about sheaves of DG-algebras in algebraic terms, i.e., as DG-algebras over the ring $\mathbb{C}[y^1, \ldots, y^n]$. From this point of view, the skyscraper sheaf with a stalk $\mathcal{A}$ is simply the DG-algebra $\mathcal{A}$ made into a $\mathbb{C}[y^1, \ldots, y^n]$-module by letting all y^i act trivially. Equivalently, it is a tensor product over $\mathbb{C}$ of the DG-algebra $\mathcal{A}$ over $\mathbb{C}$ and $\mathbb{C}$ regarded as a DG-algebra over $\mathbb{C}[y^1, \ldots, y^n]$ with a trivial action of y^i for all i and a trivial differential.

Since such a module is not flat over $\mathbb{C}[y^1, \ldots, y^n]$, to compute its derived tensor product with itself we need a flat resolution for it.[7] Consider a DG-algebra

$$\mathsf{K}_n = \big(\mathbb{C}[y^1, \ldots, y^n | \theta^1, \ldots, \theta^n], Q\big),$$

where $\theta^1, \ldots, \theta^n$ are anticommuting odd variables of degree -1, and the differential Q is the Koszul differential

$$Q = y^i \frac{\partial}{\partial \theta^i}.$$

It is quasi-isomorphic to $\mathbb{C}$ regarded as a DG-algebra over $\mathbb{C}[y^1, \ldots, y^n]$. Hence we can obtain the desired flat resolution by tensoring over $\mathbb{C}$ the

[7] We are grateful to Dima Orlov for explaining to us the content of this paragraph.

DG-algebra $\mathcal{A}$ with K_n. The derived tensor product is now computed by tensoring with $\mathcal{A}$ over $\mathbb{C}[y^1, \ldots, y^n]$. The result is a DG-algebra

$$\mathcal{A}_\theta = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\theta^1, \ldots, \theta^n], \tag{7.1}$$

with a trivial action of the variables y^i. By definition, the endomorphism category of the skyscraper is a suitable version of the derived category of modules over this DG-algebra.

The algebra $\mathbb{C}[\theta^1, \ldots, \theta^n]$ is Koszul-dual to the algebra

$$\mathbb{C}[a_1, \ldots, a_n], \tag{7.2}$$

where the variables a_i are even and have degree 2, and the differential is zero. Consequently, suitably defined derived categories of the DG-algebra (7.1) and the DG-algebra

$$\mathcal{A}_a = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[a_1, \ldots, a_n] \tag{7.3}$$

are equivalent. This agrees with what we got from the RW model and Knörrer periodicity.

Note that the resolution of the skyscraper categorical sheaf used above is in some sense Koszul-dual to the trivial fibration $\mathcal{Y} = \mathbb{A}_n[2] \times Y \times V$; the role of the Koszul differential is played by the superpotential W.

Let us consider a slightly more complicated example: a sheaf of algebras over $\mathbb{A}_1 = \operatorname{Spec} \mathbb{C}[y]$ which in algebraic terms is the algebra $\mathbb{C}[y]/y^k$ over the ring $\mathbb{C}[y]$. For $k = 1$, this is a special case of the previous example (with $n = 1$). We will argue that there exists a boundary condition in the RW model equivalent to such a sheaf of algebras.

Note first that the above sheaf of DG-algebras can be deformed into a collection of k skyscrapers by replacing y^k with $P_k(y)$, where P_k is a degree-k polynomial without multiple roots. This corresponds to the following boundary condition in the RW model: $\mathbb{Z} = \mathbb{A}_1 \times \mathbb{C}[2]$, $W = aP_k(y)$. In the limit when $P_k(y)$ degenerates to y^k, we get $W = ay^k$. Therefore we propose that the boundary condition with $\mathbb{Z} = \mathbb{A}_1 \times \mathbb{C}[2]$, $W = ay^k$, corresponds to the DG-algebra $\mathbb{C}[y]/y^k$ over $\mathbb{C}[y]$.

The category of morphisms from the distinguished boundary condition to this one is the category of $\mathbb{C}^*$-equivariant matrix factorizations of $W = ay^k$. If the proposal is correct, then this category must be equivalent to the derived category of DG-modules over $\mathbb{C}[y]/y^k$. The equivalence presumably arises from the following matrix factorization:

$$D = \begin{pmatrix} 0 & a \\ y^k & 0 \end{pmatrix} \tag{7.4}$$

Its endomorphism algebra is a DG-algebra quasi-isomorphic to $\mathbb{C}[y]/y^k$ with the zero differential. Thus we get a bimodule which defines a functor from the category of equivariant matrix factorizations to the derived category of DG-modules over $\mathbb{C}[y]/y^k$. With suitable definitions, this functor should be an equivalence of categories [11].

We note in passing that this construction allows one to think about the derived category of modules over $\mathbb{C}[y]/y^k$ as a category of B-branes in some physical theory (namely, the Landau–Ginzburg model on $\mathbb{C} \times \mathbb{C}[2]$ with the superpotential $W = ay^k$). In other words, the Landau–Ginzburg model whose target is a graded manifold allows one to give meaning to such a singular-looking theory as a sigma-model with target $\mathrm{Spec}(\mathbb{C}[y]/y^k)$. Similarly many other graded Landau–Ginzburg models can be thought of as representing sigma-models whose targets are singular schemes or even DG-schemes.

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