Unitarity in dual QCD

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In a previous paper we pointed out that dual QCD in its original form violated unitarity. In this paper we identify the cause for this violation and construct a new dual QCD Lagrangian which is unitary. We have not yet been able to determine whether this new Lagrangian leads to confinement. Finally we point out how our original dual QCD Lagrangian which successfully describes many of the aspects of the physics of confinement can be regarded as a phenomenological Lagrangian which does not violate unitarity at the classical or tree level.

I. INTRODUCTION

During the past few years we have been studying long-distance Yang-Mills theory based on a Lagrangian $\mathcal{L}(C)$ expressed in terms of dual potentials C_μ^a [1,2]. This work led to a concrete realization of the dual superconductor picture of QCD [3]. As such it yielded many of the properties of a confining theory such as the existence of (a) quantized vortices of confined color-electric flux, (b) a static potential between quarks similar to that between monopoles in a superconductor, (c) a deconfinement transition, and a chiral-symmetry transition closely related to it.

In order to write the long-distance Lagrangian $\mathcal{L}(C)$ in local form it was necessary to introduce additional degrees of freedom described by an antisymmetric tensor field $\tilde{F}_{\mu\nu}^a$ [where a is color index running from 1 to N^2-1 for SU(N)]. These fields played the role of scalar Higgs fields in a relativistic superconductor, and were essential in preventing electric-color flux from spreading out. But because of the Lorentz metric, the part of $\mathcal{L}(C)$ arising from the kinetic energy of the fields $\tilde{F}_{\mu\nu}^a$ contained terms of the wrong sign [4]. The purpose of this paper is to investigate possible violations of unitarity that these terms could produce [4].

In order to establish notation let us review the definition of dual potentials in Abelian gauge theory, describing a relativistic dielectric medium characterized by a momentum-dependent dielectric constant $\epsilon(q)$ and magnetic permeability $\mu(q)$, where

$$\mu(q)=\frac{1}{\epsilon(q)} \ . \tag{1.1}$$

The equations of motion are the source-free Maxwell's equations:

$$\nabla\cdot\mathbf{D}=0, \quad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t} \ , \tag{1.2}$$

$$\nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \ , \tag{1.3}$$

and the constitutive equations are

$$\mathbf{D}=\epsilon\mathbf{E}, \quad \mathbf{B}=\mu\mathbf{H} \ , \tag{1.4}$$

which relate the electric displacement vector $\mathbf{D}$ and the magnetic $\mathbf{H}$ vector to $\mathbf{E}$ and $\mathbf{B}$. We introduce dual potentials C_μ to solve Eqs. (1.2) by writing

$$\mathbf{D}=-\nabla\times\mathbf{C} \ , \tag{1.5}$$

$$\mathbf{H}=-\frac{\partial\mathbf{C}}{\partial t}-\nabla C_0 \ . \tag{1.6}$$

The field equations for C_μ are generated by the Lagrangian $\mathcal{L}^{(0)}$, given by

$$\mathcal{L}^{(0)}=-\tfrac{1}{4}G^{\mu\nu}\mu(q)G_{\mu\nu} \ , \tag{1.7}$$

where

$$G_{\mu\nu}\equiv\partial_\mu C_\nu-\partial_\nu C_\mu \ . \tag{1.8}$$

In the Landau gauge the resulting C_μ propagator $\Delta_{\mu\nu}^C=\langle(C_\mu,C_\nu)+\rangle$ is

$$\Delta_{\mu\nu}^C=\frac{1}{q^2\mu(q)}\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right] \ . \tag{1.9}$$

This propagator describes exactly the same physics as the ordinary A_μ propagator Δ^A, where

$$\Delta_{\mu\nu}^A=\frac{1}{q^2\epsilon(q)}\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right] \ . \tag{1.10}$$

In Yang-Mills theory we know from Mandelstam's work [5] that dual potentials can be defined and that the Yang-Mills Lagrangian as a function of the dual variables

is invariant under a non-Abelian gauge transformation of the dual potentials $C^{\mu,a}$,

$$C^{\mu,a}\to C^{\mu,a}+\mathcal{D}_C^{\mu,ab}\delta\omega^b(x)\ , \tag{1.11}$$

where

$$\mathcal{D}_C^{\mu,ab}\equiv\delta^{ab}\partial^\mu+gf_{adb}C^{\mu,d}\ . \tag{1.12}$$

Here, $g\equiv 2\pi/e$, where e is the ordinary Yang-Mills coupling constant, so that $\alpha_s=e^2/4\pi$, and f_{abc} are the SU(N) structure constants. In the non-Abelian case we do not know the explicit form of the Yang-Mills Lagrangian as a function of the dual variables, but we are interested in solving the theory only at long distances, and for this purpose, we need find only the Lagrangian $\mathcal{L}(C)$ describing the long-distance regime of Yang-Mills theory in terms of dual variables.

Our ansatz for $\mathcal{L}(C)$ is the following. $\mathcal{L}(C)$ is the minimal extension of the Abelian Lagrangian, Eq. (1.7), with

$$\mu(q)=-\frac{M^2}{q^2}+1\ , \tag{1.13}$$

which is invariant under non-Abelian gauge transformations. That is, $\mathcal{L}(C)$ is the minimal gauge-invariant extension of $\mathcal{L}^{(0)}(C)$ where

$$\mathcal{L}^{(0)}(C)=-\frac{1}{4}G_{\mu\nu}\frac{M^2}{\partial^2}G_{\mu\nu}-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\ . \tag{1.14}$$

The form Eq. (1.13) for $\mu(q)$ was motivated by the results of many authors [6], who showed that the simplest self-consistent truncation of the Schwinger-Dyson equations for the gluon propagator Δ^A leads to a Δ^A which has the behavior $\Delta^A\sim -M^2/(q^2)^2$ as $q^2\to 0$. (M is an undetermined parameter having the dimension mass.) Thus they find $\epsilon\to -q^2/M^2+\cdots$ as $q^2\to 0$. Consequently $\mu\to -M^2/q^2+\cdots$ as $q^2\to 0$. By choice of normalization of the dual potential we then obtain Eq. (1.13) as the leading infrared contribution to $\mu(q)$ in this approximation. Inserting Eq. (1.13) into Eq. (1.9), we find that this solution yields a C_μ propagator $\Delta^C_{\mu\nu}$, given by

$$\Delta^C_{\mu\nu}=\frac{1}{q^2-M^2}\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]\ . \tag{1.15}$$

Thus the infrared singular properties $\Delta^A\sim -M^2/(q^2)^2$ corresponds to an infrared nonsingular Δ^C Eq. (1.15).

The Lagrangian $\mathcal{L}^{(0)}$ yields in the Landau gauge the propagator Eq. (1.15) and describes the solution of the approximate Dyson equation in terms of dual variables. Since Δ^C is nonsingular at small q^2 it can be used as the starting point for calculating infrared contributions not included in the truncated Schwinger-Dyson equations. We assume that $\mathcal{L}(C)$, the minimal gauge-invariant extension of $\mathcal{L}^{(0)}(C)$ determines these leading infrared corrections. We will now construct $\mathcal{L}(C)$ and find the consequences of this assumption. As a first step will show that $\mathcal{L}^{(0)}(C)$ possess a tensor gauge symmetry, which plays an important role in constructing $\mathcal{L}(C)$.

II. CONSTRUCTION OF $\mathcal{L}(C)$

In order to write $\mathcal{L}^{(0)}(C)$, Eq. (1.14), in local form we introduce an antisymmetric tensor field $\tilde{F}_{\mu\nu}$. (We omit color indices since $\mathcal{L}^{(0)}$ is just a sum of terms corresponding to each of the colors.) We integrate the identity

$$\exp\left[i\left[-\frac{1}{4}G_{\mu\nu}\frac{M^2}{\partial^2}G^{\mu\nu}\right]\right]\exp\left\{i\left[\frac{1}{4}\left[\tilde{F}_{\mu\nu}+G_{\mu\nu}\frac{M}{\partial^2}\right]\partial^2\left[\tilde{F}^{\mu\nu}+\frac{M}{\partial^2}G^{\mu\nu}\right]\right]\right\}=\exp\left[i\left[\frac{M}{2}\tilde{F}_{\mu\nu}G^{\mu\nu}+\frac{1}{4}\tilde{F}_{\mu\nu}\partial^2\tilde{F}^{\mu\nu}\right]\right]\ , \tag{2.1}$$

over $\tilde{F}_{\mu\nu}$ and evaluate the integral over the left-hand side of Eq. (2.1) by translating variables:

$$\tilde{F}_{\mu\nu}\to\tilde{F}_{\mu\nu}+\frac{M}{\partial^2}G_{\mu\nu}\ . \tag{2.2}$$

