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Supermassive black hole merger rates: uncertainties from halo merger theory

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ABSTRACT

The merger of two supermassive black holes is expected to produce a gravitational-wave signal detectable by the *Laser Interferometer Space Antenna (LISA)*. The rate of supermassive-black-hole mergers is intimately connected to the halo merger rate, and the extended Press-Schechter (EPS) formalism is often employed when calculating the rate at which these events will be observed by *LISA*. This merger theory is flawed and provides two rates for the merging of the same pair of haloes. We show that the two predictions for the *LISA* supermassive-black-hole-merger event rate from EPS merger theory are nearly equal because mergers between haloes of similar masses dominate the event rate. An alternative merger rate may be obtained by inverting the Smoluchowski coagulation equation to find the merger rate that preserves the Press–Schechter halo abundance, but these rates are only available for power-law power spectra. We compare the *LISA* event rates derived from the EPS merger formalism to those derived from the merger rates obtained from the coagulation equation and find that the EPS *LISA* event rates are 30 per cent higher for a power spectrum spectral index that approximates the full Λ cold dark matter result of the EPS theory.

Key words: black hole physics – gravitational waves – galaxies: haloes – cosmology: theory.

1 INTRODUCTION

Structure formation proceeds hierarchically, with small over-dense regions collapsing to form the first dark-matter haloes. These haloes then merge to form larger bound objects. The extended Press–Schechter (EPS) formalism provides a description of ‘bottom-up’ structure formation by combining the Press–Schechter (PS) halo mass function (Press & Schechter 1974) with the halo merger rates derived by Lacey & Cole (1993). Since its inception, the EPS theory has been an invaluable tool and has been applied to a wide variety of topics in structure formation [see Benson, Kamionkowski & Hassani (2005, hereafter BKH) and references therein].

Unfortunately, the Lacey–Cole merger-rate formula, which is the cornerstone of EPS merger theory, is mathematically inconsistent (BKH). It is possible to obtain *two* equally valid merger rates for the same pair of haloes from the EPS formalism. These two merger rates are nearly equal when the masses of the two haloes differ by less than a factor of 100, but they diverge rapidly for mergers between haloes with larger mass ratios. Consequently, any application of EPS merger theory gives two answers, and if the calculation involves mergers between haloes of unequal masses, the discrepancy between these two predictions may be large.

Motivated by the ambiguity in the Lacey–Cole merger rate, BKH proposed a method to obtain self-consistent halo merger rates. Since haloes are created and destroyed through mergers, the halo merger rate determines the rate of change of the number density of haloes of a given mass. By inverting the Smoluchowski coagulation equation (Smoluchowski 1916), BKH find merger rates that predict the same halo population evolution as the time derivative of the PS mass function. In addition to eliminating the flaw that resulted in the double-valued rates in EPS theory, the BKH merger rates by definition preserve the PS halo mass distribution when used to evolve a population of haloes. The Lacey–Cole merger rate fails this consistency test as well, and the use of EPS merger trees has been constrained by this inconsistency (e.g. Menou, Haiman & Narayanan 2001).

There are three limitations to the BKH merger rates. First, they are not uniquely determined because the Smoluchowski equation does not provide sufficient constraints on the merger rate. The BKH merger rate is the smoothest, non-negative function that satisfies the coagulation equation; it exemplifies the properties of a self-consistent merger theory, but it is not a definitive result. Secondly, the inversion of the Smoluchowski equation is numerically challenging, and solutions have been obtained only for power-law density power spectra. Finally, the BKH merger rates are derived from the PS halo mass function rather than the mass functions

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obtained from N-body simulations (Sheth & Tormen 1999; Jenkins et al. 2001).

In this paper, we explore the possible quantitative consequences of our limited understanding of merger rates for one of the astrophysical applications of merger theory: the merger rate of supermassive black holes (SMBHs). Since SMBHs are believed to lie in the centre of all dark-matter haloes above some critical mass, halo mergers and SMBH mergers are intimately related. By considering only halo mergers that would result in a SMBH merger, the EPS merger rates have been used to obtain SMBH merger rates (Haehnelt 1994; Menou et al. 2001; Wyithe & Loeb 2003a; Sesana et al. 2004, 2005; Rhook & Wyithe 2005).

SMBH mergers are of great interest because they produce a gravitational-wave signal that may be detectable by the *Laser Interferometer Space Antenna (LISA)*, which is scheduled for launch in the upcoming decade. Consequently, EPS merger theory has been used to obtain estimates for the SMBH merger event rate for *LISA* (Haehnelt 1994; Menou et al. 2001; Wyithe & Loeb 2003a; Sesana et al. 2004, 2005; Rhook & Wyithe 2005). In addition to their intrinsic interest as a probe of general relativity, there is hope that *LISA*'s observations of SMBH mergers would provide a new window into astrophysics at high redshifts. Wyithe & Loeb (2003a) used EPS merger theory to derive a redshift-dependent mass function for haloes containing SMBHs and then used EPS merger theory to predict the *LISA* event rate that arises from this SMBH population. Since SMBH formation becomes more difficult after reionization due to the limitations on cooling imposed by a hot intergalactic medium, the Wyithe–Loeb SMBH mass function and corresponding *LISA* event rate are highly sensitive to the redshift of reionization. Menou et al. (2001) used EPS merger trees to demonstrate that *LISA* observes more SMBH merger events when SMBHs at redshift $z = 5$ are only found in the most massive haloes as opposed to being randomly distributed among haloes. Koushiappas & Zentner (2006) also used EPS merger trees to show that higher-mass seed black holes ($M_{\rm BH} \sim 10^5\ {\rm M}_\odot$ as opposed to $M_{\rm BH} \sim 10^2\ {\rm M}_\odot$) at high redshifts result in significantly higher *LISA* SMBH-merger event rates. Unfortunately, these ambitions of using *LISA* SMBH-merger event rates to learn about reionization and SMBH formation rest on the shaky foundation of EPS merger theory.

We first review how the rate of mergers per comoving volume translates to an observed event rate in a Λ cold dark matter (ΛCDM) universe and how the mass of the halo is related to the mass of the SMBH at its centre in Sections 2 and 3. In Section 4, we use the EPS formalism to derive an event rate for *LISA*. Throughout the calculation, we present the results derived from *both* versions of the Lacey–Cole merger rate. In Section 5, we explore the alternative merger-rate formalism proposed by BKH. Since the BKH merger rates are only available for power-law density power spectra, it is not possible to use them to make a new prediction of the SMBH merger rate and the corresponding event rate for *LISA*. Instead, in Section 6, we use the event rates for power-law power spectra derived from the EPS and BKH merger theories to gauge how the *LISA* event rates may be affected by switching merger formalisms. Finally, in Section 7, we summarize our results and discuss how these ambiguities in halo merger theory limit our ability to learn about reionization and supermassive-black-hole formation from *LISA*'s observations.

