Strongly Stable Networks*

by

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Abstract

We analyze the formation of networks among individuals. In particular, we examine the existence of networks that are stable against changes in links by any coalition of individuals. We show that to investigate the existence of such strongly stable networks one can restrict focus on a component-wise egalitarian allocation of value. We show that when such strongly stable networks exist they coincide with the set of efficient networks (those maximizing the total productive value). We show that the existence of strongly stable networks is equivalent to core existence in a derived cooperative game and use that result to characterize the class of value functions for which there exist strongly stable networks via a "top convexity" condition on the value function on networks. We also consider a variation on strong stability where players can make side payments, and examine

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situations where value functions may be non-anonymous – depending on player labels.

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1 Introduction

The importance of networks in a variety of social and economic settings is well-documented. Applications range from social networks such as friendships to more directly economically motivated ones such as trading alliances, decentralized market relationships, research partnerships, etc. Given that network relationships matter, it is important to understand which networks are likely to form and how this depends on the structure of the setting. In particular, there has been a good deal of recent research into understanding how networks form among a group of individuals (people, firms, etc.) who have the discretion to choose with whom they interact.[1]

In this paper, we continue that line of research through a careful study of the existence and properties of strongly stable networks: those networks which are stable against changes in links by any coalition of individuals. Strongly stable networks are those which are supported by strong Nash equilibria of an appropriate game of network formation.

There are many reasons for studying a strong notion of stability based on coalitional considerations. In network formation, individual or pairwise based solution concepts such as Nash equilibrium and pairwise stability (see Jackson and Wolinsky (1996)) often lead to many stable networks, so that they provide broad predictions. In some contexts this already narrows things, but in other contexts it may leave us with a large set of networks. Moreover, these networks may have very different properties and then additional considerations may help us to sort among them to produce narrower and more accurate predictions of network formation. (See Examples 1 and 2, in section 2.1 below, for illustrations.) In addition, in many contexts, there will naturally be communication among individuals that may allow a number of them to coordinate their choices of links. As such, we study strongly stable networks as a natural way for making tighter predictions using coalitional considerations. One can think of a notion such as pairwise stability as a weak stability concept which is essentially a necessary (and some times too weak) condition for stability, while strong stability is a sufficient (and some times too strong) condition for stability.

Strong stability of networks is a very demanding property, as it means that no set of players could benefit through any rearranging of the links that they are involved with (including those linking them to players outside the coalition). As such, we expect there

[1]For bibliographies on network study generally and network formation in particular, we refer the reader to Slikker and van den Nouweland (2001a), Dutta and Jackson (2003), and Jackson (2004).

to be contexts where such networks will not exist. However, strongly stable networks still exist in a number of natural settings, including some that pop up in the literature as examples of network situations. In situations where strongly stable networks exist they are quite compelling, in the sense that once formed such networks are essentially impossible to destabilize, as there is no possible reorganization that would be improving for all of the players whose consent is needed.

Another reason for examining the existence of strongly stable networks, beyond their compelling stability properties, is that such networks exhibit additional properties. For instance, as we shall show, if a network is strongly stable and has more than one component, then value must be allocated equally among members of each component, and in fact the per capita value must be equal across components. This is a very strong equity property. More importantly, strongly stable networks have strong efficiency properties. One obvious property is Pareto efficiency. But if the value of each component of a network is allocated equally among the members of that component of a network, then when strongly stable networks exist they exhibit even stronger efficiency properties. In this case, strongly stable networks maximize the overall value of the network. This statement actually takes a bit of proof as we shall show. Although it is obvious if a network consists of just one component, it is more subtle when efficient networks consist of several components.

Motivation for the study of the existence of networks that are efficient and satisfy some stability requirement comes out of the previous literature. From previous research, we know that there are a variety of contexts where the stability of networks can be at odds with efficiency. Jackson and Wolinsky (1996) show that for some settings the sets of pairwise stable networks and efficient networks do not intersect. Moreover, for some value functions they showed that this is true regardless of how value is allocated or transferred among players, provided the allocation respects component balance and anonymity (which are formally defined below). Jackson (2003) goes on to show that even a weaker form of efficiency is at odds with pairwise stability, and that in some very natural contexts even Pareto efficiency can be widely incompatible with pairwise stability.

The tension between stability and efficiency suggests several directions for further study. One is to examine whether the tension disappears if we are free to construct the allocation rule in careful and non-anonymous ways. This angle is pursued by Dutta and Mutuswami (1997) who show that careful construction of allocation rules that may be non-anonymous (on unstable networks) can restore the compatibility between efficiency and stability.[2] Another direction is to identify specific classes of value functions and/or allocation rules for which there is no tension between stability and efficiency (or at least for which there is an overlap between the two) when keeping with anonymity. That direction is pursued both in Jackson and Wolinsky (1996) and Jackson (2003), when the concept in question is pairwise stability. The current paper is in that same spirit, but moves beyond pairwise stability to strong stability. As we shall see, efficient networks and strongly stable networks will coincide when the latter exist. Of course, the existence of strongly stable networks is of interest beyond efficiency, given that such networks are robust to all kinds of deviations, as we have already discussed above.

The paper proceeds as follows. In the next section we provide definitions and examples in which we compare strong stability to pairwise stability. In Section 3 we first show that the existence of strongly stable networks requires an egalitarian allocation. Next, we characterize the existence of strongly stable networks under the component-wise egalitarian allocation rule in terms of nonemptiness of the core of a closely related cooperative game. We use this in Section 4 to obtain a characterization of the value functions for which there exist strongly stable networks, showing that a "top convexity" condition is both necessary and sufficient. We provide applications of these results to a variety of settings. In Section 5 we move on to consider side payments, showing that the characterizations in the previous sections relating to the component-wise egalitarian allocation rule are in fact necessary for any allocation rule when strong stability allows for side payments. Finally, we close the paper with some results on non-anonymous value functions in Section 6 and some concluding remarks in Section 7. The proofs of all results are collected in the appendix.

2 Definitions

Networks

There is a set $N = \{1, \dots, n\}$ of players who may be involved in network relationships.

[2]Another interesting direction, not as closely related to what we examine here, is to study situations where the allocation rule and networks are formed simultaneously and endogenously. This is explored in Currarini and Morelli (2000), Mutuswami and Winter (2002) and Slikker and van den Nouweland (2001b). As shown by Currarini and Morelli (2000), at least for some bargaining protocols, efficiency can be regained in some settings.

Non-directed graphs are used to model the network relations between players.[3] In such a graph the nodes (vertices) correspond to the players and the links (edges) correspond to bilateral relationships between players. Let g^N be the set of all subsets of N of size 2, and similarly for any $S \subseteq N$ let g^S be the set of all subsets of S of size 2. $G = \{g \mid g \subseteq g^N\}$ is the set of all possible networks or graphs on N.

The link between players i and j is denoted by ij.

A network g induces a *partition* $\Pi(g)$ of the player set N, where two players i and j are in the same partition element if and only if there exists a path[4] in the graph connecting i and j (using the convention that there is a path from each player to him or herself).

A network g is *connected* if $\Pi(g) = \{N\}$. For any $S \in \Pi(g)$, $g(S)$ denotes the subgraph of g on the set S, i.e. $g(S) = g \cap g^S$.

The *components* of a network g, denoted $C(g)$, are defined by $C(g) = \{g(S) | S \in \Pi(g), |S| \geq 2\}$. The restriction that $|S| \geq 2$ rules out empty networks as components.

The Value of a Network

The value of a network is given by a value function $v : G \to I\!R$. We normalize v so that $v(\emptyset) = 0$. The set of all such value functions is denoted V.

A value function is *anonymous* if for any permutation of the set of players π (a bijection from N to N), $v(g^\pi) = v(g)$, where $g^\pi = \{\pi(i)\pi(j) | ij \in g\}$.

Anonymity says that the value of a network is derived from the structure of the network and not the labels of the players who occupy certain positions. For many of the results we will restrict our attention to anonymous value functions, and we discuss extensions to non-anonymous value functions in a later section of the paper.

A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in G$.