Since $G_{\mu\nu}$ satisfies the kinematic identity

$$\partial^\mu\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}=0\ , \tag{2.3}$$

we may with no loss of generality integrate only over fields $\tilde{F}_{\mu\nu}$ satisfying the same identity:

$$\partial^\mu\epsilon_{\mu\nu\alpha\beta}\tilde{F}^{\alpha\beta}=0\ . \tag{2.4}$$

Using Eqs. (1.14) and (2.1) we can then replace $\mathcal{L}^{(0)}$ by the local Lagrangian

$$\mathcal{L}^{(0)}=\frac{M}{2}\tilde{F}_{\mu\nu}G^{\mu\nu}+\frac{1}{4}\tilde{F}_{\mu\nu}\partial^2\tilde{F}^{\mu\nu}-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}\ . \tag{2.5}$$

We make use of our freedom to impose the constraint (2.4) and the identity

$$\frac{1}{4}(\partial_\alpha\tilde{F}_{\beta\gamma})(\partial^\alpha\tilde{F}^{\beta\gamma})=\frac{(\partial_\alpha\tilde{F}_{\beta\gamma})(\partial^\beta\tilde{F}^{\alpha\gamma})}{2}-\frac{1}{8}(\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\tilde{F}_{\beta\gamma})^2 \tag{2.6}$$

to rewrite $\mathcal{L}^{(0)}$ as

$$\mathcal{L}^{(0)}=\frac{M}{2}\tilde{F}_{\mu\nu}G^{\mu\nu}-\frac{1}{2}\partial_\beta\tilde{F}^{\beta\gamma}\partial^\alpha\tilde{F}_{\alpha\gamma}-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}\ . \tag{2.7}$$

This Lagrangian is invariant under the Abelian vector and tensor gauge transformations:

$$\delta C_\mu=\partial_\mu\delta\omega(x)\ , \tag{2.8}$$

$$\delta\tilde{F}_{\alpha\beta}=\epsilon_{\alpha\beta\sigma\kappa}\partial^\sigma\delta\Lambda^\kappa(x)\ , \tag{2.9}$$

where $\delta\omega(x)$ and $\delta\Lambda^\kappa(x)$ are parameters characterizing the vector and tensor gauge transformations, respectively. Equation (2.4) can be regarded as a gauge-fixing condition. The second term in Eq. (2.7) is the Lagrangian for a free antisymmetric tensor gauge field. It is known to describe a massless scalar particle [7]. In the presence of the coupling term $(M/2)G_{\mu\nu}\tilde{F}^{\mu\nu}$ in $\mathcal{L}^{(0)}$, this massless

scalar degree of freedom combines with the two transverse C_μ degrees of freedom to produce a massive vector particle, as expected from Eq. (1.15). We will verify this below.

We can add to Eq. (2.7) any multiple of the last term of Eq. (2.6) without changing the physics. Such an addition becomes a gauge-fixing term which breaks the tensor gauge invariance. The Lagrangian $\mathcal{L}^{(0)}$, Eq. (2.5), which was the starting point of our previous work is an example of such a gauge-fixed Lagrangian. In this work we start with the physically equivalent Abelian Lagrangian, Eq. (2.7), in which the tensor gauge invariance is manifest. This will enable us to extend (2.7) to an interacting Lagrangian $\mathcal{L}$, which possesses a corresponding non-Abelian symmetry, which was absent in our previous interacting Lagrangian, obtained by extending the form, Eq. (2.5), of $\mathcal{L}^{(0)}$.

Equation (2.7) can be written in first-order form by introducing a further auxiliary field Z^μ:

$$\mathcal{L}^{(0)}=\frac{M}{2}\tilde{F}^{\mu\nu}W_{\mu\nu}+\frac{1}{2}M^2Z^\mu Z_\mu-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\ , \quad (2.10)$$

where

$$W_{\mu\nu}\equiv\partial_\mu(C_\nu+Z_\nu)-\partial_\nu(C_\mu+Z_\mu)\ . \quad (2.11)$$

Varying Z^μ in Eq. (2.10) gives

$$Z^\nu=\frac{1}{M}\partial_\mu\tilde{F}^{\mu\nu}\ . \quad (2.12)$$

Inserting Eq. (2.12) into Eq. (2.10) yields the original second-order expression, Eq. (2.7), for $\mathcal{L}^{(0)}$. This first-order form of $\mathcal{L}^{(0)}$ is also invariant under the gauge transformations, Eqs. (2.8) and (2.9), provided that

$$\delta Z_\mu=0\ . \quad (2.13)$$

Now we use Eq. (2.10) to determine the interacting Lagrangian $\mathcal{L}$ by the requirement that $\mathcal{L}$ be invariant under the non-Abelian gauge transformation, Eq. (1.11), of the dual potentials $C^{\mu,a}$. However $\mathcal{L}^{(0)}$ is also invariant under the Abelian tensor gauge transformation, Eq. (2.9). This guarantees that the field $\tilde{F}_{\mu\nu}$ contains only one degree of freedom. Correspondingly the Lagrangian $\mathcal{L}$ must be invariant under a non-Abelian generalization of Eq. (2.9) in order that no spurious degrees of freedom are introduced by extending $\mathcal{L}^{(0)}$ to $\mathcal{L}$.

The minimal Lagrangian $\mathcal{L}$, having the necessary symmetry, is readily constructed from Eqs. (2.10) and (2.11) as follows: Define

$$G^a_{\mu\nu}\equiv\partial_\mu C^a_\nu-\partial_\nu C^a_\mu+gf_{abc}C^b_\mu C^c_\nu\ , \quad (2.14)$$

$$Y^a_\mu\equiv C^a_\mu+Z^a_\mu\ , \quad (2.15)$$

$$W^a_{\mu\nu}\equiv\partial_\mu Y^a_\nu-\partial_\nu Y^a_\mu+gf_{abc}Y^b_\mu Y^c_\nu\ . \quad (2.16)$$

Then

$$\mathcal{L}=\frac{M}{2}\tilde{F}^a_{\mu\nu}W^{\mu\nu,a}+\frac{M^2}{2}Z^{\mu,a}Z^a_\mu-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}\ . \quad (2.17)$$

This Lagrangian describes long-distance Yang-Mills theory in terms of dual potentials C^a_μ, and the auxiliary variables $\tilde{F}^a_{\mu\nu}$ and Z^a_μ. Freedman and Townsend [8] first wrote down a Lagrangian of the form of (2.17) and studied some of its properties at the classical level.

By construction $\mathcal{L}$ is invariant under the non-Abelian vector gauge transformation:

$$\delta C^{\mu,a}(x)=\mathcal{D}^{\mu,ab}_C\delta\omega^b(x)\ , \quad (2.18)$$

$$\delta\tilde{F}^a_{\mu\nu}(x)=gf_{abc}\tilde{F}^b_{\mu\nu}\delta\omega^c(x)\ , \quad (2.19)$$

$$\delta Z^{\mu,a}(x)=gf_{abc}Z^{\mu,b}\delta\omega^c(x)\ , \quad (2.20)$$

where $\mathcal{D}^{\mu,ab}_C$ is given by Eq. (1.12). For many purposes it is convenient to consider C^a_μ and Y^a_μ as the independent fields. Equations (2.15), (2.18), and (2.20) then give

$$\delta Y^{\mu,a}(x)=\mathcal{D}^{\mu,ab}_Y\delta\omega^b(x)\ , \quad (2.21)$$

where

$$\mathcal{D}^{\mu,ab}_Y\equiv\partial^\mu\delta^{ab}+gf_{adb}Y^{\mu,d}\ . \quad (2.22)$$

Furthermore, the action is also invariant under the transformation

$$\delta\tilde{F}^a_{\alpha\beta}=\epsilon_{\alpha\beta\sigma\kappa}\mathcal{D}^{\sigma,ab}_Y\delta\Lambda^{\kappa,b}(x)\ , \quad (2.23)$$

which is the necessary non-Abelian generalization of the tensor gauge transformation. This invariance is a consequence of the kinematic identity

$$\epsilon_{\mu\nu\sigma\kappa}\mathcal{D}^{\nu,ab}_Y W^{\sigma\kappa,b}=0\ . \quad (2.24)$$

The action is invariant since the Lagrangian changes by a perfect divergence under the transformation, Eq. (2.23).

By contrast we obtained our original form of $\mathcal{L}$ by making the minimal substitution

$$\partial^\mu\to\mathcal{D}^{\mu,ab}_C,\quad G_{\mu\nu}\to G^a_{\mu\nu}\ , \quad (2.25)$$

in the Lagrangian $\mathcal{L}^{(0)}$ in the form (2.5). This led to a Lagrangian, which in this paper we denote by $\mathcal{L}_1$, given by

$$\mathcal{L}_1=\frac{M}{2}G^a_{\mu\nu}\tilde{F}^{\mu\nu,a}+\frac{1}{4}\tilde{F}^a_{\mu\nu}(\mathcal{D}^2_C)_{ab}F^{\mu\nu,b}-\frac{1}{4}G^{\mu\nu,a}G^a_{\mu\nu}\ . \quad (2.26)$$

This Lagrangian is invariant under the vector gauge transformations (2.18) and (2.19) but is not invariant under any tensor gauge transformation. As a consequence the fields $\tilde{F}^a_{\mu\nu}$ appear as physical degrees of freedom in Eq. (2.26).