2 COSMOLOGICAL EVENT RATES

The merger of two SMBHs will produce a gravitational-wave burst. The observed burst event rate depends on the number density and frequency of black-hole mergers: the number of observed gravitational-wave bursts per unit time (B) that originate from a shell of comoving radius $R(z)$ and width dR is

$$\mathrm{d}B = (1+z)^{-1}\mathcal{N}(z)\, 4\pi R^2\, \mathrm{d}R, \tag{1}$$

where $\mathcal{N}(z)$ is the SMBH merger rate per comoving volume as a function of redshift. The factor of $(1+z)^{-1}$ in equation (1) results from cosmological time dilation. In equation (1), and throughout this article, we assume a flat ΛCDM universe. Given the relation between comoving distance and redshift, $\mathrm{d}R = [c/H(z)]\ \mathrm{d}z$, equation (1) may be converted to a differential event rate per redshift interval,

$$\frac{\mathrm{d}B}{\mathrm{d}z} = (1+z)^{-1}\left\{\frac{4\pi[R(z)]^2\mathcal{N}(z)c}{H_0\sqrt{\Omega_{\rm M}(1+z)^3+\Omega_\Lambda}}\right\}, \tag{2}$$

where $\Omega_{\rm M}$ and Ω_Λ are the matter and dark-energy densities today in units of the critical density.

The observed gravitational-wave burst rate from SMBH mergers is obtained by integrating equation (2) over the redshifts from which the bursts are detectable. *LISA* will be able to detect nearly all mergers of two black holes with masses greater than $10^4\ {\rm M}_\odot$ and less than $10^8\ {\rm M}_\odot$ up to $z \lesssim 9$ (Haehnelt 1994; Rhook & Wyithe 2005; Sesana et al. 2005). Since more massive binary-black-hole systems emit gravitational radiation at lower frequencies and the observed frequency decreases with redshift, very distant ($z \sim 9$) mergers of SMBHs with masses greater than $10^8\ {\rm M}_\odot$ produce signals below *LISA*'s frequency window (Rhook & Wyithe 2005; Sesana et al. 2005). However, the number density of $10^8\ {\rm M}_\odot$ haloes is exponentially suppressed at redshifts greater than four, so it is extremely unlikely that two black holes larger than $10^8\ {\rm M}_\odot$ will merge at redshifts $z \gtrsim 4$. Thus, the upper bounds on the relevant redshift and SMBH mass intervals are determined by the population of SMBHs and not *LISA*'s sensitivity.

3 THE RELATIONSHIP BETWEEN HALO MASS AND BLACK HOLE MASS

The transition from the rate of halo mergers to the rate of detectable SMBH mergers [$\mathcal{N}(z)$ as defined in equation (1)] requires a relationship between the mass of a halo and the mass of the SMBH at its centre. Since *LISA* is sensitive to SMBH mergers at high redshifts, this $M_{\rm BH}$–$M_{\rm halo}$ relation must be applicable to high redshifts as well.

Observations of galaxies out to $z \sim 3$ reveal a redshift-independent correlation between the mass of the central black hole and the bulge velocity dispersion $\sigma_{\rm c}$ (Ferrarese & Merritt 2000; Gebhardt et al. 2000; Tremaine et al. 2002). The connection between $\sigma_{\rm c}$ and halo mass is mediated by the circular velocity $v_{\rm c}$. Using a sample of 13 spiral galaxies, Ferrarese (2002a) measured a relationship between $v_{\rm c}$ and $\sigma_{\rm c}$. Combining these measurements with the compiled relationship between SMBH mass and $\sigma_{\rm c}$ presented by Ferrarese (2002b) reveals that observations are consistent with a redshift-independent $M_{\rm BH} \propto v_{\rm c}^5$ relation.

Wyithe & Loeb (2003b) proposed a mechanism for black-hole-mass regulation that would result in a $M_{\rm BH} \propto v_{\rm c}^5$ relation between central-black-hole mass and disc circular velocity for all redshifts. They postulated that a black hole ceases to accrete when the power radiated by the accretion exceeds the binding energy of the host galactic disc divided by the dynamical time of the disc. Assuming that the accretion disc shines at its Eddington luminosity, the black hole stops growing when

$$M_{\rm BH} = 1.9\times10^8\left(\frac{F_q}{0.07}\right)\left(\frac{v_{\rm c}}{350\,{\rm km\,s^{-1}}}\right)^5 {\rm M}_\odot, \tag{3}$$

where F_q is the fraction of the radiated power which is transferred to gas in the disc. Setting F_q to 0.07 brings equation (3) into agreement with the observations presented by Ferrarese (2002a).

The final step in the determination of a halo–black-hole-mass relation is to connect the circular velocity to the halo mass via the virial velocity (Barkana & Loeb 2001). The simplest possible assumption is that the circular velocity of the disc equals the virial velocity of the halo. This assumption is made by Wyithe & Loeb (2003b), and we assume that $v_c = v_{vir}$ throughout this paper. However, different relations between v_c and v_{vir} have been proposed and can significantly impact the final M_{BH}–M_{halo} relation (see Ferrarese 2002a).

Assuming that $v_c = v_{vir}$, the halo mass then becomes a redshift-dependent function of the mass of the central black hole:

$$\frac{M_{halo}}{10^{12}\,\mathrm{M}_\odot} = 10.5\left(\frac{\Omega_M^0}{\Omega_M(z)}\frac{\Delta_c}{18\pi^2}\right)^{-1/2}(1+z)^{-3/2}\left(\frac{M_{BH}}{10^8\,\mathrm{M}_\odot}\right)^{3/5}, \tag{4}$$

where $\Omega_M(z)$ is the matter density in units of the critical density as a function of redshift, $\Omega_M^0 \equiv \Omega_M(z=0)$ and Δ_c is the non-linear over-density at virialization for a spherical top-hat perturbation in a ΛCDM universe:

$$\Delta_c = 18\pi^2 + 82[\Omega_M(z) - 1] - 39[\Omega_M(z) - 1]^2. \tag{5}$$

Fig. 1 shows the masses of haloes that contain SMBHs of several masses. Citing the fact that the largest haloes observed at low redshifts appear to contain galaxy clusters with no central black holes, Wyithe & Loeb (2003b) argue that supermassive-black-hole growth was complete by $z \sim 1$ and that local SMBH masses reflect the limiting values at that redshift. Consequently, when determining the mass of a halo that contains a black hole of a given mass, we use the $z = 1$ value of equation (4) for all redshifts less than one.