Component additivity precludes that the value of a given component of a network depends on how other components are organized. This precludes externalities across components of a network. However, it still allows for externalities within components. That is, the value of a given component, and ultimately each player's payoff, can depend on the way that the component is structured. For example, the value of $\{12, 23\}$ can differ from that of $\{12, 23, 13\}$, and so, for instance, player 2's payoff may depend on whether 1 and 3 are linked.

[3] For some analysis of network formation in directed networks see Bala and Goyal (2000) and Dutta and Jackson (2000). The general problem of strong stability in directed networks has not been studied.

[4] Formally, a path in g from i to j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

Allocation Rules

An *allocation rule* is a function $Y : G \times V \to I\!R^n$ that describes how the value of a network is distributed among the players. The payoff of player $i \in N$ in network g with a value function v under allocation rule Y is denoted $Y_i(g, v)$.

The allocation function may represent payoffs that accrue to individuals directly, or might also represent additional transfers of value among players. We can be agnostic on whether the allocation rule simply reaffirms individually-earned payoffs, is derived from some bargaining among players, or is forced by some government or other intervening party.

An allocation rule Y is *anonymous* if for any $v \in V$, $g \in G$, and permutation of the set of players π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$, where the value function v^π is defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

Anonymity of an allocation rule requires that if all that has changed is the labels of the players and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling.

An allocation rule Y is *component balanced* if $\sum_{i \in S} Y_i(g, v) = v(g(S))$ for each component additive v, $g \in G$ and $S \in \Pi(g)$.

Component balance requires that the value of a given component of a network is allocated to the members of that component in cases where the value of the component is independent of how other components are organized. This means that in situations where there are no externalities across components with respect to value, there is no cross-subsidization of components with respect to allocations either. It is a condition that an intervening planner or government would like to respect if they wish to avoid secession by components of the network.

An allocation rule Y is *component decomposable* if $Y_i(g, v) = Y_i(g(S), v)$ for each component additive v, $g \in G$, $S \in \Pi(g)$, and $i \in S$.

Component decomposability requires that in situations where v is component additive, the way in which value is allocated within a component does not depend on the structure of other components. So, in situations where there are no externalities across components, the allocation within a component is independent of the rest of the network. The idea behind component decomposability is that in situations where there are no externalities across components, a given component can bargain or decide its allocation of value without attention to the outside structure of the network as it has no effect on the given component. In fact, the members of a given component might not even be aware of the outside structure of the network, especially since this only applies in situations where the outside structure is completely inconsequential to the given component.

Some simple examples show that component balance and component decomposability are mutually independent. Suppose there are 5 individuals and consider the component additive value function according to which components with two links have value 6 and all other components have 0 value. An allocation rule for this situation that is anonymous and component balanced but not component decomposable is one that gives a payoff of 0 to every player who is not in a component with two links, that gives 2 to every player in the component with two links if this is the unique component of the network, and that in a network consisting of one component with one link and one component with two links gives 4 to the middle player in the two-link component and 1 to the two other players in this component. An allocation rule for this situation that is anonymous and component decomposable but not component balanced is one that gives 2 to every player in a two-link component, 1 to every player in a one-link component, and 0 to players who are not in a one-link or a two-link component.[5]

Given any component additive $v \in V$, the *component-wise egalitarian allocation rule* Y^{ce} is defined by

$$Y_i^{ce}(g, v) = \frac{v(g(S_i))}{|S_i|},$$

where $S_i \in \Pi(g)$ is the unique partition element containing player i. Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive.

The component-wise egalitarian rule is one where the value of each component is split equally among the members of the component; provided this can be done - i.e., within the limits of component additivity. This allocation rule is anonymous, component balanced, and component decomposable, and satisfies nice egalitarian properties in terms of equalizing payoffs.

As we shall see, this allocation rule will surface if one wishes to have strongly stable networks, and will play a key role in the characterization of value functions that allow such networks.

Efficiency and Stability Notions

A network g is *efficient* with respect to v if $v(g) \geq v(g')$ for all $g' \in G$.

[5]Note that the allocation rule described here violates 'overall balancedness' because $\sum_{i \in N} Y_i(g, v) \neq v(g)$ for some networks g. This is unavoidable, as component decomposability and 'overall balancedness' together imply component balancedness.

We denote the set of networks that are efficient with respect to value function v by $E(v)$.

Note that an efficient network always exists since there are only finitely many networks in G. This is a strong notion of efficiency as it requires the maximization of total value. It only corresponds to Pareto efficiency if the value is freely and fully transferable across all components of a network.[6]

The following definition of coalitional deviation is used in defining the strong stability notion.

A network $g' \in G$ *is obtainable from* $g \in G$ *via deviations by* S if

(i) $ij \in g'$ and $ij \notin g$ implies $ij \subseteq S$, and

(ii) $ij \in g$ and $ij \notin g'$ implies $ij \cap S \neq \emptyset$.

The above definition identifies changes in a network that can be made by a coalition S, without the need of consent of any players outside of S. (i) requires that any new links that are added can only be between players in S. This reflects the fact that consent of both players is needed to add a link between them. (ii) requires that at least one player of any deleted link be in S. This reflects that fact that either player in a link can unilaterally sever the relationship. Hence, if a single player deviates ($|S| = 1$), then this payer can break some or all of his links, but not form any new links.

A network g is *strongly stable* with respect to allocation rule Y and value function v if for any $S \subseteq N$, g' that is obtainable from g via deviations by S, and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.

We denote the set of networks that are strongly stable with respect to Y and v by $SS(Y, v)$.

The definition of strong stability we use here is slightly stronger (i.e., harder to satisfy) than that originally introduced by Dutta and Mutuswami (1997). The definition of strong stability here allows for a deviation to be valid if some members are strictly better off and others are weakly better off, while the definition in Dutta and Mutuswami (1997) considers a deviation valid only if all members of a coalition are strictly better off. For many value functions and allocation rules these definitions coincide.

There are several reasons for working with this stronger definition of strong stability. First, it implies pairwise stability, whereas the Dutta and Mutuswami (1997) version

[6]For discussion of this and some weaker notions of efficiency see Jackson (2003).

of strong stability does not quite imply pairwise stability.[7] To see this note that our definition implies that a network is not strongly stable if a single player can strictly increase his payoff by breaking some or all of his links or if a coalition of two or more players can deviate to a network in which some of its members get a strictly higher payoff while none of its members get a lower payoff. Second, this stronger definition allows for a stronger implication in Theorem 2, where we conclude that under certain conditions on the value function all efficient networks are strongly stable. Third, the converse of this statement in Theorem 2 is only true with the stronger definition of strong stability. Finally, if all members of a coalition are weakly better off and some strictly better off, then any ability of members to make even tiny transfers will result in a deviation. As we compare the definition of strong stability with what happens when transfers are possible, this slightly stronger notion of stability is more appropriate.

Such differences between weak and strong inequalities are common to definitions of Pareto efficiency, the core, strong Nash equilibrium, and coalitional stability properties; and the difference sometimes has consequences. In working with the stronger definition here, one ends up with a more attractive solution when it is non-empty, but in cases where it is empty one might also wish to examine the weaker solution.

We remark that the strongly stable networks correspond exactly to the strong Nash equilibria of the network formation game suggested by Myerson (1991). In that game players simultaneously announce the set of players with whom they wish to be linked and a link between two players forms if and only if both players have named each other.[8]

Cooperative Games and the Core

A *TU cooperative game* is a pair (N, w), where N is the set of players and $w : 2^N \to \mathbb{R}$ defines the productive value of each subset of N. In line with this interpretation

[7]Pairwise stability (from Jackson and Wolinsky (1996)) is defined as follows. A network $g \in G$ is *pairwise stable* with respect to allocation rule Y given a value function $v \in V$ if no player benefits from severing one of their links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. This last part of the definition is what makes our version of strong stability compatible with pairwise stability but the Dutta and Mutuswami version incompatible.

[8]The equivalence holds for the corresponding definition of strong Nash equilibrium which requires that there are no deviations by a coalition that make all members weakly better off and some strictly better off. There are some details to verify, as there are some strong Nash equilibria where one player names another but is not reciprocated. It is easy to check that the networks formed in such equilibria must be strongly stable networks.

