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Self-force on static charges in Schwarzschild spacetime

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Abstract. We study the self-forces acting on static scalar and electric test charges in the spacetime of a Schwarzschild black hole. The analysis is based on a direct, local calculation of the self-forces via mode decomposition and on two independent regularization procedures: a spatially extended particle model method and on a mode-sum regularization prescription. In all cases we find excellent agreement with the known exact results.

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1. Introduction and overview

The problem of calculating the gravitational wave forms generated by compact objects orbiting black holes is of crucial importance for the detection and the interpretation of observations by gravitational wave observatories such as LISA [1]. A major step towards the calculation of the wave forms is the computation of the gravitational radiation reaction forces, acting on the compact object. The generation of very accurate templates for the waveforms detected from a system of a compact object in orbit around a supermassive black hole is an extremely hard task. It is likely that one would need to have accurate templates for as many as 10^5 orbits. For such a system, accurate templates are necessary for detection, because the predicted signal-to-noise ratio for LISA is approximately of the order of 10 for a one-year integration time. Lack of accurate templates would result in a loss of a factor of roughly the square root of the number of orbits in sensitivity [2], which would result in a signal-to-noise ratio below the detectability threshold.

Several methods have been suggested for the calculation of radiation reaction. One approach follows Dirac's method for obtaining the Abraham–Lorentz–Dirac equation for an electric charge in arbitrary motion in Minkowski spacetime [3]. In that approach one imposes local conservation laws on a tube surrounding the worldline of the particle, and integrates the conservation law across the tube, thus obtaining the equations of motion, including the radiation reaction effects. In Dirac's approach the infinities which are related to the divergence of the particle's field on its worldline are removed by a simple mass renormalization. This method was used by DeWitt and Brehme [4] to generalize Dirac's analysis for a general curved background. More recently, Mino *et al* used a similar method for the case of a massive particle coupled to linearized gravity [5]. Recently, Quinn and Wald [6] have formulated an axiomatic approach for the calculation of radiation reaction. In that approach, the infinities are removed by comparing the forces in different spacetimes. However, at present it is unclear how to apply the Quinn–Wald formal approach directly to practical calculations. The main difficulty arises from the calculation of the ‘tail term’, which is difficult to compute even in the slow-motion, weak-field limit [7, 8].

Another approach is based on arguments relating to the balance of quantities, which are constants of motion in the absence of radiation reaction, specifically the energy and angular momentum. Such balance arguments involve integration of the flux of an otherwise conserved quantity over a boundary which consists of a distant sphere and the horizon of the black hole [9–16]. Although these methods are quite successful to very high relativistic order in Schwarzschild spacetime, they are problematic for the more interesting problem of motion in the spacetime of a spinning black hole, because of an inherent difficulty. For the problem of motion in the Schwarzschild spacetime the motion is completely determined by the rate of change of the energy and the angular momentum, which are additive constants of motion in the absence of radiation reaction. However, for general orbits in the spacetime of a Kerr black hole there is a third constant of motion, i.e. the Carter constant. The Carter constant is non-additive, and consequently it cannot be obtained by methods which are based on balance arguments. (Such methods can be used for circular and equatorial orbits around a Kerr black hole, because then the evolution of the Carter constant is trivial: it is given completely by the evolution of the energy and the azimuthal component of the angular momentum.) In addition, such methods also suffer from other difficulties [17]: they usually yield only the time average of the radiation reaction force, such that for any quickly evolving system they would be inherently inaccurate. In addition, they fail to obtain the conservative part of the radiation reaction force.

A different approach for the calculation of the gravitational radiation reaction is based on a direct, local calculation of the self-forces acting on the compact object. Obviously, knowledge of the instantaneous forces acting on the orbiting object would allow the calculation of the orbital evolution. Such a direct approach to the calculation of the self-force was suggested by Gal’tsov [18]. However, Gal’tsov’s approach is based on the radiative Green’s function (i.e. the ‘half-retarded minus half-advanced’ potential), which is not causal in curved spacetime, because it requires knowledge of the complete future history of the object in motion [19]. A causal approach, which is based on the retarded Green’s function rather than on the radiative one and consequently is more in the spirit of relativity theory, is much more desirable.

Recently, a local approach for the calculation of the self-force, which is based on the retarded field, and on a Fourier-harmonic mode decomposition of the field and the self-force, has been proposed [19, 20]. This approach has two very important advantages: first, when the field is decomposed into modes, each mode satisfies an ordinary differential equation rather than a partial one, and consequently the solution for each mode is considerably simpler. Second, and most importantly, each mode of the self-force turns out to be bounded, even for a point-like particle. Indeed, the total self-force, which is obtained when one sums over all modes, very frequently diverges, but this difficulty is met only at the summation over all modes step: the treatment of the individual modes is free from divergences.

This approach was used by Ori [19, 20] for the calculation of the adiabatic, orbit integrated, evolution rate of the three constants of motion in Kerr. Ori suggested a regularization prescription which is based on the assumption that the divergent piece of the self-force is proportional to the 4-acceleration of the charge. One can then use a simple mass-renormalization procedure by redefining the mass of the particle to include the divergent piece in the point-like limit. For geodesic motion the 4-acceleration vanishes, and consequently the divergent piece of the self-force, which is expected to be proportional to the 4-acceleration, also vanishes. However, when forces (including radiation reaction forces) are present, or when a point-like particle is considered *ab initio*, the self-forces would also, in general, be expected to diverge. (For non-geodesic orbits the self-force is expected to be proportional to the particle’s $(\text{charge})^2$. The correction to the orbit is therefore also of order $(\text{charge})^2$, and the correction to the self-force is consequently of order (charge)^4. When the charge of the particle is much smaller than the mass of the black hole, this correction is negligible. In this paper we shall study the self-force only to leading order, i.e. we shall study the self-force to order (charge)^2.) Therefore, a crucial ingredient for the calculation of the self-force is the regularization method which one uses. We note that in any regularization prescription in the gravitational case (i.e. when the particle has a non-zero mass) one faces a gauge problem. This difficulty, however, does not arise in the cases of scalar or electromagnetic fields, because the force in these cases is gauge independent. Therefore, consideration of scalar or electromagnetic charges is of some value, as they correspond to easier cases, where many of the difficulties related to self-forces are already present, yet one does not also have to solve the gauge problem.

In this paper we present two independent regularization procedures for the self-force, which are successful for the problem of static charges (both scalar and electric) in the spacetime of a Schwarzschild black hole. (These procedures were also found to be successful for the regularization of the radial component of the self-force for scalar or electric charge in uniform circular motion in flat spacetime [21, 22].) We hope that similar methods (or their generalizations) will also be relevant for more complicated and realistic problems, e.g. the self-forces acting on a compact object in circular motion around a Schwarzschild black hole, and ultimately, the self-force on a compact object in motion in a generic orbit around a Kerr black hole.

Let us consider first spatially extended particles (we still assume that the extension of the particles is smaller that the typical radius of curvature and the typical scale of inhomogeneity of the field). The divergent piece of the self-force, in addition to being proportional to the 4-acceleration, is also expected to be inversely proportional to the spatial extension of the particle. In the limit of a point-like particle, this is the source for the divergence of the force. One should therefore be able to obtain a regularization procedure by considering a spatially extended model for the particle, and then consider a sequence of smaller and smaller particles. The force acting on the particles would increase as the inverse of their size, and by removing this piece of the force one can expect to obtain the regularized self-force, which is independent of the assumed internal structure, in the limit of vanishing spatial extension. The question of whether the regularized force depends on the way in which the point-like limit is taken is still an open question. A similar approach was used by Ori, who calculated the self-forces acting on static scalar and electric charges in Schwarzschild and on the axis of Kerr black holes [23]. In Schwarzschild, Ori used a dumbbell model, where the axis was aligned either radially or tangentially. In Kerr, Ori used a radially aligned dumbbell model. Whereas we use a mode-decomposition approach, which does not depend on the availability of an exact solution, Ori used the exact solutions for the scalar field or the electric potential, which are available for these cases, in order to calculate the self-forces.

We also consider a second, independent, regularization prescription. We consider a point-like particle. In that case the sum over modes is expected in general to diverge. Ori has recently suggested a mode-sum regularization prescription (MSRP) for the self-force [24]. Although MSRP is not fully developed as yet, it has already been shown to be valid for simple cases, such as a static scalar charge outside a Schwarzschild black hole. MSRP can possibly also be generalized for more complicated cases, such as massive particles in orbit around a Kerr black hole. If robust, MSRP could be of great importance for the generation of templates for the detection of gravitational waves from compact objects in motion around supermassive black holes.

This paper is organized as follows. In appendix A we describe very briefly the main ideas of MSRP, applied for a scalar charge in Schwarzschild. In section 2 we discuss the self-force acting on a static scalar charge in Schwarzschild spacetime. The result has been obtained by independent methods: for a minimally coupled massless scalar field the self-force vanishes [17, 25, 26]. It is our approach which is novel: our calculation is based on a direct computation of the self-force mode by mode, followed by a summation over all modes, and finally on two independent regularization procedures. One regularization procedure is based on a spatially extended particle model. We then consider the forces acting on a sequence of such particles with decreasing spatial extensions, and remove the divergent piece of the self-force by a simple mass-renormalization procedure. The other regularization procedure is based on MSRP. We find that both methods are successful in obtaining the correct result. In section 3 we consider the analogous problem of the self-force acting on a static electric charge in Schwarzschild spacetime. Also in this case, the result is not new. This problem has been considered by several authors: DeWitt and DeWitt [7] calculated the radiation damping forces (both nonconservative and conservative) acting on a slowly moving electric charge in the far-field regime, and found that there was a repelling self-force, which lowered the much stronger gravitational pull of the black hole, and made a retrograde contribution to the periastron precession. Vilenkin [27] considered the electric charge to be very far from the black hole (specifically, he assumed the position of the charge to be at $r_0 \gg M$, where M is the mass of the Schwarzschild black hole), and again found that there was a repelling conservative self-force. Smith and Will [28] and Frolov and Zel'nikov [25, 29] were able to solve for the force exactly, for all positions of a static charge in Schwarzschild spacetime, and found that the repulsive radial self-force was $f_{\hat{r}}^{\rm exact} = e^2M/r^3$ (in the frame of a freely falling observer who is instantaneously at rest at the position of the charge). Also in this case of a static electric charge we present a direct approach for the calculation of the self-force, which is based on mode decomposition, summation over all modes and force regularization procedures similar to those we apply in the scalar case. In section 4 we summarize our methods and results.