Some calculations of the *LISA* SMBH-merger event rate impose a minimum halo virial temperature instead of a minimum black-hole mass when calculating the lower mass bound on haloes that contribute to the SMBH merger rate (Wyithe & Loeb 2003a; Rhook & Wyithe 2005). This constraint reflects the fact that SMBHs only form when the gas within dark-matter haloes can cool. However, the relation between virial temperature and virial mass (Barkana & Loeb 2001) may be used to eliminate the halo mass in equation (4) in favour of the virial temperature. The redshift-dependent terms cancel, leaving a redshift-independent relation between black-hole mass and halo virial temperature:

$$M_{BH} = (267\,\mathrm{M}_\odot)\,h^{-5/3}\left(\frac{T_{vir}}{1.98\times10^4\,\mathrm{K}}\right)^{5/2}, \tag{6}$$

where $H_0 = 100\,h$ km s^{-1} Mpc^{-1}. Therefore, defining the minimum halo mass by a minimum halo virial temperature is nearly equivalent to defining it by a minimum black-hole mass via equation (4). For example, requiring that the halo's virial temperature be significantly higher than the temperature of the intergalactic medium, $T_{vir} \gtrsim 10^5$ K (Wyithe & Loeb 2003a), corresponds to imposing a minimum black-hole mass of $2.6 \times 10^4\,\mathrm{M}_\odot$. The only discrepancy occurs when $z < 1$, because we assume that the M_{BH}–M_{halo} relation is fixed for redshifts less than one, while T_{vir} is still redshift dependent. However, we will see that nearly all SMBH mergers occur at redshifts greater than one, so this difference is negligible.

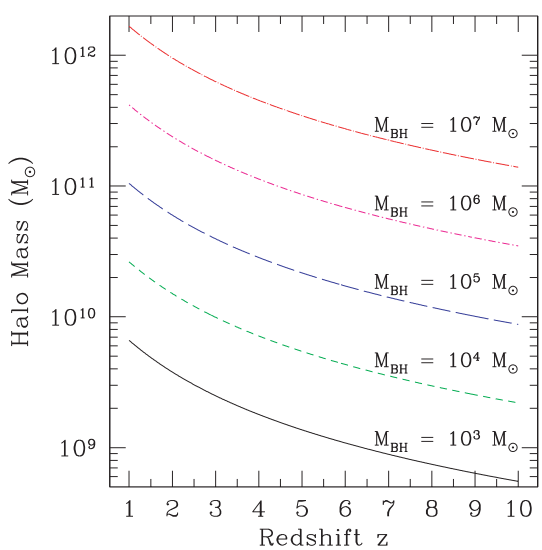

Figure 1. The masses of haloes that contain SMBHs of mass 10^3, 10^4, 10^5, 10^6 and $10^7\,\mathrm{M}_\odot$, according to the M_{BH}–M_{halo} relation proposed by Wyithe & Loeb (2003b) for a flat ΛCDM universe with $\Omega_M = 0.27$. This relation is normalized to fit local observations and assumes that the disc circular velocity equals the virial velocity.

4 EPS MERGER THEORY AND *LISA* EVENT RATES

4.1 Review of EPS merger theory

The first pillar of EPS merger theory is the PS halo mass function (Press & Schechter 1974), which gives the number of haloes with masses between M and $M + \mathrm{d}M$ per comoving volume:

$$\frac{\mathrm{d}n_{halo}}{\mathrm{d}\ln M} = \sqrt{\frac{2}{\pi}}\frac{\rho_0}{M}\left(\left|\frac{\mathrm{d}\ln\sigma}{\mathrm{d}\ln M}\right|_M\right)\frac{\delta_{coll}}{\sigma(M,z)}\exp\left[\frac{-\delta_{coll}^2}{2\sigma^2(M,z)}\right], \tag{7}$$

where ρ_0 is the background matter density today, δ_{coll} is the critical linear over-density for collapse in the spherical-collapse model and $\sigma(M, z)$ is the root variance of the linear density field at redshift z in spheres containing mass M on average. In a ΛCDM universe, δ_{coll} deviates slightly from its Einstein–de Sitter value of ~1.686 when the cosmological constant begins to dominate the energy density of the universe (Kitayama & Suto 1996; Weinberg & Kamionkowski 2003). In this work, the fitting function obtained by Kitayama & Suto (1996) was used to approximate δ_{coll}. When calculating $\sigma(M, z)$, we assumed a scale-invariant primordial power spectrum and we used the transfer function provided by Eisenstein & Hu (1998).

The second pillar of EPS merger theory is the merger probability function derived by Lacey & Cole (1993), which gives the probability that a halo of mass M_1 will become a halo of mass $M_f \equiv M_1 + M_2$ per unit time, per unit acquired mass:

$$\frac{\mathrm{d}^2p}{\mathrm{d}t\,\mathrm{d}M_2} = \frac{1}{M_f}\sqrt{\frac{2}{\pi}}\left|\frac{\dot\delta_{coll}}{\delta_{coll}} - \frac{\dot D(t)}{D(t)}\right|\left(\left|\frac{\mathrm{d}\ln\sigma}{\mathrm{d}\ln M}\right|_{M_f}\right) \times\frac{\delta_{coll}}{\sigma(M_f,z)}\left(1 - \frac{\sigma^2(M_f,z)}{\sigma^2(M_1,z)}\right)^{-3/2} \times\exp\left[\frac{-\delta_{coll}^2}{2}\left(\frac{1}{\sigma^2(M_f,z)} - \frac{1}{\sigma^2(M_1,z)}\right)\right]. \tag{8}$$

In this expression, $D(t)$ is the linear growth function, and a dot denotes differentiation with respect to time.

Equation (8) is usually interpreted as the differential probability that a given halo of mass M_1 will merge with a halo of mass between M_2 and $M_2 + \mathrm{d}M_2$ per unit time, per increment mass change. Thus, equation (8) already includes information about the abundance of haloes of mass M_2, but not the abundance of haloes of mass M_1. Following BKH, it is revealing to examine a different quantity, which does not differentiate between the two merging haloes: the rate of mergers between haloes of masses M_1 and M_2 per comoving volume,

$$R(M_1, M_2, t) \equiv \frac{\text{number of } M_1 + M_2 \text{ mergers}}{\mathrm{d}t\,\mathrm{d}(\text{comoving volume})},$$
$$= \left[\frac{\mathrm{d}n(M_1; t)}{\mathrm{d}M_1}\right]\left[\frac{\mathrm{d}^2 p}{\mathrm{d}t\,\mathrm{d}M_2}\right]\mathrm{d}M_1\,\mathrm{d}M_2. \qquad (9)$$

The EPS self-inconsistency documented by BKH manifests itself here. Although $R(M_1, M_2, t)$ must be symmetric in its mass arguments by definition, equation (9) is not symmetric under exchange of M_1 and M_2.

The mass asymmetry of EPS merger theory becomes most transparent when one defines a new function: the merger kernel. From its definition, it is apparent that $R(M_1, M_2, t)$ should be proportional to the number densities of both haloes involved in the merger. Extracting this dependence defines the merger kernel $Q(M_1, M_2, t)$:

$$R(M_1, M_2, t) \equiv \left[\frac{\mathrm{d}n(M_1; t)}{\mathrm{d}M_1}\right]\left[\frac{\mathrm{d}n(M_2; t)}{\mathrm{d}M_2}\right]$$
$$\times\, Q(M_1, M_2, t)\,\mathrm{d}M_1\,\mathrm{d}M_2. \qquad (10)$$

In addition to isolating the source of the mass-asymmetry in EPS merger theory, the merger kernel enters into the coagulation equation which is inverted to obtain BKH merger rates, as described in Section 5.