$w(\emptyset) = 0$.

As we fix N throughout our analysis, we often refer to a characteristic function $w : 2^N \to I\!R$ as a cooperative game.

An allocation $x \in I\!R^N$ is in the *core* of w if $\sum_{i \in N} x_i = w(N)$ and $\sum_{i \in T} x_i \geq w(T)$ for all $T \subseteq N$.

2.1 A Comparison of Pairwise and Strong Stability

It is easy to see how strong stability can refine pairwise stability. For instance, consider the following example.

EXAMPLE **1** *Circles*

Suppose there are $n \geq 3$ players. For any $k \geq 3$, let a k-player circle be a network of the form $\{12, 23, \ldots, ii+1, \ldots, k1\}$, or any permutation of such a network.[9] Consider a component-additive value function where the value to a component is k^2 if it is a k-person circle with $k \geq 3$, and 0 otherwise. Suppose that the value of each network is allocated according to the component-wise egalitarian allocation rule. Here, the set of pairwise stable networks includes all of the circles of any size larger than 2, networks consisting of several components that are such circles, possibly of various sizes, as well as some other networks. Every circle of size $k \geq 3$ is pairwise stable because it takes changes in links involving at least three players to change the network structure to become a larger circle, which would be the only way to increase the players' payoffs.

The set of strongly stable networks is considerably smaller, however, as only circles encompassing all n players are strongly stable. This is easily seen as the component-wise egalitarian allocation rule gives $\frac{k^2}{k} = k$ to every player in a k-player circle, and this is obviously increasing in the size of the circle. Hence, all players get the maximum payoff possible in any network when they are in a circle of maximum size, get a lower payoff in any other network, and they can obtain a circle of maximum size starting from any other network.

We make two remarks about this example. First, from any network that is pairwise stable but is not an n-player circle, there is a profitable deviation that involves exactly three players.[10] Thus, the instability of pairwise stable networks relative to coalitional

[9]Here, $ii+1$ denotes a link between player i and player $i+1$.

[10]If there are three players who are getting 0, then they can form a circle of three and all be better off. If not, then it must be that there is some circle in the pairwise stable network that is smaller than

deviations becomes evident with relatively small coalitions. Second, the number of strongly stable networks is much smaller than the number of pairwise stable networks, and this difference explodes as n grows. Thus, strong stability is a dramatic refinement.

While Example 1 illustrates the distinction between pairwise stability and strong stability, and the potential for strong stability to refine pairwise stability, the example is somewhat artificial in that only "circles" generate value. The following example is one that is derived from some primitive description of how value is generated, and also shows a refinement due to strong stability.

Example **2** *A Trading Network*

The following example is based on one from Jackson and Watts (2002).

The society consists of n individuals who get value from trading goods with each other. In particular, there are two consumption goods and individuals all have the same utility function for the two goods which is Cobb-Douglas, $u(x, y) = (xy)^{\frac{1}{2}}$. Individuals have a random endowment, which is independently and identically distributed. A individual's endowment is either (1,0) or (0,1), each with probability 1/2.

Individuals can trade with any of the other individuals in the same component of the network. For instance, in a network $g = \{12, 23, 45\}$, individuals 1, 2 and 3 can trade with each other and individuals 4 and 5 can trade with each other, but there is no trade between 123 and 45. Trade flows without friction along any path and each connected component trades to a Walrasian equilibrium. This means, for instance, that the networks $\{12, 23\}$ and $\{12, 23, 13\}$ lead to the same expected trades. However, if there are costs of forming links, then these two networks will lead to different costs.

The gains from trade in the network $g = \{12\}$ are as follows. There is a $\frac{1}{2}$ probability that one individual has an endowment of (1,0) and the other has an endowment of (0,1). They then trade to the Walrasian allocation of $(\frac{1}{2}, \frac{1}{2})$ each and so their utility is $\frac{1}{2}$ each. There is also a $\frac{1}{2}$ probability that the individuals have the same endowment and then there are no gains from trade and they each get a utility of 0. Taking expectations over these two situations leads to an expected utility of $\frac{1}{4}$.

Not accounting for the cost of links, the expected utility for an individual of being connected to one other individual is $\frac{1}{4}$. The expected utility for an individual of being

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the n-player circle. Consider a component that is the largest circle. Consider any player k not in that component and any two players i and j from that component such that ij is part of the circle. Players i, j, and k can break link ij and form links ik and jk, thereby forming a larger circle, in which each of them is better off.

connected (directly or indirectly) to two other individuals is $\frac{\sqrt{2}}{4}$. It is easily checked that the expected utility of an individual is increasing and strictly concave in the number of other individuals that she is directly or indirectly connected to, ignoring the cost of links.

For the purpose of illustration, consider a situation where $n = 3$ and the cost to a link is slightly above $\frac{1}{2}$ and split equally among all members of the relevant component. In this case, there are two types of pairwise stable networks. One type is a network with two links, so that all three players can trade with each other. The other is a network with no links. The network with no links is inefficient, and it is only pairwise stable since players only consider adding a link with one other player at a time - and the costs from doing this outweigh the gains. However, if all three players can coordinate, then adding two links makes all of them better off.[11] The networks with two links are the only networks that are strongly stable in this example.

This example is just one where strong stability provides an effective and sensible refinement of pairwise stability.

3 The Existence of Strongly Stable Networks, Efficiency and the Core

Let us begin by showing that strong stability has some particular implications about the structure of the allocation rule that must be in place.

Theorem 1 *Consider any anonymous and component additive value function $v \in V$. If Y is an anonymous, component decomposable, and component balanced allocation rule and $g \in G$ with $\Pi(g) \neq \{N\}$ is a network that is strongly stable with respect to Y and v, then $Y(g,v) = Y^{ce}(g,v)$ and $Y_i(g,v) = \frac{v(g)}{n}$ for each $i \in N$.*

Theorem 1 says that if a non-connected network is strongly stable, then the allocation must be as it would be under the component-wise egalitarian rule and in fact must involve an equal split of the total value of the network. Hence, a component-wise egalitarian allocation of value will necessarily play a prominent role in the analysis of strongly stable networks. The idea behind the proof of the theorem is quite simple and

[11] There are other possible reasons to think the two-link network might form as well, which have to do with forward looking players who can anticipate the future continuations due to their actions, as in Page, Wooders, and Kamat (2001).

very compelling. Suppose that value is not being split equally. Then, some player in some component (or perhaps completely disconnected) is getting a payoff below that of some other player in some other component and could deviate together with the other members of the second component to provide an improving deviation.

We point out that Theorem 1 only states conclusions about the allocation of value and not about the structure of strongly stable networks that have more than one component. It could be that the players share the value of the network equally, while still sitting in very different positions in the network. For example, a component of the network could be a star, in which one player has many links with each other player and the other players have only one link each (with the central player).

The following example demonstrates that it is critical to Theorem 1 that the network consist of more than one component.

Example **3** *A Connected Strongly Stable Network.*

There are three individuals. Networks with two links have value 2.5, the complete network has value 3, and other networks have 0 value. Consider the allocation rule where the middle player in a two link network (e.g., player 2 in $\{12, 23\}$) gets a payoff of .1 and the other two players get a payoff of 1.2, in the complete network each player gets 1 and in networks with at most one link each player gets 0. In this example, any network with two links is strongly stable - and these are the only strongly stable networks.

To see that networks with two links are strongly stable, consider (without loss of generality) the network $g = \{12, 23\}$. There is only one player in this network who can strictly benefit from deviating (namely player 2), as the other two players are getting the maximum that they can get in any network. But to deviate to a network in which he gets more than .1, player 2 needs other players to deviate with him. However, in any network in which player 2 gets more than .1, either player 1 or player 3 gets less than 1.2 (or both). Hence, at most one of these other players will deviate with player 2. So, consider a deviation from g by players 1 and 2. The only network in which player 2 gets more than .1 and player 1 gets at least 1.2 is $g' = \{13, 23\}$. However, g' is not obtainable from g via deviations by players 1 and 2 because players 1 and 2 cannot form the new link 13 without player 3's cooperation. To see that there are no other strongly stable networks than those with two links, note that starting from a network with fewer links, all players can strictly improve their payoffs by forming the complete network, and that starting from the complete network, any two players can strictly improve their payoffs by breaking the link between them.

