VARIABLE FLOW IN PIPES

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Abstract

The problem considered is that of finding a variable pressure gradient or forced motion of the pipe in a longitudinal direction which, at the time when it ceases to act, will have produced a prescribed distribution of velocity over the cross-section of the pipe. The subsequent changes in the distribution as the motion decays are also investigated and the cases examined point to the conclusion that an initial velocity profile with curvature of one sign will retain this property as it changes into the profiles for the different stages of the decaying motion.

Introduction

The subject of variable flow in pipes is of much importance in engineering and has been studied experimentally in the case when the motion is turbulent. The problems which arise in connection with the metering of gases have been ably discussed by Hodgson,[1] Swift,[2] Judd and Pheley.[3] Accelerated flow without pulsations has been studied by Gibson[4] and Aisenstein[5] who express the time necessary to reach a velocity v in the form

$$t=(K/2V)\log\frac{V+v}{V-v}$$

where V is the final velocity of steady flow under the force or pressure gradient which produces the constant acceleration and K is a coefficient which has one value for the Poiseuille régime and another value for the Venturi or turbulent régime, K depending in each case upon the value of the Reynolds number at time t.

We shall study here problems relating to flow in the Poiseuille régime when the acceleration is variable. For simplicity the compressibility of the fluid is neglected. This is undoubtedly a defect because in the particularly interesting case of rapid oscillations the motion will be complicated by the propagation of waves.

The propagation of waves in water has been much studied in connection with the phenomenon of water hammer in pipes and need not be discussed here. The propagation of air waves along a pipe, when viscosity is taken into consideration, has been studied theoretically by Stokes, Kirchhoff and Rayleigh.[6]

[1] Hodgson, Proc. Inst. Civil Engineers **204**, 108 (1916–17).
[2] Swift, Phil. Mag. (7) **5**, 1 (1928).
[3] Judd and Pheley, Mech. Eng., **45**, 223, 270 (1923).
[4] Gibson, "Water hammer in hydraulic pipe lines," p. 50.
[5] Aisenstein, Trans. Amer. Soc. Mech. Eng. **51**, 67 (1929).
[6] See Lamb's Hydrodynamics, pp. 612–617.

The recent experiments of Simmons and Johansen[7] indicate, however, that there is need of further mathematical investigations. The author is considering the problem of pulsations when both viscosity and compressibility are taken into consideration but the work is not sufficiently advanced to warrant publication.

1. *Free decay, motion initially prescribed.* A long straight pipe of uniform cross-section has its ends attached by flexible connections to two reservoirs containing water, the pipe itself being supposed to be at any time completely filled with water.

The pipe is now moved in the direction of its length with a varying velocity $P(t)$. If u is the velocity of the water at the point (x,y,z) at time t the direction of motion being that of the axis of x, the equation of motion of the water relative to axes fixed relative to the reservoirs is

$$\rho\frac{\partial u}{\partial t}=\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho X$$

where p is the pressure, ρ is the density of the water, μ its viscosity and X the external force per unit mass. If the water is treated as incompressible and the pipe as rigid, the equation of continuity

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0$$

reduces to $\partial u/\partial x=0$ and implies that $u=F(y,z,t)$. We shall assume moreover that ρ and μ are constants independent of position and of the time t.

If we write $u=U+P(t)$ where U is the velocity of the water relative to the pipe, the equation for U is

$$\rho\frac{\partial U}{\partial t}=\mu\left(\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)-\frac{\partial p}{\partial x}-\rho\frac{dP}{dt}+\rho X$$

and so the motion relative to the pipe is the same as if the pipe were at rest and the fluid acted upon by an additional body force $-P'(t)$ per unit mass or an equivalent pressure gradient $\rho(P't)$. When p and X are zero the free motion in an oscillating pipe is relatively the same as the forced motion in a stationary pipe when the external force or pressure gradient fluctuates in the prescribed manner.

The transfer of motion from an oscillating plane to an adjoining liquid has been studied by Stokes,[8] Rayleigh[9] and other writers. Stokes showed that when the frequency of oscillation is high the motion in the fluid rapidly diminishes in intensity as the distance from the plane increases. This indicates that if fluid bounded by parallel plane walls is set in motion by a rapidly fluctuating pressure gradient which oscillates about a zero value, the

[7] Simmons and Johansen, British Advisory Committee for Aeronautics, R & M. 957 (1925).