The EPS merger kernel $Q(M_1, M_2)$ is the probability function given by equation (8) divided by the number density of haloes of mass M_2 given by equation (7). In effect, EPS merger theory includes two distinct merger kernels, depending on the order of the mass arguments. Thus, we define two mass-symmetric merger kernels: $Q_\mathrm{M}(M_1, M_2)$ equals the EPS merger kernel with the more massive halo as the first argument, while $Q_\mathrm{L}(M_1, M_2)$ equals the EPS merger kernel with the less massive halo as the first argument. Fig. 2 illustrates the differences in the merger kernels Q_M and Q_L. Note that neither $Q_\mathrm{M}(M_1, M_2)$ nor $Q_\mathrm{L}(M_1, M_2)$ are viable candidates for the true halo merger kernel because they are not smooth functions of halo mass. They are useful because they expose the ambiguities hidden in applications of EPS merger theory.

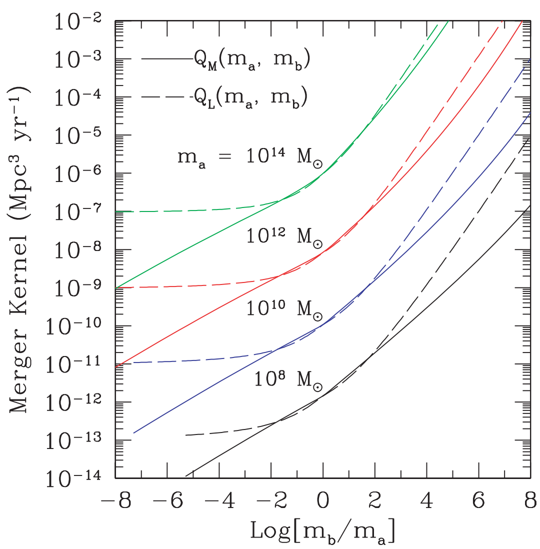

Figure 2. The two EPS merger kernels for $z = 0$. Here, Q_M is the Lacey–Cole merger kernel with the more massive halo as the first argument, and Q_L is the same kernel with the less massive halo as the first argument. Results are shown for a flat ΛCDM universe with $\Omega_\mathrm{M} = 0.27$, $h = 0.72$ and $\sigma_8 = 0.9$.

In order to avoid double counting mergers when calculating a merger rate, it is common to restrict one mass argument to be larger than the other. Using the standard expression for the Lacey–Cole merger probability function, as given by equation (8), in such calculations is equivalent to using $Q_\mathrm{M}(M_1, M_2)$ or $Q_\mathrm{L}(M_1, M_2)$. Specifically, Haehnelt (1994) effectively used Q_L to predict an event rate for *LISA*, while Wyithe & Loeb (2003a) and Rhook & Wyithe (2005) effectively used Q_M. Using the other version of the EPS merger kernel in either of these calculations would have yielded different results, as we show in Section 4.2. More generally, any application of the Lacey–Cole merger probability function uses some mixture of Q_M and Q_L, and changing the mixture will change the result of the calculation.

4.2 *LISA* event rates from EPS theory

The rate of SMBH mergers per comoving volume follows from the rate of halo mergers per comoving volume given in equation (9):

$$\mathcal{N}(z) \equiv \frac{1}{2}\int_{M_\mathrm{min}}^{\infty}\int_{M_\mathrm{min}}^{\infty}\left[\frac{\mathrm{d}n(M_1, z)}{\mathrm{d}M_1}\right]\left[\frac{\mathrm{d}n(M_2, z)}{\mathrm{d}M_2}\right]$$
$$\times\, Q(M_1, M_2, z)\,\mathrm{d}M_1\,\mathrm{d}M_2, \qquad (11)$$

where M_min is the minimum halo mass that contains a black hole massive enough to be detected when it merges with a black hole of equal or greater mass. The factor of 1/2 accounts for the double counting of mergers. Some calculations (e.g. Rhook & Wyithe 2005) only include mergers between haloes with mass ratios less than three and so integrate M_2 from $M_1/3$ to $3M_1$. This restriction is motivated by dynamical-friction calculations that indicate that when a halo merges with a halo less than a third of its size, it takes longer than a Hubble time for their central black holes to merge (Colpi, Mayer & Governato 1999). However, recent numerical simulations indicate that this restriction may be too strict; when gas dynamics are included, SMBHs with host-galaxy-mass ratios greater than three merge within a Hubble time (Kazantzidis et al. 2005). We do not impose any restrictions on the halo-mass ratios, so our event rates are upper bounds arising from the assumption that every halo merger in which both haloes contain a SMBH results in a SMBH merger.

Since *LISA* should observe mergers between two SMBHs with masses greater than $10^4\,\mathrm{M}_\odot$ out to redshifts of at least eight, we generally use this minimum black hole mass to determine M_min. The corresponding rates of SMBH mergers per comoving volume are shown in Fig. 3, as well as the rates which correspond to different choices for the minimum mass of a SMBH. Both versions of $\mathcal{N}$ are shown to illustrate the difference between the two Lacey–Cole merger kernels. The crimp in $\mathcal{N}(z)$ at $z = 1$ reflects the transition from a constant M_min (evaluated at $z = 1$) to the redshift-dependent form given by equation (4).

Once the rate $\mathcal{N}(z)$ of SMBH mergers per volume is known, equation (2) may be integrated over redshift to obtain an event rate for *LISA*. Fig. 4 shows the *LISA* event rate as a function of the minimum halo mass that contains a black hole large enough to emit an observable signal. Fig. 5 shows the event rate as a function of the maximum redshift of a detectable merger. In Fig. 5, M_min is the mass of a halo that contains a black hole more massive than

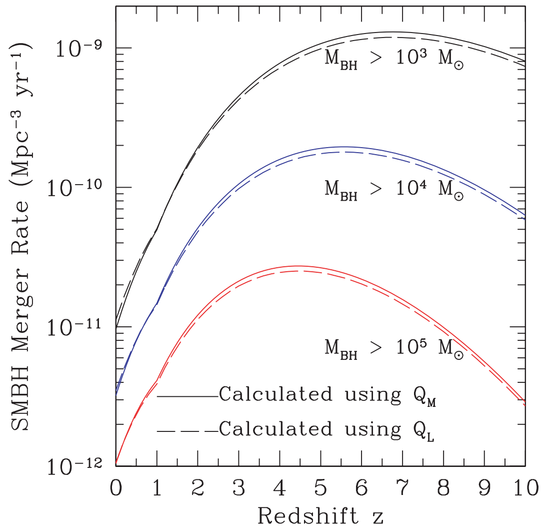

Figure 3. The rate of SMBH mergers per comoving volume where both merging black holes have a mass greater than 10^3, 10^4 and 10^5 $M_\odot$. The solid (dashed) lines show the results when the first argument of the Lacey–Cole merger kernel is the more (less) massive halo. Results are shown for a flat ΛCDM universe with $\Omega_M = 0.27$, $h = 0.72$ and $\sigma_8 = 0.9$.

