THE RADIATION OF ENERGY AND ANGULAR MOMENTUM

By H. Bateman

Abstract

When an electromagnetic field, derived from retarded potentials, is given it is generally possible to add to this a scalar field, derived from a retarded potential ψ, in such a way that the total radiation of energy to infinity is zero for each direction, the contribution of the scalar field being calculated with the aid of a tensor used in a former paper. The radiation of angular momentum to infinity is not zero for each direction unless the electromagnetic field is of a certain type. The approximate values of the field vectors at a great distance from the origin can be expressed in terms of a quantity α and when the field is of the type just mentioned α satisfies a certain partial differential equation. This result may be regarded as typical for attempts to solve the radiation problem in which the electromagnetic radiation in one of Bohr's stationary states is supposed to be balanced by radiation of a new type. Some remarks are made on the attempts which have been made to solve the problem with the aid of electromagnetic fields alone and a brief discussion is given of the radiation of angular momentum according to the classical theory.

Introduction

1. In a former paper[1] an attempt was made to justify the hypothesis of non-radiating electronic orbits by a modification of electromagnetic theory in which a scalar field, specified by a retarded potential ψ, is supposed to be associated with the ordinary electromagnetic field. It was found that in the periodic motion of a point-electron round a stationary nucleus the radiation to infinity of energy, linear momentum and angular momentum was such that when an average was taken for all directions and for a complete period, the average radiation was in each case zero. Unfortunately, this was not true for each direction and so the result was to some extent unsatisfactory. Attempts to modify the stress-energy tensor with a view to remedying this defect have so far proved unsuccessful[2] and so an attempt will now be made to alter the fields. Some alteration is, indeed, necessary, because, strictly, we are not entitled to represent an electron as a point charge. Moreover, as soon as the electron is treated as of finite size the question of the rate of spin becomes important, not only because the electron, when spinning, acts as a magnetic doublet, but also because a spin will undoubtedly alter the value of ψ.

[1] H. Bateman, Phys. Rev. **20**, 243 (1922).

[2] H. Bateman, Mess. of Math. **52**, 116 (1922); **53**, 145 (1924); **54**, 142 (1925); M. Salkover, Phys. Rev. **27**, 87 (1926).

Since the rotation and deformation of the electron are unknown we shall assume that the fields are unknown except in so far as they can be represented by retarded potentials of a certain type and shall look for the restrictions, if any, which must be laid on the electromagnetic field in order that there may be no radiation of energy and angular momentum to infinity in any direction.

THE RADIATION OF ENERGY

2. The expressions suggested originally for the components of the tensor giving the distribution of energy and momentum and also their rates of flow were subsequently modified in order that the space outside an isolated stationary electron might be entirely free from stress. The components that have now been adopted[3] for the space outside all the electric charges are of types

$$X_x=\frac{1}{3}(E_x^2+H_x^2)-\frac{1}{6}(E^2+H^2)+\frac{2}{3}\left(\frac{\partial\psi}{\partial x}\right)^2-\frac{k}{8}\frac{\partial^2}{\partial x^2}(\psi^2)-\frac{4-3k}{12}\Gamma\ ,$$

$$X_y=\frac{1}{3}(E_xE_y+H_xH_y)+\frac{2}{3}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{k}{8}\frac{\partial^2}{\partial x\partial y}(\psi^2)=Y_x\ ,$$

$$W=\frac{1}{6}(E^2+H^2)-\frac{2}{3c^2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{k}{8c^2}\frac{\partial^2}{\partial t^2}(\psi^2)-\frac{4-3k}{12}\Gamma\ ,$$

$$S_x=\frac{1}{3}c(E_yH_z-E_zH_y)+\frac{2}{3}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}-\frac{k}{8}\frac{\partial^2}{\partial x\partial t}(\psi^2)=c^2G_x\ ,$$

where

$$\Gamma=\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2$$

and $k=1$. If ψ satisfies the wave equation

$$\Box\psi\equiv\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0\ ,$$

and the field vectors $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations

$$\text{curl}\,\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\ ,\qquad \text{div}\,\mathbf{E}=0$$

$$\text{curl}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\ ,\qquad \text{div}\,\mathbf{H}=0$$

[3] H. Bateman, Phil. Mag. 49, 1 (1925). With these components the flow of energy in the field of an electron, which is moving uniformly, is at each point parallel to the line of motion of the electron. In the original paper we used the value $k=8/3$.

we have for any constant value of k

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}-\frac{\partial G_x}{\partial t}=0\ ,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+\frac{\partial S_z}{\partial z}+\frac{\partial W}{\partial t}=0\ .$$

Consequently if W represents the density of energy, S_x, S_y, S_z the components of the flow of energy, X_x, X_y, X_z, $\cdots$ the component stresses and G_x, G_y, G_z the densities of the components of linear momentum, we have conservation of energy and linear momentum in any finite portion of free space provided the flux across the boundary of the region is taken into consideration.