2. Static scalar charge

2.1. Mode decomposition of the force

Consider a point-like scalar test charge in the Schwarzschild spacetime, held fixed by some external force. Our aim here is to calculate the contribution of the self-force to the total force needed to keep it fixed. The result is well known [17, 25, 26]: the contribution of the self-force to the total force vanishes. The linearized field equation of a minimally coupled, massless scalar field Φ in the Schwarzschild geometry, which is described by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, is given by

$$\nabla_\mu \nabla^\mu \Phi(x^\alpha) = -4\pi\rho(x^\alpha), \tag{1}$$

where ∇_μ denotes covariant differentiation, and where the charge density

$$\rho = q \int_{-\infty}^{\infty} d\tau\, \frac{\delta^4[x^\mu - x_s^\mu(\tau)]}{\sqrt{-g}}. \tag{2}$$

Here, q is the charge, τ is its proper time and g is the metric determinant. The mass of the black hole is denoted by M. The worldline of the charge is given by $x_s^\mu(\tau)$. In what follows we use the usual Schwarzschild coordinates: the radial Schwarzschild coordinate is defined such that spheres of radius r have surface area $4\pi r^2$, and t is the proper time of a static observer at infinity. We take the charge to be on the equatorial plane at the coordinates $r = r_0$, $\varphi = 0$, and

$\theta = \pi/2$, without loss of generality. (Because of the symmetry of the Schwarzschild geometry the coordinates θ and φ can be rotated such that these would be the coordinates of any static charge at $r = r_0$.) Because of the staticity, the scalar field is independent of the time, and we can decompose it into modes according to

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi^l(r) Y^{lm}(\theta, \varphi), \tag{3}$$

such that the left-hand side of the wave equation (1) is given by

$$\nabla_\mu \nabla^\mu \Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[\left(1 - \frac{2M}{r}\right) \phi^l_{,rr} + \frac{2}{r^2}(r - M)\phi^l_{,r} - \frac{l(l+1)}{r^2}\phi^l \right] Y^{lm}. \tag{4}$$

The charge density is similarly decomposed into modes according to

$$\rho = q \frac{\delta(r - r_0)}{r_0^2} \frac{1}{u^t(r_0)} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^{lm*}\left(\tfrac{1}{2}\pi, 0\right) Y^{lm}(\theta, \varphi) \tag{5}$$

where u^α is the 4-velocity of the charge, and a star denotes complex conjugation. We thus find the radial equation for $\phi^l(r)$ to be

$$\left(1 - \frac{2M}{r}\right) \phi^l_{,rr} + \frac{2}{r^2}(r - M)\phi^l_{,r} - \frac{l(l+1)}{r^2}\phi^l = -4\pi q \frac{\delta(r - r_0)}{r_0^2} \frac{1}{u^t(r_0)} Y^{lm*}\left(\tfrac{1}{2}\pi, 0\right). \tag{6}$$

To solve this equation we transform to dimensionless harmonic coordinates, i.e. we define $\bar{r} \equiv (r - M)/M$. In the harmonic gauge the radial equation is nothing but the Legendre equation†. We choose the two independent solutions of the corresponding homogeneous equation to be $P_l(\bar{r})$ and $Q_l(\bar{r})$. The former is regular for $1 < \bar{r} < \bar{r}_0$, and the latter is regular for $\bar{r} > \bar{r}_0$. (Note that the horizon of the black hole is located at $\bar{r}_{\text{horizon}} = 1$.) The summation over all modes m is readily done, and we thus write the field at the point (r, θ, φ) due to a scalar charge q at the position $(r_s, \theta_s, \varphi_s)$ as

$$\Phi = \frac{q}{M}\sqrt{1 - \frac{2M}{r_s}} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \left[P_l\left(\frac{r_s - M}{M}\right) Q_l\left(\frac{r - M}{M}\right) \Theta(r - r_s) + P_l\left(\frac{r - M}{M}\right) Q_l\left(\frac{r_s - M}{M}\right) \Theta(r_s - r) \right]. \tag{7}$$

Here, $\cos\gamma = \cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\varphi - \varphi_s)$, and $\Theta(x)$ is the Heaviside step function, i.e. $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. This solution for the scalar field Φ is regular both at the black hole's event horizon and at infinity. In what follows we choose the angular coordinates such that both the origin and the evaluation point of the field lie on the equatorial plane, such that $\cos\gamma = \cos(\varphi - \varphi_s)$.

The force which a scalar field Ψ exerts on a scalar charge q' is given by $f_\alpha = q'\left(\Psi_{,\alpha} + u'_\alpha u'^\beta \Psi_{,\beta}\right)$, where the 4-velocity u'^α is that of the charge q'. The scalar field Ψ

† That this should be the transformation is most easily seen from the following consideration. In the dimensionless coordinate $x = r/(2M)$ the homogeneous equation is $(1 - x)x\phi''(x) + (1 - 2x)\phi'(x) + l(l+1)\phi(x) = 0$. This is a hypergeometric equation of the canonical form $x(1 - x)\phi'' + [c - (a + b + 1)x]\phi' - ab\phi = 0$, for $a = -l$, $b = l + 1$ and $c = 1$. As $1 - c = c - a - b$, we know from the theory of hypergeometric functions that the homogeneous equation can be transformed to Legendre's equation. The variable of the hypergeometric equation x is then related to the variable of the Legendre equation by the transformation $x = (1 + \bar{r})/2$. In view of the definition of x, we find that $\bar{r}$ is nothing but the dimensionless harmonic coordinate. We are then assured that transformation to $\bar{r}$ would yield Legendre's equation with solutions $P_l(\bar{r})$ and $Q_l(\bar{r})$ [30]. (The relations to the Legendre functions are given in equations 3.2(15) and 3.2(33) of [30].)

can be any scalar field, in particular the self-field of the charge in question itself. In a sense, this is the scalar field analogue of the electromagnetic Lorentz force [18]. Because of the staticity of our problem the only component of u'^{α} which does not vanish is the temporal component. However, the temporal derivative of the field vanishes, and consequently the force is given only by $f_\alpha = q'\Psi_{,\alpha}$. Because for scalar fields partial derivatives equal covariant derivatives, this is, in fact, the covariant equation for the force. Now consider two scalar charges, q_1 at $(r_1, \pi/2, \varphi_1)$ and q_2 at $(r_2, \pi/2, \varphi_2)$, and $r_2 > r_1$. The force that q_1 exerts on q_2 is given by

$$f_r^{12}(r_2) = \frac{q_1 q_2}{M^2}\sqrt{1-\frac{2M}{r_1}}\sum_{l=0}^{\infty}(2l+1)P_l[\cos(\varphi_2-\varphi_1)]P_l\left(\frac{r_1-M}{M}\right)Q_l'\left(\frac{r_2-M}{M}\right) \tag{8}$$

$$f_\varphi^{12}(r_2) = \frac{q_1 q_2}{M}\sqrt{1-\frac{2M}{r_1}}\sum_{l=0}^{\infty}(2l+1)P_l\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)\frac{\partial P_l[\cos(\varphi_2-\varphi_1)]}{\partial\varphi_2}. \tag{9}$$

Similarly, the force that q_2 exerts on q_1 is given by

$$f_r^{21}(r_1) = \frac{q_1 q_2}{M^2}\sqrt{1-\frac{2M}{r_2}}\sum_{l=0}^{\infty}(2l+1)P_l[\cos(\varphi_2-\varphi_1)]P_l'\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right) \tag{10}$$

$$f_\varphi^{21}(r_1) = \frac{q_1 q_2}{M}\sqrt{1-\frac{2M}{r_2}}\sum_{l=0}^{\infty}(2l+1)P_l\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)\frac{\partial P_l[\cos(\varphi_2-\varphi_1)]}{\partial\varphi_1}. \tag{11}$$

Here, a prime denotes differentiation with respect to the argument. Each of the force components is evaluated at the position of the charge on which the force is exerted. We are interested in calculating the self-force acting on a point particle. Namely, we are interested in identifying q_1 with q_2. When this is done, one naturally finds that the total force diverges (although each of the l modes of the force is still finite). Next, we describe two regularization procedures for the self-force acting on a point particle, which yield the desired result.

2.2. Regularization procedures

2.2.1. Extended particle model: inclined dumbbell. A well known classical renormalization scheme is to consider a spatially extended particle model, and then consider the limit of vanishing spatial extension, in the spirit of the classical Abraham–Lorentz–Poincaré electron models. However, as is well known [6], point-like particles are problematic in general relativity even to a greater extent than they are in electromagnetic theory because of the nonlinearity of the Einstein equations [31]. Still, in some sense, one can be hopeful that as the particle becomes smaller and smaller, the deviation of its worldline from a geodesic becomes insensitive to the particle's internal structure. The simplest particle model is a dumbbell model, consisting of two point-like charges at the two edges of an uncharged rigid rod, whose length is smaller than the typical scales of the inhomogeneities of the gravitational and scalar or electric fields (having in mind that we shall later consider the limit of vanishing spatial extension). Although this is a very simplified model for a particle, it can be simply generalized to more realistic models, bearing in mind that a general extended (classical) object can be construed as being comprised of many point-like particles, and the self-interaction of a general extended object can be obtained by summing the contributions of all pairs of point-like particles, each pair being, in fact, a dumbbell. We shall thus model the particle as a dumbbell, with two equal charges $q_1 = q_2 = e/2$, where e is the total charge of the particle. Because of the symmetry of

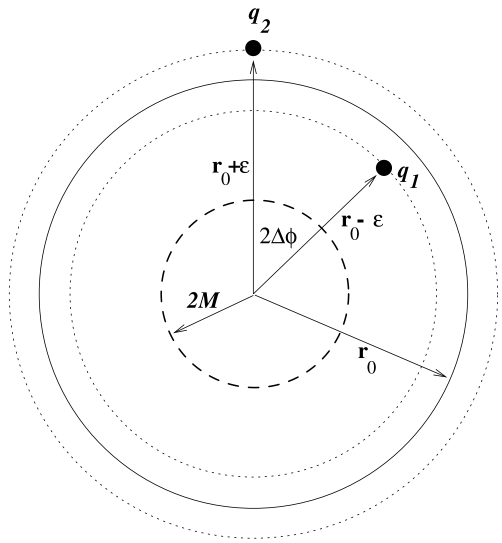

Figure 1. The geometry of the charge splitting in the equatorial plane: the charge e is split into two half charges, $q_1 = q_2 = e/2$. The charge q_1 is placed at $r_0 - \epsilon$ and the charge q_2 is placed at $r_0 + \epsilon$. The total angular separation between q_1 and q_2 is $2\Delta\varphi$. The horizon of the black hole is at $r = 2M$.

the geometry, the simplest configuration is to align the dumbbell axis in the radial direction. In that way, we still maintain axial symmetry, and the dumbbell axis is aligned along a geodesic. However, we shall see below that despite the fact that with a radial dumbbell axis one indeed recovers the known and correct result for the self-force, an important feature of a general extended particle model is missing, specifically, the mass-renormalization aspect of the force regularization procedure. This happens because the coefficient of the divergent piece of the bare force vanishes if the alignment of the dumbbell is radial (in the scalar case). As we are interested primarily in the regularization procedure, we shall consider here a more complicated case, where the dumbbell is not aligned radially. Consequently, we shall take the dumbbell axis to be inclined at some angle from the radial direction†.