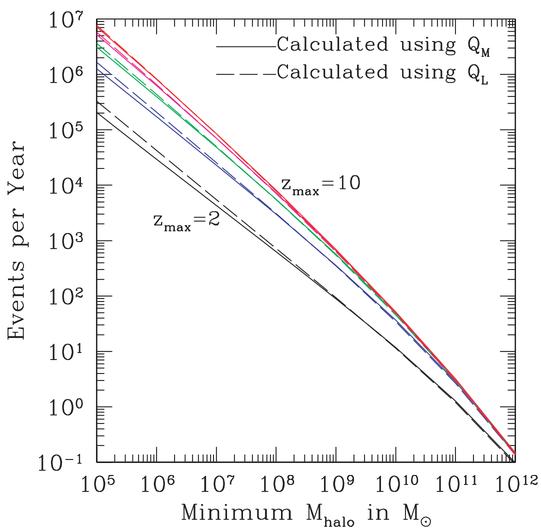

Figure 4. The gravitational-wave event rate from SMBH mergers as a function of the minimum halo mass that contains a SMBH large enough to produce a detectable signal when it merges. Mergers at redshifts up to z_{max} were included in this rate, and the five pairs of lines correspond to $z_{max} = 2$, 4, 6, 8, 10. The solid (dashed) lines show the results when the first argument of the Lacey–Cole merger kernel is the more (less) massive halo. Results are shown for a flat ΛCDM universe with $\Omega_M = 0.27$, $h = 0.72$ and $\sigma_8 = 0.9$.

10^3, 10^4, or 10^5 $M_\odot$ as determined by the M_{BH}–M_{halo} relation given by equation (4). These rates correspond to the $\mathcal{N}$ results depicted in Fig. 3. Examination of these results reveals that increasing z_{max} beyond $z_{max} = 6$ has little effect on the event rate when M_{min} is greater than 10^9 $M_\odot$, as is the case when equation (4) is used to obtain the value of M_{min} that corresponds to a minimum black-hole mass of 10^4 $M_\odot$. The levelling of the event rate for $z_{max} \gtrsim 6$ indicates that SMBH mergers are very rare at higher redshifts and that the event rate is dominated by mergers that occur at redshifts $z \lesssim 6$. Therefore, the upper bound on *LISA*'s sensitivity to larger SMBH mergers at high redshifts will have little effect on the event rate.

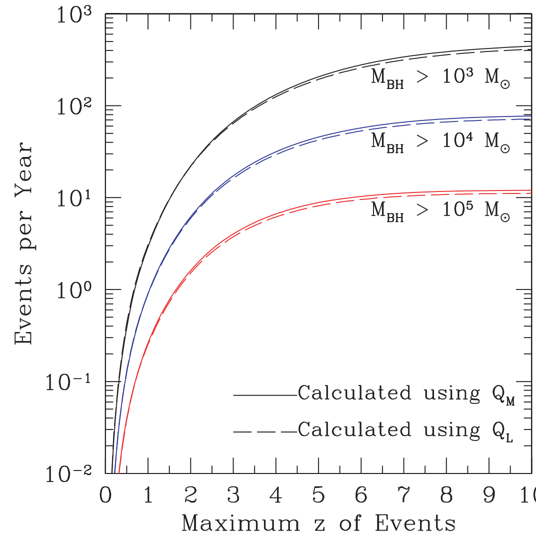

Figure 5. The gravitational-wave event rate from SMBH mergers as a function of the maximum redshift of a detectable merger. Only mergers in which both black holes have masses greater than the given lower bound are included. The solid (dashed) lines show the results when the first argument of the Lacey–Cole merger kernel is the more (less) massive halo. Results are shown for a flat ΛCDM universe with $\Omega_M = 0.27$, $h = 0.72$ and $\sigma_8 = 0.9$.

The event rates shown in Figs 4 and 5 differ significantly from those calculated by Wyithe & Loeb (2003a) and Rhook & Wyithe (2005). Our event rates are generally much higher than the event rates reported by Wyithe & Loeb (2003a) because we do not exclude mergers between haloes with mass ratios greater than three from our SMBH merger rate. The event rates calculated by Rhook & Wyithe (2005) are even lower because they do not assume that all haloes contain galaxies. The one case where our event rates are not substantially higher than those derived by Rhook & Wyithe (2005) is when the minimum black-hole mass is taken to be very high ($M_{BH} \gtrsim 10^5$ $M_\odot$). In that case, the minimum halo mass is so high that nearly all mergers involve haloes of similar masses ($M_{halo} \sim 10^{11}$ $M_\odot$), and the galaxy-occupation fraction derived by Rhook & Wyithe (2005) indicates that nearly all haloes of this size contain galaxies for redshifts greater than three, so our event rate of 12 per year is very similar to the result of the more sophisticated treatment of Rhook & Wyithe (2005).[1]

Event rates obtained from both versions of the EPS merger kernel are shown in Figs 4 and 5. The differences between these results reveal the type of mergers that dominate the calculation. As shown in Fig. 5, the event rate obtained from Q_M is slightly higher than the rate obtained from Q_L, indicating that mergers with halo mass ratios less than 10^2 are dominating the sum (see Fig. 2). For a constant value of $M_{min} = 10^5$ $M_\odot$, the difference between the two versions decreases as the maximum redshift increases, as shown in Fig. 4. This convergence indicates that the contribution from mergers between haloes of greatly unequal masses to the event rate dwindles as redshift increases. Since the lower bound on halo mass is constant with redshift, a decrease in unequal-mass mergers reflects a decrease in the population of larger haloes. The PS mass function

[1] When we attempted to reproduce the differential event rates calculated by Haehnelt (1994), we found that our rates are roughly a factor of 2 lower. After extensive review and two independent calculations, we were unable to find any errors in our analysis.

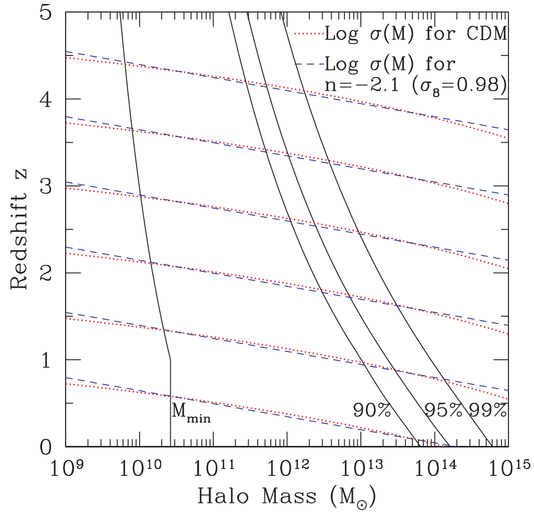

Figure 6. The halo mass range that dominates the rate of SMBH mergers per comoving volume. The three curves marked with percentages define the upper bounds of mass ranges that account for 90, 95 and 99 per cent of $\mathcal{N}$. Here, $M_{\rm min}$ is the mass of a halo that contains a SMBH of mass 10^4 M$_\odot$. The dotted curves are plots of log $\sigma(M)$ with arbitrary normalizations. Results are shown for a flat ΛCDM universe with $\Omega_{\rm M} = 0.27$ and $\sigma_8 = 0.9$. The dashed lines are plots of log $\sigma(M)$ for a power-law power spectrum with $n = -2.1$ and $\sigma_8 = 0.9843$, which is the best linear fit to log σ over the mass range between $M_{\rm min}$ and the 99 per cent curve for $z \leqslant 5$.

implies that the largest halo that is common at a given redshift is given by the function $M_*(z)$, which is defined by the relation $\sigma(M_*, z) \equiv \delta_{\rm coll}(z)$. When $M > M_*(z)$, the exponential term in equation (7) dominates, and the number density of such haloes is exponentially suppressed. $M_*(z)$ decreases with redshift, reflecting the fact that at early times, massive haloes had yet to form. Due to the exponential decline in the number density of haloes greater than $M_*(z)$, there is an effective upper bound to the integrals in equation (11), which defines $\mathcal{N}(z)$. This upper bound on halo mass follows M_* and is less than 100 times greater than $M_{\rm min}$ at the redshifts which dominate the merger rate.

