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Symbolic Algorithms for the Painlevé Test, Special Solutions, and Recursion Operators for Nonlinear PDEs.

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ABSTRACT. This paper discusses the algorithms and implementations of three MATHEMATICA packages for the study of integrability and the computation of closed-form solutions of nonlinear polynomial PDEs.

The first package, `PainleveTest.m`, symbolically performs the Painlevé integrability test. The second package, `PDESpecialSolutions.m`, computes exact solutions expressible in hyperbolic or elliptic functions. The third package, `PDERecursionOperator.m`, generates and tests recursion operators.

1. Introduction

The investigation of complete integrability of nonlinear partial differential equations (PDEs) is a nontrivial matter [**12**]. Likewise, finding the explicit form of solitary wave and soliton solutions requires tedious, unwieldy computations which are best performed using computer algebra systems. For example, the symbolic computation of solitons with Hirota's direct method and the homogenization method are covered in [**13, 16**].

Recently, progress has been made using MATHEMATICA and MAPLE in applying the inverse scattering transform (IST) method to compute solitons for the Camassa-Holm equation [**17**]. Before applying the IST method (a nontrivial exercise in analysis!), one would like to know if the PDE is completely integrable or what elementary travelling wave solutions exist. This is where the symbolic algorithms and packages presented in this paper come into play.

In this paper we introduce three algorithms and related MATHEMATICA packages [**4**] which may greatly aid the investigation of integrability and the search for exact solutions. In Section 2 we present the algorithm for the well-known Painlevé integrability test [**1, 5, 6**], which was recently implemented as `PainleveTest.m`. Section 3 outlines the algorithm behind `PDESpecialSolutions.m`, which allows one

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This is the final form of the paper.

to automatically compute exact solutions expressible in hyperbolic or elliptic functions; full details are presented in [**3**]. In Section 4 we give an algorithm for computing and testing recursion operators [**11**]; the package `PDERecursionOperator.m` automates the steps. The latter package builds on the code `InvariantsSymmetries.m` [**9**], which computes conserved densities, fluxes, and generalized symmetries for polynomial equations. Our codes are publicly available, see [**4**].

In Sections 2 and 3 we consider systems of M polynomial differential equations,

$$\boldsymbol{\Delta}(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \ldots, \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}, \tag{1.1}$$

where the dependent variable $\mathbf{u}$ has M components u_i, the independent variable $\mathbf{x}$ has N components x_j, and $\mathbf{u}^{(m)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order m. We assume that any arbitrary coefficients parameterizing the system are strictly positive and denoted by lower-case Greek letters. However, the algorithm in Section 4 only applies to a special case of (1.1), namely evolution equations in $(1 + 1)$ dimensions.

Two carefully selected examples will illustrate the algorithms. The first example is the Kaup–Kupershmidt (KK) equation (see e.g. [**8, 22**]),

$$u_t = 5u^2 u_x + \frac{25}{2} u_x u_{2x} + 5u u_{3x} + u_{5x}. \tag{1.2}$$

The second example is the Hirota–Satsuma system of coupled Korteweg-de Vries (KdV) equations (see e.g. [**1**]),

$$\begin{aligned} u_t &= \beta(6uu_x + u_{3x}) - 2vv_x, \quad \beta > 0, \\ v_t &= -3uv_x - v_{3x}. \end{aligned} \tag{1.3}$$

The computations for both examples will be done by the software.

2. The Painlevé test

The Painlevé test verifies whether a system of ODEs or PDEs satisfies the necessary conditions for having the Painlevé property. There is some variation in what is meant by the Painlevé property [**18**]. As defined in [**2**], for a PDE to have the Painlevé property, all ODE reductions of the PDE must have the Painlevé property. While [**2**] requires that all movable singularities of all solutions are poles, the more general definition used by Painlevé himself requires that all solutions of the ODE are single-valued around all movable singularities. A later version [**23**] allows testing of PDEs directly without having to reduce them to ODEs. For a thorough discussion of the Painlevé property, see [**5, 6**].

Definition 2.1. A PDE has the Painlevé property if its solutions in the complex plane are single-valued in the neighborhood of all its movable singularities.

2.1. Algorithm and implementation. Following [**23**], we assume a Laurent expansion for the solution

$$u_i(\mathbf{x}) = g^{\alpha_i}(\mathbf{x}) \sum_{k=0}^{\infty} u_{i,k}(\mathbf{x}) g^k(\mathbf{x}), \qquad u_{i,0}(\mathbf{x}) \neq 0 \quad \text{and} \quad \alpha_i \in \mathbb{Z}, \tag{2.1}$$

where at least one of the leading exponents α_i is a negative integer and $u_{i,k}(\mathbf{x})$ is an analytic function in the neighborhood of $g(\mathbf{x})$. The solution should be single-valued in the neighborhood of the non-characteristic, movable singular manifold $g(\mathbf{x}) = 0$, which can be viewed as the surface of the movable poles in the complex plane.

The algorithm for the Painlevé test is composed of the following three steps:

Step 1 (Determine the dominant behavior). It suffices to substitute

$$u_i(\mathbf{x}) = u_{i,0}(\mathbf{x}) g^{\alpha_i}(\mathbf{x}) \tag{2.2}$$

into (1.1) to determine the leading exponents α_i (one of which must be a negative integer) and the function $u_{i,0}(\mathbf{x})$. In the resulting polynomial system, equating every two possible lowest exponents of $g(\mathbf{x})$ in each equation gives a linear system to determine α_i. The linear system is then solved for α_i.

If one or more exponents α_i remain undetermined, we assign integer values to the free α_i (if applicable, up to a user-specified non-negative boundary, `DominantBehaviorMax`, with default value -1) so that every equation in (1.1) has at least two different terms with equal lowest exponents. Once α_i is known, we substitute (2.2) back into (1.1) and solve for $u_{i,0}(\mathbf{x})$.

Step 2 (Determine the resonances). For each α_i and $u_{i,0}(\mathbf{x})$, we calculate the integers r for which $u_{i,r}(\mathbf{x})$ is an arbitrary function in (2.1). We substitute

$$u_i(\mathbf{x}) = u_{i,0}(\mathbf{x}) g^{\alpha_i}(\mathbf{x}) + u_{i,r}(\mathbf{x}) g^{\alpha_i + r}(\mathbf{x}) \tag{2.3}$$

into (1.1), keeping only the lowest order terms in $g(\mathbf{x})$, and requiring that the coefficients of $u_{i,r}(\mathbf{x})$ equate to zero. This is done by computing the roots for r of $\det Q = 0$, where the $M \times M$ matrix Q satisfies

$$Q \cdot \mathbf{u}_r = \mathbf{0}, \quad \mathbf{u}_r = (u_{1,r} \ u_{2,r} \ \cdots \ u_{M,r})^{\mathrm{T}}. \tag{2.4}$$

Step 3 (Find the constants of integration and check compatibility conditions). For the system to possess the Painlevé property, the arbitrariness of $u_{i,r}(\mathbf{x})$ must be verified up to the highest resonance level. That is, all compatibility conditions must be trivially satisfied. This is done by substituting

$$u_i(\mathbf{x}) = g^{\alpha_i}(\mathbf{x}) \sum_{k=0}^{r_{\mathrm{Max}}} u_{i,k}(\mathbf{x}) g^k(\mathbf{x}) \tag{2.5}$$

into (1.1), where r_{Max} is the highest positive integer resonance. For the system to have the Painlevé property, there must be as many arbitrary constants of integration at resonance levels as resonances at that level. Furthermore, all constants of integration $u_{i,k}(\mathbf{x})$ at non-resonance levels must be unambiguously determined.