Specifically, we take $r_2 = r_0 + \epsilon$, $r_1 = r_0 - \epsilon$, $\varphi_2 = \Delta\varphi$ and $\varphi_1 = -\Delta\varphi$. For concreteness, we take $\Delta\varphi = \alpha\epsilon$, such that when we make ϵ smaller, we also reduce $\Delta\varphi$ proportionally, and we take $\epsilon \ll 2M$. Figure 1 illustrates the geometry of the charge splitting in the equatorial plane (recall that the coordinates can always be rotated such that the splitting is in the equatorial plane): the charges q_1 at $r_0 - \epsilon$ and q_2 at $r_0 + \epsilon$ are also separated angularly by an angle of $2\Delta\varphi$. When we take the limit $\epsilon \to 0$ we simultaneously take the limit $\Delta\varphi \to 0$ too, such that the point-like charge is located at the intersection of the circle of radius r_0 and the bisector of the angle between q_1 and q_2.

† A mathematical complication occurs if we take the dumbbell axis to be in the $\partial/\partial\varphi$ direction. Specifically, in that case the series expansion for the scalar field indeed converges, but not absolutely. Consequently, one is not allowed to differentiate term by term to obtain the force. This difficulty can most easily be illustrated in flat spacetime, where it already occurs: the electric scalar potential due to a static unit point charge is given by $V = \sum_{l=0}^{\infty}(r_<^l/r_>^{l+1})P_l(\cos\gamma)$ (see [32] for details). When the splitting is tangential, $r_< = r_> \equiv r$, and the potential is given simply by $V = (1/r)\sum_{l=0}^{\infty} P_l(\cos\gamma)$. It can be easily checked that this series converges (although very slowly), but because of the oscillations it does not converge absolutely. When the radial positions of the source and the evaluation point are not equal there is an additional attenuation, thanks to which the series converges absolutely.

Let us consider only the radial force which acts on the dumbbell. (Because the acceleration is purely radial, we expect only the radial component of the self-force to diverge.) The total (bare) self-force which acts on the dumbbell is made of four contributions. Schematically,

$$f_r^{\text{total}} = f_r^{12} + f_r^{21} + f_r^{11} + f_r^{22}, \tag{12}$$

where f_r^{ij} is the radial component of the force which the charge q_i exerts on the charge q_j. Let us consider this force in the point-like particle limit. Now, f_r^{total} is the self-force on a *point-like* scalar charge e. However, both f_r^{11} and f_r^{22} are just the self-forces on *point-like* scalar charges, which are identical to the original charge e in all respects, except for the fact that they each have charge $e/2$. As the self-force is proportional to the charge squared, it implies that $f_r^{11} = f_r^{22} = f_r^{\text{total}}/4$. Consequently,

$$f_r^{\text{total}} = 2\left(f_r^{12} + f_r^{21}\right). \tag{13}$$

Because we need to sum vector components, we have to perform the summation at a common point, which for symmetry we choose to be $(r_0, \pi/2, 0)$†. Specifically, we need to transport the forces $f_\mu^{12}(r_2)$ and $f_\mu^{21}(r_1)$ to $(r_0, \pi/2, 0)$ parallelly. For non-zero inclination angles the two edges of the dumbbell are not separated by a geodesic of the background geometry. The final result for the self-force should of course be independent of the artificial spatial extension we assume (i.e. independent of the internal structure of the particle), of the parallel-transport route, of the point where we sum the forces and of the specific way in which we take the point-like limit. It is still an open question as to whether the final result depends on the way in which the limit is taken. One might be worried about the introduction of ambiguities due to the arbitrariness in the choice of the parallel-transport route. Any ambiguity is of the order of the area enclosed by the two routes we compare, times the curvature. The area is of order ϵ^2 and the curvature of order M/r^3, such that the ambiguity is of order $\epsilon^2 M/r^3$. Because the Coulomb components of the individual forces cancel (see below), the leading-order term in the total force is of order ϵ^{-1}, such that the ambiguity in the total force is of order ϵ, and vanishes in the limit $\epsilon \to 0$. That is, the final result is independent of the parallel-transport route. We note that the fact that the two edges of the dumbbell are not separated by a geodesic is not a problem in principle, because for a general extended object all pairs of the object's atoms interact, and most of them are not separated by geodesics.

We perform the parallel transport of $f_\mu^{12}(r_2)$ to $(r_0, \pi/2, 0)$ in two steps: first, along the radial route $(r_2, \pi/2, \varphi_2) \to (r_0, \pi/2, \varphi_2)$, and then along the tangential route $(r_0, \pi/2, \varphi_2) \to (r_0, \pi/2, 0)$. Similarly, we parallel transport $f^{21}(r_1)$ from $(r_1, \pi/2, \varphi_1)$ first radially to $(r_0, \pi/2, \varphi_1)$ and then tangentially to $(r_0, \pi/2, 0)$. Note, that although we are eventually interested only in the radial component of the self-force, we need, in fact, to parallel transport both the radial and the tangential components of the forces in the first sections of both routes, because when one parallel transports a tangential component of a vector tangentially, it acquires a radial component (already in flat space). Another point to be made concerning the parallel transport is the following: the individual forces can be expanded in a power series in ϵ, where the leading term is proportional to ϵ^{-2}. The self-force is of the order of unity, i.e. of order ϵ^0. Therefore, one needs to perform the parallel transport accurately at least to order ϵ^2. The parallel transports along the radial routes are done as follows: the change in a covariant component of a vector in parallel transport along the $\partial/\partial x^\beta$ direction satisfies $\delta V_\alpha = \Gamma^\gamma_{\alpha\beta} V_\gamma \, \mathrm{d}x^\beta$, where $\Gamma^\gamma_{\alpha\beta}$ are the connection coefficients, which equal the Christoffel symbols of the second kind in general relativity. For the Schwarzschild

† Note that the dumbbell is not symmetric about this point, because the invariant distances from this point to the two edges are not equal. However, there is no particular need for a symmetric dumbbell, and therefore we choose the model which is the simplest mathematically.

geometry we find $\delta f_r = \Gamma^r_{rr}(r) f_r\,\mathrm{d}r$ and $\delta f_\varphi = \Gamma^\varphi_{r\varphi}(r) f_\varphi\,\mathrm{d}r$. Consequently, $\delta(\log f_r) = \mathrm{d}(\log\sqrt{1/(1-2M/r)})$ and $\delta(\log f_\varphi) = \mathrm{d}(\log r)$, such that in the radial sections of the parallel transports $f_r^{\text{new}} = f_r^{\text{old}}\sqrt{(1-2M/r^{\text{old}})/(1-2M/r^{\text{new}})}$, and $f_\varphi^{\text{new}} = f_\varphi^{\text{old}}(r^{\text{new}}/r^{\text{old}})$. In the second sections of the parallel transportation, the routes are tangential, such that $\delta f_r = \Gamma^\varphi_{\varphi r}(r) f_\varphi\,\mathrm{d}\varphi$. (We do not need to find the change in the tangential component of the force as we are interested eventually only in the radial force.) That is, we need to integrate $\delta f_r = (f_\varphi/r)\,\mathrm{d}\varphi$. This can be done straightforwardly, and we find

$$f_r^{\text{total}} = \frac{1}{2}\frac{e^2}{M^2}\left\{\sqrt{\frac{(1-2M/r_1)(1-2M/r_2)}{1-2M/r_0}}\sum_{l=0}^{\infty}(2l+1)\right.$$
$$\times\left[P_l\left(\frac{r_1-M}{M}\right)Q_l'\left(\frac{r_2-M}{M}\right)+P_l'\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)\right]P_l(\cos 2\Delta\varphi)$$
$$+M\left(\frac{1}{r_2}\sqrt{1-\frac{2M}{r_1}}+\frac{1}{r_1}\sqrt{1-\frac{2M}{r_2}}\right)\sum_{l=0}^{\infty}(2l+1)P_l\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)$$
$$\left.\times[P_l(\cos\Delta\varphi)-P_l(\cos 2\Delta\varphi)]\right\}. \tag{14}$$

In appendix B we describe briefly the numerical method we use for the evaluation of the series. We evaluate this force for various values of ϵ (recall that we take $\Delta\varphi$ to be proportional to ϵ). We find that f_r^{total} diverges like ϵ^{-1}, for very small values of ϵ. This is indeed the expected behaviour for the bare force. Classical mass renormalization can be used for the regularization of the bare force. Specifically, the divergent piece of the force is expected to be proportional to the acceleration, such that it can be absorbed in the mass of the particle. We use the exact solution for the scalar field [17]

$$\Phi = e\sqrt{1-\frac{2M}{r_0}}\left[(r-M)^2 - 2(r-M)(r_0-M)\cos\gamma + (r_0-M)^2 - M^2\sin^2\gamma\right]^{-1/2}, \tag{15}$$

and sum the mutual forces of the two charges at the dumbbell's edges at a common point, in the same way as above. Then, we expand the total force in a power series in ϵ, where the leading-order term is of order ϵ^{-1}. This leading-order term is given by

$$f_r^{\text{div}} = -e^2\frac{M}{r_0^2}\left(1-\frac{2M}{r_0}\right)^{-1}\frac{1}{\epsilon}\left\{\alpha^2\frac{r_0^2(r_0-M)}{M^3}\right.$$
$$\times\left(\alpha^2\frac{r_0^2}{M^2}+\frac{1}{1-2M/r_0}\right)^{-3/2} - \frac{2}{r_0}\left(1-\frac{2M}{r_0}\right)^{3/2}$$
$$\left.\times\left[\frac{2}{\sqrt{4+\alpha^2(r_0^2/M^2)(1-2M/r_0)}}-\frac{1}{\sqrt{1+\alpha^2(r_0^2/M^2)(1-2M/r_0)}}\right]\right\}. \tag{16}$$

