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On the Worldsheet Derivation of Large N Dualities for the Superstring

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Large N topological string dualities have led to a class of proposed open/closed dualities for superstrings. In the topological string context, the worldsheet derivation of these dualities has already been given. In this paper we take the first step in deriving the full ten-dimensional superstring dualities by showing how the dualities arise on the superstring worldsheet at the level of F terms. As part of this derivation, we show for F-term computations that the hybrid formalism for the superstring is equivalent to a $\hat{c} = 5$ topological string in ten-dimensional spacetime. Using the $\hat{c} = 5$ description, we then show that the D brane boundary state for the ten-dimensional open superstring naturally emerges on the worldsheet of the closed superstring dual.

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1. Introduction

There is by now a large class of examples in string theory that realizes the idea of 't Hooft of large N dualities for gauge theories. Most of the arguments for the existence of such dualities derive from the target space perspective: the back-reaction on the gravity modes by the D-branes. However, the original motivation of 't Hooft was a statement visible at the level of the worldsheet, namely he conjectured that somehow the holes in the large N expansions of Feynman diagrams close up and lead to a closed string expansion. Thus these dualities are expected to be visible genus by genus in the worldsheet. Understanding the large N dualities from this viewpoint is crucial because it also will teach us how the large N dualities, unlike U-dualities, are derivable from perturbative considerations of closed string theory.

A simple example of large N duality was proposed in [1] which relates large N Chern-Simons theory on S^3, which is equivalent to open topological strings [2], with topological closed strings on the resolved conifold, where the size of the blown up $\mathbf{P}^1$ is given by the 't Hooft parameter. This duality has been derived from a worldsheet perspective in [3]: Starting from the closed string side and using the linear sigma model description of the worldsheet theory [4], one discovers that in the limit of small 't Hooft parameter, the worldsheet develops a new phase (the Coulomb phase) which leads to the emergence of the open string description. The new phase of the closed string worldsheet corresponds to the 'filled holes' of the open string worldsheet.

On the other hand, motivated from the meaning of topological string computations as F term computations in an associated superstring [5], this topological string duality was embedded in superstrings [6], and extended to a relatively large class of superstring dualities (see e.g. [7]), and led to a link between $\mathcal{N}=1$ supersymmetric gauge theories and matrix models [8]. Even though the worldsheet derivation of the topological string duality would lead, by a chain of arguments, to the F term dualities in superstring context, a direct worldsheet derivation of these dualities was missing in the context of the superstring.

In this paper we aim to fill this hole, at least at the level of F terms. A $d=4$ spacetime-supersymmetric description of the superstring on Calabi-Yau threefolds is given by the hybrid formalism [9,10,11], which is related to the RNS formalism by a field redefinition. We will show that the computation of F terms using the hybrid formalism is equivalent to the computation of F terms using a ten-dimensional topological string with $\hat{c}=5$. We will then use the $\hat{c}=5$ topological string to establish the worldsheet equivalence of F terms between open and closed sides. In particular, we will find using the $\hat{c} = 5$ description that the D brane boundary state for the ten-dimensional open superstring naturally emerges on the worldsheet of the closed superstring dual.

The topological string method has been used in motivating some of the results on superpotential terms in gauge theories, for example in [12,13,14], which have then been verified by field theory methods. This paper provides a precise justification of these results from the string theory perspective. While we establish the equivalence of closed and open strings only at the level of F terms, the setup we present should be viewed as the first step in the derivation of the full duality

The organization of this paper is as follows. In section 2 we review the worldsheet derivation of large N topological string duality [3]. In section 3 we formulate topological strings directly in ten dimensions, with $\hat{c} = 5$, and show its equivalence to the hybrid formalism [9,10,11] when evaluating F terms for superstring compactifications. In section 4 we use this $\hat{c} = 5$ topological formulation of the superstring to establish the worldsheet equivalence of F terms between open and closed sides.

2. Review of Topological String Duality

In this section, we will briefly review the worldsheet derivation [3] of the duality between the A-type topological closed string on the resolved conifold and the open topological string on the deformed conifold with N A-branes wrapping on the S^3 of the conifold. The topological string coupling constants are the same on both sides of the duality and denoted by λ. The Kähler moduli t of the resolved conifold (the "size" of the $\mathbf{P}^1$) in the closed string side is mapped to the number N of the A-branes in the open string side by the relation,

$$t = iN\lambda. \tag{2.1}$$

In this sense, this is an example of the 't Hooft duality. This duality was conjectured in [1], and various evidences for the duality have been found in [15,16,17,18,19,20].

To derive the duality, we start with the closed string side and expand string amplitudes in powers of t. What is expected to emerge from the duality is a sum over open string worldsheets with each boundary weighted by the factor of $N\lambda = -it$. The target space becomes singular in the limit $t \to 0$, and the worldsheet in the limit is best described by using the linear sigma model [4]. For the resolved conifold, the linear sigma model consists of four chiral multiplets, whose scalar fields are denoted by a_1, a_2 and b_1, b_2, and one vector multiplet, whose scalar field is denoted by σ. The chiral multiplet fields a_1, a_2 carry charge $+e$ with respect to the gauge field A in the vector multiplet, and b_1, b_2 carry $-e$. After integrating out the auxiliary fields, the potential U for the bosonic fields are given as

$$U = |\sigma|^2 \left(|a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2 \right) + e^2 \left(|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 \right). \tag{2.2}$$

According to the duality relation (2.1), the Kähler moduli is pure imaginary. In this case, t appears as the theta term for the gauge field $\sim t \int dA$. If we introduce a twisted chiral superfield Σ defined from the vector superfield V as $\Sigma = \bar{D}_+ D_- V = \sigma + \cdots$, the theta term can be also written as as an F term with the superpotential

$$W = t\Sigma. \tag{2.3}$$

We will find this description in terms of Σ to be useful in the following discussion.