2.2. Examples of the Painlevé test.

Example 2.1 (Kaup–Kupershmidt). To determine the dominant behavior, we substitute (2.2) into (1.2) and pull off the exponents of $g(\mathbf{x})$ (see Table 1).

Removing duplicates and non-dominant exponents, we are left with

$$\{\alpha_1 - 5, 2\alpha_1 - 3, 3\alpha_1 - 1\}. \tag{2.6}$$

Considering all possible balances of two or more exponents leads to $\alpha_1 = -2$.

Substituting $u(\mathbf{x}) = u_{1,0}(\mathbf{x}) g^{-2}(\mathbf{x})$ into (1.2) and solving for $u_{1,0}(\mathbf{x})$ gives us

$$u_{1,0}(\mathbf{x}) = -24 g_x^2(\mathbf{x}) \quad \text{and} \quad u_{1,0}(\mathbf{x}) = -3 g_x^2(\mathbf{x}). \tag{2.7}$$

For the first branch, substituting $u(\mathbf{x}) = -24 g_x^2(\mathbf{x}) g^{-2}(\mathbf{x}) + u_{1,r}(\mathbf{x}) g^{r-2}(\mathbf{x})$ into (1.2), keeping the lowest order terms, and taking the coefficient of $u_{1,r}(\mathbf{x})$, gives

$$-2(r+7)(r+1)(r-6)(r-10)(r-12) g_x^5(\mathbf{x}) = 0. \tag{2.8}$$

TABLE 1. The exponents of $g(\mathbf{x})$ for (1.2).

Term	Exponents of $g(\mathbf{x})$ with duplicates removed
u_t	$\alpha_1 - 1$
u_{5x}	$\alpha_1 - 5, \alpha_1 - 4, \alpha_1 - 3, \alpha_1 - 2, \alpha_1 - 1$
$5uu_{3x}$	$2\alpha_1 - 3, 2\alpha_1 - 2, 2\alpha_1 - 1$
$\frac{25}{2}u_x u_{2x}$	$2\alpha_1 - 3, 2\alpha_1 - 2$
$5u^2 u_x$	$3\alpha_1 - 1$

Hence, $r = -7, -1, 6, 10, 12$. The first branch is not a principal branch due to the presence of a negative resonance other than $r = -1$. The latter is called the universal resonance and corresponds to the arbitrariness of the manifold function $g(\mathbf{x})$. In [**7**], a perturbative Painlevé approach is presented which properly treats negative integer resonances. As is common practice, our algorithm ignores them.

The constants of integration at level j are found by substituting (2.5) into (1.2), where $r_{\text{Max}} = 12$, and pulling off the coefficients of $g^j(\mathbf{x})$. The first few are

$$\text{(2.9)} \qquad \text{at } j = 1\colon u_{1,1}(\mathbf{x}) = 24 g_{2x}(\mathbf{x}),$$

$$\text{(2.10)} \qquad \text{at } j = 2\colon u_{1,2}(\mathbf{x}) = \frac{6g_{2x}^2(\mathbf{x}) - 8g_x(\mathbf{x})g_{3x}(\mathbf{x})}{g_x^2(\mathbf{x})},$$

$$\text{(2.11)} \qquad \text{at } j = 3\colon u_{1,3}(\mathbf{x}) = \frac{6g_{2x}^3(\mathbf{x}) - 8g_x(\mathbf{x})g_{2x}(\mathbf{x})g_{3x}(\mathbf{x}) + 2g_x^2(\mathbf{x})g_{4x}(\mathbf{x})}{g_x^4(\mathbf{x})}.$$

The compatibility conditions are satisfied at resonance levels 6, 10, and 12. The remaining constants of integration $u_{1,j}(\mathbf{x})$ were computed but not shown here.

Likewise, in the second branch, substitute $u(\mathbf{x}) = -3g_x^2(\mathbf{x})g^{-2}(\mathbf{x}) + u_{1,r}(\mathbf{x})g^{r-2}(\mathbf{x})$ into (1.2), and proceed as before to get $r = -1, 3, 5, 6, 7$. Thus, the second branch is a principal branch. The constants of integration at $j = 1, 2$ and 4 are again found by substituting (2.5) into (1.2) and pulling off the coefficients of $g^j(\mathbf{x})$. This gives,

$$\text{(2.12)} \qquad \text{at } j = 1\colon u_{1,1}(\mathbf{x}) = 3g_{2x}(\mathbf{x}),$$

$$\text{(2.13)} \qquad \text{at } j = 2\colon u_{1,2}(\mathbf{x}) = \frac{3g_{2x}^2(\mathbf{x}) - 4g_x(\mathbf{x})g_{3x}(\mathbf{x})}{4g_x^2(\mathbf{x})}.$$

Coefficient $u_{1,4}$ at level $j = 4$ is not shown here due to length. At the resonance levels $r = 3, 5, 6, 7$, the compatibility conditions happen to be satisfied. So, (1.2) passes the Painlevé test. It is well-known (see e.g. [**8, 22**]) that (1.2) is completely integrable.

Example 2.2 (Hirota–Satsuma). The Hirota–Satsuma system illustrates the subtleties of determining the dominant behavior.

As in Example 2.1, we substitute (2.2) into (1.3) and pull off the exponents of $g(\mathbf{x})$ (listed in Table 2).

Removing non-dominant exponents and duplicates by term, we get $\{\alpha_1 - 3, 2\alpha_1 - 1, 2\alpha_2 - 1\}$ from Δ_1 and $\{\alpha_2 - 3, \alpha_1 + \alpha_2 - 1\}$ from Δ_2.

Equating the possible dominant exponents from Δ_2 gives us $\alpha_2 - 3 = \alpha_1 + \alpha_2 - 1$ or $\alpha_1 = -2$. Unexpectedly, $\alpha_1 = -2$ balances two of the possible dominant terms

TABLE 2. The exponents of $g(\mathbf{x})$ for (1.3).