Notice that the mass-renormalization term depends on the value of the parameter α. That is, for different choices of α we rescale the mass by a different quantity. However, the renormalized, physical self-force is independent of α, as should indeed be the case. We note that this mass-renormalization procedure does not depend on the availability of an exact solution. Below, in

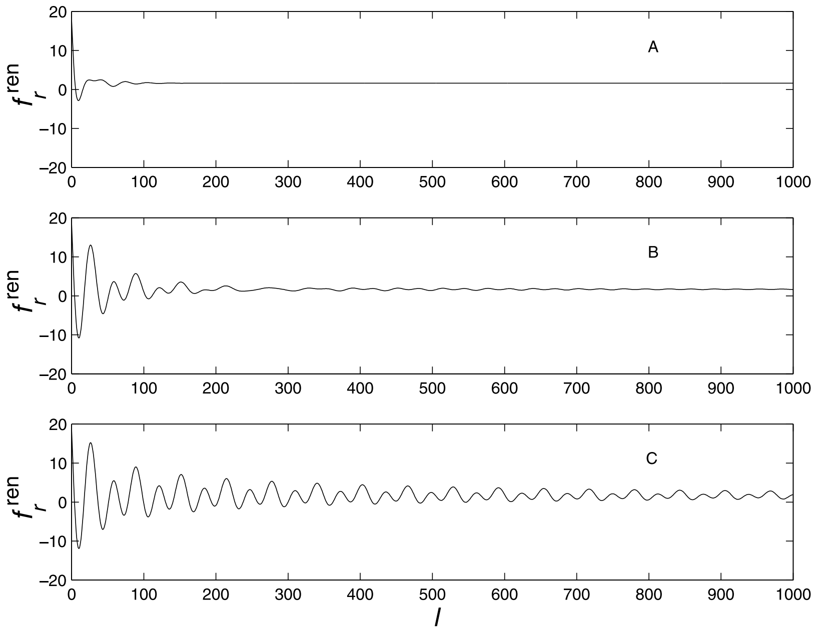

Figure 2. The behaviour of the sum over modes of the renormalized force as a function of l, for different values of the inclination parameter α. For all cases we take $r_0 = 2.1M$ and $\Delta\varphi = 0.1$. Top panel (A), $\alpha = 10M^{-1}$ (corresponding to $\epsilon = 1 \times 10^{-2}M$); middle panel (B), $\alpha = 10^2M^{-1}$ (corresponding to $\epsilon = 1 \times 10^{-3}M$); bottom panel (C), $\alpha = 10^3M^{-1}$ (corresponding to $\epsilon = 1 \times 10^{-4}M$).

section 3, when we discuss a similar mass renormalization for an electric charge, we do not use the exact solution (although it is available). Instead, we use equation (37) for calculating the divergent piece of the self-force. Even in cases where an equation analogous to equation (37) is not available, the regularization procedure can still be done. In such a case one can extract the asymptotic divergence of the force at small separation distances from the bare force, by finding the asymptotic growth rate, and remove this piece from the bare force.

We define the renormalized self-force to be

$$f_r^{\rm ren} \equiv f_r^{\rm total} - f_r^{\rm div}. \tag{17}$$

The value of $f_r^{\rm ren}$ is of course a function of ϵ, and we need to take the self-force in the limit of $\epsilon \to 0$. We find that the larger α, the greater is the number of modes over which we need to sum until $f_r^{\rm ren}$ converges and the oscillations are damped. Figure 2 displays the behaviour of the sums over modes up to a certain value of the mode number l as functions of l for several values of the inclination parameter α. It is clear from figure 2 that for large inclination parameters one needs to sum over many modes. In addition, we also find that, with fixed α, the number of modes one needs to sum over scales like ϵ^{-1}. When these two effects are combined, one finds that it is very costly numerically to consider nearly tangential splittings.

Figure 3 shows the renormalized force, i.e. $f_r^{\rm ren} \equiv f_r^{\rm total} - f_r^{\rm div}$ as a function of ϵ for a non-zero value of the inclination parameter α. Similar results were also obtained for other values of α (but the number of modes we needed to sum over depended, of course, on the value of α). The figure shows that for small spatial extension (small values of ϵ) the renormalized force is linear in ϵ, such that in the limit of vanishing spatial extension the force would equal zero. Notice that we can see deviations from the linear law for large spatial extensions. These

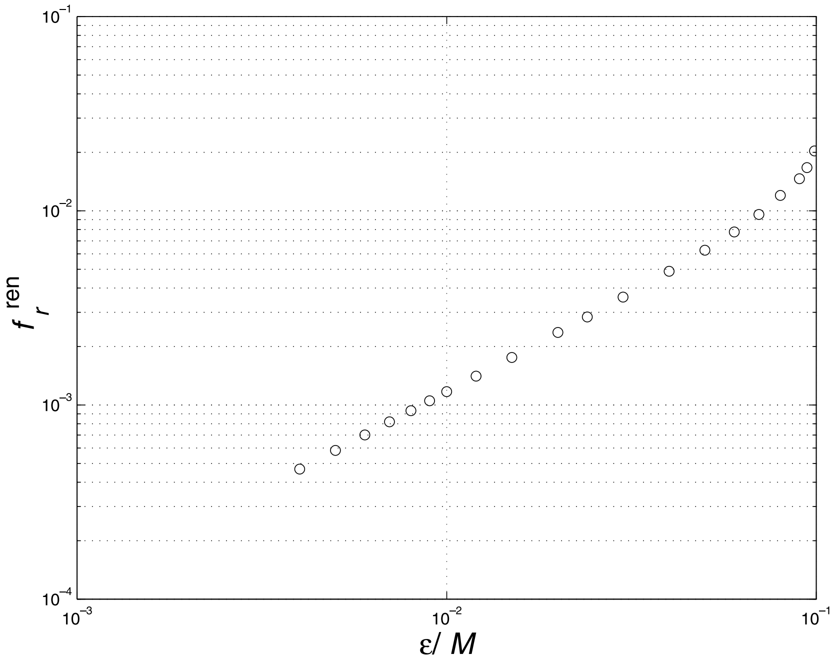

Figure 3. The renormalized self-force f_r^{ren} as a function of the spatial extension ϵ. The charge is located at $r_0 = 2.1M$, and we choose $\alpha = 0.1$. We sum the l modes here up to $l = 2.8 \times 10^3$.

deviations are expected, because the renormalized force, when expanded in a power series in ϵ, contains contributions from all non-negative powers of ϵ. The self-force is the force on a point-like particle, i.e. the force in the limit $\epsilon \to 0$. Consequently, for any non-zero value of ϵ we also have contributions from all positive values of ϵ, which are dominated by the linear term in ϵ for small values of ϵ. In the special case where the alignment of the dumbbell axis is radial ($\alpha = 0$), we find that the divergent piece of the force f_r^{div} vanishes identically, such that f_r^{total} is already renormalized. In this case we can sum the series in f_r^{total} analytically and find the self-force exactly. In fact, for any non-zero ϵ we find for a radial dumbbell axis

$$\begin{aligned} f_r^{\mathrm{total}} &= \frac{1}{2}\frac{e^2}{M^2}\sqrt{\frac{(1-2M/r_1)(1-2M/r_2)}{1-2M/r_0}}\sum_{l=0}^{\infty}(2l+1) \\ &\quad\times\left[P_l\left(\frac{r_1-M}{M}\right)Q_l'\left(\frac{r_2-M}{M}\right)+P_l'\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)\right] \\ &= 0, \end{aligned} \tag{18}$$

where we have used the summation $\sum_{l=0}^{\infty}(2l+1)[P_l(x)Q_l'(y)+P_l'(x)Q_l(y)] = 0$ for $y > x > 1$.

We thus find that if one considers a spatially extended particle model for the particle, one can obtain a finite result for the self-force in the limit of vanishing spatial extension, after performing a simple mass-renormalization procedure, which agrees with the well known exact result [17, 25].

2.2.2. Mode-sum regularization. In this section we use MSRP in order to find the self-force on a point-like static scalar charge in Schwarzschild. MSRP is described briefly in

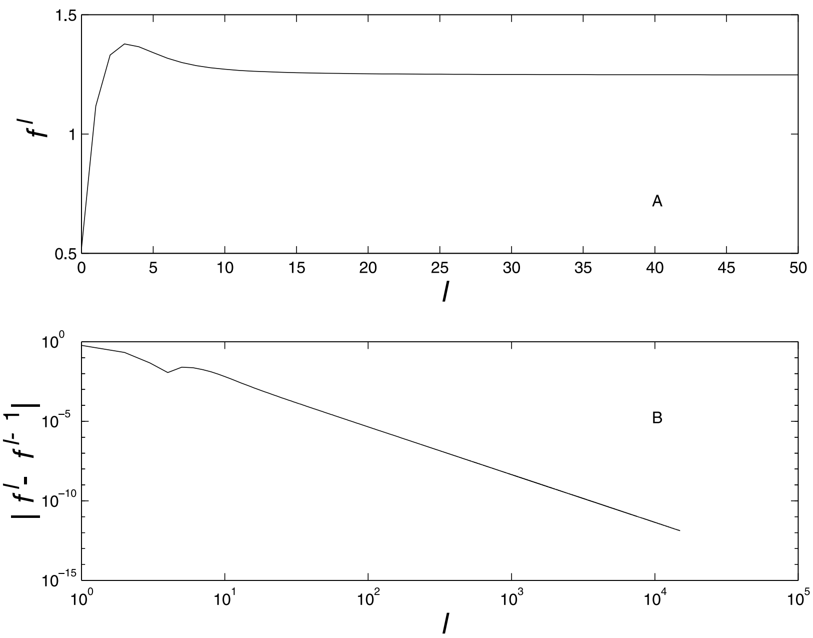

Figure 4. Behaviour of the l modes of the bare force for large values of l. Top panel (A), $|{}^{\rm bare}f_r^l|$ as a function of l; bottom panel (B), $|{}^{\rm bare}f_r^l - {}^{\rm bare}f_r^{l-1}|$ as a function of l. The scalar charge is located at $r = 2.1M$.

appendix A, where the notation and definitions of the MSRP parameters are given. We note that our discussion here serves a dual purpose: first, it applies MSRP for a specific case, and obtains non-trivial physical results. Secondly, because our results can be compared with the final results for the self-forces, which are already known, it predicts values for the MSRP parameters, which can then be tested analytically.