[8] Stokes, Cambr. Phil. Trans. **9** (1850).

[9] Rayleigh, Phil. Mag. (6) **21**, 697 (1911).

distribution of velocity over the cross-section will be approximately uniform except in the immediate neighbourhood of the walls, where rapid variations are to be expected. This result has been extended by S. F. Grace[10] to the case of flow in a pipe of circular section. For a pipe of radius 1 cm there is approximate uniformity in the distribution of velocity over a large part of the cross-section if the period of oscillation is less than six seconds. The velocity moreover, lags 90° in phase behind the force producing the motion.

The existence of a phase lag indicates that the water may still be moving when the driving force has ceased to act. If, then, we go back to the case of the moving pipe we can say that the water may still be moving when the pipe is brought to rest. The velocity distribution at this time ($t=0$, say) may, indeed, be prescribed more or less arbitrarily so long as the velocity at the boundary is zero. Some interesting questions now arise with regard to the manner in which the velocity profile changes as the motion dies down.

Suppose, for instance, that the velocity profile has initially curvature of only one sign so that, when the velocity at each point of the cross-section is represented by an ordinate MQ, the locus of Q is everywhere concave to the plane of the section. It is interesting to inquire whether at some later time t the velocity profile can be partly convex to the plane of the section This question is an interesting one because in his studies of the stability of. an inviscid incompressible fluid in rectilinear motion between parallel planes Lord Rayleigh[11] found that a steady motion, without changes in the sign of the curvature of the velocity profile, was stable for small oscillations but that a profile with changes in the sign of the curvature might be unstable.

The question has been studied for the case of flow between parallel planes $z=\pm a$, the free motion being represented by

$$u=c_1e^{-s}\cos\theta+c_3e^{-9s}\cos 3\theta+c_5e^{-25s}\cos 5\theta+\cdots$$

where $s=\nu\pi^2t/4a^2$ and $\theta=\pi z/2a$. The expansion was limited to the first three terms and the coefficients c_1, c_3, c_5 were chosen so that the coefficients of z^2 and z^4 in the Taylor series for u were initially zero. This gives an initial distribution of velocity with curvature of one sign and a nearly constant value for a large part of the cross-section. The numerical work gave no indication of a change in sign of the curvature at any point of the profile as it changed with the time. The profile, in fact, gradually approached the shape of the cosine curve $u=k\cos\theta$ which differs only slightly in shape from the parabola characteristic of steady motion.

The question was also examined by assuming that at time $t=0$

$$d^2u/dz^2=-\cos\theta\ \left[(\alpha\cos^2\theta+\beta)^2+\gamma^2\right]$$

where α, β and γ are constants. Again there was no indication that at a later time t it would be legitimate to write

[10] Grace, Phil. Mag. (7) **5**, 933 (1928).

[11] Rayleigh, Proc. London Math. Soc. **10**, 4 (1879); **11**, 57 (1880); **19**, 67 (1887); **27**, 5 (1895); Phil. Mag. **34**, 59 (1892); **26**, 1001 (1913); **28**, 609 (1914). See also, L. Prandtl, Ziets. f. Angew. Math. und Mech. **1**, 431 (1921).

$$d^2u/dz^2 = -\sigma \cos\theta \,[\cos^2\theta - \cos^2\theta_1][\cos^2\theta - \cos^2\theta_2]$$

where θ_1 and θ_2 are real angles.

In the work of Grace the oscillations were undamped. To discuss a type of variable motion which ceases at time $t=0$ we first consider the case in which

$$P(t) = (-\nu t)^{-1/2} \exp(m^2/4\nu t) \quad t<0,$$
$$P(t) = 0 \quad t>0,$$

where $\nu = \mu/\rho$ is the kinematic viscosity. Since $\partial u/\partial t = \nu \partial^2 u/\partial z^2$ and

$$P(t) = K \int_0^\infty e^{\xi^2 \nu t} \cos(m\xi) d\xi, \quad t<0,$$

where $K = 2/\sqrt{\pi}$, an appropriate solution is

$$u = K \int_0^\infty e^{\xi^2 \nu t} \cos(m\xi) \cosh(z\xi) \operatorname{sech}(a\xi)\, d\xi \quad t<0.$$