3. At a very great distance r from the center of an atom, which we take as our origin, the electromagnetic field may be written in the form

$$\mathbf{H}=-\frac{1}{c}\frac{\partial}{\partial t}\text{ curl }\mathbf{Q}+\text{ curl curl }\mathbf{P}\ ,\quad \mathbf{E}=-\frac{1}{c}\frac{\partial}{\partial t}\text{ curl }\mathbf{P}-\text{ curl curl }\mathbf{Q}$$

where the vectors $\mathbf{Q}$ and $\mathbf{P}$ are given by the series

$$\mathbf{Q}=\frac{\mathbf{q}}{r}+\frac{\partial}{\partial x}\left(\frac{\mathbf{q}_1}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\mathbf{q}_2}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\mathbf{q}_3}{r}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\mathbf{q}_{11}}{r}\right)+\cdots$$
$$+2\frac{\partial^2}{\partial y\partial z}\left(\frac{\mathbf{q}_{23}}{r}\right)+\cdots$$

$$\mathbf{P}=\frac{\mathbf{p}}{r}+\frac{\partial}{\partial x}\left(\frac{\mathbf{p}_1}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\mathbf{p}_2}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\mathbf{p}_3}{r}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\mathbf{p}_{11}}{r}\right)+\cdots$$
$$+2\frac{\partial^2}{\partial y\partial z}\left(\frac{\mathbf{p}_{23}}{r}\right)+\cdots\ .$$

in which quantities $\mathbf{q}$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_{11}$, $\cdots$ $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_{11}$, $\cdots$ are all functions of $\tau=t-r/c$. These quantities are supposed to be finite for all values of τ and their derivatives with respect to τ are supposed to be likewise finite. We shall imagine them to be simple trigonometrical functions but this restriction is not necessary. Since these quantities are all bounded we shall disregard them or treat them as constants in estimating the order of magnitude of an expression.

In calculating the radiation of energy and linear momentum we need only retain terms of order $1/r$ in the expressions for $\mathbf{E}$, $\mathbf{H}$ and ψ. Consequently, to this order we may write

$$\mathbf{H}=\mathbf{s}\times\mathbf{b}+\mathbf{s}\times(\mathbf{s}\times\mathbf{a})\ ,\qquad \mathbf{E}=-(\mathbf{s}\times\mathbf{H})$$

where $\mathbf{s}$ is the unit vector with components $(x/r,\ y/r,\ z/r)$ and where

$$\mathbf{a}=\frac{1}{c^2r}\mathbf{p}''-\frac{1}{c^3r^2}(x\mathbf{p}_1'''+y\mathbf{p}_2'''+z\mathbf{p}_3''')+\cdots$$

$$\mathbf{b}=\frac{1}{c^2r}\mathbf{q}''-\frac{1}{c^3r^2}(x\mathbf{q}_1'''+y\mathbf{q}_2'''+z\mathbf{q}_3''')+\cdots$$

In these equations primes denote differentiations with respect to the argument $t-r/c$. The approximate expressions for $\mathbf{E}$ and $\mathbf{H}$ give

$$\mathbf{E}\times\mathbf{H}=\mathbf{s}[\mathbf{a}^2+\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{s})^2-(\mathbf{b}\cdot\mathbf{s})^2+2\mathbf{s}\cdot(\mathbf{a}\times\mathbf{b})]$$