Had we made the minimal substitution, (2.25) in the new form (2.10) of $\mathcal{L}^{(0)}$ we would have an interacting Lagrangian, which although distinct from Eq. (2.26), is also not invariant under a tensor gauge transformation. This is because it lacks an essential term

$$\frac{gM}{2}\tilde{F}^a_{\mu\nu}f_{abc}Z^b_\mu Z^c_\nu\ , \quad (2.27)$$

which is present in (2.17).

Indeed if we write out the expression for $W^a_{\mu\nu}$ appearing in (2.17) as

$$W^a_{\mu\nu}=G^a_{\mu\nu}+\partial_\mu Z^a_\nu-\partial_\nu Z^a_\mu+gf_{abc}(C^b_\mu Z^c_\nu+Z^b_\mu C^c_\nu)+gf_{abc}Z^b_\mu Z^c_\nu\ , \quad (2.28)$$

the first four terms in Eq. (2.28) are obtained by making the minimal substitution in $W_{\mu\nu}$. The last term in Eq. (2.28) when multiplied by $\tilde{F}^a_{\mu\nu}$, yields (2.27) which is present in $\mathcal{L}$ and missing in $\mathcal{L}_1$.

III. THE HAMILTONIAN

Let us next turn to the Hamiltonian in order to elucidate some of the features of the theory described by $\mathcal{L}$. We will show that in the case $g=0$, $\mathcal{L}$ describes a system of noninteracting vector particles of mass M.

It is convenient to use a three-dimensional notation for the components of the tensor $\tilde{F}_{\mu\nu}$. We define vectors $\mathbf{E}^a$ and $\mathbf{B}^a$ as

$$B^a_k \equiv -M\tilde{F}^a_{0k}, \quad E^a_k \equiv -\frac{M}{2}\epsilon_{klm}\tilde{F}^a_{lm} \ . \tag{3.1}$$

Choosing Y^a_μ and C^a_μ as the independent variables we can write the integral of $\mathcal{L}$ in Hamiltonian form:

$$\int \mathcal{L}\, d\mathbf{x} = \int d\mathbf{x}(-\mathbf{H}^a\cdot\dot{\mathbf{C}}^a - \mathbf{B}^a\cdot\dot{\mathbf{Y}}^a - \mathcal{H}) \ , \tag{3.2}$$

where the Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{\mathbf{H}^{a^2}}{2} + \frac{\mathbf{D}^{a^2}}{2} + \frac{M^2}{2}(\mathbf{Z}^{a^2} - Z_0^{a^2}) + Z_0^a \mathcal{D}_C^{ab}\cdot\mathbf{H}^b - Y_0^a(\mathcal{D}_C^{ab}\cdot\mathbf{H}^b + \mathcal{D}_Y^{ab}\cdot\mathbf{B}^b) + \mathbf{W}^a\cdot\mathbf{E}^a \ . \tag{3.3}$$

The fields $\mathbf{D}^a$ and $\mathbf{W}^a$ appearing in Eq. (3.3) are defined as

$$\mathbf{W}^a = -\nabla\times\mathbf{Y}^a + \frac{g}{2}f_{abd}\mathbf{Y}^b\times\mathbf{Y}^d \ , \tag{3.4}$$

$$\mathbf{D}^a = -\nabla\times\mathbf{C}^a + \frac{g}{2}f_{abd}\mathbf{C}^b\times\mathbf{C}^d \ , \tag{3.5}$$

i.e.,

$$W^a_k \equiv \tfrac{1}{2}\epsilon_{kmn}W^a_{mn} \quad \text{and} \quad D^a_k = \tfrac{1}{2}\epsilon_{kmn}G^a_{mn} \ .$$

Equation (3.5), defining the color-electric displacement vector $\mathbf{D}^a$, is the non-Abelian generalization of Eq. (1.5). Since $-\mathbf{H}^a$ is the momentum conjugate to $\mathbf{C}^a$, $\mathbf{H}^a$ must be the color-magnetic $\mathbf{H}$ vector. We can see this explicitly by varying $\mathcal{L}$ with respect to $\mathbf{H}^a$. This gives the equation of motion

$$\mathbf{H}^a = -\dot{\mathbf{C}}^a - \mathcal{D}_C^{ab}C^{0b} \ , \tag{3.6}$$

which is just the non-Abelian generalization of Eq. (1.6).

Varying $\mathcal{L}$ with respect to C^a_0 with Y^a_0 fixed gives

$$\mathcal{D}_C^{ab}\cdot\mathbf{H}^b = M^2 Z^a_0 \equiv M^2(Y^a_0 - C^a_0) \ . \tag{3.7}$$

Equation (3.7), which is the dual of Gauss' law, shows that $M^2Z^a_0$ is the monopole charge density. Using Eq. (3.7) to eliminate the dependent variable C^a_0 in the Hamiltonian we obtain

$$\mathcal{H} = \frac{\mathbf{H}^{a^2}}{2} + \frac{\mathbf{D}^{a^2}}{2} + \frac{(\mathcal{D}_C^{ab}\cdot\mathbf{H}^b)^2}{2M^2} + \frac{M^2}{2}\mathbf{Z}^{a^2} - Y_0^a(\mathcal{D}_C^{ab}\cdot\mathbf{H}^b + \mathcal{D}_Y^{ab}\cdot\mathbf{B}^b) + \mathbf{W}^a\cdot\mathbf{E}^a \ . \tag{3.8}$$

The variables Y^a_0 and $\mathbf{E}^a$ appear as Lagrange multipliers and their variation in $\mathcal{L}$ leads to the equations of constraint

$$\mathcal{D}_C^{ab}\cdot\mathbf{H}^b + \mathcal{D}_Y^{ab}\cdot\mathbf{B}^b = 0 \ , \tag{3.9}$$

$$\mathbf{W}^a = 0 \ . \tag{3.10}$$

Varying $\mathbf{C}^a$ in $\mathcal{L}$ gives the equation

$$\mathcal{D}_C^{0ab}\mathbf{H}^b + \mathcal{D}_C^{ab}\times\mathbf{D}^b = -M^2\mathbf{Z}^a \ , \tag{3.11}$$

where the variable C^a_0 appearing in Eq. (3.11) through $\mathcal{D}_C^{0ab}$ is determined from Eq. (3.7). Equation (3.11), which is the dual of Ampere's law, shows that $M^2\mathbf{Z}^a$ is the monopole current density. Equations (3.7) and (3.11) yield monopole current conservation:

$$\mathcal{D}_C^{0ab}Z^b_0 + \mathcal{D}_C^{ab}\cdot\mathbf{Z}^b = 0 \ . \tag{3.12}$$

Varying $\mathbf{B}^a$ in $\mathcal{L}$ gives

$$\dot{\mathbf{Y}}^a + \mathcal{D}_Y^{ab}Y^{0b} = 0 \ . \tag{3.13}$$

Equation (3.6), (3.11), and (3.13) are dynamical equations for $\mathbf{C}^a$, $\mathbf{H}^a$, and $\mathbf{Y}^a$, respectively, which are decoupled from the equation for $\mathbf{B}^a$. The Lagrange multiplier field Y^a_0 appearing in Eq. (3.13) is not determined by the dynamics. To specify the solution we must impose a gauge condition. For example, we can choose

$$\nabla\cdot\mathbf{C}^a = 0 \ . \tag{3.14}$$

The equation for $\mathbf{B}^a$, obtained by varying $\mathbf{Y}^a$ in $\mathcal{L}$, is

$$\mathcal{D}_Y^{0,ab}\mathbf{B}^b + \mathcal{D}_Y^{ab}\times\mathbf{E}^b = M^2\mathbf{Z}^a \ . \tag{3.15}$$

The Lagrange multiplier field $\mathbf{E}^a$ appearing in Eq. (3.15) is not fixed by the dynamics and we must impose further gauge conditions. We can choose

$$\nabla\times\mathbf{B}^a = 0 \ . \tag{3.16}$$

Note Eqs. (3.7) and (3.9) yield

$$\mathcal{D}_Y^{ab}\cdot\mathbf{B}^d = -M^2 Z^a_0 \ . \tag{3.17}$$

Equations (3.15) and (3.17) are the dual Yang-Mills equations for the fields $\mathbf{E}^a$ and $\mathbf{B}^a$ with gauge potential Y^a_μ in the presence of a monopole current density $M^2Z^a_\mu$.