The relevant mass range may be quantified by considering the ratio,

$$C(U) \equiv \frac{\frac{1}{2}\int_{M_{\rm min}}^{U} {\rm d}M_1 \int_{M_{\rm min}}^{U} {\rm d}M_2 \left(\frac{{\rm d}n}{{\rm d}M_1}\right)\left(\frac{{\rm d}n}{{\rm d}M_2}\right) Q(M_1, M_2)}{\frac{1}{2}\int_{M_{\rm min}}^{\infty} {\rm d}M_1 \int_{M_{\rm min}}^{\infty} {\rm d}M_2 \left(\frac{{\rm d}n}{{\rm d}M_1}\right)\left(\frac{{\rm d}n}{{\rm d}M_2}\right) Q(M_1, M_2)},$$

where the z-dependence of all quantities has been suppressed. Using the standard Lacey–Cole merger kernel when evaluating $C(U)$ is equivalent to using the arithmetic mean of $Q_{\rm M}$ and $Q_{\rm L}$. Fig. 6 shows the values of U for $C = 0.9$, 0.95 and 0.99. Since the dominant halo mass range is so narrow, it is possible to find a power-law power spectrum that accurately approximates the value of $\sigma(M)$ over this mass range, and both the exact and the approximate $\sigma(M)$ are shown in Fig. 6. This approximation will allow us to apply BKH merger theory to the calculation of *LISA* event rates in Section 6.

5 BKH MERGER THEORY

5.1 Solving the coagulation equation

A merger kernel that preserves the PS halo mass distribution must satisfy the Smoluchowski coagulation equation (Smoluchowski 1916), which simply states that the rate of change in the number of haloes of mass M equals the rate of creation of such haloes through mergers of smaller haloes minus the rate at which haloes of mass M merge with other haloes. Adopting the shorthand $n(M)$ for the PS halo number density per interval mass and suppressing the redshift dependence of all terms, the coagulation equation is

$$\frac{\rm d}{{\rm d}t}n(M) = \frac{1}{2}\int_0^M n(M')n(M-M')Q(M', M-M')\,{\rm d}M' - n(M)\int_0^\infty n(M')Q(M, M')\,{\rm d}M', \tag{12}$$

where $Q(M_1, M_2, z)$ is the desired merger kernel. The first term on the right-hand side is the rate of mergers per comoving volume that create a halo of mass M. The second term is the rate of mergers involving a halo of mass M per comoving volume – these mergers effectively destroy haloes of mass M.

BKH numerically invert the coagulation equation for Q for power-law density power spectra $P(k) \propto k^n$. When the density power spectrum is a power law, $\sigma(M)$ is proportional to $M^{-(3+n)/6}$. Since the redshift-dependence of the PS mass function enters via the ratio $\delta_{\rm coll}(z)/\sigma(M, z) = (M/M_*)^{(3+n)/6}$, the z-dependence of the PS mass function may be eliminated by expressing the masses in units of $M_*(z)$. For a judicious choice of time variables, differentiating the PS mass function introduces no z-dependence, and the coagulation equation becomes redshift-invariant. Consequently, the coagulation equation only has to be inverted once, for the resulting merger kernel $Q(M_1/M_*, M_2/M_*)$ is applicable to all redshifts. This simplification is only possible when the power spectrum is a power law. For more complicated spectra, the coagulation equation will have to be solved at multiple redshifts.

When they numerically solve the coagulation equation on a discrete grid, BKH require that the merger kernel be symmetric in its two mass arguments. However, this restriction is not sufficient to determine Q uniquely from the coagulation equation. On an $N \times N$ mass grid, the coagulation equation becomes N equations for the N possible values of M. Meanwhile, the symmetric Q matrix on the grid, $Q_{ij} = Q(M_i, M_j)$, has $N(N+1)/2$ independent components. To break the degeneracy, BKH impose a regularization condition. By minimizing the second derivatives of Q, they find the smoothest, non-negative kernel that solves the coagulation equation.

5.2 BKH merger rates for power-law power spectra

In Section 4.2, we demonstrated that the rate of SMBH mergers per comoving volume is dominated by mergers between haloes in a very limited mass range, as shown in Fig. 6. The $\sigma(M)$ curves in Fig. 6 show that it is possible to accurately approximate $\sigma(M)$ over the relevant mass range as originating from a power-law power spectrum. We consider a power-law fit for $\sigma(M)$ that extends over all masses that fall within the 99 per cent mass range at any redshift less than five. The fit has a lower mass bound of 5.44×10^9 M$_\odot$, which is the value of $M_{\rm min}$ at $z = 5$, and extends to a mass of 4.26×10^{14} M$_\odot$. Over this range, $\sigma(M)$ is best fit by spectral index $n = -2.1$ normalized so that $\sigma_8 = 0.9843$, as shown by the dashed lines in Fig. 6. This $n = -2.1$ power-law approximation of $\sigma(M)$ is accurate to within 16 per cent over this mass range. We chose to fit the mass range for $z \lesssim 5$ because the SMBH merger rate peaks at redshifts less than five when the minimum black-hole mass is greater than 10^4 M$_\odot$, as shown in Fig. 3. Also, when the mass range is lowered, the best-fitting spectral index decreases, and BKH merger rates have not been obtained for $n < -2.2$.

The density power spectrum enters the EPS merger kernel only through $\sigma(M)$, so any power-law approximation that accurately models $\sigma(M)$ for M_1, M_2 and $M_f = M_1 + M_2$ will accurately model the Lacey–Cole merger kernel $Q(M_1, M_2, z)$. Unfortunately, the same is not necessarily true for the BKH merger kernels obtained by inverting the coagulation equation. Since the coagulation equation (equation 12) involves integrals over all masses and is solved for all masses on the grid, the solution $Q(M_1, M_2, z)$ is dependent on $\sigma(M)$ over all masses and not just the arguments of the kernel. Therefore, while the power-law approximation accurately reflects the full ΛCDM result for EPS merger theory, the BKH merger rates obtained for the same power law may differ greatly from the merger rates that solve the coagulation equation for a ΛCDM universe. However, since the coagulation equation has not been solved for a ΛCDM power spectrum, we compare the EPS merger rates to the BKH merger rates for the same power law. This comparison demonstrates how the BKH merger rates differ from the EPS rates, but should not be considered a definitive description of merger rates in a ΛCDM universe.