Assuming that

$$\psi=\frac{\kappa}{r}+\frac{\partial}{\partial x}\left(\frac{\kappa_1}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\kappa_2}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\kappa_3}{r}\right)+\cdots$$

$$=\frac{\kappa}{r}-\frac{1}{cr^2}(x\kappa_1'+y\kappa_2'+z\kappa_3')+\cdots$$

approximately, we find that the flow of energy in the direction of $\mathbf{s}$ is $\mathbf{S}$ where

$$\mathbf{S}=\frac{1}{3}c(\mathbf{E}\times\mathbf{H})+\frac{2}{3}\nabla\psi\frac{\partial\psi}{\partial t}-\frac{k}{4}\nabla\psi\frac{\partial\psi}{\partial t}-\frac{k}{4}\psi\nabla\frac{\partial\psi}{\partial t}$$

$$=\frac{1}{3}c\mathbf{s}\left[\mathbf{a}^2+\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{s})^2-(\mathbf{b}\cdot\mathbf{s})^2+2\mathbf{s}\cdot(\mathbf{a}\times\mathbf{b})\right.$$

$$\left.-\left(2-\frac{3k}{4}\right)\frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{3k}{4c^2}\psi\frac{\partial^2\psi}{\partial t^2}\right]$$

approximately. Now ψ can generally be chosen so that this expression vanishes. This is evidently possible when $k=0$ for then we have simply to choose ψ so that $(\partial\psi/\partial\tau)^2$ is a given positive function. When $k\neq0$ we have to solve an equation of type

$$\left(2-\frac{3k}{4}\right)\left(\frac{\partial\psi}{\partial\tau}\right)^2-\frac{3k}{4}\psi\frac{\partial^2\psi}{\partial\tau^2}=\frac{1}{r^2}\left[f\left(\frac{x}{r},\frac{y}{r},\frac{z}{r},\tau\right)\right]^2$$

$$=[F(\tau)]^2 \text{ say}.$$

It seems likely that a solution exists only when $F(\tau)$ is subject to certain restrictions. A general existence theorem for this differential equation is hard to find but it is easy to see that there are many cases in which the equation possesses a real solution. To see this let us put

$$\psi=e^\sigma,\qquad F(\tau)=e^\sigma G(\tau)$$

When σ is given we can find F if it is possible to choose a real function G such that

$$\left(2-\frac{3k}{2}\right)\left(\frac{\partial\sigma}{\partial\tau}\right)^2-\frac{3k}{4}\frac{\partial^2\sigma}{\partial\tau^2}=[G(\tau)]^2 \tag{1}$$

The quantity on the left-hand side must thus be positive. Let us now alter the problem a little and endeavor to determine σ when G is given. We then have to solve a Riccatian equation

$$\left(2-\frac{3k}{2}\right)\rho^2-\frac{3k}{4}\frac{\partial\rho}{\partial\tau}=[G](\tau)^2$$

for $\rho=\partial\sigma/\partial\tau$. Putting

$$\rho=-\frac{3k}{8-6k}\cdot\frac{1}{u}\frac{du}{d\tau}$$

we have the linear differential equation

$$\frac{d^2u}{d\tau^2}=\frac{8u}{9k^2}(4-3k)[G(\tau)]^2$$

for the determination of u and this equation generally possesses a real solution when $G(\tau)$ is an analytic function. Since $G(\tau)$ is practically arbitrary the function $F(\tau)$ is also of a very general nature. The restriction which is introduced by the circumstance that $F(\tau)=e^{\sigma}G(\tau)$, where σ is a solution of (1) does not seem to be very great.

The Radiation of Angular Momentum

3. When the radiation of energy is zero for each direction the radiation of linear momentum is also zero for each direction since the components of linear momentum are proportional to the components of the flow of energy and the momentum travels in a radial direction.

The rate of flow in a radial direction of the z-component of angular momentum is

$$-\frac{1}{r}[x(xY_x-yX_x)+y(xY_y-yX_y)+z(xY_z-yX_z)]\ .$$

This will be zero for each direction if

$$E_r(xE_y-yE_x)+H_r(xH_y-yH_x)+\left(2-\frac{3}{4}k\right)\frac{\partial\psi}{\partial r}\left(x\frac{\partial\psi}{\partial y}-y\frac{\partial\psi}{\partial x}\right)$$
$$-\psi\frac{3k}{4r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}-1\right)\left(x\frac{\partial\psi}{\partial y}-y\frac{\partial\psi}{\partial x}\right)=0\ ,$$

where the suffix r is used to denote the radial component. Introducing the polar coordinates

$$z = r\cos\theta\ , \qquad x = r\sin\theta\cos\phi\ , \qquad y = r\sin\theta\sin\phi\ ,$$

we may write

$$x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\phi}\ ,$$

$$x\left(z\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial z}\right) - y\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right) = r\sin\theta\frac{\partial\psi}{\partial\theta}\ .$$