In the case of a point-like particle, we find from equation (14) that the bare force is given by

$$
{}^{\rm bare}F_r = \frac{1}{2}\frac{e^2}{M^2}\sqrt{1-\frac{2M}{r_0}}\sum_{l=0}^{\infty}(2l+1)\left[P_l\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\right.
\left.+P_l'\left(\frac{r_0-M}{M}\right)Q_l\left(\frac{r_0-M}{M}\right)\right]. \tag{19}
$$

Obviously, when this series is summed naively, the bare force diverges. In order to check the applicability of MSRP we first observe numerically that the l modes of this force, ${}^{\rm bare}f_r^l$, approach a non-zero constant as $l \to \infty$, which we denote by ${}^{\rm bare}f_r^{\infty}$. Figure 4 shows the convergence of the l mode of the bare force to a constant, as $l \to \infty$. The top panel of figure 4 shows ${}^{\rm bare}f_r^l$ as a function of l, for the first few values of the latter, and the bottom panel shows the difference between two consecutive l modes of the force as a function of l. We find that this difference scales like l^{-3} for large values of l, which implies that the series indeed converges to a constant. This behaviour implies that the MSRP parameters a_r and c_r vanish. (A non-zero value of c_r implies that the difference between two consecutive modes should scale like l^{-2}.)

Because the l modes of the bare force approach a non-zero constant as $l \to \infty$, it is clear that the sum over all modes diverges to infinity. That is, the source for the divergence

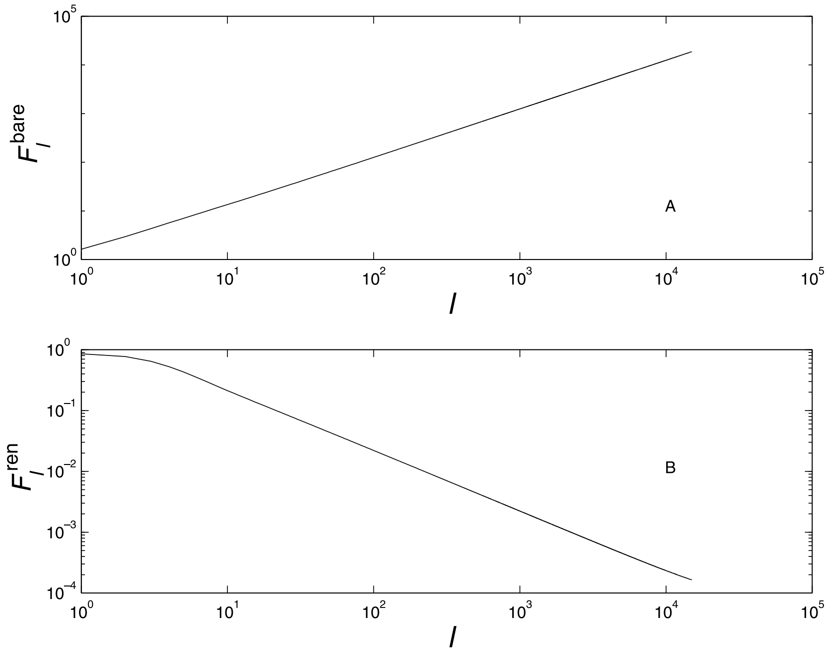

Figure 5. The bare force and the renormalized force as functions of the l up to which we sum over the modes. Top panel (A), ${}^{\rm bare}F_r^l(r_0)$ as a function of l; bottom panel (B), ${}^{\rm ren}F_r^l(r_0)$ as a function of l. For the renormalization procedure we use $l' = 4.5 \times 10^4$. The scalar charge is located at $r_0 = 2.1M$.

comes from the contributions of the large-l modes. Let us assume now that this divergence can be regularized by removing the large-l contributions. That is, we assume that the large-l contributions to the regularized self-force die off with l. The only sensible way to do that is to subtract the asymptotic value of the modes (as $l \to \infty$) from all the modes of the bare force. Although this procedure yields a finite result for the self-force, it is not *a priori* clear whether that is the correct, physical result. However, because the self-force is already known, this can be checked, and we can predict a value for a possible finite additional term for the regularization procedure, which can then be tested analytically using MSRP. For obvious practical reasons, we do the summation over the modes only up to a finite value of l. We denote the approximations of the bare and regularized forces (which are obtained by summing over a finite number of modes) by ${}^{\rm bare}F_r^l(r_0)$ and ${}^{\rm ren}F_r^l(r_0)$, respectively. Then, we represent ${}^{\rm bare}f_r^{\infty}(r_0)$ by the l' mode of the force, for l' much larger than the l up to which we sum the series. In practice, we find that $l' \approx 3l$ suffices to a very good accuracy. Specifically,

$$ {}^{\rm tail}F_r \approx {}^{\rm ren}F_r^l(r_0) = \sum_{j=0}^{l} \left[{}^{\rm bare}f_r^j(r_0) - {}^{\rm bare}f_r^{l'}(r_0) \right]. \tag{20} $$

Figure 5 shows the bare force ${}^{\rm bare}F_r^l(r_0)$ and the renormalized force ${}^{\rm ren}F_r^l(r_0)$ as functions of l. The bare force of course diverges for large values of l. However, figure 5 implies that the renormalized force vanishes as l^{-1} for large l. Recall that the self-force for this case is already known to be zero [17, 25]. Consequently, we infer that the value of the possible additional term for the regularization procedure is zero. Indeed, MSRP yields for this particular case $d_r = 0$,

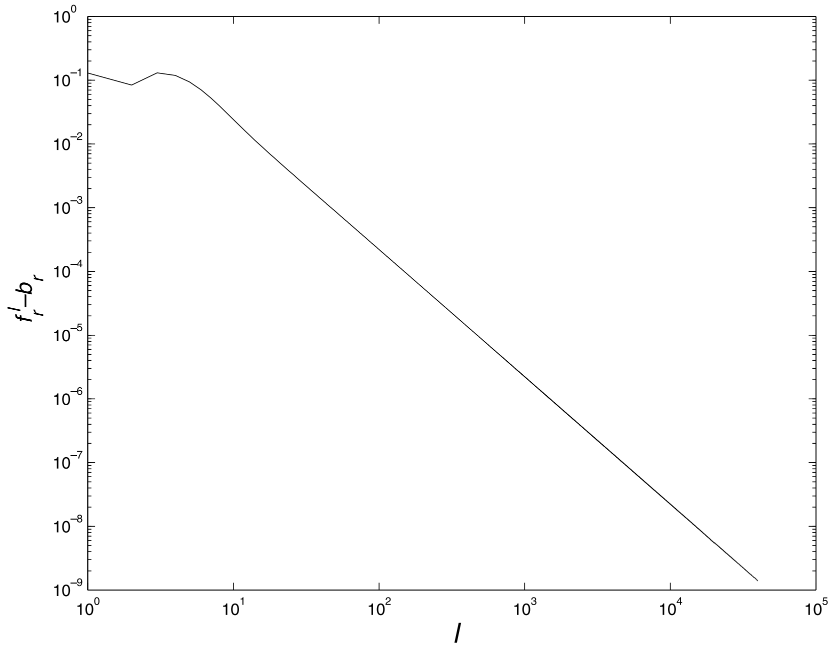

Figure 6. The quantity ${}^{\rm bare}f^l_r(r_0) - b_r(r_0)$ as a function of l. The scalar charge is located at $r_0 = 2.1M$.

which agrees with our result (see appendix A). Because $d_r = 0$, the regularized self-force is given by MSRP to be simply

$$ {}^{\rm tail}F_r = \sum_{l=0}^{\infty} \left[{}^{\rm bare}f^l_r(r_0) - b_r(r_0) \right], \tag{21} $$

where $b_r = {}^{\rm bare}f^{l\to\infty}_r$, i.e. the regularization procedure is reduced to subtracting the asymptotic value of the modes of the bare force from all of its modes, and then summation over all the modes.

We can also check the prediction of MSRP for the exact value of b_r. Recall that in this case $b_r = -[q^2/(2r^2)](1 - M/r)/(1 - 2M/r)$, and that, with $a_r = 0$, MSRP predicts ${}^{\rm bare}f^l_r \to b_r$ as $l \to \infty$. Figure 6 displays the difference between ${}^{\rm bare}f^l_r$ and b_r as a function of l. This difference behaves like l^{-2} for large values of l. This asymptotic behaviour again implies that $a_r = 0$ and $c_r = 0$, as we found above. For $r_0 = 2.1M$, we find this difference to be 1.39×10^{-9} for $l = 4 \times 10^4$. This agreement between the analytical prediction for b_r and the value to which the modes of the bare force approach at large values of the mode number provides a strong support for the validity of MSRP.

3. Static electric charge

3.1. Mode decomposition of the force

An interesting case to study with our method is the case of a static electric test charge in Schwarzschild spacetime. This is interesting because it is known that in this case the radial self-force does not vanish. This can give us two benefits. First, we can see whether our method can yield a correct non-zero result (a zero result cannot reveal a wrong factor, say), and second, we can use the exact expression for the result to evaluate the error in our calculation. The exact result for the self-force in this case was found by Smith and Will [28] and by Zel'nikov and Frolov [25]. The field of a static electric charge in the Schwarzschild spacetime was found in terms of a series expansion solution by Cohen and Wald [33] (see also [17, 34].)