When $t \neq 0$, the linear sigma model flows in the infrared limit to the non-linear sigma model for the conifold. The theta term lifts the Coulomb branch and constrains $\sigma = 0$. Since $e \to \infty$ in the infrared limit, the chiral multiplet fields should obey $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$ modulo the gauge symmetry, $(a_{1,2}, b_{1,2}) \to (e^{i\theta} a_{1,2}, e^{-i\theta} b_{1,2})$. We recognize this quotient is the conifold geometry.[1] In this limit, σ is identified with the chiral primary field associated to the element of $H^{1,1}$ dual to the $\mathbf{P}^1$.

When we expand around $t = 0$, however, we need to take into account a new flat direction where σ can be non-zero. Due to the potential (2.2), the chiral multiplet fields are now constrained to vanish, $a_{1,2} = b_{1,2} = 0$. We call this flat direction as the C branch. In comparison, the branch where $\sigma = 0$ is called the H branch. When we quantize the linear sigma-model, we need to integrate over both C and H branches. It is useful to think that the worldsheet is divided into C and H domains, where the fields take values in the C and H branches respectively. Performing the functional integral involves summing over all possible configurations of these two branches.

We expect that quantization of the H branch still leads to the sigma-model on the conifold away from the conifold point. How to remove the conifold point would depend on how we divide the integral over σ between the two branches. On the other hand, the

[1] The gauge invariant combinations, $z_{ij} = a_i b_j$, obey the relation $z_{11}z_{22} - z_{12}z_{21} = 0$ defining the conifold geometry. For a given set of z_{ij}, the original fields a_i and b_i are determined modulo $(a_{1,2}, b_{1,2}) \to (e^{\rho} a_{1,2}, e^{-\rho} b_{1,2})$, which is taken into account by the gauge symmetry and the constraint $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$.

C branch is non-geometric since $a_{1,2}, b_{1,2}$, which are coordinates for the conifold, become massive. We regard C domains as holes on the worldsheet and claim that this is how open strings emerge from the closed string theory. For this interpretation to work, we need that:

(1) Every C domain has the topology of the disk.

Contributions from all other topologies should vanish in string amplitudes.

(2) Each disk in the C branch contributes the factor of $-it = N\lambda$.

It was shown in [3] that both of these statements are true.

To show (1), it was noted that each C domain contributes to a topological string amplitude as

$$\oint d\sigma_0 \frac{\partial}{\partial \sigma_0} \mathcal{F}^{(C)}(\sigma_0), \tag{2.4}$$

where $\mathcal{F}^{(C)}(\sigma)$ is the partition function for the C domain with the boundary condition $\sigma = \sigma_0$. The action of $\oint d\sigma_0 \partial/\partial\sigma_0$ is due to a Jacobian factor that is needed to trade a part of the functional integral into an integral over configurations of the C domain. By the topological BRST symmetry, $\mathcal{F}^{(C)}(\sigma_0)$ is holomorphic in σ_0. This means that the contribution (2.4) would vanish if $\mathcal{F}^{(C)}$ is a single-valued function of σ_0. This is the case when the C domain has a handle or more than one boundaries. The only exception is the case when the C domain has the topology of the disk. The string amplitude on the disk is not well-defined unless we have some punctures, and $\mathcal{F}^{(C)}(\sigma_0)$ can have a monodromy around $\sigma_0 = 0$, which can be picked up by the integral in (2.4).

To evaluate (2.4), we note that the C domain has a description as a Landau-Ginzburg model with the superpotential W being given by (2.3). The disk amplitude is then given by an integral of $\exp(-W)$. The only subtlety is the measure factor of σ^{-2} which arises from the integral over $a_{1,2}, b_{1,2}$, which are massive in this domain. Taking this into account, we find,

$$\mathcal{F}^{(C)}(\sigma_0) = \int^{\sigma_0} \frac{d\sigma}{\sigma^2} \exp(-t\sigma).$$

This show that the disk amplitude is indeed multivalued around $\sigma_0 = 0$ as $\mathcal{F}^{(C)}(\sigma_0) \sim -\sigma_0^{-1} - t \log \sigma_0 + \cdots$. Therefore the contribution of the C domain of the disk topology is given by

$$\oint d\sigma_0 \frac{\partial}{\partial \sigma_0} \mathcal{F}^{(C)}(\sigma_0) = \oint \frac{d\sigma_0}{\sigma_0^2} \exp(-t\sigma) \sim -it = N\lambda.$$

This shows that (1) and (2) are indeed true for the closed string theory.

We have found that the closed string amplitude, when expanded in powers of t, can be expressed as a sum over holes on the worldsheet with the power of t keeping track of the number of holes. Namely the closed string theory is indeed equivalent to an open string theory with some boundary condition. Is the boundary condition exactly what we expect from the large N duality? Since the worldsheet variables $a_{1,2}, b_{1,2}$ become massive in the C domain, near the interface of the C and H domains, they stay near the tip of the conifold. Their precise behavior depend on how we divide the σ integral between the two branches. On the other hand, the A brane for the open string is supposed to wrap on the S^3 of the deformed conifold. Its size is undetermined since changing the radius is a BRST trivial deformation. When the radius is small, the S^3 is near the tip of the conifold. Therefore, modulo the ambiguities that exist in both sides of the duality, the boundary of the C domain correctly reproduces the A brane boundary condition in the open string dual.

3. Equivalence of $\hat{c} = 5$ and Hybrid Computation of F Terms

In this section we introduce the concept of topological strings in ten dimensions with $\hat{c} = 5$, generalizing the topological strings often used in the context of Calabi-Yau three-folds, and establish its direct equivalence to the hybrid formalism for certain F term computations in type II superstrings.

In the first subsection, we will show that states in the G^+ cohomology in the $\hat{c} = 5$ topological string include supersymmetry multiplets containing massless compactification moduli as well as the multiplet containing the self-dual graviphoton field strength. In the second subsection, we will give a $\hat{c} = 5$ topological prescription for computing tree and loop scattering amplitudes involving these states which will contribute only to F terms in the low-energy effective action. And in the third subsection, we will show how to describe these states using the hybrid formalism and will prove that the hybrid prescription for their scattering amplitudes agrees with the $\hat{c} = 5$ topological string prescription.