So far as terms of order $1/r$ are concerned we may also write

$$\frac{\partial\psi}{\partial r} = -\frac{1}{c}\frac{\partial\psi}{\partial\tau}$$

Now E_r and H_r are both of order $1/r^2$ consequently we may retain only terms of order $1/r$ in calculating $(xE_y - yE_x)/r$ and $(xH_y - yH_x)/r$. In particular we may write

$$xE_y - yE_x = rH_z\ , \qquad xH_y - yH_x = -rE_z$$

$$x(zE_x - xE_z) - y(yE_z - zE_y) = z(xE_x + yE_y + zE_z) - r^2E_z$$

$$= -r^2E_z$$

$$x(zH_x - xH_z) - y(yH_z - zH_y) = -r^2H_z$$

Writing

$$H_z = -H_\theta\sin\theta\ , \qquad E_z = -E_\theta\sin\theta$$

$$H_\theta = E\cos\omega\ , \qquad E_\theta = E\sin\omega$$

$$rH_r = u\cos\omega + v\sin\omega\ , \qquad rE_r = u\sin\omega - v\cos\omega$$

we find that the radiation of energy and angular momentum will be zero for each direction if the following equations are satisfied.

$$E^2 = \left(2 - \frac{3}{4}k\right)\frac{1}{c^2}\left(\frac{\partial\psi}{\partial\tau}\right)^2 - \frac{3}{4c^2}k\psi\frac{\partial^2\psi}{\partial\tau^2}$$

$$Eu = \left(2 - \frac{3}{4}k\right)\frac{1}{c}\frac{\partial\psi}{\partial\theta}\frac{\partial\psi}{\partial\tau} - \frac{3}{4c}k\psi\frac{\partial^2\psi}{\partial\theta\partial\tau} \qquad (2)$$

$$Ev\sin\theta = \left(2 - \frac{3}{4}k\right)\frac{1}{c}\frac{\partial\psi}{\partial\phi}\frac{\partial\psi}{\partial\tau} - \frac{3}{4c}k\psi\frac{\partial^2\psi}{\partial\phi\partial\tau}$$

We shall find it convenient to write

$$2 - \frac{3}{4}k = \lambda\ , \qquad \frac{3}{4}k = \mu\ , \qquad \psi = c\alpha\ .$$

When we take into consideration the fact that each of the quantities E_θ, E_ϕ, H_θ, H_ϕ, rE_r, rH_r is of the form $f(\theta, \phi, \tau)/r$ Maxwell's equations in polar coordinates may be written

$$\frac{1}{c}\sin\theta\frac{\partial}{\partial t}(rE_r) = \frac{\partial}{\partial\theta}(\sin\theta\, H_\phi) - \frac{\partial H_\theta}{\partial\phi}$$

$$\frac{1}{c}\sin\theta\frac{\partial}{\partial t}(rE_\theta) = \frac{\partial H_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta\, H_\phi) \equiv \frac{1}{c}\sin\theta\frac{\partial}{\partial t}(rH_\phi)$$

$$\frac{1}{c}\frac{\partial}{\partial t}(rE_\phi) = \frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta} \equiv -\frac{1}{c}\frac{\partial}{\partial t}(rH_\theta)$$

. . . .

and give simply

$$E_\theta = H_\phi\ , \qquad E_\phi = -H_\theta$$

$$\frac{1}{c}\sin\theta\frac{\partial}{\partial\tau}(rE_r) = \frac{\partial}{\partial\theta}(\sin\theta\cdot E_\theta) - \frac{\partial H_\theta}{\partial\phi}$$

$$\frac{1}{c}\sin\theta\frac{\partial}{\partial\tau}(rH_r) = \frac{\partial}{\partial\theta}(\sin\theta\cdot H_\theta) + \frac{\partial E_\theta}{\partial\phi}$$