As $t \to 0$, $u \to u_0$, where

$$u_0 = K \int_0^\infty \cos(m\xi) \cosh(z\xi) \operatorname{sech}(a\xi) d\xi$$
$$= \frac{(2K/a) \cosh(m\pi/2a) \cos(z\pi/2a)}{\cosh(m\pi/a) + \cos(z\pi/a)}$$

When $\sinh^2(m\pi/2a) = \frac{1}{2} u_0$ is approximately constant over a considerable part of the cross-section. If, however, we take $\pi^2 m^2/4a^2 = 1/32$, we have the result that u_0 is approximately proportional to

$$\frac{65 \cos\theta}{4 + 128 \cos^2\theta}.$$

When $z=0$ this fraction is equal to 65/132 and when $z=2a/3$ the fraction is equal to 65/72. It should be noticed that the maximum value of $P(t)$ occurs when $2\nu t = -m^2$, the maximum value being $(m^2 e/2)^{-1/2}$. Hence a small value of m corresponds to a large maximum velocity of the walls.

The present result confirms our statement that the fluid may still be moving when the pipe has ceased to move.

To discuss the decay of the motion when $t>0$ we expand u_0 in the form

$$u_0 = H \sum_{s=0}^{\infty} (-)^s e^{-mS} \cos(zS)$$

where $S = \pi(2s+1)/2a$ and $H = 4\sqrt{\pi}/a$. The velocity at time t is then

$$u = H \sum_{s=0}^{\infty} (-)^s e^{-mS - \nu S^2 t} \cos(zS).$$

Since u satisfies the equation $\partial u/\partial t = \nu\partial^2 u/\partial z^2$ the velocity $u^{(n)}$ corresponding to a function

$$P^{(n)}(t) = \frac{\partial^n}{\partial t^n} P(t)$$

is

$$\begin{aligned} u^{(n)} &= \nu^n \frac{\partial^{2n} u}{\partial z^{2n}} \\ &= K\nu^n \int_0^\infty e^{\xi^2 \nu t} \xi^{2n} \cos(m\xi) \cosh(z\xi) \operatorname{sech}(a\xi) d\xi \quad (t<0) \\ &= H\nu^n \sum_{s=0}^{\infty} (-)^{s+n} S^{2n} e^{-mS-\nu S^2 t} \cos(zS) \quad (t>0) . \end{aligned}$$

An initial distribution of velocity

$$\bar{u}_0 = H \sum_{s=0}^{\infty} (-)^s \phi(-\nu S^2) e^{-mS} \cos(zS)$$

will, with a suitable form of the function ϕ, be produced by a velocity of the walls varying up to time 0 according to the law

$$\bar{P}(t) = \phi\left(\frac{\partial}{\partial t}\right) P(t) .$$

2. *Forced motion.* We next consider the equation

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial z^2} - \frac{\partial p}{\partial x}, \quad (\mu = \rho\nu)$$

for the rectilinear flow of an incompressible viscous fluid of constant density ρ when the pressure gradient $\partial p/\partial x$ varies with the time. For convenience we write

$$-\frac{\partial p}{\partial x} = \left(\frac{2\mu}{\pi}\right) \int_0^\infty e^{-\nu m^2 t} \cos ma \cdot dm \int_0^\infty \cos mv\, \phi(v) dv$$

where $\phi(v)$ is an even function for which Fourier's integral formula may be used with confidence. The corresponding expression for u is then

$$u = (2/\pi) \int_0^\infty e^{-\nu m^2 t} (\cos mz - \cos ma)(dm/m^2) \int_0^\infty \cos mv\, \phi(v) dv.$$

It should be noticed that at time $t=0$ we have

$$-\frac{\partial p_0}{\partial x} = \mu\phi(a)$$

$$u_0 = \int_z^a ds \int_0^s \phi(v) dv.$$

If, on the other hand we write $f\ (m)$ instead of $\int_0^\infty \cos mv\,\phi\,(v)dv$ and take $f(m)=1$ we have

$$-\frac{\partial p}{\partial x}=\mu(\pi\nu t)^{-1/2}e^{-a^2/4\nu t}, \quad \text{and}$$

$$u=(\pi\nu t)^{-1/2}[F(a,t)-F(z,t)], \quad \text{where}$$

$$F(z,t)=\int_0^z (z-s)e^{-(s^2/4\nu t)}ds.$$

A short table of values of $F(z,\ t)/2\nu t$ is given below. For convenience n has been written for $z/2(^{\nu}t)^{1/2}$ and values of e^{-n^2} are also included. The fifth figure in the decimal may be slightly incorrect.