3.1. Chiral states using the $\hat{c} = 5$ description

The worldsheet fields in the $\hat{c} = 5$ formalism include the $d = 4$ variable x^m for $m = 0$ to 3, the left-moving chiral superspace variables θ^α and its conjugate momentum p_α for $\alpha = 1$ to 2, and an $\mathcal{N} = 2$ $\hat{c} = 3$ superconformal field theory for the internal compactification manifold. Unlike the superstring in the hybrid formalism, the $\hat{c} = 5$ formalism does not involve dotted superspace variables $\theta^{*\dot{\alpha}}$ or its conjugate momenta $p^*_{\dot{\alpha}}$, and also does not contain the chiral boson ρ. For the type II superstring, the $\hat{c} = 5$ formalism also includes the right-moving fermionic variables $\bar{\theta}^\alpha$ and its conjugate momenta $\bar{p}_\alpha$, but does not involve $\bar{\theta}^{*\dot{\alpha}}$ or $\bar{p}^{*\dot{\alpha}}$. (We will reserve barred notation throughout this paper to denote right-moving variables, and will use the $*$ superscript to denote dotted spinor variables.) For the formalism to be Hermitian, one therefore needs to Wick-rotate to either signature $(4,0)$ or $(2,2)$ so that θ^α is real. Although the reality conditions for spacetime fields in these signatures are not the standard ones, it is straightforward to Wick-rotate back to the standard Minkowski reality conditions after computing scattering amplitudes and determining the corresponding F terms in the effective action.

In the $\mathcal{N}=2$ $\hat{c}=5$ formalism, the worldsheet action is

$$S = \int d^2 z (p^\alpha \bar{\partial}\theta_\alpha + \bar{p}^\alpha \partial \bar{\theta}_\alpha + \frac{1}{2}\epsilon_{\alpha\beta}\partial x^{\alpha\dot{+}}\bar{\partial}x^{\beta\dot{-}}) + S_{CY}$$

and the left and right-moving twisted $\mathcal{N}=2$ generators are

$$\begin{aligned}
&T = p_\alpha \partial\theta^\alpha + \frac{1}{2}\epsilon_{\alpha\beta}\partial x^{\alpha\dot{+}}\partial x^{\beta\dot{-}} + T_{CY}, \\
&G^+ = \theta_\alpha \partial x^{\alpha\dot{+}} + G^+_{CY}, \quad G^- = p_\alpha \partial x^{\alpha\dot{-}} + G^-_{CY}, \\
&J = \theta^\alpha p_\alpha + J_{CY}. \\
&\bar{T} = \bar{p}_\alpha \bar{\partial}\bar{\theta}^\alpha + \frac{1}{2}\epsilon_{\alpha\beta}\bar{\partial} x^{\alpha\dot{+}}\bar{\partial} x^{\beta\dot{-}} + \bar{T}_{CY}, \\
&\bar{G}^+ = \bar{\theta}_\alpha \bar{\partial} x^{\alpha\dot{+}} + \bar{G}^+_{CY}, \quad \bar{G}^- = \bar{p}_\alpha \bar{\partial} x^{\alpha\dot{-}} + \bar{G}^-_{CY}, \\
&\bar{J} = \bar{\theta}^\alpha \bar{p}_\alpha + \bar{J}_{CY},
\end{aligned} \tag{3.1}$$

where $x^{\alpha\dot{\alpha}} = x^m \sigma_m^{\alpha\dot{\alpha}}$ and $\dot{\alpha} = (\dot{+}, \dot{-})$, S_{CY} and $\{\, T_{CY}, G^+_{CY}, G^-_{CY}, J_{CY} \,\}$ are the worldsheet action and twisted $\mathcal{N}=2$ $\hat{c}=3$ generators for the internal compactification manifold, and G^+ and G^- carry conformal weight $+1$ and $+2$ respectively. In the traditional description of the topological string, one treats $(x^{\alpha\dot{+}}, \theta^\alpha, \bar{\theta}^\alpha)$ as holomorphic coordinates on $\mathbf{C}^2 = \mathbf{R}^4$ and their superpartners and $(x^{\alpha\dot{-}}, p^\alpha, \bar{p}^\alpha)$ as anti-holomorphic coordinates and their partners. The four-dimensional part of the twisted $\mathcal{N}=2$ theory is then the topological B model whose target space is $\mathbf{C}^2$. Note that the $\mathcal{N}=2$ generators of (3.1) only preserve a $U(1) \times SU(2)$ (or $GL(1) \times SL(2)$) subgroup of $SO(4)$ (or $SO(2,2)$) Lorentz invariance in the signature $(4,0)$ (or $(2,2)$). For simplicity, we will usually restrict our attention to the left-moving sector.

Since $\oint G^+ = \oint(\theta_\alpha \partial x^{\alpha\dot{+}} + G^+_{CY})$ plays the role of a BRST operator in the topological $\mathcal{N}=2$ string, it is natural to compute its cohomology. Since $\theta_\alpha \partial x^{\alpha\dot{+}}$ and G^+_{CY} involve different worldsheet fields, states V in the cohomology of $\oint G^+$ can be written as $V =$

$\sum_i \Phi^i \sigma_i$ where Φ^i is constructed from the four-dimensional fields $\{x^m, \theta^\alpha, p_\alpha\}$ and is in the cohomology of $\oint \theta_\alpha \partial x^{\alpha\dot{+}}$, and σ_i is constructed from compactification-dependent fields and is in the cohomology of $\oint G^+_{CY}$. Using the standard quartet argument, states in the cohomology of $\oint \theta_\alpha \partial x^{\alpha\dot{+}}$ can depend only on the zero modes of θ^α and $x^{\alpha\dot{+}}$. So the most general state in the cohomology of $\oint G^+$ is

$$V = \sum_i \Phi^i(x^{\alpha\dot{+}}, \theta^\beta, \bar{\theta}^\gamma)\ \sigma_i \tag{3.2}$$

where σ^i is in the cohomology of $\oint G^+_{CY}$. Such states will be called "chiral" states.