Let us now count the number of independent degrees of freedom. For each color we started with six degrees of freedom described by the six pairs of canonical variables $(\mathbf{C}^a,\mathbf{H}^a)$, $(\mathbf{Y}^a,\mathbf{B}^a)$. We have one constraint in Eq. (3.9), and three constraints in (3.10), only two of which are independent because of the identity

$$\mathcal{D}_Y\cdot\mathbf{W} = 0 \ . \tag{3.18}$$

The constraints fix the transverse components $\mathbf{Y}^{a^T}$ of the $\mathbf{Y}^a$ in terms of their longitudinal components $\mathbf{Y}^{a^L}$. The gauge conditions (3.16) eliminate the transverse components of $\mathbf{B}^a$, while the constraints (3.9) eliminate the longitudinal components $\mathbf{H}^{a^L}$ of $\mathbf{H}^a$. For each color we are left with three pairs of independent canonical variables $(\mathbf{C}^{a^T},\mathbf{H}^{a^T})$ and $(\mathbf{Y}^{a^L},\mathbf{B}^{a^L})$ corresponding to three degrees of freedom for each color. For SU(N) we are thus left with the three (N^2-1) independent degrees of freedom necessary to describe (N^2-1) massive vector

particles.

Now let us apply this discussion to the Abelian theory described by $\mathcal{L}^{(0)}$, obtained by setting $g=0$ in $\mathcal{L}$. In this case the constraint (3.10) reduces to

$$\nabla\times\mathbf{Y}=0 \ . \tag{3.19}$$

Then from Eqs. (3.5) and (3.19) we obtain

$$\mathbf{D}=\nabla\times\mathbf{Z} \ . \tag{3.20}$$

Using Eq. (3.20) in Eq. (3.11) then gives

$$\dot{\mathbf{H}}+\nabla\times(\nabla\times\mathbf{Z})=-M^2\mathbf{Z} \ . \tag{3.21}$$

Adding Eqs. (3.6) and (3.13) gives

$$\mathbf{H}=\dot{\mathbf{Z}}+\nabla Z^0 \ . \tag{3.22}$$

Equation (3.21) and (3.22) are two coupled equations for $\mathbf{Z}$ and $\mathbf{H}$. The dependent field Z^0 is determined in terms of $\mathbf{H}$ by Gauss' law, Eq. (3.7), which for $g=0$ assumes the form

$$Z^0=\frac{\nabla\cdot\mathbf{H}}{M^2} \ . \tag{3.23}$$

Note that we have made no use of the gauge-fixing conditions to obtain Eqs. (3.21) and (3.22). This is because we have obtained an equation for Z^μ which transforms like a current and not like a gauge field. Inserting the time derivative of Eq. (3.22) into Eq. (3.21) and using the equation of continuity, Eq. (3.12), $\partial_\mu Z^\mu=0$, we obtain

$$(\partial^2+M^2)\mathbf{Z}=0 \ . \tag{3.24}$$

Taking the divergence of Eq. (3.22) gives the same equation for Z_0. Thus the monopole current density Z^μ is a free massive vector field.

Finally we evaluate the Hamiltonian $H=\int d\mathbf{x}\,\mathcal{H}(\mathbf{x})$. Using Eq. (3.3) and the equations of motion, (3.21) and (3.22), we obtain

$$H=\int d\mathbf{x}[-Z^{\mu(-)}(x)\omega Z_\mu^{(+)}(x)] \ , \tag{3.25}$$

where

$$Z_\mu^{(+)}(x)=\frac{1}{\sqrt{2\omega}}(\dot{Z}_\mu\mp i\omega Z_\mu) \ , \tag{3.26}$$

and

$$\omega=\sqrt{-\nabla^2+M^2} \ . \tag{3.27}$$

Equation (3.25) is the usual Hamiltonian for a system of noninteracting vector particles of mass M. Furthermore, the canonical commutation relations of $\mathbf{C}$ and $\mathbf{H}$ yield the usual massive vector field commutation relations of Z_μ and $\dot{Z}_\mu$. This demonstrates the result [8] that the coupled Maxwell antisymmetric tensor Lagrangian, Eq. (2.10), provides a gauge-invariant description of a massive vector particle.

IV. UNITARITY AND OTHER PROPERTIES OF $\mathcal{L}$

In this section we discuss further properties, including unitarity, of the interacting Lagrangian, Eq. (2.17), describing long-range Yang-Mills theory in terms of dual potentials. For the purposes of the remaining discussion, it is convenient to use matrix notation: Let us use SU(N) generators T_a, normalized so that

$$2\,\mathrm{Tr}T_aT_b=\delta_{ab} \ , \tag{4.1}$$

and define matrices Z_μ, C_μ, $\tilde{F}_{\mu\nu}$, and Y_μ, $G_{\mu\nu}$, and $W_{\mu\nu}$, by the equations

$$\begin{aligned}
&Z_\mu=\sum_a Z_\mu^a T_a, \quad C_\mu=\sum_a C_\mu^a T_a \ ,\\
&\tilde{F}_{\mu\nu}=\sum_a \tilde{F}_{\mu\nu}^a T_a, \quad Y_\mu=C_\mu+Z_\mu \ ,\\
&G_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu-ig[C_\mu,C_\nu] \ ,\\
&W_{\mu\nu}=\partial_\mu Y_\nu-\partial_\nu Y_\mu-ig[Y_\mu,Y_\nu] \ .
\end{aligned} \tag{4.2}$$

Then the Lagrangian can be written as

$$\mathcal{L}=2\,\mathrm{tr}\left[\frac{M}{2}\tilde{F}_{\mu\nu}W^{\mu\nu}+\frac{M^2}{2}Z^\mu Z_\mu-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right] \ . \tag{4.3}$$

The matrix form of the vector gauge transformations is

$$C_\mu\to\Omega^{-1}C_\mu\Omega+\frac{i}{g}\Omega^{-1}\partial_\mu\Omega \ , \tag{4.4}$$

$$Z_\mu\to\Omega^{-1}Z_\mu\Omega \ , \tag{4.5}$$

$$\tilde{F}_{\mu\nu}\to\Omega^{-1}\tilde{F}_{\mu\nu}\Omega \ , \tag{4.6}$$

where Ω is an SU(N) matrix. Note that under these transformations

$$Y_\mu\to\Omega^{-1}Y_\mu\Omega+\frac{i}{g}\Omega^{-1}\partial_\mu\Omega \ , \tag{4.7}$$

and hence

$$W_{\mu\nu}\to\Omega^{-1}W_{\mu\nu}\Omega \ . \tag{4.8}$$

Furthermore the action $S\equiv\int dx\,\mathcal{L}$ is invariant under the tensor gauge transformation, which in matrix notation takes the form

$$C_\mu\to C_\mu, \quad Z_\mu\to Z_\mu \ , \tag{4.9}$$

$$\tilde{F}_{\alpha\beta}\to\tilde{F}_{\alpha\beta}+\epsilon_{\alpha\beta\gamma\delta}\mathcal{D}_Y^\gamma\delta\Lambda^\delta(x) \ , \tag{4.10}$$

where

$$\mathcal{D}_Y^\alpha\equiv\partial^\alpha-ig[Y^\alpha,] \ . \tag{4.11}$$

Under the transformation (4.10), $\mathcal{L}\to\mathcal{L}+\delta\mathcal{L}$ where

$$\delta\mathcal{L}=2\,\mathrm{tr}\left[\frac{M}{2}[\epsilon_{\alpha\beta\gamma\delta}\mathcal{D}_Y^\gamma\delta\Lambda^\delta(x)]W^{\alpha\beta}\right]=2\,\mathrm{tr}\left[-\frac{M^2}{2}\delta\Lambda^\delta(x)\epsilon_{\alpha\beta\gamma\delta}\mathcal{D}_Y^\gamma(x)W^{\alpha\beta}+\partial^\gamma\left[\frac{M}{2}\epsilon_{\alpha\beta\gamma\delta}\delta\Lambda^\delta(x)W^{\alpha\beta}\right]\right] \ . \tag{4.12}$$

The first term in the right-hand side of Eq. (4.12) vanishes identically while the second term gives no contribution to the action.

Freedman and Townsend [8] showed that the self-interacting antisymmetric tensor field Lagrangian obtained by setting $C_\mu=0$ in Eq. (4.3) is classically equivalent to the nonlinear σ model. Later Slavnov and Frolov [9] quantized this interacting antisymmetric tensor theory and demonstrated that it was unitary. They also proved its quantum-mechanical equivalence to the nonlinear σ model. Kimura [7] and Hata and Kugo [7] had previously shown that when suitable gauge fixing and ghost terms are added to the pure antisymmetric tensor Lagrangian, the tensor gauge symmetry is replaced by a global Beechi-Rovet-Stora-Tyutin (BRST) invariance. Clark, Lee, and Love [10] showed the Lagrangian in Eq. (4.3) also possessed a BRST invariance. This guarantees that the theory defined by the Lagrangian, Eq. (4.3), is unitarity.