BKH merger kernels for a power-law power spectrum with $n = -2.1$ were obtained by inverting the coagulation equation on a 91×91 grid of logarithmically spaced M/M_* values ranging from 10^{-12} to 3000. For M/M_* values greater than 10^{-8}, the merger kernel values are not dependent on grid resolution, which indicates that the kernel is a numerically robust solution of the discretized coagulation equation for masses above $10^{-8}\ M_*$. The $M_{\rm BH}$ –$M_{\rm halo}$ relation (equation 4) implies that SMBHs with masses greater than $10^3\ \mathrm{M}_\odot$ and redshifts less than 10 reside in haloes with masses greater than $10^8\ \mathrm{M}_\odot$, while the $z = 0$ value of M_* for the $n = -2.1$ power-law power spectrum is $6 \times 10^{12}\ \mathrm{M}_\odot$. Therefore, for all haloes that contain SMBHs capable of producing a gravitational-wave signal detectable by *LISA*, $M/M_* \gtrsim 10^{-5}$, so the lower mass bound on reliable kernel values is of no concern.

Unfortunately, the same is not true for the upper bound on M/M_*. The upper bound on the halo masses which contribute to the SMBH merger rate $\mathcal{N}$ in EPS theory, shown in Fig. 6, extends to $M/M_* \gtrsim 10^5$ for $z \gtrsim 5$. However, extending the mass grid to higher values of M/M_* introduces numerical noise that prevents the kernels from converging as grid resolution is increased. Therefore, we must extrapolate the BKH kernel to higher masses. We bilinearly extrapolate the logarithm of the kernel with respect to the logarithms of its mass arguments. When used to extrapolate from a grid with $M/M_* < 100$, this recovers the kernel to within a factor of 2. Moreover, ignoring mergers of haloes with $M/M_* > 3000$ only slightly affects the gravitational-wave event rate calculated from the BKH merger rates: the event-rate reduction is less than 3 per cent. Therefore, the errors introduced by our extrapolation of the BKH merger kernel are negligible.

The differences between the BKH merger kernel and both versions of the EPS merger kernel are illustrated by Fig. 7. The BKH merger kernel is less than both EPS kernels when the masses of the merging haloes are similar, and the difference increases as the haloes get smaller. For mergers between haloes with mass ratios greater than 10^2, the BKH merger kernel is nearly equal to the EPS kernel with the least-massive halo as the first argument ($Q_{\rm L}$) for all masses. Therefore, for an $n = -2.1$ power-law power spectrum, $Q_{\rm L}$ comes closer to solving the coagulation equation than $Q_{\rm M}$.

6 COMPARISON OF *LISA* EVENT RATES FROM BKH AND EPS MERGER THEORIES

Since the BKH merger kernels for haloes of nearly equal masses are smaller than the EPS kernels for the same spectral index, applying EPS merger theory may over-estimate the *LISA* event rate. Fig. 8 shows the rate $\mathcal{N}$ of SMBH mergers per comoving volume for the power-law model discussed in the previous section. For comparison, the EPS results for a ΛCDM universe are also shown as dotted curves. (These are the arithmetic means of the corresponding solid and dashed curves in Fig. 3.) However, it is important to remember that although the power-law models may accurately approximate the ΛCDM results in the EPS theory, the same should not be assumed for the BKH merger rates. The BKH merger rates should only be compared to the EPS rates for the same power law.

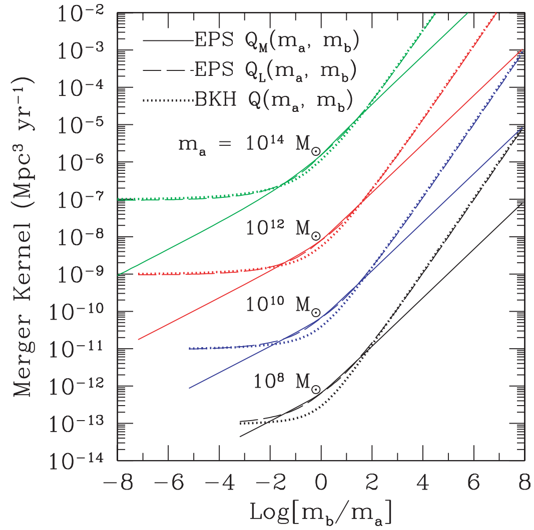

Figure 7. The two EPS merger kernels and the BKH merger kernel for a $n = -2.1$ power-law power spectrum at $z = 0$. Here, $Q_{\rm M}$ is the Lacey–Cole merger kernel with the more massive halo as the first argument, and $Q_{\rm L}$ is the same kernel with the less massive halo as the first argument. Results are shown for $\Omega_{\rm M} = 0.27$, $h = 0.72$ and $\sigma_8 = 0.98$. The low-mass cut-off of the curves arises from the $M/M_* \gtrsim 10^{-8}$ bound on the BKH merger kernel.

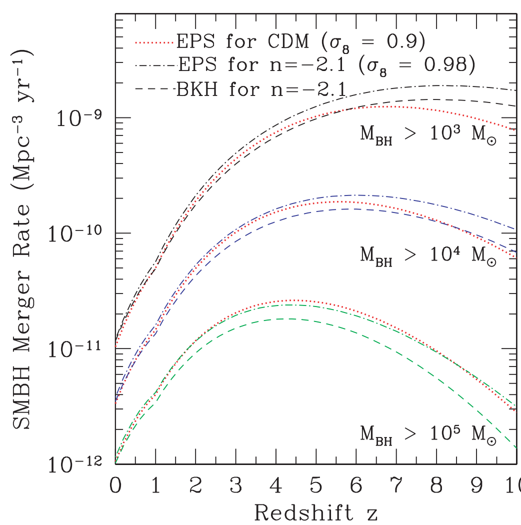

Figure 8. The rate of SMBH mergers per comoving volume where both merging black holes have a mass greater than 10^3, 10^4, or $10^5\ \mathrm{M}_\odot$. The dotted line shows the EPS merger kernel for a ΛCDM power spectrum with $\sigma_8 = 0.9$. The dot–dashed curves are the results derived from EPS theory for a power-law approximation with $n = -2.1$ and $\sigma_8 = 0.98$. The dashed curves are the BKH results for the same power law and normalization. These results all assume a flat ΛCDM universe with $\Omega_{\rm M} = 0.27$ and $h = 0.72$.