In this paper, we shall only consider chiral states where σ_i contains either $+1$ or zero $U(1)$ charge with respect to the left and right-moving internal J_{CY}. (σ_i carrying zero internal $U(1)$ charge correspond to the identity operator.) Note that the $U(1)$ charge in the $d = 4$ sector is unconstrained in the chiral states considered here.

For the Type IIA (or Type IIB) superstring, chiral states carrying $+1$ left and right-moving $U(1)$ charge in the internal sector correspond to massless multiplets associated with Kähler (or complex) moduli of the Calabi-Yau space. The associated chiral moduli vertex operator is

$$V = \sum_i \Phi^i(x^{\alpha\dot{+}}, \theta, \bar{\theta})\ \sigma_i \tag{3.3}$$

where σ_i is a chiral primary of (left,right)-moving charge $(+1, +1)$ associated with the internal $\mathcal{N} = 2$ $\hat{c} = 3$ superconformal field theory. The $\theta = \bar{\theta} = 0$ component of Φ^i is the chiral modulus field and the $\theta = \bar{\theta} = 0$ component of $D_\alpha \bar{D}_\beta \Phi^i$ is the self-dual Ramond-Ramond (R-R) flux associated with this modulus.

For both the Type IIA and IIB superstring, chiral states carrying zero $U(1)$ charge in the internal sector correspond to a multiplet containing the self-dual graviphoton. The associated self-dual graviphoton vertex operator is

$$V = R(x^{\alpha\dot{+}}, \theta, \bar{\theta}) \tag{3.4}$$

where the self-dual graviphoton field strength $F_{\alpha\beta}$ is the $\theta = \bar{\theta} = 0$ component of $\partial_{\alpha\dot{+}} \partial_{\beta\dot{+}} R$ and the self-dual Riemann tensor $R_{\alpha\gamma\beta\delta}$ is the $\theta = \bar{\theta} = 0$ component of $\partial_{\alpha\dot{+}} \partial_{\beta\dot{+}} D_\gamma \bar{D}_\delta R$.

Although the chiral states of (3.3) and (3.4) do not have fixed charge with respect to the $U(1)$ charges $\int dz J$ and $\int d\bar{z} \bar{J}$ of (3.1), they can be defined to have fixed charge with respect to

$$\int dz(J + K) + \int d\bar{z}(\bar{J} + \bar{K}) \tag{3.5}$$

where $K = \frac{1}{2}\epsilon_{\alpha\beta}x^{\alpha\dot{+}}\partial x^{\beta\dot{-}} - \theta^\alpha p_\alpha$ and $\bar{K} = \frac{1}{2}\epsilon_{\alpha\beta}x^{\alpha\dot{+}}\bar{\partial}x^{\beta\dot{-}} - \bar{\theta}^\alpha \bar{p}_\alpha$. Note that $\int dz K + \int d\bar{z}\bar{K}$ is a conserved charge which commutes with the $\mathcal{N}=2$ generators of (3.1). When (3.3) is independent of $x^{\alpha\dot{+}}$ and (3.4) is quadratic in $x^{\alpha\dot{+}}$ (i.e. when $F_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ are constants), these chiral states all have charge $+2$ with respect to (3.5).

3.2. Scattering amplitudes using the $\hat{c}=5$ formalism

To compute scattering amplitudes of chiral states using the $\hat{c}=5$ formalism, we shall use the topological $\mathcal{N}=2$ prescription where $\oint G^+$ is treated as the BRST charge and G^- is treated as the b ghost. For M-point g-loop Type II scattering amplitudes, the $\mathcal{N}=2$ topological prescription is

$$A_{g,M} = \left\langle | \prod_{j=1}^{3g-3+M} \int dm^j \int \mu_j G^- |^2 \quad \prod_{r=1}^{M} V_r(z_r) \right\rangle \tag{3.6}$$

where μ_j denotes the $(3g-3+M)$ Beltrami differentials associated with the worldsheet moduli m_j, and $|\ |^2$ signifies the product of left and right-moving terms. Since $\hat{c}=5$, this amplitude vanishes by charge conservation unless

$$5(1-g) = \sum_{r=1}^{M} J_r - (3g-3+M), \tag{3.7}$$

where J_r is the $U(1)$ charge of V_r. So the sum of the $U(1)$ charges of the vertex operators must be equal to $(2-2g+M)$ both in the left and right-moving sectors.

The M-point g-loop amplitudes considered here will involve $(M-2g)$ chiral moduli described by the vertex operators of (3.3) and $2g$ self-dual graviphoton vertex operators described by the vertex operators of (3.4). With this choice, the charge conservation equation of (3.7) implies that $+2$ left and right-moving $U(1)$ charge must come from the $d=4$ sector of the formalism. As will be seen below, this $d=4$ $U(1)$ charge comes from the zero modes of θ^α and $\bar{\theta}^\alpha$. Although it might be interesting to consider more general scattering amplitudes in the $\hat{c}=5$ formalism, it is not clear if more general $\hat{c}=5$ scattering amplitudes will be $d=4$ super-Poincaré invariant like the amplitudes considered here.