The Lagrangian (4.3) is nonrenormalizable. This should produce no problems in applications of dual QCD where we are restricted to low-momentum phenomena.

Next note that adding to $\mathcal{L}$ polynomials involving products of traces of $Z_\mu, G_{\alpha\beta}$ and their covariant derivatives $\mathcal{D}_C^\mu Z^\nu$ and $\mathcal{D}_C^\mu G_{\alpha\beta}$ will yield a Lagrangian which remains invariant under the transformations (4.4)–(4.10). The covariant derivative $\mathcal{D}_C^\mu$ can be written as

$$\mathcal{D}_C^\mu \equiv \partial^\mu - ig[C^\mu,\ ] = \mathcal{D}_Y^\mu + ig[Z^\mu,\ ]\ . \tag{4.13}$$

We can write $W^{\mu\nu}$ as

$$W^{\mu\nu} = \mathcal{D}_C^\mu Z^\nu - \mathcal{D}_C^\nu Z^\mu + G^{\mu\nu} - ig[Z^\mu, Z^\nu]\ . \tag{4.14}$$

Equations (4.13) and (4.14) can be used to eliminate $\mathcal{D}_Y^\mu$ and $W^{\mu\nu}$ from any gauge-invariant additions to $\mathcal{L}$.

We had previously constructed $\mathcal{L}$ as the minimal gauge-invariant extension of the Abelian theory (2.7) based on the $M^2/(q^2)^2$ propagator. $\mathcal{L}$ already contains the dimension-4 operator $G_{\mu\nu}G^{\mu\nu}$. There are clearly other gauge-invariant dimension-4 operators, such as, for example, $W_{\mu\nu}W^{\mu\nu}$ or $[Z_\mu, Z_\nu]^2$ which we could have added to $\mathcal{L}$. We did not include these terms in Sec. II since nothing essential would have changed at that point. Nevertheless such terms do have effects which we will describe below. Note that we cannot add additional terms involving the tensor field $\tilde{F}_{\alpha\beta}$, since there are no other forms involving $\tilde{F}_{\alpha\beta}$ which are invariant under the tensor gauge transformation Eq. (4.10).

In this section we discuss only the minimal Lagrangian (4.3), supplemented by a particular dimension-4 operator. We add to (4.3) a term $-V(Z_\mu)$ where $V(Z_\mu)$ has the Skyrme form [11]

$$V(Z_\mu) = -\frac{\lambda}{8}\mathrm{tr}[Z_\mu, Z_\nu]^2\ . \tag{4.15}$$

Then making the replacement

$$\mathcal{L} \to \mathcal{L} - V(Z_\mu)\ , \tag{4.16}$$

we have

$$\mathcal{L} = 2\,\mathrm{tr}\left[\frac{M}{2}\tilde{F}_{\mu\nu}W^{\mu\nu} + \frac{M^2}{2}Z^\mu Z_\mu - \frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{\lambda}{16}[Z_\mu, Z_\nu]^2\right]\ . \tag{4.17}$$

We chose $V(Z_\mu)$ to have the Skyrme form in order to have a positive-definite Hamiltonian. The inclusion of the additional term $-V(Z_\mu)$ in $\mathcal{L}$, Eq. (4.17), does not essentially change the construction of the Hamiltonian density $\mathcal{H}$ carried out in Sec. III. We obtain

$$\mathcal{H} = 2\,\mathrm{tr}\left[\frac{\mathbf{H}^2}{2} + \frac{\mathbf{D}^2}{2} + \frac{M^2}{2}(\mathbf{Z}^2 + Z^{02}) - \frac{\lambda}{16}([Z^k, Z^l]^2 + 2[Z^0, Z^k]^2)\right]\ . \tag{4.18}$$

The fields $\mathbf{H}$ and $\mathbf{D}$ appearing in Eq. (4.18) are just the matrix forms of $\mathbf{H}^a$ and $\mathbf{D}^a$, Eqs. (3.5) and (3.6). With $\lambda > 0$ the Hamiltonian is positive definite.

Next let us obtain the equations of motion generated from (4.17). Varying $\tilde{F}_{\mu\nu}$ gives

$$W_{\mu\nu} = 0\ , \tag{4.19}$$

while varying C_μ gives

$$\mathcal{D}_C^\mu G_{\mu\nu} - M\mathcal{D}_Y^\mu \tilde{F}_{\mu\nu} = 0\ . \tag{4.20}$$

Finally varying Z^μ yields

$$-M\mathcal{D}_Y^\mu \tilde{F}_{\mu\nu} + M^2 Z_\nu - \frac{\delta V}{\delta Z^\nu} = 0\ . \tag{4.21}$$

Subtracting Eq. (4.20) from Eq. (4.21) then gives

$$\mathcal{D}_C^\mu G_{\mu\nu} = M^2 Z_\nu - \frac{\delta V}{\delta Z^\nu}\ . \tag{4.22}$$

Equation (4.22) is the covariant form of Eqs. (3.7) and (3.11), modified to account for the addition of $-V(Z_\mu)$ to $\mathcal{L}$. Thus we see that $M^2 Z^0 - \delta V/\delta Z^0$ is the monopole charge density while $M^2\mathbf{Z} + \delta V/\delta \mathbf{Z}$ is the monopole current density.

The Lagrangian (4.3) is equivalent to a gauged nonlinear σ model [8,10]. To express the Lagrangian (4.17) as a nonlinear σ model we first note that the general solution to the equation of motion, Eq. (4.19), is

$$Y_\mu = \frac{i}{g}U\partial_\mu U^{-1}\ , \tag{4.23}$$

where U is an SU(N) matrix. We now use Eq. (4.23) to eliminate $Z_\mu \equiv Y_\mu - C_\mu$ from the Lagrangian Eq. (4.17). We have

$$Z_\mu = \frac{i}{g}U\partial_\mu U^{-1} - C_\mu = -\frac{i}{g}(D_\mu U)U^{-1}\ , \tag{4.24}$$

where

$$D_\mu U \equiv \partial_\mu U - igC_\mu U\ . \tag{4.25}$$

Using Eqs. (4.17), (4.19), and (4.24) we obtain

$$\mathcal{L}=2\,\mathrm{tr}\left[\frac{M^2}{2g^2}(D_\mu U)(D_\mu U)^\dagger-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right]-\frac{\lambda}{8g^4}\mathrm{tr}[(D_\mu U)U^{-1},(D_\nu U)U^{-1}]^2 . \tag{4.26}$$

Equation (4.26) has the form of a gauged nonlinear σ model supplemented by a gauged Skyrme term.

We had seen for $g=0$ and $\lambda=0$ that the Lagrangian (4.17) describes a system of N^2-1 noninteracting particles of mass M, which we call dual gluons. This noninteracting dual theory is equivalent to an Abelian theory having an infrared singular A_μ propagator Δ_A, which has the behavior $\Delta_A \sim -M^2/(q^2)^2$ as $q^2\to 0$. As a consequence it produces a linear potential between quarks. However the electric-color flux between the quarks spreads out. There is no simple form for $V(Z)$ which confines this flux at the classical level.

V. STABLE SPHERICALLY SYMMETRIC CLASSICAL SOLUTIONS

The perturbation solution of the theory defined by (4.17) describes a system of interacting dual gluons. To have any possibility of describing confinement one must study the Lagrangian (4.17) or, equivalently, (4.26) nonperturbatively. This has been done to some extent by Ambjørn and Rubakov, [12], Eilam, Klabucar, and Stern [13], and Brihaye and Kunz [14], who found stable static spherically symmetric solutions of the classical equations of motion generated by $\mathcal{L}$ for the case that the gauge group is SU(2). We begin by classifying all the known classical solutions [12–14], i.e., the unstable ones as well. First note that for $\lambda=0$, Eq. (4.26) describes the limit of the Higgs sector of the SU(2) piece of the standard model as $M_{\mathrm{Higgs}}\to\infty$. For finite $M_{\mathrm{Higgs}}<12M_W$, this theory is known to have an unstable spherically symmetric classical solution (the sphaleron). Yaffe [15] has shown that as M_{Higgs} is increased new unstable solutions appear ("deformed sphalerons"), whose masses are less than that of the sphaleron. In the limit $M_{\mathrm{Higgs}}\to\infty$, described by the Lagrangian (4.26) with $\lambda=0$, there then appear an infinite sequence of unstable classical solutions [14], with masses which approach the sphaleron mass from below. The lowest mass in this sequence is finite.