The Maxwell equations in curved spacetime are given by

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu \tag{22}$$

where the Maxwell field strength tensor is given in terms of the 4-vector potential by $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and where $j^\mu = \rho u^\mu$ is the 4-current density, ρ being the charge density. Because of the staticity of the problem (both the charge and the fixed background geometry are static), all spatial components of both the vector potential and the current density vanish. The temporal component of equation (22) becomes

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, g^{\nu\alpha} g^{tt} A_{t,\alpha}\right)_{,\nu} = -4\pi j^t. \tag{23}$$

In Schwarzschild coordinates this equation is explicitly written as

$$\left(r^2 A_{t,r}\right)_{,r} + \left(1 - \frac{2M}{r}\right)^{-1}\left[\frac{1}{\sin\theta}(\sin\theta A_{t,\theta})_{,\theta} + \frac{1}{\sin^2\theta} A_{t,\varphi\varphi}\right] = 4\pi r^2 j^t. \tag{24}$$

We next assume a series expansion of the form

$$A_t(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R^l(r) Y^{lm}(\theta,\varphi) \tag{25}$$

and decompose the current density j^μ into modes

$$j^t(r,\theta,\phi) = q\frac{\delta(r-r_0)}{r_0^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^{lm*}\left(\tfrac{1}{2}\pi, 0\right) Y^{lm}(\theta,\varphi). \tag{26}$$

This current density corresponds to a total charge q, as is evident from

$$q = \int j^t(x^i)\sqrt{-g}\, \mathrm{d}^3x^i. \tag{27}$$

We thus find the radial equation to be

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}R_l(r)}{\mathrm{d}r}\right] - l(l+1)\left(1-\frac{2M}{r}\right)^{-1} R_l(r) = 4\pi q\delta(r-r_0) Y^{lm*}\left(\tfrac{1}{2}\pi, 0\right). \tag{28}$$

The basic functions which solve the corresponding homogeneous equation, with a convenient choice of normalization, are given by [35, 36]

$$R_l^\infty(r) = -\frac{(2l+1)!}{2^l(l+1)!l!M^{l+2}}(r-2M)Q_l'\left(\frac{r-M}{M}\right) \tag{29}$$

$$R_l^0(r) = \frac{2^l l!(l-1)!M^{l-1}}{(2l)!}(r-2M)P_l'\left(\frac{r-M}{M}\right) \qquad (l \neq 0), \tag{30}$$

and $R_0^0(r) = 1$. The Wronskian determinant of these two basic solutions is [33] $W_l(r) = -(2l+1)/r^2$. The solution of the inhomogeneous equation (28) is thus

$$R_l(r) = \frac{f(r_0)R_l^\infty(r_0)}{W_l(r_0)} R_l^0(r)\Theta(r_0 - r) + \frac{f(r_0)R_l^0(r_0)}{W_l(r_0)} R_l^\infty(r)\Theta(r - r_0) \tag{31}$$

where

$$f(r_0) = 4\pi q \frac{1}{r_0^2} Y^{lm*}\left(\tfrac{1}{2}\pi, 0\right). \tag{32}$$

The function $R_l(r)$ is regular both at infinity and at the black hole's event horizon. The summation over all modes m is straightforward, and we find that the l mode A_t^l satisfies

$$A_t^l = \frac{q}{M^3}\frac{2l+1}{l(l+1)}(r-2M)(r_0-2M)P_l(\cos\gamma)\left[P_l'\left(\frac{r-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\Theta(r_0-r) + P_l'\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r-M}{M}\right)\Theta(r-r_0)\right] \qquad (l \neq 0). \tag{33}$$

For the monopole term ($l = 0$) we find $A_t^0 = -(q/r)\Theta(r-r_0) - (q/r_0)\Theta(r_0-r)$. Also in this case an exact solution is known [17], which is

$$A_t = \frac{q}{r_0 r}\left[M + \frac{(r-M)(r_0-M) - M^2\cos\gamma}{\sqrt{(r-M)^2 - 2(r-M)(r_0-M)\cos\gamma + (r_0-M)^2 - M^2\sin^2\gamma}}\right]. \tag{34}$$

The total covariant temporal component of the 4-vector potential is obtained by summing over all l modes. The expression we thus find for A_t is identical to the expression given in [33] and [17]. For the calculation of the force we need only the gradient of A_t with respect to r, which we simplify with the differential equation which the Legendre functions satisfy. We find that

$$\begin{aligned} A_{t,r} = {} & \frac{q}{r^2}\Theta(r-r_0) - \frac{q}{M^3}\frac{(r-2M)(r_0-2M)}{r} \\ & \times \sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\left[P_l'\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r-M}{M}\right)\Theta(r-r_0) \right. \\ & \left. + P_l'\left(\frac{r-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\Theta(r_0-r)\right]P_l(\cos\gamma) \\ & + \frac{q}{M^2}\frac{r_0-2M}{r}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\gamma) \\ & \times\left[P_l'\left(\frac{r_0-M}{M}\right)Q_l\left(\frac{r-M}{M}\right)\Theta(r-r_0)\right. \\ & \left. + P_l\left(\frac{r-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\Theta(r_0-r)\right]. \end{aligned} \tag{35}$$

We note that we did not include in this expression the terms proportional to a delta function for the following reason. When the field is evaluated at any point which is not the position of the charge, these terms are zero. When the evaluation point is at the position of the charge, the sum of the terms proportional to a delta function vanishes. From this expression the force is calculated according to the Lorentz force formula, specifically, $f^\mu = qF^\mu{}_\nu u^\nu$. Here, the only non-zero component of the Maxwell field strength tensor is $F_{rt} = A_{t,r}$, and the only non-zero component of the force is therefore the radial component.

3.2. Regularization procedures

3.2.1. Extended particle model: radial dumbbell. Let us now assume for simplicity that the charges q_1 and q_2 are separated only radially (there is no need for the more complicated splitting we did in the scalar case, because in the electric case we still need to perform mass regularization even for radial splitting). As before, we sum the forces at a common point at r_0. After parallel transporting the forces radially to the common point r_0, in the same way as was done above for the scalar case, we find that

$$
{}^{\text{bare}}f_r = \frac{e^2}{2}\frac{1}{\sqrt{1-2M/r_0}}\left\{\frac{1}{r_2^2} - \sum_{l=1}^{\infty}(2l+1)\left[\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{(r_1-2M)(r_2-2M)}{M^3 l(l+1)}\right.\right.
\times P_l'\left(\frac{r_1-M}{M}\right)Q_l'\left(\frac{r_2-M}{M}\right) - \frac{r_1-2M}{r_2M^2}P_l'\left(\frac{r_1-M}{M}\right)Q_l\left(\frac{r_2-M}{M}\right)
\left.\left. - \frac{r_2-2M}{r_1M^2}P_l\left(\frac{r_1-M}{M}\right)Q_l'\left(\frac{r_2-M}{M}\right)\right]\right\}. \tag{36}
$$

We do not regularize this bare force in the limit $\epsilon \to 0$ with the help of the exact solution because of the following. For an electric dumbbell in arbitrary acceleration in flat spacetime, the divergent piece of the self-force is well known [37], and is given by

$$
\boldsymbol{f}^{\text{div}} = -\frac{e^2}{2d}\left[\boldsymbol{a} + \left(\boldsymbol{a}\cdot\hat{d}\right)\hat{d}\right]. \tag{37}
$$

Here, $\boldsymbol{a}$ is the 3-acceleration, $\hat{d}$ is a unit 3-vector in the direction of the dumbbell axis and d is the length of the dumbbell axis. Note a factor of two between this expression and equation (57) of [37], which is due to the fact that according to equation (13) the total force is *twice* the sum of the two forces. One would expect a similar expression to also hold in curved spacetime. In our problem, the dumbbell axis is aligned in the radial direction, such that instead of $\boldsymbol{a}+\left(\boldsymbol{a}\cdot\hat{d}\right)\hat{d}$ we would have $2a_r$. (The acceleration is only radial.) We find

$$
a_r = \frac{M}{r_0^2}\left(1-\frac{2M}{r_0}\right)^{-1},
$$

such that

$$
{}^{\text{inst}}f_r = -\frac{e^2}{2}\frac{M}{r_0^2}\frac{1}{\sqrt{1-2M/r_0}}\frac{1}{\epsilon}. \tag{38}
$$

Note, that the length of the dumbbell axis is given by the invariant distance between r_1 and r_2. As in the scalar case, we perform mass renormalization by subtracting this divergent piece of the force from the total force given by equation (36). Figure 7 displays the renormalized force as a function of ϵ. We find that the renormalized force approaches the *correct* finite value of $f_r^{\text{exact}} = e^2M/(r^3\sqrt{1-2M/r})$ [25, 28] like ϵ, as indeed we expect.

3.2.2. Mode-sum regularization. As in the scalar case, we can also construe the charge as point-like, and find from equation (36) the total bare radial self-force to be given by

$$
f_r(r_0) = \frac{e^2}{2}\left(1-\frac{2M}{r_0}\right)^{-1/2}\left\{r_0^{-2} - 2\frac{(r_0-2M)^2}{r_0M^3}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l'\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\right.
+\frac{r_0-2M}{r_0M^2}\sum_{l=1}^{\infty}(2l+1)\left[P_l'\left(\frac{r_0-M}{M}\right)Q_l\left(\frac{r_0-M}{M}\right)\right.
\left.\left.+P_l\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\right]\right\}. \tag{39}
$$

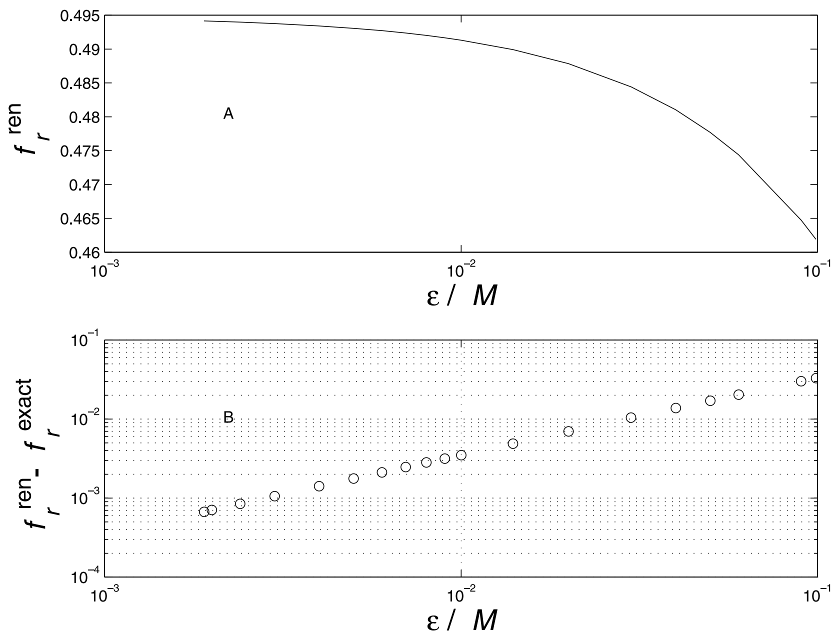

Figure 7. Top panel, the renormalized force as a function of ϵ; bottom panel, $|f_r^{\rm ren} - f_r^{\rm exact}|$ as a function of ϵ/M. The charge is located at $r = 2.1M$.