This shows that a connected network can be strongly stable without having its allocation being egalitarian, which shows why Theorem 1 requires that the network have more than one component. This example also shows that connected strongly stable networks can be inefficient when the allocation rule is not component-wise egalitarian.

The following example illustrates the role of component decomposability in Theorem 1.

Example 4 *The Role of Component Decomposability.*

There are $n = 6$ players. Let v be defined by $v(\{12, 23\}) = 10$ and for any other structure of a component (that has one link or three or more links) we let v have a value of 0. Also, let v be anonymous and component additive, so that $v(\{12, 23, 45, 56\}) = 20$ and permutations of the above networks have the corresponding values.

On efficient networks such as $\{12, 23, 45, 56\}$, set $Y_1(\{12, 23, 45, 56\}) = Y_3(\{12, 23, 45, 56\}) = Y_4(\{12, 23, 45, 56\}) = Y_6(\{12, 23, 45, 56\}) = 4$, and $Y_2(\{12, 23, 45, 56\}) = Y_5(\{12, 23, 45, 56\}) = 2$. Also, set $Y_1(\{12, 23, 45\}) = Y_1(\{12, 23\}) = Y_3(\{12, 23\}) = Y_3(\{12, 23, 45\}) = 5$, $Y_2(\{12, 23, 45\}) = Y_2(\{12, 23\}) = 0$. Set Y elsewhere to respect anonymity and component balance.

Consider network $g = \{12, 23, 45, 56\}$. It is impossible for two players i and j to deviate to a network $\{ik, jk\}$ or $\{ik, jk, lm\}$ in which they both would get 5, as this would involve forming links with or breaking links between players who are not deviating (note that any of the other players would not assist in such a deviation as their payoffs would decrease to 0). Next, consider a deviation by a coalition of players from network g to a network that is a permutation of g in such a way that at least one devating player strictly improves his payoff (from 2 to 4). Then the payoff of at least one player i must decrease from 4 to 2. This player i is involved in one link in network g and is involved in two links after the deviation, which implies that i's assistance is needed to obtain the new network. This shows that there is no deviation from network g that strictly improves the payoff of at least one deviating player without decreasing the payoff of another deviating player. We conclude that $g = \{12, 23, 45, 56\}$ is strongly stable with respect to Y and v.

The allocation rule Y differs from the component-wise egalitarian rule for the non-connected strongly stable network $\{12, 23, 45, 56\}$. In particular, Y is adjusted on $\{12, 23\}$ depending on how 4, 5, and 6 are linked, if at all. We have done this in such a way as to preclude blocking by a coalition involving some players from $\{1, 2, 3\}$ and players from $\{4, 5, 6\}$. However, the allocation rule Y violates component decomposability.

Example 4 illustrates the difference between strong stability and the weaker notion of strong stability that was introduced in Dutta and Mutuswami (1997), where a deviation is only valid if *all* deviating players get a strictly higher payoff. Consider the value function in Example 4 and the component-decomposable allocation rule Y that gives middle players in components with two links 2 while giving the other two players in such a component 4 each. All players in other components get 0. Networks such as $\{12, 23, 45, 56\}$ are stable in the weaker sense of Dutta and Mutuswami, but are not strongly stable in our sense. Clearly, Y differs from the component-wise egalitarian rule on any network that contains a component with two links. In fact, it seems Y does not have any particular properties besides anonymity, component decomposability, and component balancedness and this suggests that a characterization similar to that in Theorem 1 will be impossible to obtain when the weaker notion of strong stability is used.

If we consider the weaker notion of strong stability and in addition allow for arbitrarily small transfers among players, then we recover the result in Theorem 1. To see this, suppose that g is a network that is not strongly stable in our sense. Consider a deviation by a coalition S to a network g' that leaves no member of S worse off and improves the payoffs of some of its members. By transferring an arbitrarily small positive amount ε from a player who has a strictly higher payoff in g' to those players whose payoffs remain unchanged, the payoffs of all deviating players can be increased strictly. This shows that in a setting where small transfers are feasible, a network that is strongly stable in the weaker sense is also strongly stable in the our sense. Hence, the conclusions of Theorem 1 hold if we replace our notion of strong stability with the weaker notion coupled with some arbitrarily small transfers.

Theorem 1 shows that the component-wise egalitarian allocation rule will necessarily play a prominent role in the analysis of strongly stable networks and therefore we focus on this allocation rule in what follows. This is with a loss of generality, as Theorem 1 does not imply that the allocation of value must be egalitarian on networks that are not strongly stable or in which all players are in a single component. For instance, it is possible that an allocation rule splits value equally on strongly stable networks but not on other networks. Nevertheless, the component-wise egalitarian rule is an interesting one from many perspectives and we shall see that it has some nice properties. We return to consider more general allocation rules when we discuss transfers.

Given a value function v, let the cooperative game (N, w^v) be defined by

$$w^v(S) = \max_{g \in g^S} v(g).$$

Thus, every value function $v \in V$ defines a cooperative game where the value of a coalition is the maximum value it can obtain by arranging its members in a network.

Note that if v is anonymous, then w^v is symmetric (so $w^v(S) = w^v(T)$ whenever $|S| = |T|$). Also, if v is component additive, then w^v is superadditive. That is, $w^v(S \cup T) \geq w^v(S) + w^v(T)$ whenever $S \cap T = \emptyset$.

THEOREM **2** *Consider any anonymous and component additive value function $v \in V$. Some efficient $g \in G$ with respect to v is strongly stable with respect to $Y^{ce}(\cdot, v)$ if and only if the core of w^v is nonempty. Moreover, $SS(Y^{ce}, v) \neq \emptyset$ if and only if $E(v) = SS(Y^{ce}, v)$.*

Theorem 2 shows that our interest in guaranteeing that a society forms efficient networks is closely tied to the non-emptiness of the core of a related cooperative game. This allows us to make use of the substantial knowledge on core existence in cooperative game theory to analyze the efficiency of network formation.

Let us make a couple of remarks on the conditions in Theorem 2. First, Example 3 demonstrates that for allocation rules other than Y^{ce} it is possible to find strongly stable networks with one component that are not efficient. Thus, while Theorem 1 argued that egalitarian allocations are the right ones to consider on non-connected networks, when we consider connected networks the restriction to the component-wise egalitarian rule is important to the conclusions. Second, the equivalence between the core and strong stability depends on the specifics of the definition of strong stability. That equivalence does not hold if one replaces our definition of strong stability with the stability notion in Dutta and Mutuswami (1997), where a deviation is valid only if all members of a coalition are strictly better off. An example shows this. Consider 5 players and the following anonymous and component additive value function v. A component with 3 players has value of 7 and a component of 2 players has value 3. By component additivity, a network that consists of two 2-player components and one isolated player has value 6 and a network that consists of one 2-player component and one 3-player component has value 10. All other networks have value 0. In this setting, an efficient network consists of two components, one encompassing 2 players and the other 3. Under the component-wise egalitarian rule, such a network is stable as defined by Dutta and Mutuswami (1997). However, such a network is not strongly stable in our sense, and it follows by standard game-theoretic arguments that the core of w^v is empty.

On a superficial level Theorem 2 seems obvious, since both strong stability and the core notion allow for deviations by arbitrary subsets of players. However, there are several levels on which Theorem 2 is not obvious (which can also be seen from the proof). Moreover, these less obvious points are those which result in some of the theorem's power and usefulness, as we shall see later. Some of the differences between the core and strong stability notions are as follows. Strong stability allows for a deviating coalition to maintain links with non-deviating players (and keeps the rest of the network intact), while the core notion requires complete separation by a deviating coalition. This gives better opportunities for a coalition to improve under the strong stability notion. Working in the other direction is that the core allows for transfers to be made among players in a deviating coalition regardless of how that coalition derives its value, while under component balance a deviating coalition under the strong stability notion cannot make transfers across components of a new network that is formed. With these two critical differences, there is no obvious reason to expect the relationship outlined in the theorem to hold in general. Moreover, the last part of the theorem shows that it is not simply that there exists a network that is strongly stable with respect to $Y^{ce}(\cdot, v)$, but that the efficient networks and strongly stable networks with respect to $Y^{ce}(\cdot, v)$ coincide.