TABLE I

n	$F(z,\ t)/2\nu t$	$\exp\ (-n^2)$
0	0	1
0.1	0.00999	0.99005
0.2	0.05974	0.96079
0.3	0.08866	0.91393
0.4	0.15587	0.85214
0.5	0.24009	0.77880
0.6	0.33992	0.69768
0.7	0.45361	0.61263
0.8	0.57959	0.52729
0.9	0.71611	0.44486
1.0	0.86156	0.36788
1.1	1.01436	0.29820
1.2	1.17313	0.23693
1.3	1.33668	0.18452
1.4	1.50399	0.14086
1.5	1.67401	0.10540
1.6	1.84608	0.07731
1.7	2.02000	0.05557
1.8	2.19489	0.03916
1.9	2.37055	0.02705
2.0	2.54666	0.01832

For the case in which $a=2$ the distribution of velocity has been calculated at times t for which $2(\nu t)^{1/2}$ has the values 1, 2, 4, respectively and compared with the parabolic distribution for steady flow. In each case the velocity is compared with the axial value c. The third time is beyond that at which the pressure gradient is a maximum. It will be noticed that while the pressure gradient is increasing the distribution of velocity differs only slightly from the distribution for steady flow. When t is large we have $F(z,\ t)=z^2/2-z^4/48\nu t$, approximately, and so in the final stages of the motion the distribution of velocity again differs only slightly from the parabolic distribution.

TABLE II
z/a

$2(\nu t)^{1/2}$	0	0.2	0.4	0.6	0.8	1	
1	1	0.9384	0.7724	0.5393	0.2751	0	u/c
2	1	0.9307	0.8191	0.6054	0.3273	0	u/c
4	1	0.9584	0.7512	0.6307	0.3508	0	u/c
Parabola	1	0.96	0.79	0.64	0.36	0	u/c

3. The analysis for rectilinear flow in a tube of circular section is very similar to that for flow between two parallel walls. The equation of motion is now

$$\rho \frac{\partial u}{\partial t} = \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \frac{\partial p}{\partial x},$$

where r is the distance of a point from the axis of the tube. We shall commence with a study of forced motion.

If

$$-\frac{\partial p}{\partial x} = \mu \int_0^\infty e^{-\nu m^2 t} J_0(ma) m dm \int_0^\infty J_0(mv) \psi(v) v dv$$

where $\psi(v)$ is a function for which the inversion formula of Hankel may be used with confidence, the corresponding expression for u is

$$u = \int_0^\infty e^{-\nu m^2 t} [J_0(mr) - J_0(ma)] (dm/m) \int_0^\infty J_0(mv) \psi(v) v dv$$

and at time $t=0$ we have

$$-\frac{\partial p}{\partial x} = \mu \psi(a)$$

$$u_0 = \int_r^a (ds/s) \int_0^s v\psi(v) dv.$$

Another particular case of interest is that in which

$$-\partial p/\partial x = (\mu/2\nu t) \exp\,[-a^2/4\nu t] = \mu \int_0^\infty e^{-\nu m^2 t} J_0(ma) m dm$$

$$u = \int_t^\infty (d\tau/2\tau) [\exp\,(-r^2/4\nu t) - \exp\,(-a^2/4\nu t)].$$

Next, taking the case of free motion, let us suppose that the velocity of the pipe at time t is given by the function $P(t)$, where

$$P(t) = (-1/2\nu t) \exp\,(g^2/4\nu t) \quad t<0$$

$$P(t) = 0 \quad t>0$$

Since

$$P(t) = \int_0^\infty e^{\nu m^2 t} J_0(mg) m dm \quad t<0$$

an appropriate solution is

$$u = \int_0^\infty e^{\nu m^2 t} J_0(mg) m dm [I_0(mr)/I_0(ma)], \quad t<0$$

with the usual notation for the Bessel function with imaginary argument. As $t \to 0$, $u \to u_0$, where

$$u_0 = \int_0^\infty J_0(mg) m dm [I_0(mr)/I_0(ma)]$$