Term	Exponents of $g(\mathbf{x})$
u_t	$\alpha_1 - 1$
$-\beta u_{3x}$	$\alpha_1 - 3, \alpha_1 - 3, \alpha_1 - 3, \alpha_1 - 2, \alpha_1 - 2, \alpha_1 - 1$
$-6\beta u u_x$	$2\alpha_1 - 1$
$2vv_x$	$2\alpha_2 - 1$
v_t	$\alpha_2 - 1$
v_{3x}	$\alpha_2 - 3, \alpha_2 - 3, \alpha_2 - 3, \alpha_2 - 2, \alpha_2 - 2, \alpha_2 - 1$
$3uv_x$	$\alpha_1 + \alpha_2 - 1$

in Δ_1, and we are free to choose α_2 such that

$$2\alpha_1 - 1 \leq 2\alpha_2 - 1 \quad \text{or} \quad -2 \leq \alpha_2. \tag{2.14}$$

Using the default value -1 for `DominantBehaviorMax`, we have $-2 \leq \alpha_2 \leq -1$. Hence, $\alpha_2 = -1$ or $\alpha_2 = -2$.

Using the two solutions for α_i and solving for $u_{i,0}$ results in

$$\begin{cases} \alpha_1 = -2, & u_{1,0}(\mathbf{x}) = -4g_x^2(\mathbf{x}), \\ \alpha_2 = -2, & u_{2,0}(\mathbf{x}) = \pm 2\sqrt{6\beta}\, g_x^2(\mathbf{x}), \end{cases} \tag{2.15}$$

$$\begin{cases} \alpha_1 = -2, & u_{1,0}(\mathbf{x}) = -2g_x^2(\mathbf{x}), \\ \alpha_2 = -1, & u_{2,0}(\mathbf{x}) \text{ arbitrary}. \end{cases} \tag{2.16}$$

We substitute (2.3) into (1.3) while using the results for α_i and $u_{i,0}(\mathbf{x})$. For (2.15), substituting $u(\mathbf{x}) = -4g_x^2(\mathbf{x})g^{-2}(\mathbf{x}) + u_{1,r}(\mathbf{x})g^{r-2}(\mathbf{x})$ and $v(\mathbf{x}) = \pm 2\sqrt{6\beta}\, g_x^2(\mathbf{x})g^{-2}(\mathbf{x}) + u_{2,r}(\mathbf{x})g^{r-2}(\mathbf{x})$ into (1.3) and keeping the most singular terms gives

$$Q \cdot \mathbf{u}_r = \begin{pmatrix} -(r-4)(r^2 - 5r - 18)\beta g_x^3(\mathbf{x}) & \pm 12\sqrt{6\beta}\, g_x^3(\mathbf{x}) \\ \mp 4(r-4)\sqrt{6\beta}\, g_x^3(\mathbf{x}) & (r-2)(r-7)r g_x^3(\mathbf{x}) \end{pmatrix} \begin{pmatrix} u_{1,r}(\mathbf{x}) \\ u_{2,r}(\mathbf{x}) \end{pmatrix} = \mathbf{0}.$$

Setting

$$\det Q = -\beta(r+2)(r+1)(r-3)(r-4)(r-6)(r-8)g_x^6(\mathbf{x}) = 0$$

yields $r = -2, -1, 3, 4, 6, 8$. This is not a principal branch.

As in the previous example, the constants of integration at level j are found by substituting (2.5) into (1.3) and pulling off the coefficients of $g^j(\mathbf{x})$. At $j = 1$,

$$\begin{pmatrix} -66\beta g_x^3(\mathbf{x}) & \mp 12\sqrt{6\beta}\, g_x^3(\mathbf{x}) \\ \mp 12\sqrt{6\beta}\, g_x^3(\mathbf{x}) & 6g_x^3(\mathbf{x}) \end{pmatrix} \begin{pmatrix} u_{1,1}(\mathbf{x}) \\ u_{2,1}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} -120\beta g_x^3(\mathbf{x})g_{2x}(\mathbf{x}) \\ \mp 60\sqrt{6\beta}\, g_x^3(\mathbf{x})g_{2x}(\mathbf{x}) \end{pmatrix}. \tag{2.17}$$

Thus,

$$u_{1,1}(\mathbf{x}) = 4g_{2x}(\mathbf{x}), \quad u_{2,1}(\mathbf{x}) = \pm 2\sqrt{6\beta}\, g_{2x}(\mathbf{x}). \tag{2.18}$$

At $j = 2$,

$$
\begin{aligned}
u_{1,2}(\mathbf{x}) &= \frac{3g_{2x}^2(\mathbf{x}) - g_x(\mathbf{x})\big(g_t(\mathbf{x}) + 4g_{3x}(\mathbf{x})\big)}{3g_x^2(\mathbf{x})}, \\
u_{2,2}(\mathbf{x}) &= \pm\frac{3\beta g_{2x}^2(\mathbf{x}) - 4\beta g_x(\mathbf{x})g_{3x}(\mathbf{x}) - (1+2\beta)g_t(\mathbf{x})g_x(\mathbf{x})}{\sqrt{6}\beta g_x^2(\mathbf{x})}.
\end{aligned}
\tag{2.19}
$$

The remaining constants of integration are omitted due to length. The compatibility conditions at $r = 3$ and 4 are satisfied. At $r = 6$ and $r = 8$, the compatibility conditions require $\beta = \frac{1}{2}$.

Likewise, for (2.16), the resonances following from the substitution of $u(\mathbf{x}) = -2g_x^2(\mathbf{x})g^{-2}(\mathbf{x}) + u_{1,r}(\mathbf{x})g^{r-2}(\mathbf{x})$ and $v(\mathbf{x}) = u_{2,0}(\mathbf{x})g^{-2}(\mathbf{x}) + u_{2,r}(\mathbf{x})g^{r-1}(\mathbf{x})$ into (1.3) are $r = -1, 0, 1, 4, 5, 6$. This is a principal branch. The zero resonance explains the arbitrariness of $u_{2,0}(\mathbf{x})$. Similarly, we computed all constants of integration, but ran into a compatibility conditions at $r = 5$ and $r = 6$, which requires $\beta = \frac{1}{2}$. Therefore, (1.3) passes the Painlevé test if $\beta = \frac{1}{2}$, a fact confirmed by other analyses of integrability [**1**].

3. Travelling wave solutions in hyperbolic or elliptic functions

The traveling wave solutions of many nonlinear ODEs and PDEs from soliton theory (and elsewhere) can be expressed as polynomials of hyperbolic or elliptic functions. For instance, the bell shaped sech-solutions and kink shaped tanh-solutions model wave phenomena in fluid dynamics, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions, bio-genetics, etc. A discussion is given in [**15**], while a multitude of references to tanh-based techniques and applications can be found in [**3, 19**].

In this section we discuss the tanh-, sech-, cn- and sn-methods as they apply to nonlinear polynomial systems of ODEs and PDEs in multi-dimensions.

3.1. Algorithm and implementation. All four flavors of the algorithm share the same five basic steps.

Step 1 (Transform the PDE into a nonlinear ODE). We seek solutions in the traveling frame of reference,

$$\xi = \sum_{j=1}^{N} c_j x_j + \delta, \tag{3.1}$$

where the components c_j of the wave vector and the phase δ are constants.