Application to Communication Networks and Convex Games

A special class of value functions are those derived from cooperative games with communication networks (see Slikker and van den Nouweland (2001a)). Let z be a characteristic function that indicates the productive value $z(S)$ of each coalition S, which the players in this coalition can achieve if they can communicate with each other. Communication between players can only take place along links in a communication network g and each link in this network incurs a cost c. This then allows one to define a value function that assigns to each network g the productive value that the players can obtain when they have the communication channels in g available, minus the cost of the network.

To be specific: a given cooperative game z and cost per link c lead to the value function $v^{z,c} \in V$ defined by

$$v^{z,c}(g) = \sum_{S \in \Pi(g)} z(S) - \sum_{ij \in g} c.$$

In order to obtain a value function $v^{z,c}$ with $v^{z,c}(\emptyset) = 0$, we limit ourselves to characteristic functions z that are *zero-normalized*, i.e., $z(\{i\}) = 0$ for each $i \in N$. Also, to obtain an anonymous value function we limit ourselves to so-called symmetric cooperative games z, in which the productive value of a coalition depends only on the size of that coalition. We write z as a function of coalition size and denote $z_k = z(S)$ where $|S| = k$. Let Z denote the class of zero-normalized symmetric cooperative games.

A cooperative game $z \in Z$ is *convex* if

$$\forall k \geq 2 : z_k - z_{k-1} \geq z_{k-1} - z_{k-2}.$$

We can use Theorem 2 to show that for value functions derived from convex games in the manner described above, strongly stable networks exist and are efficient.

Corollary **1** *Consider any convex cooperative game $z \in Z$ and any cost per link $c \geq 0$. Then $E(v^{z,c}) = SS(Y^{ce}, v^{z,c})$.*

Corollary 1 shows that Theorem 2 has powerful implications, as the class of communication games with convex production and costly links is a wide class.

The proof of Corollary 1 uses Theorem 2 and the cooperative game $w^{v^{z,c}}$ derived from the value function $v^{z,c}$. Hence, $w^{v^{z,c}}(S)$ is the maximum value that a coalition S can obtain by building a communication network among its members and thereby enabling them to achieve productive value. It can be shown that the cooperative game $w^{v^{z,c}}$ is convex and thus has a non-empty core. This is not immediate since although z is convex, one needs to show that the induced game is still convex when link costs are accounted for.

The scope of Corollary 1 does not extend arbitrarily to a class of games that is larger than the class of convex games. We demonstrate this in the following example.

Example **5** *A Non-Convex Game.*

Consider the cooperative 5-player game (N, z) defined by $z(S) = |S|$ if $|S| \geq 2$ and $z(S) = 0$ otherwise. This game is obtained from an additive game in which each player contributes 1 to every coalition by setting the worth of one-player coalitions equal to 0. Suppose that $0 < c < 1$. The value function $v^{z,c}$ is component additive and assigns a value $2-c$ to components with one link (and two players), a value $3-2c$ to components with two links (and three players), a value $3-3c$ to components with three links and three players, a value $4-3c$ to components with three links and four players, and so on. Since links are costly, an efficient network will in each component connect players with a minimum number of links. It is now easily seen that an efficient network g consists of two components, one with two players connected by a link and the other with three players connected by two links. However, such a network is not strongly stable with respect to $Y^{ce}(\cdot, v^{z,c})$ as two of the players in the three-player component can deviate to form a component with one link, thereby increasing each their payoffs from $\frac{3-2c}{3}$ to $\frac{2-c}{2}$. A network that is strongly stable with respect to $Y^{ce}(\cdot, v^{z,c})$ partitions the player set into three parts: two components that each have two players connected by one link and one isolated player. Hence, no network that is efficient with respect to $v^{z,c}$ is strongly stable with respect to $Y^{ce}(\cdot, v^{z,c})$ and vice versa. In fact, it can be shown that for *any* anonymous and component balanced allocation rule Y it holds that $E(v^{z,c}) \cap SS(Y, v^{z,c}) = \emptyset$.

4 Primitive Conditions on Value Functions

While the non-emptiness of the core of the associated cooperative game w^v is an interesting and useful condition, as illustrated at the end of the last section, we are also interested in direct conditions on v which characterize the strong stability of efficient networks. Theorem 2 is still useful in this regard, as the characterization of v's that allow for strongly stable networks to exist (and then coincide with efficient networks) can be obtained through the conditions on w^v.

A value function v is *top convex* if some efficient network also maximizes the per-capita value among individuals.[12] Formally, let $p(v, S) = \max_{g \in g^S} \frac{v(g)}{|S|}$.

[12]A related condition is called "domination by the grand coalition," as defined in the context of a cooperative game by Chatterjee, Dutta, Ray, and Sengupta (1993). That condition requires that the per capita value of the grand coalition be at least that of any sub-coalition. Shubik (1982, page 149) shows that for symmetric cooperative games this condition is a necessary and sufficient condition for nonemptiness of the core. The top convexity condition we identify here is defined for the network setting, but is equivalent to requiring that w^v be dominated by the grand coalition. In a bargaining context, Chatterjee, Dutta, Ray and Sengupta show that this condition is equivalent to existence of a sequence of limiting efficient stationary equilibria for each bargaining protocol in a wide class.

The value function v is *top convex* if $p(v, N) \geq p(v, S)$ for all S.

An implication of top convexity is that all components of an efficient network must generate the same per-capita value. To see this note that if some component $g(S)$ of an efficient network led to a per capita value $\frac{g(S)}{|S|}$ that is lower than $p(v, N)$, then some other component $g(T)$ would have to lead to a per capita value $\frac{g(T)}{|T|}$ that is higher than $p(v, N)$, which would imply that $p(v, T) > p(v, N)$, a contradiction of top convexity.

The following theorem shows that top convexity plays a key role in the existence of strongly stable networks.

THEOREM **3** *Consider any anonymous and component additive value function v. The core of w^v is nonempty if and only if v is top convex. Thus, $E(v) = SS(Y^{ce}, v)$ (or $SS(Y^{ce}, v) \neq \emptyset$) if and only if v is top convex.*

We remark that Example 3 shows that for connected networks if one is not working with the component-wise egalitarian rule, then it is possible to have inefficient networks be strongly stable, even when the value function is top convex. Thus, top convexity does not guarantee efficiency of strongly stable networks if we look at connected networks and non-egalitarian allocations. In addition, top convexity is not a necessary condition for the existence of strongly stable networks when connected networks are considered and the allocation rule can be different from Y^{ce}.[13]

Theorem 3 shows that one needs strong conditions on the value function v in order to have existence and/or efficiency of the set of networks that are strongly stable with respect to the component-wise egalitarian allocation rule. Despite the strength of top-convexity, it is satisfied by many v's, and we now point out several such value functions.

EXAMPLE **6** *The Symmetric Connections Model*

The symmetric connections model of Jackson and Wolinsky (1996) is one where links represent social relationships between individuals; for instance friendships.[14] These

[13]This point is illustrated in the following example. There are four players. The anonymous component-additive value function v is such that 3-player circles have value 3, 4-player stars have value 3.6, and all other components have value 0. This value function is not top convex, as the maximal per-capita value in 3-player networks is 1, whereas it is .9 in 4-player networks. Consider the allocation rule where each player in a 3-player circle gets 1, the central player in a 4-player star gets .3 and each of the other 3 players in such a star get 1.1, and all players in other components get 0. Any 4-player star is strongly stable (as well as efficient).