For calculation of the bare force, equation (39) can be rewritten as

$$ {}^{\rm bare}F_r(r_0) = \frac{e^2}{\sqrt{1-2M/r_0}}\left[\frac{1}{r_0^2} - \frac{(r_0-2M)^2}{r_0M^3}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l'\left(\frac{r_0-M}{M}\right)Q_l'\left(\frac{r_0-M}{M}\right)\right], \tag{40} $$

which simplifies the calculation. However, equation (40) mixes the contributions of the different modes. Although the regularization procedure also works with this mixing, we shall consider below the regularization procedure with the force as given by equation (39). We first check the behaviour of the modes ${}^{\rm bare}f_r^l(r_0)$ as $l \to \infty$. Figure 8 shows that indeed ${}^{\rm bare}f_r^l(r_0)$ approaches a constant, and that the difference between two consecutive modes scales like l^{-3} for large values of l, in a similar way to the behaviour of the modes for the scalar case. Consequently, also for this case, we infer that $a_r = 0$ and that $c_r = 0$. We emphasize that for the case of an electric charge these parameters have not been calculated analytically, whereas in the scalar case they have. In this case we also do not have prior knowledge about the value of the parameter d_r. This is in general a serious problem, because without knowledge of d_r, the final result for the self-force is not unambiguous. However, in this case we do have the final result from independent approaches, such that we can, in fact, predict the value of d_r. Work still remains to be done to compute the values of a_r, b_r, c_r and d_r analytically for this case.

As in the scalar case, we approximate the bare and the renormalized forces by the sum over a finite number of modes, and denote them by ${}^{\rm bare}F_r^l$ and ${}^{\rm ren}F_r^l$, respectively. We again define ${}^{\rm ren}F_r^l$ as in the scalar case, by subtracting f_r^{∞} from each mode of the series. Figure 9 shows the renormalized force ${}^{\rm ren}F_r^l$ and its difference from $f_r^{\rm exact}$ as functions of the mode number l. We find that ${}^{\rm ren}F_r^l - f_r^{\rm exact}$ approaches zero like l^{-1} for large values of l, such that we recover the results of [17, 28], i.e. we find that the self-force is a repelling force, which

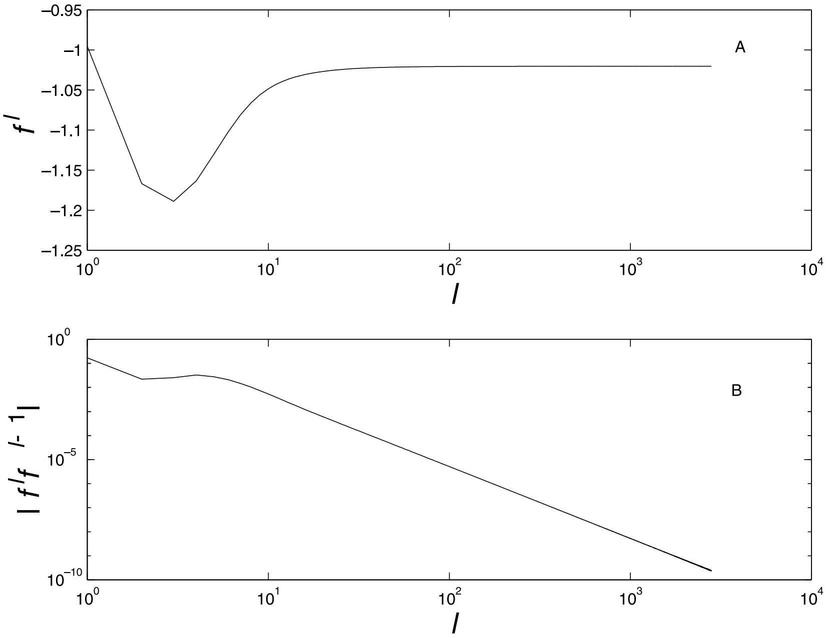

Figure 8. Behaviour of the l modes of the bare force for large values of l. Top panel (A), the behaviour of ${}^{\mathrm{bare}}F_r^l(r_0)$ as a function of l; bottom panel (B), $|{}^{\mathrm{bare}}F_r^l(r_0) - {}^{\mathrm{bare}}F_r^{l-1}(r_0)|$ as a function of l. The charge is located at $r = 2.1M$.

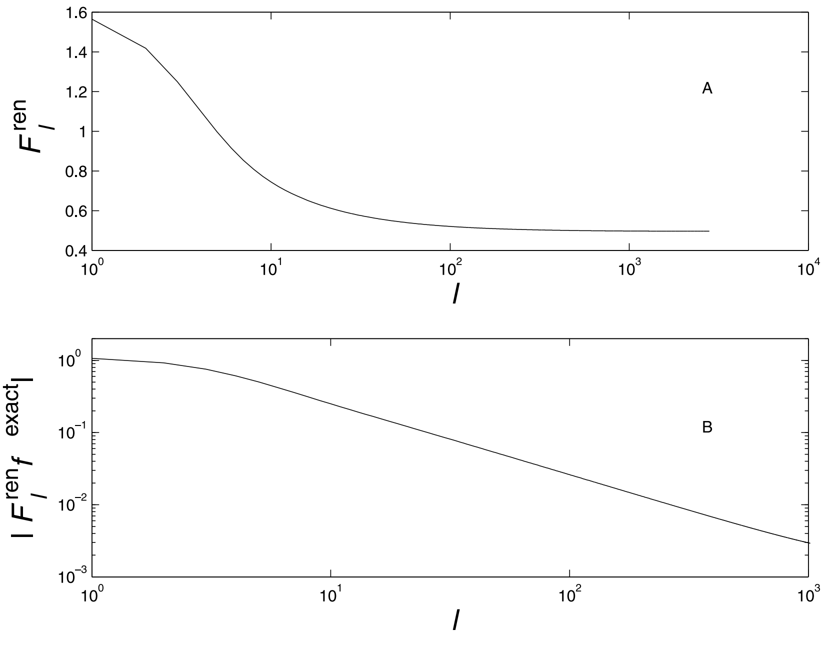

Figure 9. Top panel (A), the renormalized force ${}^{\mathrm{ren}}F_r^l$ as a function of l; bottom panel (B), $|F_l^{\mathrm{ren}} - f^{\mathrm{exact}}|$ as a function of l. The charge is located at $r = 2.1M$. The regularization procedure is performed with $l' = 2.8 \times 10^3$.

is given by $f_r = q^2 M/(r^3\sqrt{1-2M/r})$. The asymptotic agreement of ${}^{\rm ren}F_r^l$ and $f_r^{\rm exact}$ imply that also for this case $d_r = 0$. This prediction can be tested analytically.

4. Summary

We have presented a direct calculation of the self-forces acting on two types of static charges in Schwarzschild spacetime: a scalar charge and an electric charge. In both cases the boundary conditions were chosen such that the scalar field and the potential, correspondingly, would be regular both at infinity and at the black hole's event horizon. Our method is based on decomposition of the field and the force into modes. Each mode satisfies an ordinary differential equation which we solve exactly in terms of Legendre functions (in the scalar case) or derivatives of the Legendre functions (in the electric case). We find the total bare forces by summing over all modes numerically. This total force typically diverges. We then regularize the divergent self-force with two independent procedures: first, we model the point-like particle to be spatially extended, and then consider a sequence of such particles, letting the spatial extension decrease. The divergent piece of the force is removed by a mass-renormalization procedure (i.e. it is used to redefine the mass of the particle), and the remaining force approaches the self-force in the limit of vanishing extension. Second, we use Ori's mode-sum regularization prescription, and remove the divergent piece of the force by studying the behaviour of the bare force at large values of the mode number, and subtracting the value of the bare force at the limit of infinite mode numbers from all modes. Both regularization procedures recovered the well known results for static charges in the spacetime of a Schwarzschild black hole: a zero self-force in the scalar case and a repelling radial self-force in the electric case.

When one compares the relative effectiveness of the two regularization procedures, one finds that their effectivenesses are comparable. Specifically, for comparable values of l up to which we sum the series, we find that for both regularization schemes we obtain similar deviations of the computed regularized forces from the exact solutions, with roughly the same computation time.

Evidently, more work is needed for both regularization prescriptions. In particular, it is not presently understood how to apply MSRP for more complicated cases, e.g. it is not presently fully understood whether there are cases with non-vanishing parameters c_μ, and whether the formalism can be extended to handle such cases (the radial component c_r was shown to be zero only for a scalar charge, although for all orbits in Schwarzschild). Also, it is not clear when non-zero functions d_μ should be expected [38]. A generalization of MSRP to also include the gravitational case is also needed, a case for which the inherent gauge problem should be solved. We are currently using MSRP to study more complicated cases, in particular the self-force acting on a scalar charge in circular orbit around a Schwarzschild black hole [39].

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Appendix A. Mode-sum regularization prescription

In this appendix we describe very briefly the main ideas behind Ori's method for regularizing the mode-sum (MSRP) [24] for a scalar charge in Schwarzschild.

We emphasize that the work on this method is still in progress. However, for the case of a static scalar charge in Schwarzschild, the regularization scheme has been developed in full.

As was pointed out by Quinn and Wald [6], the physical self-force (in vacuum) is the sum of two parts: (a) a local, Abraham–Lorentz–Dirac-type term and (b) a 'tail' term ${}^{\rm tail}F_\mu$, associated with the tail part of the Green's function. The local term is trivial to calculate (and anyway it vanishes in the static case considered in this paper). Therefore we shall only consider the tail term. This term may be expressed as

$$ {}^{\rm tail}F_\mu \equiv \lim_{\epsilon\to 0^-} {}_\epsilon F_\mu, \tag{A1} $$

where ${}_\epsilon F_\mu$ denotes the contribution to the force (evaluated at $\tau = 0$) from the part $\tau \leqslant \epsilon$ of the particle's worldline. Decomposing this expression into ℓ-modes, one finds

$$ {}^{\rm tail}F_\mu = \lim_{\epsilon\to 0^-} \sum_\ell {}_\epsilon f^\ell_\mu = \lim_{\epsilon\to 0^-} \sum_\ell ({}^{\rm bare}f^\ell_\mu - \delta_\epsilon f^\ell_\mu). \tag{A2} $$

Here, ${}_\epsilon f^\ell_\mu$, $\delta_\epsilon f^\ell_\mu$ and ${}^{\rm bare}f^\ell_\mu$ denote the force from the ℓ-multipole of the field sourced by the interval $\tau \leqslant \epsilon$, the interval $\tau > \epsilon$ and the entire worldline, respectively. The force ${}^{\rm bare}f^\ell_\mu$ may be identified with the sum over m and ω of the contributions from all stationary Teukolsky modes ℓ, m, ω for a given ℓ (recall that in calculating a stationary field's mode ℓ, m, ω one takes the source term to be the *entire* worldline). Since we are using the retarded Green's function, the part $\tau > 0$ does not contribute. However, the interval from ϵ to 0^+ does contribute. Essentially, it is this part which is responsible for the instantaneous, divergent piece of the Green's function, which should be removed from the expression for ${}^{\rm tail}F_\mu$.