In the tanh method, we seek polynomial solutions expressible in hyperbolic tangent, $T = \tanh\xi$. Based on the identity $\cosh^2\xi - \sinh^2\xi = 1$,

$$\tanh'\xi = \operatorname{sech}^2\xi = 1 - \tanh^2\xi, \tag{3.2}$$

$$\tanh''\xi = -2\tanh\xi + 2\tanh^3\xi, \text{ etc.} \tag{3.3}$$

Therefore, the first and all higher-order derivatives of $T = \tanh\xi$ are polynomial in T. Consequently, repeatedly applying the chain rule,

$$T = \tanh(\xi)\colon \frac{\partial\bullet}{\partial x_j} = \frac{\partial\xi}{\partial x_j}\frac{dT}{d\xi}\frac{d\bullet}{dT} = c_j(1-T^2)\frac{d\bullet}{dT} \tag{3.4}$$

transforms (1.1) into a coupled system of nonlinear ODEs.

TABLE 3. Values for $R(F)$ in (3.11).

F	$R(F)$
T	0
S	$1 - S^2$
CN	$(1 - \mathrm{CN}^2)(1 - m + m\,\mathrm{CN}^2)$
SN	$(1 - \mathrm{SN}^2)(1 - m\,\mathrm{SN}^2)$

Similarly, using the identity $\tanh^2\xi + \operatorname{sech}^2\xi = 1$, we get

$$\operatorname{sech}'\xi = -\operatorname{sech}\xi\tanh\xi = -\operatorname{sech}\xi\sqrt{1-\operatorname{sech}^2\xi}. \tag{3.5}$$

Likewise, for Jacobi's elliptic functions with modulus m, we use the identities

$$\mathrm{sn}^2(\xi;m) = 1 - \mathrm{cn}^2(\xi;m), \quad \text{and} \quad \mathrm{dn}^2(\xi;m) = 1 - m + m\ \mathrm{cn}^2(\xi;m), \tag{3.6}$$

to write, for example, cn' in terms of cn :

$$\mathrm{cn}'(\xi;m) = -\mathrm{sn}(\xi;m)\,\mathrm{dn}(\xi;m) = -\sqrt{\left(1-\mathrm{cn}^2(\xi;m)\right)\left(1-m+m\,\mathrm{cn}^2(\xi;m)\right)}. \tag{3.7}$$

Then, repeatedly applying the chain rules,

$$S = \operatorname{sech}(\xi)\colon\ \frac{\partial\bullet}{\partial x_j} = -c_j S\sqrt{1-S^2}\frac{d\bullet}{dS}, \tag{3.8}$$

$$\mathrm{CN} = \mathrm{cn}(\xi;m)\colon\ \frac{\partial\bullet}{\partial x_j} = -c_j\sqrt{(1-\mathrm{CN}^2)(1-m+m\,\mathrm{CN}^2)}\frac{d\bullet}{d\mathrm{CN}}, \tag{3.9}$$

$$\mathrm{SN} = \mathrm{sn}(\xi;m)\colon\ \frac{\partial\bullet}{\partial x_j} = c_j\sqrt{(1-\mathrm{SN}^2)(1-m\mathrm{SN}^2)}\frac{d\bullet}{d\mathrm{SN}}, \tag{3.10}$$

we transform (1.1) into a coupled system of nonlinear ODEs of the form

$$\mathbf{\Gamma}(F, \mathbf{u}(F), \mathbf{u}'(F), \ldots) + \sqrt{R(F)}\mathbf{\Pi}(F, \mathbf{u}(F), \mathbf{u}'(F), \ldots) = \mathbf{0}, \tag{3.11}$$

where F is either T, S, CN, or SN, and $R(F)$ is defined in Table 3.

Step 2 (Determine the degree of the polynomial solutions). Since we seek polynomial solutions

$$U_i(F) = \sum_{j=0}^{M_i} a_{ij}F^j, \tag{3.12}$$

the leading exponents M_i must be determined before the a_{ij} can be computed. The process for determining M_i is quite similar to the one for finding α_i in Section 2.

Substituting $U_i(F)$ into (3.11), the coefficients of every power of F in every equation must vanish. In particular, the highest degree terms must vanish. Since the highest degree terms only depend on F^{M_i} in (3.12), it suffices to substitute $U_i(F) = F^{M_i}$ into (3.11). Doing so, we get

$$\mathbf{P}(F) + \sqrt{R(F)}\mathbf{Q}(F) = 0, \tag{3.13}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are polynomials in F. Equating every two possible highest exponents in each P_i and Q_i gives a linear system for determining M_i. The linear system is then solved for M_i.

If one or more exponents M_i remain undetermined, we assign a strictly positive integer value to the free M_i, so that every equation in (3.11) has at least two different terms with equal highest exponents in F.

Step 3 (Derive the algebraic system for the coefficients a_{ij}). To generate the system for the unknown coefficients a_{ij} and wave parameters c_j, substitute (3.12) into (3.11) and set the coefficients of F^j to zero. The resulting nonlinear algebraic system for the unknown a_{ij} is parameterized by the wave parameters c_j and the parameters in (1.1), if any.

Step 4 (Solve the nonlinear parameterized algebraic system). The most difficult aspect of the method is solving the nonlinear algebraic system. To solve the system we designed a customized, yet powerful, nonlinear solver.

The nonlinear algebraic system is solved under the following assumptions:

(1) all parameters (the lower-case Greek letters) in (1.1) are strictly positive. (Vanishing parameters may change the exponents M_i in Step 2). To compute solutions corresponding to negative parameters, reverse the signs of the parameters in (1.1).
(2) the coefficients of the highest power terms ($a_{i\,M_i}$, $i = 1, \ldots, M_i$) in (3.12) are all nonzero (for consistency with Step 2).
(3) all c_j are nonzero (demanded by the physical nature of the solutions).

Step 5 (Build and test solutions). Substitute the solutions from Step 4 into (3.12) and reverse Step 1 to obtain the explicit solutions in the original variables. It is prudent to test the solutions by substituting them into (1.1), which we do.

3.2. Examples of travelling wave solutions.

Example 3.1 (Kaup–Kupershmidt). While the tanh-, sech-, cn- and sn-methods find solutions for (1.2), we demonstrate the steps of the algorithm using the tanh-method. After which, we summarize the results for the other methods.

First, transform (1.2) into a nonlinear ODE by repeatedly applying chain rule (3.4). The resulting ODE is

$$
\begin{aligned}
(3.14)\quad & 2c_2U' \\
& + c_1\Big[10U^2U' + c_1^2\Big[25(T^2-1)U'[2TU' + (T^2-1)U'' \\
& \qquad + 10U\big[(6T^2-2)U' + 6T(T^2-1)U'' + (T^2-1)^2U''' \\
& \qquad\quad + c_1^2(16(15T^4-15T^2+2)U' + 40(T^2-1)\big(6T(2T^2-1)U'' \\
& \qquad\qquad + (6T^4-7T^2+1)U''' + T(T^2-1)^2U^{(4)}\big) + (T^2-1)^4U^{(5)}\big)\big]\Big]\Big] = 0,
\end{aligned}
$$

where $T = \tanh(\xi)$ and $U = U(T)$.