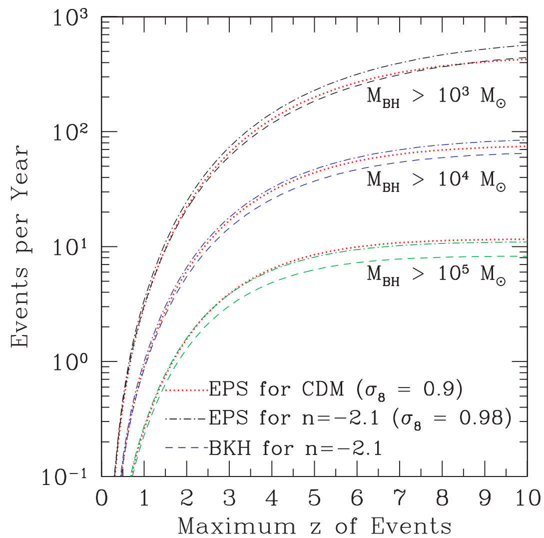

Figure 9. The gravitational-wave event rate from SMBH mergers as a function of the maximum redshift of a detectable merger. Only mergers in which both black holes have a mass greater than 10^3, 10^4, or 10^5 $M_\odot$ are included. The dotted line shows the EPS merger kernel for a ΛCDM power spectrum with $\sigma_8 = 0.9$. The dot–dashed curves are the results derived from EPS theory for a power-law approximation: $n = -2.1$ and $\sigma_8 = 0.98$. The dashed curves are the BKH results for the same power law and normalization. These results all assume a flat ΛCDM universe with $\Omega_M = 0.27$ and $h = 0.72$.

The discrepancy between the power-law EPS results and the ΛCDM curves at high redshifts is attributable to the power-law halo number density, which is much greater than the ΛCDM halo number density for masses below 10^{11} $M_\odot$ at these redshifts. The same mass function is used to calculate the merger rate in BKH theory, so when the power-law merger rate is higher than the ΛCDM rate in EPS theory, it is reasonable to assume that the same is true for the rate derived from BKH theory. Fig. 8 also shows that the predictions for the SMBH merger rate from the BKH and EPS merger theories diverge with increasing redshift. In Section 4.2, we showed that as redshift increases, nearly equal-mass halo mergers dominate the event rate. The differences between the BKH merger kernel and the EPS kernel are greatest when the masses of the merging haloes are nearly equal, so as these mergers dominate the event rate at high redshifts, the BKH and EPS merger rates diverge.

Fig. 9 illustrates the potential consequences BKH merger theory has for the SMBH merger event rate observed by *LISA*. The difference between the BKH and EPS merger kernels for the same spectral index leads to a fairly substantial difference in the resulting event rates for *LISA*. For realistic values of the maximum redshift of a detectable merger ($z_{\max} \gtrsim 5$), the EPS prediction is about 30 per cent higher than the BKH prediction for the $n = -2.1$ power-law approximation. If the BKH merger kernel for a full ΛCDM power spectrum preserves the ratio of the BKH and EPS event rates for this spectral index, the *LISA* event rate from SMBH mergers would be reduced as well. Rhook & Wyithe (2005) used EPS merger theory to predict that *LISA* will have approximately 15 SMBH-merger detections per year at a signal-to-noise ratio of greater than five. (They only consider mergers with $M_{BH} \gtrsim 10^5$ $M_\odot$.) These comparisons of EPS and BKH event rates indicate that *LISA*'s event rate may be closer to 10, with all other assumptions held fixed.

7 SUMMARY AND DISCUSSION

The EPS merger theory used to predict supermassive-black-hole merger rates is mathematically inconsistent because it contains two merger rates for the same pair of haloes. When the EPS formalism is used to derive supermassive-black-hole merger rates and the corresponding event rate for *LISA*, there are two potential results; the EPS predictions are ambiguous. We have found that mergers between haloes whose masses differ by less than a factor of 10^2 dominate the SMBH merger rate, even when all mergers between SMBH-containing haloes are included. The difference between the EPS merger rates for mass ratios in this range is small, so the two merger rates predicted by EPS theory are nearly equal.

The concordance between the two EPS predictions for the SMBH merger rate is an artefact of the relative paucity of haloes with masses larger than 10^{11} $M_\odot$. It is not an indication that the EPS merger formalism may be trusted to give realistic merger rates. In addition to its mass-asymmetry, the Lacey–Cole merger rate fails to give the same evolution of the halo population as the time derivative of the PS mass function. Both of these flaws justify the search for a new theory of halo mergers. BKH inverted the coagulation equation to find merger rates that preserve the PS halo mass function for power-law power spectra. They found that these merger rates differ significantly from the EPS rates for the same power spectrum.

The limited range of halo masses that contribute to the SMBH merger rate makes it possible to find a power-law power spectrum that accurately fits the mass variance $\sigma(M)$ in this region. We consider such a power-law approximation with spectral index $n = -2.1$. Since the EPS merger formula depends only on the values of $\sigma(M)$ for the two halo masses that are merging and the mass of the resulting halo, the power-law approximation accurately describes the result obtained from the ΛCDM power spectrum. The same correspondence cannot be assumed for the BKH merger rates because they are dependent on $\sigma(M)$ at all masses.

Nevertheless, it is illuminating to compare the SMBH merger rates derived from BKH merger theory to those derived from EPS theory for the same spectral index. When $n = -2.1$, the BKH merger rates are lower than the corresponding EPS rates for nearly equal-mass halo mergers, which dominate the rate of SMBH mergers. This discrepancy is a clear demonstration of how the EPS rates fail to solve the coagulation equation and therefore fail to preserve the PS halo mass function. It also indicates how BKH theory may predict a different SMBH-merger event rate for *LISA*, since the difference in merger rates results in an equally large difference in event rates. Comparing the event rates derived from EPS and BKH merger theories for this spectral index indicates that the *LISA* event-rate predictions that employ EPS merger theory may over-estimate the event rate by 30 per cent.

Fortunately, the ambiguity carried into the SMBH-merger event-rate predictions for *LISA* from the uncertainty surrounding halo merger theory does not appear to immediately preclude extracting information regarding reionization or black-hole formation from *LISA*'s observations of SMBH mergers. Wyithe & Loeb (2003a) showed that the *LISA* SMBH-merger event rate with reionization occurring at $z = 7$ is about 2.4 times higher than if reionization occurred earlier, at $z = 12$. This difference is larger than the uncertainties in the event rate revealed by our comparisons of BKH and EPS predictions, so it may be possible to constrain the reionization redshift from the *LISA* SMBH-merger event rate without a definitive theory of halo mergers. The 30 per cent uncertainty implied by these halo-merger-theory comparisons is also less than the difference in event rates for different SMBH seeding found by Menou et al. (2001). However, a 30 per cent uncertainty in the SMBH merger rate will significantly loosen the constraints *LISA*'s observations of SMBH mergers could place on reionization and SMBH formation. More concerning is the fact that there is no guarantee that the merger rate derived from the merger kernel that satisfies the coagulation equation for a ΛCDM universe does not differ from the EPS merger rate by more than 30 per cent.

Clearly, solving the coagulation equation for a ΛCDM power spectrum is imperative. Any application of EPS merger theory to astrophysical phenomena has a flawed foundation and the resulting predictions are unreliable. Specifically, we have shown that the differences between EPS merger theory and BKH merger theory for power-law power spectra indicate that switching merger theories could significantly alter the *LISA* SMBH-merger event rate. This theoretical uncertainty should be resolved before *LISA*'s measurements of SMBH merger rates are used to constrain cosmological models.

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