Substituting the values of E_θ, H_θ, rE_r, rH_r in terms of u, v, E and ω, we obtain

$$\left.\begin{aligned}\frac{1}{c}\sin\theta\left(\frac{\partial u}{\partial\tau}+v\frac{\partial\omega}{\partial\tau}\right) &= E\cos\theta+\sin\theta\frac{\partial E}{\partial\theta}+E\frac{\partial\omega}{\partial\phi}\\ \frac{1}{c}\sin\theta\left(\frac{\partial v}{\partial\tau}-u\frac{\partial\omega}{\partial\tau}\right) &= \frac{\partial E}{\partial\phi} - E\sin\theta\frac{\partial\omega}{\partial\theta}\end{aligned}\right\} \tag{3}$$

Writing $\lambda\log\alpha-\mu\log\partial\alpha/\partial\tau=\beta$ and making use of the relations (2) we obtain the following differential equations for ω.

$$\begin{aligned}\frac{1}{\sin\theta}\frac{\partial(\omega,\beta)}{\partial(\phi,\tau)} &= \frac{1}{2}\frac{\partial(\log\partial\beta/\partial\tau,\beta)}{\partial(\theta,\tau)}+\frac{\partial(\log\alpha,\log\partial\alpha/\partial\tau)}{\partial(\theta,\tau)}-\frac{\partial\beta}{\partial\tau}\cot\theta\\ \frac{\partial(\omega,\beta)}{\partial(\theta,\tau)} &= -\frac{1}{\sin\theta}\left[\frac{1}{2}\frac{\partial(\log\partial\beta/\partial\tau,\beta)}{\partial(\phi,\tau)}+\frac{\partial(\log\alpha,\log\partial\alpha/\partial\tau)}{\partial(\phi,\tau)}\right]\end{aligned} \tag{4}$$

When $k=0$ we have $\beta=2\log\alpha$ and the equations for ω reduce to

$$\frac{\partial(\omega,\beta)}{\partial(\phi,\tau)} = -\frac{\partial\beta}{\partial\tau}\cos\theta\ , \qquad \frac{\partial(\omega,\beta)}{\partial(\theta,\tau)}=0\ ,$$

The last equation gives

$$\omega=F(\beta,\phi)\ ,$$

and when we substitute in the first equation we find that

$$\left(\frac{\partial F}{\partial \phi} + \cos\theta\right)\frac{\partial \beta}{\partial \tau} = 0.$$

This equation gives a value of β which is independent of τ and so when $k=0$ there is no solution which is of physical interest. When $k\neq 0$ we may eliminate ω from the equations

$$\frac{\partial(\omega,\beta)}{\partial(\phi,\tau)} = \Theta\,, \qquad \frac{\partial(\omega,\beta)}{\partial(\theta,\tau)} = -\Phi \tag{5}$$

and obtain a partial differential equation for α. The result is

$$\Theta\frac{\partial^2\beta}{\partial\theta\partial\tau} + \Phi\frac{\partial^2\beta}{\partial\phi\partial\tau} = \frac{\partial\beta}{\partial\tau}\left[\frac{\partial\Theta}{\partial\theta} + \frac{\partial\Phi}{\partial\phi}\right] - \frac{\partial\beta}{\partial\theta}\,\frac{\partial\Theta}{\partial\tau} - \frac{\partial\beta}{\partial\phi}\,\frac{\partial\Phi}{\partial\tau}$$
$$+\frac{\dfrac{\partial^2\beta}{\partial\tau^2}}{\dfrac{\partial\beta}{\partial\tau}}\left[\Theta\frac{\partial\beta}{\partial\theta} - \Phi\frac{\partial\beta}{\partial\phi}\right]$$

where Θ and Φ have the values obtained by substituting (5) in (4). The general solution of this partial differential equation will probably be difficult to obtain. A particular solution may be obtained at once by assuming that α and ω are independent of ϕ. The equations (4) then give

$$\omega = f(\beta)$$

$$\frac{1}{2}\,\frac{\partial(\log \partial\beta/\partial\tau, \beta)}{\partial(\theta,\tau)} + \frac{\partial(\log\alpha, \log(\partial\alpha/\partial\tau)}{\partial(\theta,\tau)} = \frac{\partial\beta}{\partial\tau}\cos\theta$$

In this case the axis of z is an axis of symmetry for the fields and so from the point of view of Bohr's theory of the atom the fields are not of much physical interest.