Next, to compute the degree of the polynomial solution(s), substitute $U(T) = T^{M_1}$ into (3.14) and pull off the exponents of T (see Table 4). Remove duplicates and non-dominant exponents, to get

$$(3.15)\qquad \{3M_1 - 1, 2M_1 + 1, M_1 + 3\}.$$

TABLE 4. The exponents of T after substituting $U(T) = T^{M_1}$.

Term	Exponents of T with duplicates removed
u_t	$M_1 - 1$
$u^2 u_x$	$3M_1 - 1$
$\frac{25}{2} u_x u_{2x}$	$2M_1 + 1, 2M_1 - 1, 2M_1 - 3$
$5uu_{3x}$	$2M_1 + 1, 2M_1 - 1, 2M_1 - 3$
u_{5x}	$M_1 + 3, M_1 + 1, M_1 - 1, M_1 - 3, M_1 - 5$

Consider all possible balances of two or more exponents to find $M_1 = 2$.

Substitute

$$U(T) = a_{10} + a_{11}T + a_{12}T^2 \tag{3.16}$$

into (3.14) and equate the coefficients of T^j to zero (where $i = 0, 1, \ldots, 5$) to get

$$\begin{gathered}
(a_{12} + 3c_1^2)(a_{12} + 24c_1^2) = 0, \\
a_{11}(5a_{12}^2 + 55a_{12}c_1^2 + 24c_1^4) = 0, \\
a_{11}(5a_{10}^2 c_1 - 10a_{10}c_1^3 + 25a_{12}c_1^3 + 16c_1^5 + c_2) = 0, \\
a_{11}(a_{11}^2 + 6a_{10}a_{12} + 6a_{10}c_1^2 - 48a_{12}c_1^2 - 24c_1^4) = 0, \\
4a_{11}^2 a_{12} + 4a_{10}a_{12}^2 + 11a_{11}^2 c_1^2 + 24a_{10}a_{12}c_1^2 - 56a_{12}^2 c_1^2 - 192a_{12}c_1^4 = 0, \\
10a_{10}a_{11}^2 c_1 + 10a_{10}^2 a_{12}c_1 - 35a_{11}^2 c_1^3 \\
\qquad - 80a_{10}a_{12}c_1^3 + 50a_{12}^2 c_1^3 + 272a_{12}c_1^5 + 2a_{12}c_2 = 0.
\end{gathered} \tag{3.17}$$

Solve the nonlinear algebraic system with the assumption that a_{12}, c_1, and c_2 are all nonzero. Two solutions are obtained:

$$\begin{cases} a_{10} = 16c_1^2, & a_{11} = 0, \\ a_{12} = -24c_1^2, & c_2 = -176c_1^5, \end{cases} \quad \text{and} \quad \begin{cases} a_{10} = 2c_1^2, & a_{11} = 0, \\ a_{12} = -3c_1^2, & c_2 = -c_1^5, \end{cases} \tag{3.18}$$

where c_1 is arbitrary.

Substitute the solutions into (3.16) and return to $u(x,t)$ to get

$$u(x,t) = 16c_1^2 - 24c_1^2 \tanh^2(c_1 x - 176c_1^5 t + \delta), \tag{3.19}$$

$$u(x,t) = 2c_1^2 - 3c_1^2 \tanh^2(c_1 x - c_1^5 t + \delta). \tag{3.20}$$

Using the sech-method, one finds

$$u(x,t) = -8c_1^2 + 24c_1^2 \operatorname{sech}^2(c_1 x - 176c_1^5 t + \delta), \tag{3.21}$$

$$u(x,t) = -c_1^2 + 3c_1^2 \operatorname{sech}^2(c_1 x - c_1^5 t + \delta). \tag{3.22}$$

Alternatively, the latter solutions can be found directly from the tanh-solutions by using the identity $\tanh^2 \xi = 1 - \operatorname{sech}^2 \xi$.

Using the sn-method, one gets

$$u(x,t) = 8c_1^2\big[1 + m - 3m\,\mathrm{sn}^2\big(c_1 x - 176c_1^5(m^2 - m + 1)t + \delta; m\big)\big], \tag{3.23}$$

$$u(x,t) = c_1^2\big[1 + m - 3m\,\mathrm{sn}^2\big(c_1 x - c_1^5(m^2 - m + 1)t + \delta; m\big)\big]. \tag{3.24}$$

The cn-solutions computed with the cn-method are not explicitly shown since they can be obtained from the sn-solutions using the identity $\mathrm{sn}^2\,\xi = 1 - \mathrm{cn}^2\,\xi$.

Example 3.2 (Hirota–Satsuma). As in the previous example, the tanh-, sech-, cn- and sn-methods all find solutions for (1.3). In this example, however, we will illustrate the steps using the sech-method.

Transform (1.3) into a coupled system of ODEs, apply the chain rule (3.8) and cancel the common $S\sqrt{1-S^2}$ factors to get

$$\begin{aligned}(3.25)\quad & c_2U_1' - 6\beta c_1U_1U_1' - \beta c_1^3[(1-6S^2)U_1' \\ &\qquad + 3S(1-2S^2)U_1'' + S^2(1-S^2)U_1'''] + 2c_1U_2U_2' = 0, \\ & c_2U_2' + 3c_1U_1U_2' + c_1^3[(1-6S^2)U_2' + 3S(1-2S^2)U_2'' + S^2(1-S^2)U_2'''] = 0.\end{aligned}$$

To find the degree of the polynomials, substitute $U_1(S) = S^{M_1}$, $U_2(S) = S^{M_2}$ into (3.25) and equate the highest exponents from Δ_2 to get

$$M_1 + M_2 - 1 = 1 + M_2 \quad \text{or} \quad M_1 = 2. \tag{3.26}$$

The maximal exponents coming from Δ_1 are $2M_1 - 1$ (from the U_1U_1' term), $M_1 + 1$ (from U_1'''), and $2M_2 - 1$ (from U_2U_2').