In computing these special scattering amplitudes, it will be convenient to choose $2g$ of the $(3g-3+M)$ Beltrami differentials to be associated with the locations of the graviphoton vertex operators. So the formula of (3.6) becomes

$$A_{g,M} = \left\langle | \prod_{j=1}^{g-3+M} \int dm^j \int \mu_j G^-_{CY} |^2 \quad \prod_{r=1}^{M-2g} \Phi_r^{i_r}\sigma_{i_r}(z_r) \prod_{s=1}^{2g} \int d^2 z_s W_s(z_s) \right\rangle \tag{3.8}$$

where

$$
\begin{aligned}
W_s = &\oint G^- \oint \bar{G}^- R(x,\theta,\bar{\theta}) \\
=&(p^\alpha \partial_{\alpha\dot{+}} + \partial x_{\alpha\dot{+}} \frac{\partial}{\partial\theta_\alpha})(\bar{p}^\beta \partial_{\beta\dot{+}} + \bar{\partial} x_{\beta\dot{+}} \frac{\partial}{\partial\bar{\theta}_\beta}) R(x,\theta,\bar{\theta}),
\end{aligned}
\tag{3.9}
$$

and $\oint G^- \oint \bar{G}^- R$ signifies the single pole of G^- and $\bar{G}^-$ with R. It will be useful to note that since $(3-3g)$ $U(1)$ charge is needed from the internal sector, only the G^-_{CY} term in G^- contributes in $\int \mu_j G^-$.

In order that the $\mu_j G^-$ integrals in (3.6) reproduce the correct Faddeev-Popov measure for integration over worldsheet metrics, it is usually required that the vertex operators V_r have no double (or higher-order) poles with G^-. This condition guarantees that $\oint G^- V$ has no singularities with G^- which, together with $\oint G^+ V = 0$, implies that V is an $\mathcal{N}=2$ chiral primary. For chiral states of the two types considered here, this would imply that

$$
\frac{\partial}{\partial\theta_\alpha} \partial_{\alpha\dot{+}} \Phi^i = 0 \quad \text{and} \quad \frac{\partial}{\partial\theta_\alpha} \partial_{\alpha\dot{+}} R = 0. \tag{3.10}
$$

However, for the amplitudes considered here, these conditions are unnecessary since only the G^-_{CY} term contributes in $\int \mu_j G^-$. So there is no problem with reproducing the Faddeev-Popov measure if the vertex operators in (3.8) have singularities with the $d=4$ part of G^-, and there is therefore no need to impose (3.10) for consistency of these scattering amplitudes.

Furthermore, the fact that only G^-_{CY} contributes to $\int \mu_j G^-$ implies that the amplitude is spacetime supersymmetric. To show this, define the spacetime supersymmetry generators in the $\hat{c}=5$ formalism as

$$
q_\alpha = \oint p_\alpha, \quad q^*_{\dot{\alpha}} = \oint \theta^\alpha \partial x_{\alpha\dot{\alpha}}, \tag{3.11}
$$

which anticommute to the usual supersymmetry algebra

$$
\{q_\alpha, q_\beta\} = 0, \quad \{q^*_{\dot{\alpha}}, q^*_{\dot{\beta}}\} = 0, \quad \{q_\alpha, q^*_{\dot{\beta}}\} = \oint \partial x_{\alpha\dot{\beta}}.
$$

Note that these supersymmetries preserve the $\oint G^+$ cohomology when acting on states that carry no $P^{\alpha\dot{+}}$ momentum since $\{q^*_{\dot{\alpha}}, \oint G^+\} = 0$ and $\{q_\alpha, \oint G^+\} = \int \partial x^{\alpha\dot{+}}$. Finally, note that $\{q_\alpha, G^-\} = \{q^*_{\dot{\alpha}}, G^-_{CY}\} = 0$ and $\{q^*_{\dot{\alpha}}, G^-_{4d}\} = \delta^{\dot{+}}_{\dot{\alpha}} T_{4d}$ where G^-_{4d} and T_{4d} are the four-dimensional contributions to G^- and T. Since G^-_{4d} appears only in the integrated graviphoton vertex operator of (3.9), the anticommutator $\{q^*_{\dot{\alpha}}, G^-_{4d}\} = \delta^{\dot{+}}_{\dot{\alpha}} T_{4d}$ can be ignored since it only shifts the graviphoton vertex operator by a surface term.

To obtain the supersymmetric F term associated with the amplitude of (3.8), integrate over the zero modes of $(x^m, \theta^\alpha, \bar\theta^\alpha)$ and use the graviphoton vertex operator of (3.9) to absorb the zero modes of p^α. In terms of the self-dual graviphoton superfield $F_{\alpha\beta} = \partial_{\alpha\dot+}\partial_{\beta\dot+}R$, one finds

$$\begin{aligned} A_{g,M} = \int d^4x \int d^2\theta \int d^2\bar\theta \prod_{r=1}^{M-2g} \Phi_r^{i_r}(x,\theta,\bar\theta) \prod_{s=1}^{2g} F_{s\ \alpha\beta}(x,\theta,\bar\theta) \\ \times \left\langle | \prod_{j=1}^{g-3+M} \int dm^j \int \mu_j G^-_{CY} |^2 \prod_{r=1}^{M-2g} \sigma_{i_r}(z_r) \right\rangle_{CY} \end{aligned} \tag{3.12}$$

where $\langle\ \rangle_{CY}$ denotes a functional integral over the internal compactification-dependent fields and the $2g$ α indices and $2g$ β indices in $\prod_{s=1}^{2g} F_{s\ \alpha\beta}$ are contracted with each other in all possible combinations. So the F term associated with this scattering amplitude is

$$S = f_{i_1 \ldots i_{M-2g}} \int d^4x \int d^2\theta \int d^2\bar\theta \left(F_{\alpha\beta}(x,\theta,\bar\theta) F^{\alpha\beta}(x,\theta,\bar\theta)\right)^g \prod_{r=1}^{M-2g} \Phi^{i_r}(x,\theta,\bar\theta) \tag{3.13}$$

where the coefficient $f_{i_1\ldots i_{M-2g}}$ is defined by the $\mathcal{N}=2$ $\hat c = 3$ topological amplitude

$$f_{i_1 \ldots i_{M-2g}} = \left\langle | \prod_{j=1}^{g-3+M} \int dm^j \int \mu_j G^-_{CY} |^2 \prod_{r=1}^{M-2g} \sigma_{i_r}(z_r) \right\rangle_{CY}.$$