For $\lambda\neq 0$ each of the $\lambda=0$ solutions develop into a branch of solutions which vary smoothly with λ for values of λ less than some critical value. All these solutions remain unstable except the branch of solutions that begins at the lowest mass $\lambda=0$ solution. In this case solutions also continue to exist and to remain unstable as λ is increased as long as the parameter ξ defined as

$$\xi\equiv g^2/\lambda \tag{5.1}$$

remains greater than $\xi_{\mathrm{crit}}\approx 10.35$. There are no solutions for $\xi<\xi_{\mathrm{crit}}$. However for $\xi>\xi_{\mathrm{crit}}$ a new branch of solutions develop which are stable and which have a lower mass than the unstable branch for all values of ξ for which they exist, i.e., for

$$\infty>\xi>\xi_{\mathrm{crit}}\approx 10.35 . \tag{5.2}$$

It is these stable solutions (solitons) that could be relevant to our work, and we now describe the work of Ambjørn and Rubakov [12] who found these solitons and studied their properties. They choose the gauge

$$U=1 . \tag{5.3}$$

Then, from Eq. (4.23),

$$Y_\mu=0 ,$$

or

$$C_\mu=-Z_\mu .$$

Therefore we can eliminate C_μ from the Hamiltonian density $\mathcal{H}$, Eq. (4.18). The fields $\mathbf{D}$ and $\mathbf{H}$, Eqs. (3.5) and (3.6) appearing in $\mathcal{H}$ are now functions of Z_μ. For static Z_μ all terms in $\mathcal{H}$ containing Z_0 are quadratic and nonnegative. Hence stable static classical solution of the field equations must have

$$Z_0=0 . \tag{5.4}$$

The magnetic $\mathbf{H}$ vector Eq. (3.6) vanishes and $\mathcal{H}$ reduces to

$$\mathcal{H}=2\,\mathrm{tr}\left[\frac{\mathbf{D}^2}{2}+\frac{M^2}{2}\mathbf{Z}^2-\frac{\lambda}{16}[Z^k,Z^l]^2\right] , \tag{5.5}$$

where the color electric displacement vector $\mathbf{D}$ is given in matrix form by

$$\mathbf{D}=\nabla\times\mathbf{Z}-i\frac{g}{2}[\mathbf{Z}\times,\mathbf{Z}] . \tag{5.6}$$

We introduce the following rescaled fields Z'_μ and coordinates x'_μ:

$$Z'_\mu=\frac{\sqrt{\lambda}}{M}Z_\mu,\quad x'_\mu=\frac{Mg}{\sqrt{\lambda}}x_\mu . \tag{5.7}$$

Then

$$\mathbf{D}=\frac{M^2g}{\lambda}\mathbf{D}' , \tag{5.8}$$

where

$$\mathbf{D}'\equiv\nabla'\times\mathbf{Z}'-i\frac{g}{2}[\mathbf{Z}'\times\mathbf{Z}'] . \tag{5.9}$$

We then obtain

$$\mathcal{H}=\frac{M^4}{\lambda}\mathcal{H}'(\xi) , \tag{5.10}$$

where

$$\mathcal{H}'(\xi)=2\,\mathrm{tr}\left[\frac{\xi\mathbf{D}'^2}{2}+\frac{1}{2}\mathbf{Z}'^2-\frac{1}{16}[\mathbf{Z}'^k,\mathbf{Z}'^l]^2\right] , \tag{5.11}$$

and ξ is defined in Eq. (5.1). The Hamiltonian H is given by

$$H=\int d\mathbf{x}\,\mathcal{H}=\frac{M^4}{\lambda}\left[\frac{\sqrt{\lambda}}{Mg}\right]^3\int d\mathbf{x}'\mathcal{H}'(\xi)=\frac{M}{g^2}\frac{H'(\xi)}{\sqrt{\xi}} , \tag{5.12}$$

where

$$H'(\xi)\equiv\int d\mathbf{x}'\mathcal{H}'(\xi)\ . \tag{5.13}$$

Now let $\mathbf{Z}'_\xi(\mathbf{x}')$ be a local minimum of $H'(\xi)$, and let $E'(\xi)$ be the corresponding minimum value of the scaled Hamiltonian $H'(\xi)$. Then from Eq. (5.12) the mass $\overline{M}$ of this solution is

$$\overline{M}=\frac{M}{g^2}\frac{E'(\xi)}{\sqrt{\xi}}\ . \tag{5.14}$$

Ambjørn and Rubakov [12] found such solitons $\mathbf{Z}'_\xi(\mathbf{x}')$ for all ξ in the range $\infty>\xi>\xi_{\text{crit}}$. Thus the dual QCD Lagrangian $\mathcal{L}$ has stable solutions when $g^2/\lambda>\xi_{\text{crit}}$ with a mass $\overline{M}$ given by Eq. (5.14). To see what these solutions look like and to find $E'(\xi)$, first consider the limit $\xi\to\infty$. Then the term $\xi\mathbf{D}'^2$ dominates $\mathcal{H}'(\xi)$, Eq. (5.11), unless

$$\mathbf{D}'=0\ . \tag{5.15}$$

Hence from Eq. (5.9) in order to minimize $H'(\xi)$ as $\xi\to\infty$, $\mathbf{Z}'$ must be a pure gauge, i.e.,

$$\mathbf{Z}'=iU_0^{-1}\nabla' U_0\ . \tag{5.16}$$

The function U_0 is determined by substituting the form (5.16) for $\mathbf{Z}'$ into Eq. (5.11) and minimizing the Hamiltonian H', Eq. (5.13). Only the second and third terms in Eq. (5.11) are nonvanishing and H' reduces to the Skyrme Hamiltonian [11] for a nonlinear σ model field U_0 without gauge interactions. The local minimum of H' then satisfies Skyrme's equation.

The solutions of Skyrme's equation are characterized by their winding number $n(\mathbf{Z}')$ given by

$$n(\mathbf{Z}')=\frac{1}{24\pi^2}\int d\mathbf{x}'\,\mathrm{Tr}\mathbf{Z}'\cdot(\mathbf{Z}'\times\mathbf{Z}')\ , \tag{5.17}$$

where $\mathbf{Z}'$ has the form Eq. (5.16). The minimal energy nontrivial solution U_0 of Skyrme's equation, called the Skyrmion U_S, is spherically symmetric and has winding number one. Then setting

$$U_0=U_S \tag{5.18}$$

in Eq. (5.16) gives

$$\mathbf{Z}_S(\mathbf{x}')=iU_S^{-1}\nabla' U_S\ . \tag{5.19}$$

Equation (5.19) yields a $\mathbf{Z}_S(\mathbf{x}')$ having a spherically symmetric structure mixing space and color variables like that of a monopole potential. The corresponding value of the energy E' is the Skyrmion mass M_S where, with our dimensionless units,

$$M_S\approx 315\ . \tag{5.20}$$

Hence

$$\lim_{\xi\to\infty}\mathbf{Z}'_\xi(\mathbf{x}')=\mathbf{Z}_S(\mathbf{x}') \tag{5.21}$$

and

$$\lim_{\xi\to\infty}E'(\xi)=M_S\ . \tag{5.22}$$

Furthermore Ambjørn and Rubakov find that their solutions for $\mathbf{Z}'_\xi(\mathbf{x}')$ changes very smoothly from its limiting value Eq. (5.21) as ξ is reduced. For example for $\xi=15$, $E'(\xi)$ is reduced by about 15% from its limiting value Eq. (5.22) and the "winding number" $n(\mathbf{Z}')$ is reduced to 0.8 from its limiting value one. Of course, for finite ξ, $\mathbf{Z}'$ is no longer a pure gauge so $n(\mathbf{Z}')$ has no topological significance and these solutions are not topologically stable. The value of $n(\mathbf{Z}'_\xi)$, like that of $E'(\xi)$, gives us a measure of how much the solution $\mathbf{Z}'_\xi$ changes with ξ. Ambjørn and Rubakov present curves for $\mathbf{Z}'_\xi(\mathbf{x}')$ for different values of ξ which show that for all values of ξ for which stable solutions exist, i.e., for ξ satisfying Eq. (5.2), $\mathbf{Z}'_\xi(\mathbf{x}')$ remains qualitatively similar to its limiting Skyrmion form $\mathbf{Z}_S(\mathbf{x}')$. Indeed if we denote the soliton size in dimensionless units by $R'(\xi)$, then $R'(\xi)$ is of the order unity. (It increases somewhat less than 50% as ξ decreases from ∞ to $\xi_{\text{crit}}\approx 10.35$.)