A clarification is required here concerning the meaning of the last equality in equation (A2): let r_0 denote the value of r at the evaluation point. Then, ${}_\epsilon f^\ell_\mu$ is well defined at $r = r_0$. The situation with ${}^{\rm bare}f^\ell_\mu$ and $\delta_\epsilon f^\ell_\mu$ is more involved, however. Each of these quantities has a well defined value at the limit $r \to r_0^-$, and a well defined value at the limit $r \to r_0^+$. Generically, for the r-component (and in some cases also for other components) these two one-sided values are not the same. Equation (A2) should thus be viewed as an equation for either the limit $r \to r_0^-$ of the relevant quantities (i.e. ${}^{\rm bare}f^\ell_\mu$ and $\delta_\epsilon f^\ell_\mu$), or the limit $r \to r_0^+$ of these quantities. Obviously, this equation is also valid for the *averaged* force, i.e. the average of these two one-sided values. In what follows we shall always consider the averaged force. Of course, the final result of the calculation, ${}^{\rm tail}F_\mu$ (which has a well defined value at the evaluation point), is the same regardless of whether it is derived from its one-sided limit $r \to r_0^-$, or from $r \to r_0^+$, or from their average.

Next, we seek an ϵ-independent function h^ℓ_μ, such that the series $\sum_\ell ({}^{\rm bare}f^\ell_\mu - h^\ell_\mu)$ converges. Once such a function is found, then equation (A2) becomes

$$ {}^{\rm tail}F_\mu = \sum_\ell ({}^{\rm bare}f^\ell_\mu - h^\ell_\mu) - \lim_{\epsilon\to 0^-} \sum_\ell (\delta_\epsilon f^\ell_\mu - h^\ell_\mu). \tag{A3} $$

In principle, h^ℓ_μ can be found by investigating the asymptotic behaviour of ${}^{\rm bare}f^\ell_\mu$ as $\ell \to \infty$. It is also possible, however, to derive h^ℓ_μ from the large-ℓ asymptotic behaviour of $\delta_\epsilon f^\ell_\mu$ (the latter and ${}^{\rm bare}f^\ell_\mu$ must have the same large-ℓ asymptotic behaviour, because their difference yields a convergent sum over ℓ). In addition to h^ℓ_μ, the investigation of $\delta_\epsilon f^\ell_\mu$ should also yield the parameter $d_\mu \equiv \lim_{\epsilon\to 0^-} \sum_\ell (\delta_\epsilon f^\ell_\mu - h^\ell_\mu)$, required for the calculation of ${}^{\rm tail}F_\mu$ in equation (A3).

Since we only need the asymptotic behaviour of $\delta_\epsilon f^\ell_\mu$ for arbitrarily small $|\epsilon|$, it is possible to analyse it using local analytic methods. In particular, we can apply a perturbation analysis to the ℓ-mode field equation (in the time domain). That is, we express the ℓ-mode effective potential $V^\ell(r)$ as a small perturbation $\delta V^\ell(r)$ over the value of $V^\ell(r)$ at the evaluation point, $V_0^\ell \equiv V^\ell(r = r_0)$. Expressing $G^\ell[x^\mu, x_s^\mu(\tau)]$, the ℓ-mode Green's function, as a function of τ and $z \equiv \tau\ell$, the perturbation analysis provides an expression for G^ℓ as a power series in τ (with z-dependent coefficients). Only terms up to order τ^2 are required for the calculation of the self-force (recall that eventually we take the limit $\epsilon \to 0$), and the perturbation analysis yields explicit expressions for the required three expansion coefficients of G^ℓ (as functions of z). Constructing $\delta_\epsilon f_\mu^\ell$ from G^ℓ (essentially by integrating the latter's gradient from ϵ to $\tau = 0$), it can be shown that the large-ℓ asymptotic behaviour of $\delta_\epsilon f_\mu^\ell$ takes the form $\delta_\epsilon f_\mu^\ell = a_\mu \ell + b_\mu + c_\mu \ell^{-1} + \mathrm{O}(\ell^{-2})$, in which the parameters a_μ, b_μ, c_μ are independent of ℓ and ϵ (though they depend on the orbit and evaluation point). (It can also be shown that there is no logarithmic divergence of h_μ^ℓ.) The regularization function h^ℓ thus takes the form $h_\mu^\ell = a_\mu \ell + b_\mu + c_\mu \ell^{-1}$, and the tail part of the self-force is given by

$$ {}^{\mathrm{tail}}F_\mu = \sum_\ell \left({}^{\mathrm{bare}}f_\mu^\ell - a_\mu \ell - b_\mu - c_\mu \ell^{-1} \right) - d_\mu. \tag{A4} $$

In the case of a static scalar particle in Schwarzschild, one can show that $a_\mu = c_\mu = d_\mu = 0$ [24]. (a_μ and c_μ are likely to vanish for all orbits in Schwarzschild, but so far this has been shown explicitly for the radial component only.) The self-force for a static particle then takes the simple form

$$ {}^{\mathrm{tail}}F_\mu = \sum_\ell \left({}^{\mathrm{bare}}f_\mu^\ell - b_\mu \right). \tag{A5} $$

Namely, in this simple case the regularization procedure is reduced to subtracting ${}^{\mathrm{bare}}f_\mu^{\ell\to\infty}$, the large-$\ell$ limit of the ℓ multipole of the bare force, from each multipole ℓ (note that since $a_\mu = 0$, $b_\mu \equiv {}^{\mathrm{bare}}f_\mu^{\ell\to\infty}$). For the particular case of a static scalar charge in Schwarzschild, Ori [24] also obtained analytically the value of this large-ℓ limit of the force: $b_\mu = -[q^2/(2r^2)](1 - M/r)/(1 - 2M/r)\delta_\mu^r$.

This regularization prescription takes a trivial form in the cases of static scalar or electric charges in Minkowski spacetime. In these cases it is easy to verify that all the ℓ modes of the bare force are equal (i.e. independent of ℓ), specifically ${}^{\mathrm{bare}}f_r^\ell = -q^2/(2r^2) =$ constant (this can be obtained easily directly from a decomposition of the field). When one sums over all modes, the bare force of course diverges. However, subtracting this constant term from each mode yields a new series, where all modes are zero, such that the total force vanishes, which is the well known result in Minkowski spacetime. We note that MSRP turns out to also be effective for the cases of scalar or electric charges in circular orbits in Minkowski spacetime [21].

We emphasize that whereas the parameters a_μ, b_μ and c_μ can be found from the behaviour of ${}^{\mathrm{bare}}f_\mu^\ell$ at large values of the mode number ℓ, the parameter d_μ can only be calculated according to its definition. In the simple case of a scalar charge in circular orbit around Schwarzschild, which includes as a special case a static scalar charge, this calculation is not hard to do, and the exact value of d_μ was found (for this case $d_\mu = 0$). However, it might be the case that for more complicated cases d_μ is more difficult to find. Then, one can still regularize the force, but the result would not be free from the ambiguity which results from ignorance of the exact value of d_μ.

MSRP involves integration over the entire worldline of the orbiting object. In that respect, it is especially suitable for periodic, or near periodic, orbits. For aperiodic orbits, such as the final plunge of the object into the black hole, one can perhaps use a different approach, where one integrates only over the past worldline, excluding the position of the particle itself, and thus avoids the singular contribution. Of course, the closer to the particle one integrates, the more modes one would need to sum over in order to obtain convergence. In fact, the number of modes is inversely proportional to the proper-time difference from the event up to which one integrates and the position of the particle. It has been shown recently by Wiseman [40] that in the far-field limit (i.e. to leading order in the ratio of the black hole mass to the radius of the orbit), the contribution of the near neighbourhood of the past worldline is negligible, such that one may need to sum only over a relatively small number of modes. In stronger fields, this approach can perhaps be combined with a normal-neighbourhood expansion [40, 41] to obtain the self-force.

Appendix B. Numerical evaluation of the Legendre functions

All the series we need to evaluate very accurately involve the Legendre functions of the first and second kinds, and their derivatives. The degree of the functions is very high. For example, in the numerical summations reported here we evaluated the series up to $l = 4.5 \times 10^4$. This requires us to use a very accurate algorithm for the calculation of the Legendre functions. In fact, we find that the Legendre functions of the first kind can be computed very accurately by using the recursion relations, such as

$$P_{l+1}(x) = \frac{1}{l+1}[(2l+1)xP_l(x) - lP_{l-1}(x)] \tag{B1}$$

and

$$P_l'(x) = \frac{l}{x^2-1}[xp_l(x) - P_{l-1}(x)]. \tag{B2}$$

Although similar relations also hold for the Legendre function of the second kind, they are not practical for the following reason: the functions $Q_l(x)$ approach zero very quickly for fixed $x > 1$ when the degree becomes very large. The subtraction which is inherent to the recursive expression becomes numerically inaccurate very rapidly. The functions Q_l can also be considered as the sum of two series, one being a polynomial and the other a polynomial multiplied by a common logarithmic factor. Each of the polynomials satisfy the same recursive formula as the Legendre functions, but with different initial terms for $l = 0$ and 1. Each of the two series grow very quickly with l, but their difference becomes very small. Therefore, this method would also not be very accurate numerically. A way to avoid these difficulties is to use the integral representation of the functions $Q_l(x)$. This is given by

$$Q_l(x) = \frac{1}{2^{l+1}} \int_{-1}^{1} \mathrm{d}t \, \frac{(1-t^2)^l}{(x-t)^{l+1}}. \tag{B3}$$

The integrand does not have any pathologies over the entire interval of integration, and also the boundaries are regular. We perform this integral using Romberg integration, which proves to be very efficient and accurate [42]. The derivatives of the functions $Q_l(x)$ can still be computed by the relation given above for $P_l'(x)$. Another improvement on the numerical evaluation of both $P_l(x)$, $Q_l(x)$ and their derivatives is the following. In all the expressions we have, we need only compute the *product* of two Legendre functions or their derivatives, one factor involving P_l (or its derivative), and the other involving Q_l (or its derivative). Because we are not interested in the value of the Legendre functions themselves, but only in such products, we can disregard the factor of $2^{-(l+1)}$ in the integral representation of $Q_l(x)$. This would mean that each of the functions we compute is too large by a factor of 2^{l+1}. If we then compute, instead of the functions $P_l(x)$, a new function, which is smaller than $P_l(x)$ by the same factor, the product of the two new functions would be unchanged. This can also be done for the derivatives of the Legendre functions. It is advantageous to do this, because for a given floating point arithmetic this procedure increases the maximal value of l for which accurate computations can be performed by an order of magnitude.

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