If we denote the Kähler (complex) moduli by t_i and denote the topological string amplitude at genus g by $F_g(t_i)$, then

$$f_{i_1 \ldots i_{M-2g}} = \partial_{i_1} \ldots \partial_{i_{M-2g}} F_g(t_i).$$

3.3. Hybrid description of chiral states

It will be shown here that the scattering amplitudes of chiral moduli states and self-dual graviphoton states computed in (3.12) using the $\hat c = 5$ formalism agree with those computed using the hybrid formalism. Note that hybrid scattering amplitudes involving only self-dual graviphoton states were computed previously in [10]. As discussed in [9,10,11], the hybrid formalism is related to the RNS formalism by a field redefinition. In the hybrid formalism, physical superstring states are described by chiral primary fields of $+1$ $U(1)$ charge with respect to the twisted $\mathcal{N}=2$ $\hat c = 2$ generators

$$\begin{aligned} &T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha + p^*_{\dot\alpha}\partial\theta^{*\dot\alpha} + \frac{1}{2}\partial\rho\partial\rho + \frac{1}{2}\partial^2\rho + T_{CY}, \\ &G^+ = e^{-\rho}(d^*)^2 + G^+_{CY}, \quad G^- = e^{\rho}(d)^2 + G^-_{CY}, \\ &J = \partial\rho + J_{CY}, \end{aligned} \tag{3.14}$$

where

$$d_\alpha = p_\alpha + i\theta^{*\dot\alpha}\partial x_{\alpha\dot\alpha} - (\theta^*)^2\partial\theta_\alpha, \quad d^*_{\dot\alpha} = p^*_{\dot\alpha},$$

and $\{\, T_{CY}, G^+_{CY}, G^-_{CY}, J_{CY} \,\}$ are the same twisted $\mathcal{N}=2$ $\hat{c}=3$ generators as before. Note that ρ is a negative-energy chiral boson satisfying the OPE

$$\rho(y)\rho(z) \sim -\log(y-z)$$

and d_α and $d^*_{\dot\alpha}$ are defined such that they anticommute with the supersymmetry generators

$$q_\alpha = \oint p_\alpha, \quad q^*_{\dot\alpha} = \oint (p^*_{\dot\alpha} - i\theta^\alpha \partial x_{\alpha\dot\alpha})$$

and satisfy the OPE's

$$d_\alpha(y) d^*_{\dot\alpha}(z) \sim \frac{1}{y-z}(\partial x_{\alpha\dot\alpha} + i\theta^*_{\dot\alpha}\partial\theta_\alpha). \tag{3.15}$$

To compare scattering amplitudes using the hybrid formalism with those of (3.12), one first needs the hybrid version of the vertex operators for the chiral moduli and graviphoton multiplets. The superstring states corresponding to compactification moduli multiplets are described in the hybrid formalism by the vertex operators

$$V = \sum^{i} \Phi^i(x,\theta,\bar\theta)\sigma_i, \tag{3.16}$$

where σ_i is the same compactification-dependent field as in the $\hat{c}=5$ description and carries $+1$ left and right moving $U(1)$ charge. One can easily check that V is chiral (*i.e.* is annihilated by $\oint G^+$ and $\oint \bar{G}^+$) if $D^{*\dot\alpha}\Phi^i = \bar{D}^{*\dot\alpha}\Phi^i = 0$ and is a chiral primary (*i.e.* has no double poles with G^-) if $D_\alpha D^\alpha \Phi^i = \bar{D}_\alpha \bar{D}^\alpha \Phi^i = 0$.

Because of the additional condition $D^\alpha D_\alpha \Phi^i = \bar{D}^\alpha \bar{D}_\alpha \Phi^i = 0$, the $\hat{c}=5$ vertex operator $V = \sum_i \Phi^i\, \sigma_i$ is not necessarily a chiral primary vertex operator in the hybrid formalism. However, as will be seen later in this subsection, the condition $D^\alpha D_\alpha \Phi^i = \bar{D}^\alpha \bar{D}_\alpha \Phi^i = 0$ will not be necessary for consistency of hybrid scattering amplitudes involving only chiral states. This is because, just as in the $\hat{c}=5$ formalism, only the G^-_{CY} term will contribute in G^- for these scattering amplitudes in the hybrid formalism. So there is no problem if the vertex operators have singularities with the four-dimensional $d^2 e^\rho$ term in G^-. This implies that one can prove equivalence of scattering amplitudes even for chiral states such as $V = (\theta-\bar\theta)^\alpha(\theta-\bar\theta)_\alpha\ \sigma$ which are not $\mathcal{N}=2$ primary fields in the hybrid formalism and therefore do not correspond to on-shell superstring states. This vertex operator V, which corresponds to a supersymmetric combination of the R-R and NS-NS fluxes associated to the moduli σ, will play an important role in the next section.

The superstring state corresponding to the self-dual graviphoton multiplet will be described in the hybrid formalism by the vertex operator

$$V = e^{-\rho} p^{*\dot{+}} e^{-\bar{\rho}} \bar{p}^{*\dot{+}} R(x, \theta, \bar{\theta}). \tag{3.17}$$

This vertex operator is chiral if $D^*_{\dot{\alpha}} R = \bar{D}^*_{\dot{\alpha}} R = 0$ and is primary if $D_\alpha \partial^{\alpha\dot{+}} R = \bar{D}_\alpha \partial^{\alpha\dot{+}} R = 0$. Although this vertex operator carries zero $U(1)$ charge in the internal sector, it carries $+1$ left and right-moving $U(1)$ charge in the four-dimensional sector because of its ρ dependence. Using the OPE's of (3.15), one finds that the integrated form of the graviphoton vertex operator is

$$\begin{aligned} &\oint G^- \oint \bar{G}^- V \\ &= \int d^2 z \left(d_\alpha \partial^{\alpha\dot{+}} + (\partial x^{\alpha\dot{+}} + \theta^{*\dot{+}} \partial \theta^\alpha) D_\alpha \right) \left(\bar{d}_\beta \bar{\partial}^{\beta\dot{+}} + (\bar{\partial} x^{\beta\dot{+}} + \bar{\theta}^{*\dot{+}} \partial \bar{\theta}^\beta) \bar{D}_\beta \right) R. \end{aligned} \tag{3.18}$$

So if one sets $\theta^*_{\dot{\alpha}} = \bar{\theta}^*_{\dot{\alpha}} = 0$, this expression coincides with the $\hat{c} = 5$ expression of (3.9).