Since $M_1 = 2$ balances at least two of the possible dominant exponents in Δ_1, namely $2M_1 - 1$ and $M_1 + 1$, one is again left with $1 \le M_2 \le M_1 = 2$, or

$$\begin{cases} M_1 = 2, & U_1(S) = a_{10} + a_{11}S + a_{12}S^2, \\ M_2 = 1, & U_2(S) = a_{20} + a_{21}S. \end{cases} \tag{3.27}$$

$$\begin{cases} M_1 = 2, & U_1(S) = a_{10} + a_{11}S + a_{12}S^2, \\ M_2 = 2, & U_2(S) = a_{20} + a_{21}S + a_{22}S^2. \end{cases} \tag{3.28}$$

To derive the algebraic system for a_{ij}, substitute (3.27) into (3.25), cancel common numerical factors, and organize the equations (according to complexity):

$$\begin{aligned} a_{11}a_{21}c_1 &= 0, \\ \beta a_{11}c_1(3a_{12} - c_1^2) &= 0, \\ \beta a_{12}c_1(a_{12} - 2c_1^2) &= 0, \\ a_{21}c_1(a_{12} - 2c_1^2) &= 0, \\ a_{21}(3a_{10}c_1 + c_1^3 + c_2) &= 0, \\ 6\beta a_{10}a_{11}c_1 - 2a_{20}a_{21}c_1 + \beta a_{11}c_1^3 - a_{12}c_2 &= 0, \\ 3\beta a_{11}^2c_1 + 6\beta a_{10}a_{12}c_1 - a_{21}^2c_1 + 4\beta a_{12}c_1^3 - a_{12}c_2 &= 0. \end{aligned} \tag{3.29}$$

Similarly, after substitution of (3.28) into (3.25), one gets

$$
\begin{aligned}
&a_{22}c_1(a_{12}-4c_1^2)=0,\\
&a_{21}(3a_{10}c_1+c_1^3+c_2)=0,\\
&c_1(a_{12}a_{21}+2a_{11}a_{22}-2a_{21}c_1^2)=0,\\
&c_1(3\beta a_{11}a_{12}-a_{21}a_{22}-\beta a_{11}c_1^2)=0,\\
&c_1(3\beta a_{12}^2-a_{22}^2-6\beta a_{12}c_1^2)=0,\\
&6\beta a_{10}a_{11}c_1-2a_{20}a_{21}c_1+\beta a_{11}c_1^3-a_{11}c_2=0,\\
&3a_{11}a_{21}c_1+6a_{10}a_{22}c_1+8a_{22}c_1^3+2a_{22}c_2=0,\\
&3\beta a_{11}^2c_1+6\beta a_{10}^2a_{12}c_1-a_{21}^2c_1-2a_{20}a_{22}c_1+4\beta a_{12}c_1^3-a_{12}c_2=0.
\end{aligned}
\tag{3.30}
$$

Since a_{12}, a_{21}, β, c_1, and c_2, are nonzero, the solution of (3.29) is

$$
\begin{cases}
a_{10}=-(c_1^3+c_2)/(3c_1), & a_{20}=0,\\
a_{11}=0, & a_{21}=\pm\sqrt{4\beta c_1^4-2(1+2\beta)c_1c_2},\\
a_{12}=2c_1^2.
\end{cases}
\tag{3.31}
$$

For a_{12}, a_{22}, β, c_1, and c_2 nonzero, the solution of (3.30) is

$$
\begin{cases}
a_{10}=-(4c_1^3+c_2)/(3c_1), & a_{20}=\pm[4\beta c_1^3+(1+2\beta)c_2]/(c_1\sqrt{6\beta}),\\
a_{11}=0, & a_{21}=0,\\
a_{12}=4c_1^2, & a_{22}=\mp 2c_1^2\sqrt{6\beta}.
\end{cases}
\tag{3.32}
$$

The solutions of (1.3) involving sech are then

$$
\begin{aligned}
u(x,t)&=-\frac{c_1^3+c_2}{3c_1}+2c_1^2\,\mathrm{sech}^2(c_1x+c_2t+\delta),\\
v(x,t)&=\pm\sqrt{4\beta c_1^4-2(1+2\beta)c_1c_2}\,\mathrm{sech}(c_1x+c_2t+\delta),
\end{aligned}
\tag{3.33}
$$

and

$$
\begin{aligned}
u(x,t)&=-\frac{4c_1^3+c_2}{3c_1}+4c_1^2\,\mathrm{sech}^2(c_1x+c_2t+\delta),\\
v(x,t)&=\pm\frac{4\beta c_1^3+(1+2\beta)c_2}{c_1\sqrt{6\beta}}\mp 2c_1^2\sqrt{6\beta}\,\mathrm{sech}^2(c_1x+c_2t+\delta).
\end{aligned}
\tag{3.34}
$$

In both sets of solutions, c_1, c_2, β, and δ are arbitrary.

4. Recursion Operators

The previous two algorithms work for (1.1). The algorithm for recursion operators presented in Section 4.2 uses the concepts of dilation invariance, densities, and symmetries. Thus, in contrast to more general approaches [**20**], our method only works for polynomial systems of evolution equations in $(1+1)$ dimensions,

$$
\mathbf{u}_t=\mathbf{F}(\mathbf{u},\mathbf{u}_x,\mathbf{u}_{2x},\ldots,\mathbf{u}_{mx}),
\tag{4.1}
$$

where $\mathbf{u}(x,t)$ has M components u_i and $\mathbf{u}_{mx}=\partial^m\mathbf{u}/\partial x^m$. For brevity, we write $\mathbf{F}(\mathbf{u})$, although $\mathbf{F}$ typically depends on $\mathbf{u}$ and its x-derivatives up to order m. If present, any parameters in (4.1) are strictly positive and denoted by lower-case Greek letters.

The algorithm in Section 4.2 will use the concepts of dilation invariance, densities, and symmetries.

4.1. Scaling invariance, densities, symmetries. A PDE is *dilation invariant* if it is invariant under a dilation symmetry.

Example 4.1 (Kaup–Kupershmidt). As an example, (1.2) is invariant under the dilation (scaling) symmetry

$$(t, x, u) \rightarrow (\lambda^{-5} t, \lambda^{-1} x, \lambda^{2} u), \tag{4.2}$$

where λ is an arbitrary parameter, leaving λ^7 as a common factor upon scaling.

To find the dilation symmetry, set the *weight* of the x-derivative to one, $w(D_x) = 1$, and require that all terms in (4.1) have the same weight. For (1.2), we have

$$w(u) + w(D_t) = 3w(u) + 1 = 2w(u) + 3 = 2w(u) + 3 = w(u) + 5. \tag{4.3}$$

So, $w(u) = 2$ and $w(D_t) = 5$, Consequently, in (1.2) the sum of the weights or *rank* of each term is 7.

A *generalized symmetry*, $\mathbf{G}(\mathbf{u})$, leaves (4.1) invariant under the replacement $\mathbf{u} \rightarrow \mathbf{u} + \epsilon \mathbf{G}$ within order ϵ. Hence, $\mathbf{G}$ must satisfy the linearized equation [**21**]

$$D_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}], \tag{4.4}$$

on solutions of (4.1). $\mathbf{F}'(\mathbf{u})[\mathbf{G}]$ is the Fréchet derivative of $\mathbf{F}$ in the direction of $\mathbf{G}$.

A *recursion operator*, $\mathcal{R}$, is a linear integro-differential operator which links generalized symmetries [**21**]

$$\mathbf{G}^{(j+s)} = \mathcal{R}\mathbf{G}^{(j)}, \quad j \in \mathbb{N}, \tag{4.5}$$

where s is the seed and $\mathbf{G}^{(j)}$ is the j-th symmetry. The symmetries are linked consecutively if $s = 1$. This happens in most, but not all, cases.