[14]For further study of variations on the connections model, see Johnson and Gilles (2000), Watts (2001), and Jackson (2003).

relationships offer benefits in terms of favors, information, etc., and also involve some costs. Moreover, individuals also benefit from indirect relationships. A "friend of a friend" also results in some benefits, although of a lesser value than a "friend," as do "friends of a friend of a friend" and so forth. The benefit deteriorates with the "distance" of the relationship. For instance, in the network $g = \{12, 23, 34\}$ individual 1 gets a benefit $\delta < 1$ from the direct connection with individual 2, an indirect benefit δ^2 from the indirect connection with individual 3, and an indirect benefit δ^3 from the indirect connection with individual 4. As $\delta < 1$, this leads to a lower benefit from an indirect connection than a direct one. Individuals only pay costs, however, for maintaining their direct relationships.

Formally, the payoff player i receives from network g is

$$u_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j: ij \in g} c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). The value in the connections model of a network g is simply $v(g) = \sum_i u_i(g)$.

It is easily seen that v is top convex for all values of $\delta \in [0, 1)$ and $c \geq 0$, so that all networks that are strongly stable with respect to Y^{ce} and v are efficient with respect to v.[15]

We remark that $Y_i^{ce}(g, v) \neq u_i(g)$ for some networks g. Thus, our result is not in contradiction with the finding of Jackson and Wolinsky (1996) that sometimes none of the pairwise stable networks (under u_i) are efficient in the connections model. Here the reallocation of value under the component-wise egalitarian rule guarantees stability of the efficient network.[16] This makes clear the role of the allocation rule in the existence of strongly stable networks.

Example **7** *The Co-Author Model*

The co-author model (from Jackson and Wolinsky (1996)) is described as follows. Each individual is a researcher who spends time working on research projects. If two

[15] The proof of Proposition 1 in Jackson and Wolinsky (1996) provides some hints to the interested reader on how to fill in omitted details. Most importantly, for intermediate cost ranges the per capita value of the (efficient) star network is growing in the number of players in the star.

[16] See Jackson and Wolinsky (1996) for some additional study of the component-wise egalitarian rule.

researchers are connected, then they are working on a project together. The amount of time researcher i spends on a given project is inversely related to the number of projects, n_i, that he is involved in. Formally, i's production is represented by

$$u_i(g) = \sum_{j:ij\in g} \left(\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right)$$

for $n_i > 0$, and $u_i(g) = 0$ if $n_i = 0$.[17] The total value is $v(g) = \sum_i u_i(g)$.

Provided that n is even, it is easily seen that v is top-convex as the efficient network always involves pairs of players who are only linked to each other. Thus strongly stable networks exist in this situation (under Y^{ce}), and correspond to the networks with evenly matched pairs. The reallocation of value under the component-wise egalitarian allocation rule forces players to take into account the negative externalities that they inflict upon their current co-authors when they start new projects. This prevents them from engaging in too many projects from an efficiency point of view, as happens when the allocation rule u is used.

If n is odd, then top convexity is violated since dropping any player increases the per capita value obtainable. Thus, Theorem 3 implies that if n is odd then no strongly stable network exists.[18]

5 Strong Stability with Side Payments

Once we allow for coalitional deviations, so presumably coalitions can coordinate their actions, in many contexts it is reasonable to assume that they will also be able to reallocate value. This leads to the formulation of an even stronger stability concept.

Say that g is SSS (strongly stable with side payments) relative to an allocation rule Y and value function v if $\sum_{i\in S} Y_i(g,v) \geq \sum_{i\in S} Y_i(g',v)$ for any $S \subseteq N$ and g' obtainable from g by S. We denote this set $SSS(Y,v)$.

[17]It might also make sense to set $u_i(g) = 1$ when an individual has no links, as the person can still produce research. This is not in keeping with the normalization of $v(\emptyset) = 0$, but it is easy to simply subtract 1 from all productivities and then view u as the extra benefits above working alone.

[18]Note that if instead of our current definition, which only requires one player to be strictly better off in the deviation, we were to require both players to be better off in a deviation, then there would exist a stable network in the co-author model with an odd number of players. However, this existence of strongly stable networks would no longer hold if we introduce any sort of side payments, as we discuss next.

THEOREM **4** *Let $v \in V$ be component additive and anonymous. The following statements are equivalent:*

(i) *there exists a component balanced allocation rule Y such that $SSS(Y, v) \neq \emptyset$,*

(ii) *there exists a component balanced allocation rule Y such that $SSS(Y, v) = E(v)$,*

(iii) $E(v) = SS(Y^{ce}, v)$,

(iv) $E(v) = SSS(Y^{ce}, v)$.

Theorem 4 reinforces the implications of Theorem 1 that component-wise egalitarian allocation of value plays a key role in the existence of strongly stable networks, this time including the possibility of side payments. So beyond Y^{ce}'s natural appeal in terms of egalitarian properties, we find that it is a key allocation rule to understand when it comes to finding existence of strongly stable networks and strongly stable networks with side payments.

An example shows that the result is not true if one changes SSS to SS in part (i) or (ii) of Theorem 4.

EXAMPLE **8** *Strong Stability with Side Payments*

There are six players. A circle encompassing all six players has value 6 and a star encompassing four players has value 5. All other networks have value 0. For the allocation rule Y that we describe momentarily the efficient networks (circles) are exactly the strongly stable networks. According to Y each player gets 1 if they are in a circle. If g is a four-person star, then the player who is the center of the star gets 0 and the three outside players in the star each get $\frac{5}{3}$. Players get 0 according to Y otherwise. For this Y, it holds that $E(v) = SS(Y, v) \neq \emptyset$. Under the component-wise egalitarian rule, however, the circle is not strongly stable. Hence, $E(v) \cap SS(Y^{ce}, v) = \emptyset$ and the equivalence in Theorem 4 would not hold.

If a network is SSS then it is stable in a very strong sense and so Theorem 4, together with our other results, shows that any top convex value function v (and only such value functions!) will have networks that are stable in very strong ways.

6 Non-anonymous Value Functions

So far, we have limited our attention to anonymous value functions. Let us consider the extent to which similar results hold for non-anonymous value functions.

THEOREM **5** *Let $v \in V$ be a component additive value function. The following statements are equivalent:*

(i) the core of w^v is nonempty,

(ii) there exists a component balanced allocation rule Y such that $SSS(Y, v) \neq \emptyset$,

(iii) there exists a component balanced allocation rule Y such that $E(v) = SSS(Y, v)$.

Moreover, top convexity of v implies each of the above and also implies that $E(v) = SS(Y^{ce}, v)$.

Unlike in a setting of anonymous value functions, for non-anonymous value functions top convexity of v, nonemptiness of the core of w^v, and $E(v) = SS(Y^{ce}, v)$ are no longer equivalent. We demonstrate this in two examples. Example 9 shows that it is possible to have the core of w^v be nonempty, while both $E(v) \neq SS(Y^{ce}, v)$ and v is not top convex.

EXAMPLE **9** *The Component-Wise Egalitarian Rule for a Non-Anonymous Value Function.*

Consider a setting with 3 players. The value function v is defined by $v(\{13\}) = v(\{23\}) = 1$ and $v(g) = 0$ for all other $g \subseteq g^N$.[19] This value function is not top convex, as the per-capita value in network $\{13\}$ on the 2-player coalition consisting of players 1 and 3 is higher than the per-capita value of the best network on the coalition consisting of all three players. Yet, the core of w^v is nonempty, as it contains the allocation (0,0,1). In addition, the efficient networks are $E(v) = \{\{13\}, \{23\}\}$ while there are no strongly stable networks with respect to the component-wise egalitarian

[19] Note that we can interpret this example as a special case of the bilateral bargaining model of Corominas-Bosch (2004). Here, player 3 would be the seller of a single indivisible object, and players 1 and 2 are potential buyers. A buyer and seller generate value 2 by trading the good, and trades can only be made across linked players. Here each link between a buyer and seller costs 1. Player 3 is essential to the transaction, while players 1 and 2 are perfect substitutes for each other.

rule ($SS(Y^{ce}, v) = \emptyset$). Thus, no efficient network is strongly stable (or strongly stable with side payments) with respect to the component-wise egalitarian rule.

This example makes it clear that the component-wise egalitarian rule fails to respect the asymmetries of this example, where player 3 has a special role as the (only) critical player for generating value. Thus, in a sense, Y^{ce} allocates too much to players 1 and 2 and not enough to player 3. This shows us why the relationship between the core and strong stability now needs to be stated with respect to other allocation rules.

