Entropy of Small Black Holes

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I will describe the relation between the entropy of extremal black holes and the topological string partition function.

Black holes are classical solutions to the Einstein equation. When a sufficiently large amount of energy is localized in a small region, an event horizon will emerge. Once one crosses the event horizon, one cannot send information back to the outside world. This leads to various interesting puzzles. For example, presence of an event horizon apparently violates the second law of thermodynamics. We can take any object that carries non-zero entropy and simply throw it inside of the horizon. The entropy outside of the horizon can be decreased in this way. In order to save the second law, Jacob Bekenstein speculated that a black hole carries an intrinsic entropy proportional to the area of the horizon and that it is the sum of the thermodynamic entropy outside of the horizon and the intrinsic entropy of black holes that obeys the second law. In fact, if we throw matter into a black hole, the black hole would gain extra mass and its horizon would grow. Thus, if we can include the area of the horizon as a part of the entropy, the total entropy could still increase.

If a black hole carries an entropy S, the standard thermodynamical relation:

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

implies the black hole with energy E also has temperature T. Indeed Steven Hawking found that vacuum polarization near the horizon generates thermal radiation from the black hole and its temperature is given precisely by this formula if we set the black hole entropy to be

$$S_{BH} = \frac{1}{4}\frac{A}{l_P^2},$$

where A is the area of the horizon of the black hole (which is a function of E) and l_P is the Planck constant given by $l_P = \sqrt{Gh/2\pi c^3}$ with G being the Newton constant. From now on, I will use the natural units where I set all these constants to be equal to one.

Due to the Hawking radiation, the black hole loses its mass. If the black hole carries electric or magnetic charges, there is a lower bound for its mass. When the black hole hits this bound, the Hawking temperature vanishes and we are left with a stable black hole with non-zero entropy as a function of its electric and magnetic charges. Such a black hole is called an extremal black hole. In statistical mechanics, an entropy has a microscopic interpretation as a logarithm of a number of quantum states of the system in question. If this applies to the black hole entropy, the Bekenstein-Hawking formula would mean that the dimensions of the Hilbert space of the extremal black hole is given by this formula:

$$\dim \text{Hilb} = \exp\left(\frac{1}{4}A\right).$$

One of the important tests for quantum theory of gravity is whether one can reproduce this formula by quantizing black holes.

One should note that this formula is supposed to hold only when the charges are very large. The mass of the black hole and the area of its horizon are increasing functions of the charges. When the charges are large, the black hole is big and the spacetime curvature near the horizon is weak. In that case, we can trust the Einstein equation, on which the Bekenstein-Hawking formula is based. However, as the charges get small, the curvature near the horizon becomes strong and we cannot ignore quantum corrections to the Einstein equation which take the form of higher derivative terms. So, the Bekenstein-Hawking formula should be corrected:

$$\dim \text{Hilb} = \exp\left(\frac{1}{4}A + (\text{quantum corrections})\right).$$

How do we go about computing quantum corrections to the entropy? The claim of fame of string theory is that it is a consistent theory of quantum gravity, so one should be able to calculate quantum effects near the horizon and get some finite answer. It turned out that, for a class of extremal black holes, quantum corrections can be computed exactly and explicitly to all order in the perturbative expansion using the method of *topological string*.

In my talk at the YKIS conference, I first discussed various mathematical properties of topological string amplitudes. The topological string theory was initially introduced in Ref. 1) as a "toy model" of string theory to learn general lessons on string theory on a Calabi-Yau manifold. During the past decades, various powerful techniques such as the holomorphic anomaly equation[2)] and the topological vertex[3)] have been invented to compute these explicitly.

It turned out that topological string is not only interesting as a toy model, but it has important physical applications. It was shown in Ref. 2) that the topological string amplitudes can be used to compute a certain class of superpotential terms in the four-dimensional effective theory which arises from type II superstring theory compactified on a Calabi-Yau manifold.[*)] The authors of Ref. 4) used this result to compute quantum corrections to the black hole entropy to all order in the perturbative expansion.

There is, however, a subtlety in computing perturbative corrections. As I mentioned in the above, the Bekenstein-Hawking formula is valid in the limit of large charges, and quantum corrections come in the form of an expansion in inverse powers of the charges. This means that we are evaluating finite volume corrections, whereas the computation of Ref. 4) assumes the thermodynamic limit. Inspired by a mathematical structure in the formula of Ref. 4), we came up with the following conjecture in Ref. 5). Suppose $\Omega(p, q)$ is the number of ground states of the Hilbert space of the

[*)] The connection at the string tree level had been noted earlier.

black hole with electric charges q and their dual magnetic charges p. (In general, a four-dimensional theory coming from the Calabi-Yau compactification contains several $U(1)$ gauge fields.) Then the conjecture states that its Laplace transform with respect to the electric charges is given by the topological string partition function to all order in the perturbative expansion in the following form,

$$\sum_q \Omega(p,q) e^{-\pi q \phi} = \left| \exp\left(\sum_g F_g(p + i\phi) \right) \right|^2 ,$$

where F_g is a g-loop contribution to the partition function (that is the integral over space of holomorphic maps from genus-g Riemann surfaces to the given Calabi-Yau manifold), and F_g is a function of the holomorphic combination $p+i\phi$ of the magnetic charges p and the electric potential ϕ.

In my talk, I discussed cases when $\Omega(p,q)$ can be computed independently and exactly using gauge theories on D-branes, and showed how this conjecture holds in these cases.[6),7)] I also noted that this formula is supposed to be valid to all order in the perturbative expansion, but there are non-perturbative corrections which may become important when the charges are small. As an example of non-perturbative effects, I discussed the possibility that the black hole horizon can fragment into small pieces and described how this effect can be seen from the point of view of gauge theories on D-branes. For details, I would like to refer to the papers.[8),9)]

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