Expressing these results in terms of the unscaled variables $\mathbf{Z}(\mathbf{x})$ via Eq. (5.7), we obtain

$$\mathbf{Z}(\mathbf{x})=\frac{M}{g}\sqrt{\xi}\mathbf{Z}'_\xi(M\sqrt{\xi}\mathbf{x})\ . \tag{5.23}$$

Then Eqs. (5.21) and (5.23) give

$$\lim_{\xi\to\infty}\mathbf{Z}(\mathbf{x})=\frac{M}{g}\sqrt{\xi}\mathbf{Z}_S(M/\sqrt{\xi}\mathbf{x})\ . \tag{5.24}$$

Equation (5.14) and (5.22) then yield the following limiting value for the corresponding soliton mass $\overline{M}$:

$$\lim_{\xi\to\infty}\overline{M}=\frac{M}{g^2\sqrt{\xi}}M_S\ , \tag{5.25}$$

where M_S is the dimensionless Skyrmion mass (5.22). Furthermore Eq. (5.24) gives a good qualitative description of the soliton field $\mathbf{Z}(\mathbf{x})$ and Eq. (5.25) gives the soliton mass to within 20% for all $\xi>\xi_{\text{crit}}$. From Eq. (5.24) we see that radius R of the soliton is of the order

$$R\sim\frac{1}{M\sqrt{\xi}}=\frac{\sqrt{\lambda}}{Mg}\ , \tag{5.26}$$

and from Eq. (5.25) its mass $\overline{M}$ is of the order

$$\tilde{M}\sim\frac{M}{g^2}\left[\frac{\lambda}{g^2}\right]^{1/2}M_S\ . \tag{5.27}$$

From Eq. (5.27) we see that as $\lambda\to 0$ for fixed g^2, the mass $\overline{M}$ and the radius R of the soliton both vanish.

For small λ the mass $\overline{M}$ of the soliton becomes much smaller than the elementary vector particle mass M. Although not topologically stable except in the limit $\xi\to\infty$, these solitons have a monopolelike structure and are classically stable. They have small mass and we can speculate that they could condense in the vacuum so as to change essentially the vacuum structure of the theory. To see whether this could occur requires quantizing these monopolelike solutions and determining nonperturbatively their effect on the vacuum. We have not carried out such a calculation. It is essential to do this since using the perturbative vacuum leads to the existence of a physical dual gluon of mass M, which probably does not exist.

Finally it is clear that there are no classical flux-tube solutions based on the perturbative vacuum. To see this

suppose there was a two-dimensional static configuration $\mathbf{Z}_m(\mathbf{x})$ which minimized the Hamiltonian obtained from $\mathcal{H}$, Eq. (5.5). Then the configuration $\mathbf{Z}_\beta=\beta\mathbf{Z}_m(\beta\mathbf{x})$ would yield a value H_β of the Hamiltonian of the form $H_\beta=a\beta^2+b$ where a and b are positive. Then $dH_\beta/d\beta|_{\beta=1}>0$, i.e., $\mathbf{Z}_m$ could not have minimized H (this is just the usual Derrick scaling argument).

VI. PREVIOUS WORK ON DUAL QCD BASED ON LAGRANGIAN $\mathcal{L}_1$

We now discuss the implications of the present work on our original treatment of long-distance QCD based on the Lagrangian $\mathcal{L}_1$, Eq. (2.26). To obtain the final form of $\mathcal{L}_1$ we must make two additions. First we make the replacement

$$\mathcal{L}_1\to\mathcal{L}_1+\psi_i^{a\dagger}(\mathcal{D}_C^2)_{ab}\psi_i^b\equiv\mathcal{L}_1 , \tag{6.1}$$

where ψ_i^a and $\psi_i^{a\dagger}$ are a set of ghosts fields and the index i runs from one to three. The functional integral over ψ_i and $\psi_i^{a\dagger}$ then cancels the determinant arising from the functional integration over $\tilde{F}_{\mu\nu}^a$ in Eq. (2.1). Second we replace

$$\mathcal{L}_1\to\mathcal{L}_1-W(F,\psi,\tilde{\psi})\equiv\mathcal{L}_1 , \tag{6.2}$$

where W is a fourth-order polynomial in $\tilde{F}_{\mu\nu}$, ψ_i, and $\psi_i^\dagger$. The polynomial W is the counter term necessary for renormalization. The Lagrangian $\mathcal{L}_1$ is renormalizable, unlike $\mathcal{L}$. W has the form

$$W=-\frac{\mu^2N}{2}\tilde{F}^2+\frac{N\lambda}{4}W_4(\tilde{F}) + \text{terms involving } \psi_i \text{ and } \psi_i^\dagger , \tag{6.3}$$

where

$$\tilde{F}^2=\tilde{F}_{\mu\nu}^a\tilde{F}^{\mu\nu a}=-2(\mathbf{B}^{a2}-\mathbf{E}^{a2}) , \tag{6.4}$$

and $W_4(\tilde{F})$ is a quartic function of $\tilde{F}_{\mu\nu}^a$. The parameters μ^2 and λ determine the strength of the counterterms. The color-magnetic field $\mathbf{B}^a$ and the color-electric field $\mathbf{E}^a$ are defined by Eq. (3.1). The variables $\mathbf{E}^a$ and $\mathbf{B}^a$ appearing in $\mathcal{L}_1$ are independent of the dual potentials C_μ^a and serve as a convenient relabeling of the components of $\tilde{F}_{\mu\nu}^a$. With the identification, Eq. (3.1), the equations of motion obtained by varying $\tilde{F}_{\mu\nu}$ in the noninteracting Lagrangian become the constitutive relations, Eq. (1.4), where $\epsilon=\mu^{-1}=\partial^2/M$. For this reason we denote $\mathbf{E}^a$ and $\mathbf{B}^a$ as color-electric and -magnetic fields. These variables appear automatically in $\mathcal{L}_1$, via Eq. (2.1), as auxiliary fields necessary to render the Lagrangian local.

The structure of W is compatible with the vector gauge symmetry of $\mathcal{L}_1$. Note such a counterterm could not appear as a nonminimal addition to the Lagrangian $\mathcal{L}$, Eq. (2.17), because W is not invariant under the tensor gauge symmetry, Eq. (2.23). On the other hand, since $\mathcal{L}_1$ is not invariant under this transformation, the term W is compatible with the invariances of $\mathcal{L}_1$. The $\tilde{F}_{\mu\nu}^a$ then play the role of Higgs fields, and W that of a Higgs potential. Since $\lambda>0$ (because of stability) the minimum of W always occurs at a nonvanishing value $\tilde{F}_{0\mu\nu}$ of $\tilde{F}_{\mu\nu}$, and there is always spontaneous symmetry breaking. If $\mu^2<0$, then $-\tilde{F}_0^2=2(\mathbf{B}^{a2}-\mathbf{E}^{a2})>0$ and the QCD vacuum is magnetic. Let us call M_g the mass of the gluon. Then as a consequence of spontaneous symmetry breaking ($\tilde{F}_{0\mu\nu}\neq0$) the gluon mass M_g is given by

$$M_g^2=M^2+M_1^2 , \tag{6.5}$$

where [4]

$$M_1^2\equiv-\frac{g^2\tilde{F}_0^2}{4} . \tag{6.6}$$

M is the mass of the dual gluon in the perturbative vacuum and M_1 is the contribution to the gluon mass via the Higgs mechanism in the nonperturbative vacuum ($\tilde{F}_{0\mu\nu}\neq0$). Furthermore since the "Higgs fields" lie in the adjoint representation, there exist two-dimensional static solutions of the equations of motion generated by $\mathcal{L}_1$ corresponding to vortex tubes of quantized color electric flux. The flux tube is confined by the "magnetic pressure" produced by the nonvanishing vacuum expectation value $\mathbf{B}_0^a$ of $\mathbf{B}^a$. The vacuum expectation value of $\mathbf{E}^a$ vanishes. Thus

$$-\tilde{F}_0^2=2\mathbf{B}_0^{a2} . \tag{6.7}$$

This flux tube then produces a linear potential between quarks just as the magnetic vortices in a superconductor produce a linear potential between monopoles.

The flux tube solution becomes particularly simple if we let $M\to0$. In this case the gluon mass M_g^2 is given by

$$M_g^2=-\frac{g^2\tilde{F}_0^2}{4}=-\frac{g^2\mathbf{B}_0^{a2}}{2} . \tag{6.8}$$

The mass M_g of the dual gluon arises entirely from the nonvanishing vacuum expectation of the color magnetic field, just as the photon mass in a relativistic superconductor arises from the nonvanishing vacuum expectation value of the Higgs field. Furthermore in the limit $M\to0$, the color-electric fields $\mathbf{E}^a$ decouple from the flux tube equations, and hence $\mathbf{E}^a$ vanishes everywhere, not only asymptotically. The string tension is of order $-\tilde{F}_0^2$ and is insensitive to the value of M. All our results in dual QCD are likewise insensitive to the value of M and we have carried out all our calculations with $\mathcal{L}_1$ setting $M=0$.