Further discussion of the differential equation must be postponed until the nature of the solutions is better understood. Another differential equation which is more easily treated is obtained by considering the case in which

$$E^2 = \frac{1}{c^2}\,\frac{\partial\Omega}{\partial t}\,, \qquad Eu = \frac{1}{c}\,\frac{\partial\Omega}{\partial\theta}\,, \qquad Eu\sin\theta = \frac{1}{c}\,\frac{\partial\Omega}{\partial\phi}\,.$$

Substituting in equations (3) we find that if $\chi = \log(\operatorname{cosec}^2\theta \cdot \partial\Omega/\partial\tau)$

$$\operatorname{cosec}\theta \frac{\partial(\omega,\Omega)}{\partial(\phi,\tau)} = \frac{1}{2}\frac{\partial(\chi,\Omega)}{\partial(\theta,\tau)}$$

$$\frac{\partial(\omega,\Omega)}{\partial(\theta,\tau)} = -\frac{1}{2}\operatorname{cosec}\theta \frac{\partial(\chi,\Omega)}{\partial(\phi,\tau)}$$

Writing

$$\omega + \frac{1}{2} i \log\left(\frac{\partial\Omega}{\partial\tau}\operatorname{cosec}^2\theta\right) = \alpha\ , \qquad \omega - i\chi = \beta$$

$$\sigma = \log\tan\frac{1}{2}\theta + i\phi\ , \qquad \rho = \log\tan\frac{1}{2}\theta - i\phi$$

we have

$$\frac{\partial(\alpha,\Omega)}{\partial(\sigma,\tau)} = 0\ , \qquad \frac{\partial(\beta,\Omega)}{\partial(\rho,\tau)} = 0\ ,$$

These equations are satisfied by

$$\alpha = i\log f[\Omega,\sigma]$$

$$\beta = -i\log f[\Omega,\rho]$$

where f is an arbitrary function of its two arguments. We finally obtain

$$\partial\Omega/\partial t = \sin^2\theta\, f[\Omega, \log\tan\tfrac{1}{2}\theta + i\phi] f[\Omega, \log\tan\tfrac{1}{2}\theta - i\phi]\ ,$$

This may be regarded as a partial differential equation for the determination of Ω. This partial differential equation may be solved in a large number of cases.

The electromagnetic field determined by a function Ω of this type is particularly interesting because the rate of radiation of energy is a time derivative and so the total radiation in any interval of time may be expressed as the difference of the initial and final values of a certain quantity. The rate of radiation of the z-component of angular momentum may, too, be expressed in a simple form, for in the integral

$$\int_0^\pi \int_0^{2\pi} (\partial\Omega/\partial\phi)\sin\theta\, d\theta\, d\phi$$

the integration with respect to ϕ may be performed at once. If the function Ω is single-valued in ϕ, there is no radiation of this component.

If it ever becomes possible to describe the properties of radiation on the basis of the classical electromagnetic theory alone this type of field may be of some interest.

Remarks on the Classical Theory

4. A new attempt to solve the radiation problem on the basis of Maxwell's electromagnetic equations has been made recently by N. v. Raschevsky.[4] His work possesses some good features inasmuch as an attempt is made to find a field which first gives no radiation in a stationary state, then passes over into a state in which there is radiation and finally returns to another stationary state in which there is no radiation. His field which gives no radiation is represented by means of integrals of Kirchhoff's type including surface integrals one of which is practically the potential of a double layer. Now in ordinary potential theory the potential of a double layer is discontinuous at the layer itself. Consequently one is left with the impression that Raschevsky's field is not really a continuous field. This is a defect but may not be a fatal objection to the theory because the discontinuity may simply mean that the ordinary Maxwell field-equations break down in the immediate vicinity of an atom when the fields are very strong. There may be equations something like Maxwell's but the relations

$$\mathbf{D}=\mathbf{E}\,,\qquad \mathbf{B}=\mathbf{H}$$

expressing a simple proportionality between stress and strain may no longer hold. Like the solution proposed by Dr. H. A. Lorentz, in his lectures at Pasadena, Raschevsky's solution has the advantage of giving no radiation in any direction at any instant and since the field vectors are of order $1/r^2$ at a great distance from the origin, there is no radiation of angular momentum.