A *conservation law* [**21**],

$$D_t \rho(x, t) + D_x J(x, t) = 0, \tag{4.6}$$

valid for solutions of (4.1), links a conserved density $\rho(x, t)$ with the associated flux $J(x, t)$.

If (4.1) is scaling invariant, then its conserved densities, fluxes, generalized symmetries, and recursion operators are also dilation invariant. One could say they 'inherit' the scaling symmetry of the original PDE. The existence of an infinite number of symmetries and an infinite number of conservation laws are good indicators for complete integrability [**21**]. Such predictors, albeit useful, do not provide information about the actual analytic behavior of solutions of PDEs which is also at the core of complete integrability [**1**].

4.2. Algorithm and implementation.

Step 1 (Compute conserved densities and generalized symmetries). For details about the algorithms and symbolic computation of conserved densities and generalized symmetries, see [**8–10**].

Step 2 (Determine the rank of the recursion operator). The rank of the recursion operator is determined by the difference in ranks of the generalized symmetries it links,

$$\operatorname{rank} \mathcal{R}_{ij} = \operatorname{rank} \mathbf{G}_i^{(k+s)} - \operatorname{rank} \mathbf{G}_j^{(k)}, \tag{4.7}$$

where $\mathcal{R}$ is an $M \times M$ matrix and $\mathbf{G}$ has M components.

Step 3 (Determine the form of the recursion operator). The recursion operator naturally splits into two pieces [**11**],

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \tag{4.8}$$

where $\mathcal{R}_0$ is a differential operator and $\mathcal{R}_1$ is an integral operator. The differential operator, $\mathcal{R}_0$, is a linear combination (with constant coefficients) of terms of type $D_x^i u^j$ $(i, j \in \mathbb{Z}^*)$, which must be of the correct rank. To standardize the form of $\mathcal{R}_0$, propagate D_x to the right, for example, $D_x^2 u = u_{2x} I + u_x D_x + u D_x^2$.

The integral operator, $\mathcal{R}_1$, is composed of the terms

$$\sum_i \sum_j \tilde{c}_{ij} \mathbf{G}^{(i)} D_x^{-1} \otimes \rho^{(j)\prime} \tag{4.9}$$

of the correct rank, where $\otimes$ is the matrix outer product and $\rho^{(j)\prime}$ is the covariant (Fréchet derivative of $\rho^{(j)}$). To standardize the form of $\mathcal{R}_1$, propagate D_x to the left, for example, $D_x^{-1} u_x D_x = u_x I - D_x^{-1} u_{2x} I$.

Step 4 (Determine the coefficients). To determine the coefficients in the form of the recursion operator, we substitute $\mathcal{R}$ into the defining equation [**11, 22**],

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}) - \mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = 0, \tag{4.10}$$

where $\circ$ denotes a composition of operators, $\mathcal{R}'[\mathbf{F}]$ is the Fréchet derivative of $\mathcal{R}$ in the direction of $\mathbf{F}$, and $\mathbf{F}'(\mathbf{u})$ is the Fréchet derivative with entries

$$\mathbf{F}'_{ij}(\mathbf{u}) = \sum_{k=0}^{m} \left(\frac{\partial F_i}{\partial (u_j)_{kx}} \right) D_x^k. \tag{4.11}$$

4.3. Examples of scalar and matrix recursion operators.

Example 4.2 (Kaup–Kupershmidt). Using the weights $w(u) = 2$, $w(D_x) = 1$, and $w(D_t) = 5$, we find

$$\begin{aligned} &\operatorname{rank} G^{(1)} = 3, \qquad &&\operatorname{rank} G^{(2)} = 7, \qquad &&\operatorname{rank} G^{(3)} = 9, \\ &\operatorname{rank} G^{(4)} = 13, \qquad &&\operatorname{rank} G^{(5)} = 15, \qquad &&\operatorname{rank} G^{(6)} = 19. \end{aligned} \tag{4.12}$$

We guess that $\operatorname{rank} \mathcal{R} = 6$ and $s = 2$, since $\operatorname{rank} G^{(2)} - \operatorname{rank} G^{(1)} \neq \operatorname{rank} G^{(3)} - \operatorname{rank} G^{(2)}$ but $\operatorname{rank} G^{(3)} - \operatorname{rank} G^{(1)} = \operatorname{rank} G^{(4)} - \operatorname{rank} G^{(2)} = 6$.

Thus, taking all $D_x^i u^j$ $(i, j \in \mathbb{Z}^*)$ such that $\operatorname{rank} D_x^i u^j = 6$ gives

$$\begin{aligned} \mathcal{R}_0 = {} & c_1 D_x^6 + c_2 u D_x^4 + c_3 u_x D_x^3 + c_4 u^2 D_x^2 + c_5 u_{2x} D_x^2 \\ & + c_6 u u_x D_x + c_7 u_{3x} D_x + c_8 u^3 I + c_9 u_x^2 I + c_{10} u u_{2x} I + c_{11} u_{4x} I. \end{aligned} \tag{4.13}$$

Using the densities $\rho^{(1)} = u$ and $\rho^{(2)} = 3u_x^2 - 4u^3$, and the symmetries $G^{(1)} = u_x$, and $G^{(2)} = F(u) = 5u^2 u_x + \frac{25}{2} u_x u_{2x} + 5 u u_{3x} + u_{5x}$ from (1.2), we compute

$$\begin{aligned} \mathcal{R}_1 &= \tilde{c}_{12} G^{(1)} D_x^{-1} \rho^{(2)\prime} + \tilde{c}_{21} G^{(2)} D_x^{-1} \rho^{(1)\prime} \\ &= \tilde{c}_{12} u_x D_x^{-1} (6 u_x D_x - 12 u^2 I) + \tilde{c}_{21} G^{(2)} D_x^{-1} I \\ &= c_{12} u_x [D_x^{-1} (u_{2x} I + 2u^2 I) - u_x I] + c_{13} G^{(2)} D_x^{-1}. \end{aligned} \tag{4.14}$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ and $G^{(2)} = F$ into (4.10) gives us 49 linear equations for c_i. Solving, we find

$$\begin{aligned} &c_1 = \frac{4c_9}{69}, \quad c_2 = \frac{8c_9}{23}, \quad c_3 = \frac{24c_9}{23}, \quad c_4 = \frac{12c_9}{23}, \quad c_5 = \frac{98c_9}{69}, \quad c_6 = \frac{40c_9}{23}, \\ &c_7 = \frac{70c_9}{69}, \quad c_8 = \frac{16c_9}{69}, \quad c_{10} = \frac{82c_9}{69}, \quad c_{11} = \frac{26c_9}{69}, \quad c_{12} = \frac{2c_9}{69}, \quad c_{13} = \frac{4c_9}{69}, \end{aligned} \tag{4.15}$$

where c_9 is arbitrary. Taking $c_9 = 69/4$, we find the recursion operator in [**22**]:

$$\begin{aligned} \mathcal{R} = {} & D_x^6 + 6uD_x^4 + 18u_xD_x^3 + 9u^2D_x^2 \\ & + \frac{49}{2}u_{2x}D_x^2 + 30uu_xD_x + \frac{35}{2}u_{3x}D_x + 4u^3I + \frac{69}{4}u_x^2I \\ & \quad + \frac{41}{2}uu_{2x}I + \frac{13}{2}u_{4x}I + \frac{1}{2}u_xD_x^{-1}(u_{2x} + 2u^2)I + G^{(2)}D_x^{-1}. \end{aligned} \tag{4.16}$$