To compute scattering amplitudes in the hybrid formalism, one first extends the $\hat{c} = 2$ $\mathcal{N} = 2$ generators of (3.14) to a set of $\mathcal{N} = 4$ generators

$$\{ \, T, G^+, \tilde{G}^+, G^-, \tilde{G}^-, J^{++}, J, J^{--} \, \}$$

by defining

$$J^{++} \equiv \exp\left(\int^z J \right), \qquad J^{--} \equiv \exp\left(- \int^z J \right)$$

to form an $SU(2)$ set of generators together with J, and by defining

$$\tilde{G}^- \equiv \left[\oint J^{--}, G^+ \right], \quad \tilde{G}^+ \equiv \left[\oint J^{++}, G^- \right],$$

to transform together with G^+ and G^- as two doublets under this $SU(2)$. As discussed in [10], the M-point g-loop amplitude is defined by the formula

$$\begin{aligned} &A_{M,g}(u_1, u_2, \bar{u}_1, \bar{u}_2) \\ &= \prod_{i=1}^{g} \int d^2 v_i \left\langle | \prod_{i=1}^{g-1} \widehat{\tilde{G}^+}(v_i) J(v_g) \prod_{j=1}^{3g-3+M} dm^j \int \mu_j \widehat{G^-} |^2 V_1 ... V_M \right\rangle, \end{aligned} \tag{3.19}$$

where

$$\widehat{G^-} = u_1 G^- + u_2 \tilde{G}^-, \quad \widehat{\tilde{G}^+} = u_1 \tilde{G}^+ + u_2 G^+,$$

and

$$A_{g,M}(u_1, u_2, \bar{u}_1, \bar{u}_2)$$
$$= \sum_{P=2-2g-M}^{2g-2} \sum_{\bar{P}=2-2g-M}^{2g-2} (u_1)^{P+2g-2+M} (u_2)^{2g-2-P} (\bar{u}_1)^{P+2g-2+M} (\bar{u}_2)^{2g-2-\bar{P}} A_{g,M,P,\bar{P}}$$

is a polynomial of degree $(4g-4+M, 4g-4+M)$ in $(u, \bar{u})$. The different components $A_{g,M,P,\bar{P}}$ correspond to amplitudes which violate (left,right)-moving R-charge by $(P, \bar{P})$. Note that R-charge in the hybrid formalism is equivalent to picture in the RNS formalism.

For scattering amplitudes corresponding to F terms with $(M-2g)$ chiral moduli and $2g$ graviphoton superfields, R-charge is violated by $(g-1, g-1)$. This is because chiral moduli superfields carry zero R-charge, self-dual graviphoton superfields carry $(\frac{1}{2}, \frac{1}{2})$ R-charge, and F terms carry $(-1,-1)$ R-charge from the $d^2\theta d^2\bar{\theta}$ integration. So we are interested in computing the component which violates R-charge by $(P, \bar{P}) = (g-1, g-1)$. To compute the $A_{g,M,g-1,g-1}$ component of $A_{g,M}$ using the formula of (3.19), first note that all terms in this component contain an equal number of $\tilde{G}^-$ and $\tilde{G}^+$ operators. To compare with the $\hat{c} = 5$ prescription of (3.8), it will be useful to first turn all pairs of $(\tilde{G}^+, \tilde{G}^-)$ operators into pairs of (G^+, G^-) operators by performing the appropriate contour deformations.

For example, suppose one has a pair of $\tilde{G}^+(y_1)\tilde{G}^-(y_2)$ operators at y_1 and y_2. First write $\tilde{G}^- = [\oint G^+, J^{--}(y_2)]$ and deform the $\oint G^+$ contour off of $J^{--}(y_2)$ until it hits the $J(v_g)$ operator, turning it into $G^+(v_g)$. Secondly, write $\tilde{G}^+(y_1) = [\int \tilde{G}^+, J(y_1)]$ and deform the $\int \tilde{G}^+$ contour off of $J(y_1)$ until it hits the $J^{--}(y_2)$ operator, turning it into $G^-(y_2)$. Finally, write $G^+(v_g) = [\oint G^+, J(v_g)]$ and deform the $\oint G^+$ contour off of $J(v_g)$ until it hits the $J(y_1)$ operator, turning it into $G^+(y_1)$. So this procedure has turned $\tilde{G}^+(y_1)\tilde{G}^-(y_2)$ into $G^+(y_1)G^-(y_2)$.

In performing these contour deformations, we have ignored possible surface terms on the moduli space of the worldsheet coming from the commutator $[\oint G^+, \int \mu_j G^-] = \int \mu_j T$, where $\int \mu_j T$ produces a total derivative on the moduli space. However, for the scattering amplitudes discussed here, one can show that internal $U(1)$ charge conservation implies that these surface terms do not contribute. As in the $\hat{c} = 5$ computation, internal $U(1)$ conservation implies that the $d = 4$ part of G^- only contributes to the scattering amplitude when it acts on the graviphoton vertex operator. Also, one can argue by internal $U(1)$ conservation that only the $d = 4$ part of G^+ contributes. So the only possibility of producing a surface term comes from $[\oint G^+_{4d}, \int \mu_j G^-_{4d}] = \int \mu_j T_{4d}$ where the subscript $4d$ denotes the four-dimensional contribution to these generators and μ_j is associated with the location of the graviphoton vertex operator. But this type of surface term is harmless since it does not involve the $(3g-3)$ worldsheet moduli whose boundary describes degeneration of the genus g surface.