The next example shows that the other equivalence that held with anonymous value functions, namely that $E(v) = SS(Y^{ce}, v)$ if and only if the core of w^v is nonempty, or equivalently that v is top convex, fails with non-anonymous value functions.

Example **10** *Non-Anonymity and Top Convexity*

Consider 4 players and define $g^1 = \{12\}$, $g^2 = \{34\}$, $g^3 = \{13, 34\}$, and $g^4 = \{23, 34\}$. The non-anonymous value function v is defined by $v(g^1) = 4$, $v(g^2) = 8$, $v(g^1 \cup g^2) = 12$, $v(g^3) = v(g^4) = 11$, and $v(g) = 0$ for all other $g \subseteq g^N$. Then, network $g^1 \cup g^2 = \{12, 34\}$ is the unique efficient network and it is also the unique network that is strongly stable with respect to Y^{ce} and v. This shows that $E(v) = SS(Y^{ce}, v)$. However, the core of w^v is empty because any core element x would have to simultaneously satisfy the requirements $x_1 + x_2 = 4$, $x_3 + x_4 = 8$, $x_1 + x_3 + x_4 \geq 11$, and $x_2 + x_3 + x_4 \geq 11$, which is impossible. The value function also fails top-convexity.

7 Concluding Remarks

Our main results may be summarized as follows. First, Theorem 1 showed that the component-wise egalitarian rule plays a prominent role in the study of the existence strongly stable networks. This was reinforced in some of the other results which are, for anonymous value functions, all captured in the following theorem.

Theorem **6** *Let v be component additive and anonymous. The following statements are equivalent:*[20]

(i) $SS(Y^{ce}, v) \neq \emptyset$,

(ii) $SS(Y^{ce}, v) = E(v)$,

[20]Note that (v) was not included in our earlier statements, but is easily seen to be equivalent given that it is implied by (vi) and implies (i).

(iii) the core of w^v *is nonempty,*

(iv) v *is top convex,*

(v) $SSS(Y^{ce}, v) \neq \emptyset$*,*

(vi) $SSS(Y^{ce}, v) = E(v)$*,*

(vii) there exists a component balanced allocation rule Y *such that* $SSS(Y, v) \neq \emptyset$*,*

(viii) there exists a component balanced allocation rule Y *such that* $SSS(Y, v) = E(v)$*.*

Throughout our analysis in this paper we have focused our attention on component additive value functions. These are natural in the context of some social relationships, exchange relationships, etc., but are not so natural when different components of the network might be in competition with each other (e.g., political or trade alliances). On one level, once we move beyond component additive value functions, Y^{ce} exhibits even stronger properties. That is because under our definitions, Y^{ce} can split value completely evenly among all players (even across components as now component balance is no longer relevant given the lack of component additivity) and thus result in exactly the set of efficient networks always being strongly stable. Thus strongly stable networks always exist and coincide with the efficient networks.

This conclusion, however, depends on how one defines component balance when v is not component additive. If one has further information about the value accruing to each component when v is not component additive, then one could require that Y allocate the value of each component to that component even when there exist externalities.[21] With externalities, how players are arranged when some group deviates matters in determining the value of the deviating coalition. This changes the nature of stable networks under a variety of different stability concepts, as is nicely demonstrated in a new paper by Currarini (2002). The general existence of strongly stable networks in such settings is a difficult and open problem.[22]

Finally, once one opens the door to coalitional considerations there are a variety of questions that one has to deal with. For instance, what about immunity to further

[21]The argument for doing this in the presence of externalities is not quite as clear cut as in the case where no externalities are present, unless one assumes that no transfers are made at all.

[22]The problem has some similarities to the existence of core stable partitions in coalition formation games when there are externalities. See Bloch (2001) for some discussion of that problem.

deviations of subcoalitions, as in coalition-proof Nash equilibrium? What about reactions from players not in the coalition? There are a host of such questions that have clear analogs in defining core and coalition based equilibrium concepts, and so we do not rehash them here. We simply mention that it will be worthwhile to investigate what new issues they raise in the network context because the network structure adds new features to the problem as we have already seen.

Appendix

Proof of Theorem 1: Consider an anonymous and component additive v and any anonymous, component decomposable, and component balanced allocation rule Y. Consider $g \in G$ that is not connected and is strongly stable. It follows from component balance of Y that $\sum_{i \in N} Y_i(g,v) = v(g)$. Consider any S and $S' \in \Pi(g)$ such that $S \neq S'$. Without loss of generality, assume that $\max_{i \in S} Y_i(g,v) \geq \max_{i \in S'} Y_i(g,v)$. Find $j \in \text{argmax}_{i \in S} Y_i(g,v)$ and $k \in \text{argmin}_{i \in S'} Y_i(g,v)$. To prove that $Y_i(g,v) = \frac{v(g)}{n}$ for all i, we need only show that $Y_j(g,v) = Y_k(g,v)$. Suppose, to the contrary that $Y_j(g,v) > Y_k(g,v)$. Consider a deviation by $S \cup \{k\} \setminus \{j\}$ so that k severs all links under g, $S \setminus \{j\}$ severs all links with j, and $S \cup \{k\} \setminus \{j\}$ form a component h' that is a duplicate of $g(S)$ with k replacing j. By component decomposability and anonymity it follows that $Y_i(h',v) = Y_i(g,v)$ for all $i \in S \setminus \{j\}$ and $Y_k(h',v) = Y_j(g,v) > Y_k(g,v)$. This contradicts the strong stability of g via a deviation by $S \cup \{k\} \setminus \{j\}$. Thus our supposition was incorrect. Given that Y is component balanced and $Y_i(g,v) = \frac{v(g)}{n}$ for all i, it follows that $Y_i^{ce}(g,v) = \frac{v(g)}{n}$ for all i. ∎

Proof of Theorem 2: The following lemma is useful.

Lemma 1 *Consider an anonymous and component additive value function $v \in V$. If the core of w^v is nonempty, then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v.*

Proof of Lemma 1: Given the symmetry of w^v (implied by the anonymity of v), the core of w^v is symmetric. The core is also convex by standard arguments. The statement of the lemma follows from the convexity and symmetry of the core of w^v, as taking any x in the core and averaging all of its permutations leads to identical payoffs of $\frac{w^v(N)}{n}$.[23] ∎

To complete the proof of Theorem 2, we prove that for any anonymous and component additive value function v the following statements are equivalent

(1) $SS(Y^{ce}, v) \neq \emptyset$,

(2) $SS(Y^{ce}, v) = E(v)$,

(3) the core of w^v is nonempty.

[23]A similar proof in a different context appears in Shubik (1982, page 149).

It is clear that (2) implies (1). We start by showing that (1) implies (3).

Suppose to the contrary that g is strongly stable with respect to $Y^{ce}(\cdot, v)$, and that the core of w^v is empty. Since by supposition the core is empty, we know that $Y^{ce}(g, v)$ is not a core element. Because $g \in SS(Y^{ce}, v)$, it holds that $Y_i^{ce}(g, v) = \frac{v(g)}{n}$ for each i (this follows by Theorem 1 when there is more than one component, and directly otherwise). Thus, there exists a $T \subseteq N$ such that $w^v(T) > \sum_{i \in T} \frac{v(g)}{n}$, which implies that $\frac{w^v(T)}{|T|} > \frac{v(g)}{n}$. By the definition of w^v it then follows that there exists some $S \subseteq T$ and g' with $S \in \Pi(g')$ such that $\frac{v(g'(S))}{|S|} > \frac{v(g)}{n}$. This contradicts the strong stability of g. So, our supposition was incorrect and the conclusion is established.

Next, let us show that (3) implies (1).