At first sight it might seem surprising that one can take the limit $M\to0$ without losing all the physics of the $M^2/(q^2)^2$ propagator from which we started. However, the role that this propagator plays in obtaining $\mathcal{L}_1$ is twofold. First of all it necessitates the introduction of the auxiliary fields $\tilde{F}_{\mu\nu}$, via Eq. (2.1). This then leads to the equivalent local Abelian Lagrangian Eq. (2.5). Second the $(M/2)\tilde{F}_{\mu\nu}G^{\mu\nu}$ interaction between the fields $\tilde{F}_{\mu\nu}$ and C_μ in Eq. (2.5) gives a mass M to the C_μ field. If we let $M\to0$ at this stage we lose all the physics we started with. However once the interactions are introduced so that the Lagrangian Eq. (2.5) is replaced by $\mathcal{L}_1$, Eqs. (2.26) and (6.2), then $\tilde{F}_{\mu\nu}$ obtains a nonvanishing vacuum expectation value. This leads to a nonzero gluon mass,

Eq. (6.8), even as $M \to 0$. Thus the essential physics of the $M^2/(q^2)^2$ propagator, which is the existence of a massive dual gluon, is left intact if $M \to 0$ after the interactions have been introduced. Furthermore this situation is analogous to superconductivity in that the mass of the gluon would be determined solely by the Higgs mechanism.

There are also fundamental reasons why we consider only the case $M=0$. This is because $\mathcal{L}_1$ with $M \neq 0$ violates unitarity already at the tree level in CC scattering. The reason is as follows: Because of the Lorentz metric, the kinetic energy term in $\mathcal{L}_1$ involving the fields $\tilde{F}^a_{\mu\nu}$ has the wrong sign for the fields $\mathbf{E}^a$. As a consequence there is a unitarity violating pole with a negative residue in any amplitude (such as CC scattering) which can couple to an intermediate $\mathbf{E}^a$ state. The CCE coupling comes from the term $\frac{1}{2}M\tilde{F}^a_{\mu\nu}G^{\mu\nu a}$ in $\mathcal{L}_1$, which contains the interaction $gMf_{abc}\tilde{F}^a_{\mu\nu}C^{\mu b}C^{\nu c}$. This coupling vanishes when $M \to 0$. There is then no unitarity-violating contribution to CC scattering. The same is true for any amplitude at the tree level in the $M=0$ theory. However we will see that in the nonperturbative vacuum where $\mathbf{B}^a_0 \neq 0$, this result does not extend to loop graphs. $\mathcal{L}_1$ therefore leads to unitarity violations in higher-order processes. However these unitarity violations occur at a level beyond any of our applications. $\mathcal{L}_1$ with $M=0$ provides a phenomenological description of long-distance QCD compatible with tree-level unitarity. From now on when we refer to $\mathcal{L}_1$, we imply also that $M=0$.

Next note that the only coupling of the space-time indices $\mu\nu$ of $\tilde{F}_{\mu\nu}$ to a Lorentz tensor is via the $(M/2)\tilde{F}^a_{\mu\nu}G^{\mu\nu a}$ term in $\mathcal{L}_1$. Since $M=0$, this term is absent. $\mathcal{L}_1$ becomes invariant under Lorentz transformations of the coordinates x_μ and potentials C_μ without transforming the $\tilde{F}^a_{\mu\nu}$. (This means that $\tilde{F}_{\mu\nu}$ transforms like a scalar under Lorentz transformations, in which case the physical interpretation of $\tilde{F}_{\mu\nu}$ as electric and magnetic fields cannot be maintained.) Thus the nonperturbative vacuum with $\mathbf{B}^a = \mathbf{B}^a_0$ and $\mathbf{E}^a = 0$ does not spontaneously break Lorentz invariance as it would if M did not vanish.

Now we turn to the question of the unitarity of $\mathcal{L}_1$ beyond the tree approximation. It can be shown that since $M=0$, $\mathcal{L}_1$ possesses a BRST symmetry which mixes the $\tilde{F}_{\mu\nu}$ fields and the ψ_i and $\psi_i^\dagger$ fields introduced in Eq. (6.1). As a consequence of this symmetry the contribution arising from $\tilde{F}_{\mu\nu}$ internal lines cancels a corresponding contribution from ψ_i lines provided that the vacuum is perturbative, i.e., $\tilde{F}_{0\mu\nu}=0$. In this case the theory is unitarity but uninteresting. On the other hand if $\tilde{F}_{0\mu\nu} \neq 0$, as it is in the nonperturbative vacuum, then there is no such cancellation between $\tilde{F}_{\mu\nu}$ and ψ_i intermediate-state contributions. This is due to the fact that the vacuum is no longer invariant under the BRST transformation mentioned above and hence unitarity is violated.

Thus at this stage our understanding of the physics of confinement comes from the phenomenological Lagrangian $\mathcal{L}_1$. This description is Lorentz invariant and there are no violations of unitarity at the level where the applications are made. Furthermore, $\mathcal{L}_1$ provides a specific link between long-distance Yang-Mills theory and dual superconductivity. We conclude by making a few comments on this relation.

In the Abelian Higgs model description of superconductivity the photon becomes massive and there is a linear potential between monopoles. In dual QED described by $\mathcal{L}_1$ the dual gluon develops a mass $-(g^2/2)\mathbf{B}^{a2}_0$ and there is a linear potential between quarks. Recall that g is the inverse of the Yang-Mills coupling constant. Its value can be estimated from the $1/R$ contribution to the phenomenologically determined potential between heavy quarks [16]. The fit of Ref. [16] gives

$$g^2 \approx 8 \ . \tag{6.9}$$

The value of $-\tilde{F}^2_0$ is determined from string tension κ. The fit of Ref. [4] gives

$$-F^2_0 \equiv 2\mathbf{B}^{a2}_0 \approx \kappa \approx (427 \text{ MeV})^2 \ . \tag{6.10}$$

Equations (6.8), (6.9), and (6.10) then give the following value of the gluon mass M_g:

$$M_g \approx 604 \text{ MeV} \ . \tag{6.11}$$

The gluon mass M_g determines the scale of dual QCD. Energies less than M_g gave the important contributions in all our applications of dual QCD such as the calculation of chiral-symmetry-breaking phenomena and the deconfining transitions [17]. However the domain of applicability of dual QCD cannot be extended up to and beyond M_g, since it would then predict the existence of an octet of strongly interacting particles at a mass of about 0.6 GeV for which there is no evidence.

VII. CONCLUSION

Finally we raise the general question of how closely the mechanism of confinement should correspond to the dual of the mechanism of superconductivity. To clarify this problem let us recall the original arguments of 't Hooft and Mandelstam. 't Hooft [3] defined a dual Wilson loop and proved that it satisfied a perimeter law if the original Wilson loop satisfied an area law. Mandelstam then used the dual Wilson loop to define dual potentials C_μ which are related to the dual Wilson loop in the same way that the original Yang-Mills potentials A_μ are related to the ordinary Wilson loop. We then have two alternatives in a confining theory.

(a) The propagator for the dual potential is that of a weakly coupled massive particle. Then the dual Wilson loop would satisfy a perimeter law. In this case one would expect that the long-distance part of the dual Lagrangian is a simple function of the dual potentials, which are weakly coupled at long distances. The description of dual QCD by the Lagrangian $\mathcal{L}_1$ is the concrete realization of this picture. It leads the simple dual superconducting picture of confinement in which the dual photon acquires a mass via a dual Higgs mechanism, but its predictions are limited to energies below the mass of the

dual gluon. This version bears some similarity to the Landau-Ginzburg treatment of superconductivity in which the role of the Higgs field was recognized before its dynamical origin was understood.

(b) In the second alternative the perimeter law for the dual Wilson loop must arise from a more complicated mechanism. In this case there are two possibilities.

(i) The dual Lagrangian is a simple function of the dual potentials, but the solution is nonperturbative so that the physical spectrum bears little resemblence to the perturbative spectrum. The unitary dual Lagrangian $\mathcal{L}$ that we have constructed in this paper could be a concrete realization of this possibility. The perturbative solution of the theory described by $\mathcal{L}$ consists of interacting massive vector mesons and there is no evidence for confinement. However the theory possesses stable classical solutions which could have small mass and affect the vacuum structure. The result could be a physical spectrum which is essentially different from the perturbative spectrum and the physical vacuum could have confining properties. In the absence of a concrete calculation, this remains speculation. The other possibility is the following.

(ii) The dual potentials defined indirectly by Mandelstam are extremely complicated objects and that there is no simple Lagrangian describing long-distance Yang-Mills theory in terms of local dual fields. This is the alternative to the hypothesis that all our work has been based on.

To summarize, we know that $\mathcal{L}_1$ describes many aspects of the physics of confinement. The formation of quantized color electric flux tubes with the usual phenomenological applications all occur naturally at the classical level. Since the Lagrangian $\mathcal{L}$ is obtained more directly from the $M^2/(q^2)^2$ dynamics of the Dyson equations, it is perhaps more fundamental. Nevertheless at the classical level there is no sign of any behavior that might be associated with a dual superconductor. On the other hand it possesses a rich spectrum of solutions which could in principle be ingredients for constructing a physical vacuum state. The value of $\mathcal{L}$ for understanding confinement depends upon the success of this construction.

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