5. Like most writers Raschevsky takes a vibrating electric dipole as a model of an electromagnetic radiator. The radiation of angular momentum in the field of a dipole has been calculated by several writers[5] and Rubinowicz has used the results to formulate a selection principle in quantum theory.

A point that seems rather puzzling is that when the dipole rotates like a rigid body the angular momentum about the z-axis is radiated not in the direction of the axis, the direction in which the light is circularly polarized, but chiefly in the neighborhood of the plane of rota-

[4] N. v. Raschevsky, Zeits. f. Physik **35**, 100 (1925).

[5] For the literature see W. H. Westphal, Jahrb. d. Radioaktivität. **18**, 105 (1921).

tion and in directions in which the light is linearly polarized. As Ehrenfest pointed out, the theory of the Hertzian dipole gives no support to Poynting's idea that circularly polarized light is accompanied by angular momentum.

The radiation of angular momentum in the field of the vibrating diplole is seen to depend on the fact that the flow of energy is strictly not in a radial direction and that consequently there is a moment arm from the origin to the line of flow. The length of this perpendicular may be found as follows:

Let us take $\mathbf{P}=0$, $r\mathbf{Q}=\mathbf{q}(t-r/c)$, then, retaining terms of orders $1/r^2$ and $1/r^3$ we have

$$E^2=H^2=\frac{r^2\mathrm{q}''^2-(\mathbf{r}\cdot\mathrm{q}'')^2}{c^4r^4}+2\,\frac{r^2(\mathrm{q}'\cdot\mathrm{q}'')-(\mathbf{r}\cdot\mathrm{q}')(\mathbf{r}\cdot\mathrm{q}'')}{c^3r^5}$$

$$\mathbf{E}\times\mathbf{H}=\mathbf{s}\left[E^2-\frac{2(\mathbf{r}\cdot\mathrm{q}')(\mathbf{r}\cdot\mathrm{q}'')}{c^3r^5}\right]+\frac{2\mathrm{q}''(\mathbf{r}\cdot\mathrm{q}')}{c^3r^4}$$

Using ϖ to denote the length of the perpendicular from the origin to the line of flow, we have

$$\varpi^2=\frac{[\mathbf{r}\times(\mathbf{E}\times\mathbf{H})]^2}{E^2}=\frac{4(\mathbf{r}\cdot\mathrm{q}')^2}{c^6r^8E^4}[\mathbf{r}^2q''^2-(\mathbf{r}\cdot\mathrm{q}'')^2]$$

Replacing E^2 by its approximate value

$$\frac{1}{c^4r^4}[r^2q''^2-(\mathbf{r}\cdot\mathrm{q}'')$$

we find that

$$\varpi^2=\frac{4c^2(\mathbf{r}\cdot\mathrm{q}')^2}{r^2q''^2-(\mathbf{r}\cdot\mathrm{q}'')^2}$$

When

$$q_x=a\cos 2\pi\nu\tau\ ,\qquad q_y=a\sin 2\pi\nu\tau\ ,\qquad q_z=0\ ,$$

we have

$$\varpi^2=\frac{c^2}{\pi^2\nu^2}\ \frac{(y\cos 2\pi\nu\tau-x\sin 2\pi\nu\tau)^2}{z^2+(y\cos 2\pi\nu\tau-x\sin 2\pi\nu\tau)^2}$$

This formula indicates that ϖ lies between zero and $c/\pi\nu$; it is zero on the axis of z when $x=y=0$ and has its maximum value in the plane $z=0$.

A line drawn through the point (x, y, z) in the direction of Poynting's vector will, to the present order of approximation, meet the plane of xy in a point whose coordinates are

$$X = J \cos 2\pi\nu\tau\ , \qquad Y = J \sin 2\pi\nu\tau$$

where

$$J = \frac{cr}{\pi\nu} \, \frac{(y \cos 2\pi\nu\tau - x \sin 2\pi\nu\tau)}{z^2 + (y \cos 2\pi\nu\tau - x \sin 2\pi\nu\tau)^2}$$

approximately. Eliminating τ we obtain the equation

$$\frac{\pi\nu z^2}{cr}(X^2+Y^2) + (yX - xY)\left(yX - xY - \frac{cr}{\pi\nu}\right) = 0$$

The lines of motion which pass through the point (x, y, z) thus generate a quadric cone.

Department of Mathematics,
California Institute of Technology,
Pasadena, California,
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