Example 4.3 (Hirota–Satsuma). Only when $\beta = \frac{1}{2}$ does (1.3) have infinitely many densities and symmetries. The first few are

$$\begin{aligned} &\rho^{(1)} = u, \quad \rho^{(2)} = 3u^2 - 2v^2, \\ &\mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \mathbf{G}^{(2)} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \beta(6uu_x + u_{3x}) - 2vv_x \\ -(3uv_x + v_{3x}) \end{pmatrix}. \end{aligned} \tag{4.17}$$

We also computed the $\mathbf{G}^{(3)}$ and $\mathbf{G}^{(4)}$, but they are not shown due to length. Solving the equations for the weights,

$$\begin{cases} w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3 = 2w(v) + 1, \\ w(v) + w(D_t) = w(u) + w(v) + 1 = w(v) + 3, \end{cases} \tag{4.18}$$

yields $w(u) = w(v) = 2$ and $w(D_t) = 3$. Based on these weights, $\operatorname{rank}\rho^{(1)} = 2$, $\operatorname{rank}\rho^{(2)} = 4$, and

$$\begin{aligned} &\operatorname{rank}\mathbf{G}^{(1)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \operatorname{rank}\mathbf{G}^{(2)} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \\ &\operatorname{rank}\mathbf{G}^{(3)} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}, \quad \operatorname{rank}\mathbf{G}^{(4)} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}. \end{aligned} \tag{4.19}$$

We would first guess that $\operatorname{rank}\mathcal{R}_{ij} = 2$ and $s = 1$. If indeed the symmetries were linked consecutively, then

$$\mathcal{R}_0 = \begin{pmatrix} c_1D_x^2 + c_2uI + c_3vI & c_4D_x^2 + c_5uI + c_6vI \\ c_7D_x^2 + c_8uI + c_9vI & c_{10}D_x^2 + c_{11}uI + c_{12}vI \end{pmatrix}. \tag{4.20}$$

Using (4.17), we have

$$\mathcal{R}_1 = \tilde{c}_{11}\mathbf{G}^{(1)}D_x^{-1} \otimes \rho^{(1)\prime} = \tilde{c}_{11}\begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} I & 0 \end{pmatrix} = c_{13}\begin{pmatrix} u_xD_x^{-1} & 0 \\ v_xD_x^{-1} & 0 \end{pmatrix}.$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (4.10), we find $c_1 = \cdots = c_{13} = 0$. Therefore, either the form of $\mathcal{R}$ is incorrect or the system does not have a recursion operator. Let us now repeat the steps taking $s = 2$, so $\operatorname{rank}\mathcal{R}_{ij} = 4$. Then,

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R}_0)_{11} & (\mathcal{R}_0)_{12} \\ (\mathcal{R}_0)_{21} & (\mathcal{R}_0)_{22} \end{pmatrix} + \tilde{c}_{12}\mathbf{G}^{(1)}D_x^{-1} \otimes \rho^{(2)\prime} + \tilde{c}_{21}\mathbf{G}^{(2)}D_x^{-1} \otimes \rho^{(1)\prime}, \tag{4.21}$$

with $(\mathcal{R}_0)_{ij}$ a linear combination of $\{D_x^4, uD_x^2, vD_x^2, u_xD_x, v_xD_x, u^2, uv, v^2, u_{2x}, v_{2x}\}$. For instance,

$$(4.22)\quad (\mathcal{R}_0)_{12} = c_{11}D_x^4 + c_{12}uD_x^2 + c_{13}vD_x^2 + c_{14}u_xD_x + c_{15}v_xD_x + c_{16}u^2I + c_{17}uvI + c_{18}v^2I + c_{19}u_{2x}I + c_{20}v_{2x}I.$$

Using (4.17), the first term of $\mathcal{R}_1$ in (4.21) is

$$\mathcal{R}_1^{(1)} = \tilde{c}_{12}\mathbf{G}^{(1)}D_x^{-1} \otimes \rho^{(2)'} = \tilde{c}_{12}\begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} 6uI & -4vI \end{pmatrix} = c_{41}\begin{pmatrix} 3u_xD_x^{-1}uI & -2u_xD_x^{-1}vI \\ 3v_xD_x^{-1}uI & -2v_xD_x^{-1}vI \end{pmatrix}.$$

The second term of $\mathcal{R}_1$ in (4.21) is

$$\mathcal{R}_1^{(2)} = \tilde{c}_{21}\mathbf{G}^{(2)}D_x^{-1} \otimes \rho^{(1)'} = \tilde{c}_{21}\begin{pmatrix} F_1(\mathbf{u}) \\ F_2(\mathbf{u}) \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} I & 0 \end{pmatrix} = c_{42}\begin{pmatrix} F_1(\mathbf{u})D_x^{-1} & 0 \\ F_2(\mathbf{u})D_x^{-1} & 0 \end{pmatrix}.$$

Substituting the form of $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 = \mathcal{R}_0 + \mathcal{R}_1^{(1)} + \mathcal{R}_1^{(2)}$ into (4.10), the linear system for c_i has a non-trivial solution. Solving the linear system, we finally obtain

$$(4.23)\quad \mathcal{R} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix},$$

where

$$\mathcal{R}_{11} = D_x^4 + 8uD_x^2 + 12u_xD_x + 16u^2I + 8u_{2x}I - \frac{16}{3}v^2I + 4u_xD_x^{-1}uI + 12uu_xD_x^{-1} + 2u_{3x}D_x^{-1} - 8vv_xD_x^{-1},$$

$$\mathcal{R}_{12} = -\frac{20}{3}vD_x^2 - \frac{16}{3}v_xD_x^1 - \frac{16}{3}uvI - \frac{4}{3}v_{2x}I - \frac{8}{3}u_xD_x^{-1}vI$$

$$\mathcal{R}_{21} = -10v_xD_x^1 - 12v_{2x}I + 4v_xD_x^{-1}uI - 12uv_xD_x^{-1} - 4v_{3x}D_x^{-1},$$

$$\mathcal{R}_{22} = -4D_x^4 - 16uD_x^2 - 8u_xD_x^1 - \frac{16}{3}v^2I - \frac{8}{3}v_xD_x^{-1}vI.$$

A similar algorithm for the symbolic computation of recursion operators of systems of differential-difference equations (DDEs) is given elsewhere in these proceedings [**14**].

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