We show the stronger statement that if the core of w^v is nonempty, then $E(v) \subseteq SS(Y^{ce}, v)$. Suppose that the core of w^v is nonempty and let g be efficient with respect to v. Define x by $x_i = \frac{w^v(N)}{n}$ for each i. Then $\sum_{S \in \Pi(g)} v(g(S)) = v(g) = \sum_{i \in N} x_i = \sum_{S \in \Pi(g)} \sum_{i \in S} x_i$. Also, Lemma 1 tells us that x is in the core of w^v, and so $\sum_{i \in S} x_i \geq w^v(S) \geq v(g(S))$ for each $S \in \Pi(g)$. Hence, all weak inequalities must hold with equality, so that $\sum_{i \in S} x_i = v(g(S))$ for each $S \in \Pi(g)$. Define a component balanced allocation rule Y by $Y_i(g', v) = x_i \frac{v(g'(S))}{\sum_{j \in S} x_j}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. With this construction, it follows that $x_i \geq Y_i(g', v)$ for each $g' \in g^N$ and $i \in N$; and also that $Y_i(g, v) = x_i$ for any $g \in E(v)$ and $i \in N$. This implies that $Y_i(g, v) = x_i \geq Y_i(g', v)$ for each $g \in E(v)$, $S \subseteq N$, $g' \in g^N$ reachable from g by S, and $i \in S$; which proves that $g \in SS(Y, v)$. However, note that $Y(\cdot, v)$ coincides with $Y^{ce}(\cdot, v)$, because $Y_i(g', v) = x_i \frac{v(g'(S))}{\sum_{j \in S} x_j} = \frac{v(g'(S))}{|S|}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. We therefore conclude that $g \in SS(Y^{ce}, v)$.

To complete the proof, let us show that (1) implies (2).

We have shown above that $E(v) \cap SS(Y^{ce}, v) \neq \emptyset$ implies (3) and that (3) implies that $E(v) \subseteq SS(Y^{ce}, v)$. Thus, we know that $E(v) \cap SS(Y^{ce}, v) \neq \emptyset$ implies $E(v) \subseteq SS(Y^{ce}, v)$. Next, we argue that (1) implies $\emptyset \neq SS(Y^{ce}, v) \subseteq E(v)$. Consider a strongly stable g. If it is not efficient, then there exists g' such that $v(g') > v(g)$. It follows that there exists some $S \in \Pi(g')$ such that $\frac{v(g'(S))}{|S|} > \frac{v(g)}{n}$. Since, as argued above $Y_i^{ce}(g, v) = \frac{v(g)}{n}$ for all i, this contradicts the strong stability of g and so we conclude that g must be efficient. Thus, (1) implies both $SS(Y^{ce}, v) \subseteq E(v)$ and $E(v) \subseteq SS(Y^{ce}, v)$, which is (2). ∎

Proof of Corollary 1: We show that $w^{v^{z,c}}$ is convex and then the result follows from Theorem 2 as the core of a convex game is non-empty. In what follows, we fix z and c

and so we write w to indicate $w^{v^{z,c}}$, and v to indicate $v^{z,c}$.

It follows directly from the definition of w and the symmetry and zero-normalization of z that w is symmetric and zero-normalized. Thus, we can also write w as a function w_k. For each $k \leq n$, let $v(k) = v(g)$ where $g = \{12, 23, \ldots, k-1k\}$. Thus $v(k)$ is the value of a coalition of size k connected in a network that is a line. The function $v(k)$ can also be viewed as a zero-normalized symmetric cooperative game. Let $X(k) = \{X \subseteq \{1, \ldots, k\}^k \mid k = \sum_{k' \in X} k'\}$. We think of breaking k into a set of integers that sum to k, and $X(k)$ is the set of such decompositions. We can write

$$w_k = max_{X \in X(k)} \sum_{k' \in X} v(k'). \tag{1}$$

Since $v(k) = z_k - (k-1)c$ for $k \geq 1$, it follows from convexity of z that

$$v(k) - v(k-1) \geq v(k-1) - v(k-2) \tag{2}$$

for every $k \geq 3$. So, v is "almost" convex, except possibly that it may be that $v(2) = v(2) - v(1) < v(1) - v(0) = 0$. However, by standard arguments inequality (2) still implies that if $v(k') > 0$ then $v(k' + k'') \geq v(k') + v(k'')$ for any k''. This combined with equation (1) implies that

$$w_k = \max\{0, v(k)\}. \tag{3}$$

It then follows directly from (2) and (3) that w is convex. ∎

Proof of Theorem 3: Suppose that the core of w^v is nonempty. Then by Lemma 1, x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v. Hence, for every $S \subseteq N$ we have $\sum_{i \in S} x_i = |S| \frac{w^v(N)}{n} \geq w^v(S) = |S| p(v, S)$. This results in $p(v, N) = \frac{w^v(N)}{n} \geq p(v, S)$, so that v is top convex.

Now suppose that v is top convex. It is a straightforward exercise to show that then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v. ∎

Proof of Theorem 4: It is clear that (iv) implies (ii) and (ii) implies (i). So we need only show that (i) implies (iii) implies (iv). To show that (i) implies (iii), first, note that for any component balanced Y, $SSS(Y, v) \subseteq E(v)$. So, consider Y and g such that $g \in SSS(Y, v) \subseteq E(v)$. This implies that the vector $Y(g, v)$ is in the core of w^v. From Theorem 2, it then follows that (iii) holds.

Next, let us show that (iii) implies (iv). Let $g \in E(v) = SS(Y^{ce}, v)$. Since we know by Theorem 3 that v must be top-convex, it follows that $Y_i^{ce}(g, v) \geq Y_i^{ce}(g', v)$

for all i and g'. Thus, $\sum_{i\in S} Y_i^{ce}(g,v) \geq \sum_{i\in S} Y_i^{ce}(g',v)$ for any S and g', and so $g \in SSS(Y^{ce},v)$. So we have shown that $E(v) \subseteq SSS(Y^{ce},v)$. Pairing this with $SSS(Y^{ce},v) \subseteq E(v)$, it follows that $SSS(Y^{ce},v) = E(v)$. ▮

Proof of Theorem 5: First, let us show the equivalence that (i) implies (iii) implies (ii) implies (i).

Let us show that (i) implies (iii). It is clear that for any component balanced Y, $SSS(Y,v) \subseteq E(v)$ simply from considering deviations by N. Thus, we need only show that (i) implies that there exists a Y such that $E(v) \subseteq SSS(Y,v)$. Let $g \in E(v)$ and let x in the core of w^v. Define a component balanced allocation rule Y by $Y_i(g',v) = x_i \frac{v(g'(S))}{\sum_{j\in S} x_j}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. With this construction, it follows analogously to the part of the proof of Theorem 2 where it is proved that (3) implies (1), that for $S \subseteq N$ and $g' \in g^N$ reachable from g by S we have $\sum_{i\in S} Y_i(g,v) = \sum_{i\in S} x_i \geq \sum_{i\in S} Y_i(g',v)$. This proves that $g \in SSS(Y,v)$.

It is clear that (iii) implies (ii).

We complete the equivalence proof by showing that (ii) implies (i). Let Y be a component balanced allocation rule such that $SSS(Y,v) \neq \emptyset$. Since $SSS(Y,v) \subseteq E(v)$, we can find $g \in E(v) \cap SSS(Y,v)$. It follows directly that $Y(g,v)$ is in the core of w^v.

Next, let us show the remaining statements of the theorem. If v is top convex, then it is a straightforward exercise to show that then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v.

Finally, let us show that if v is top convex and component additive, then $E(v) = SS(Y^{ce},v)$. Let $g \in E(v)$. Then $\frac{v(g)}{n} = p(v,N) = \max_{S\subseteq N} p(v,S)$ and, hence, $\frac{v(g(S))}{|S|} = \frac{v(g)}{n}$ for each $S \in \Pi(g)$. Then, for each $i \in N$ we have $Y_i^{ce}(g,v) = p(v,N)$, the maximum a player can get in any network. Hence, $g \in SS(Y^{ce},v)$. This shows that $E(v) \subseteq SS(Y^{ce},v)$. To show the reverse inclusion, take $g \notin E(v)$. Then $Y_i^{ce}(g,v) \leq p(v,N)$ for all $i \in N$ with strict inequality for at least one $i \in N$. A $g' \in E(v)$ is reachable from g by N, and $Y_i^{ce}(g',v) = p(v,N)$ for each $i \in N$. This shows that $g \notin SS(Y^{ce},v)$. ▮

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