After replacing all $(\tilde{G}^+, \tilde{G}^-)$ pairs with (G^+, G^-) pairs and choosing $2g$ of the Beltrami differentials to be associated with the locations of the graviphoton vertex operators, one obtains the formula

$$
\begin{aligned}
A_{M,g} = & \prod_{i=1}^{g} \int d^2 v_i \left\langle \big| \prod_{i=1}^{g-1} G^+(v_i) J(v_g) \prod_{j=1}^{g-3+M} \int dm^j \int \mu_j G^- \big|^2 \right. \\
& \left. \times \prod_{r=1}^{M-2g} \Phi_r^{i_r} \sigma_{i_r}(z_r) \prod_{s=1}^{2g} \int d^2 z_s W_s(z_s) \right\rangle_H
\end{aligned}
\tag{3.20}
$$

where W_s is defined in (3.18) and $\langle \ \rangle_H$ denotes the functional integral using the hybrid formalism which includes the $(\theta^*_{\dot\alpha}, p^*_{\dot\alpha})$ and ρ fields.

To compare this formula with the $\hat{c} = 5$ formula of (3.8), insert the identity operator $1 = [\oint G^+, \theta^*_{\dot\alpha} \theta^{*\dot\alpha} e^{\rho}(w)]$ in (3.20) and pull the $\oint G^+$ contour off of $\theta^*_{\dot\alpha} \theta^{*\dot\alpha} e^{\rho}(w)$ until it hits $J(v_g)$ to give the formula

$$
\begin{aligned}
A_{M,g} = & \prod_{i=1}^{g} \int d^2 v_i \left\langle \big| (\theta^*_{\dot\alpha} \theta^{*\dot\alpha} e^{\rho})(w) \prod_{i=1}^{g} (p^{*\dot\alpha} p^*_{\dot\alpha} e^{-\rho})(v_i) \prod_{j=1}^{g-3+M} \int dm^j (\int \mu_j G^-_{CY}) \big|^2 \right. \\
& \left. \times \prod_{r=1}^{M-2g} \Phi_r^{i_r} \sigma_{i_r}(z_r) \prod_{s=1}^{2g} \int d^2 z_s W_s(z_s) \right\rangle_H .
\end{aligned}
\tag{3.21}
$$

To derive (3.21), we have used that $U(1)$ charge conservation implies that only G^-_{CY} contributes in the $\mu_j G^-$ terms and that only G^+_{4d} contributes to $G^+(v_i)$.

Finally, one needs to do the functional integral over the worldsheet fields $(\theta^*_{\dot\alpha}, p^*_{\dot\alpha}, \rho)$ which are present in the hybrid formalism but not in the $\hat{c} = 5$ formalism. Since all $p^*_{\dot\alpha}$ variables in $G^+(v_i)$ must be used to soak up the $2g$ zero modes of $p^*_{\dot\alpha}$, none of the $\theta^*_{\dot\alpha}$ variables in the vertex operators can contribute and the $\theta^*_{\dot\alpha} \theta^{*\dot\alpha}(w)$ soaks up the zero modes of $\theta^*_{\dot\alpha}$. Because the ρ chiral boson has negative energy (like the ϕ chiral boson in the RNS formalism which comes from fermionizing the (β, γ) ghosts), it is subtle to define its functional integral. However, for the amplitudes being considered here, the ρ field always appears together with the $(\theta^*_{\dot{+}}, p^{*\dot{+}})$ fields in the combination $\theta^*_{\dot{+}} e^{\rho}$ or $p^{*\dot{+}} e^{-\rho}$. For this reason, the functional integral over the ρ chiral boson precisely cancels the functional integral over the $(\theta^*_{\dot{+}}, p^{*\dot{+}})$, even for the zero modes. So after performing the functional integral over the $(\theta^*_{\dot{\alpha}}, p^*_{\dot{\alpha}}, \rho)$ fields, one obtains the amplitude

$$A_{M,g} = \left\langle \Big| \prod_{j=1}^{g-3+M} \int dm^j \Big(\int \mu_j G^-_{CY} \Big) \Big|^2 \quad \prod_{r=1}^{M-2g} \Phi_r^{i_r} \sigma_{i_r}(z_r) \prod_{s=1}^{2g} \int d^2 z_s W_s(z_s) \right\rangle , \qquad (3.22)$$

which agrees with the $\hat{c} = 5$ formula of (3.8).

4. Large N Duality in Superstring

It was pointed out in [6] that the duality between the open and closed topological string theories can be uplifted to the type IIA superstring on the conifold times $\mathbf{R}^4$ with N D5 branes wrapping on the $\mathbf{P}^1$ of the conifold and extended in the $\mathbf{R}^4$ direction to another compactification with N units of R-R flux and without D branes. As far as the F terms are concerned, this superstring duality is inferred from the topological string duality combined with the relation between the superpotential terms and the topological string amplitudes [5,15]. This duality is supposed to hold beyond the superpotential computation, along the line of construction described in the closely related papers [21,22]. A derivation of the full duality would require controlling back-reactions of the R-R fluxes to the metric and understanding worldsheet dynamics in such a background, and it would be tantamount to proving the AdS/CFT correspondence. In this section, we will make the first step in this direction by giving a direct worldsheet derivation of the duality restricted to the superpotential computation, where the back-reaction to the metric can be ignored as being a BRST trivial deformation of the background.

As we saw in the last section, the $\hat{c} = 5$ formalism allows us to compute superpotential terms as topological string amplitudes. In this formalism, in addition to the $\hat{c} = 3$ model discussed in section 2, we have four bosons $x_{\alpha\dot{\alpha}}$ and four pairs of fermions $(p_\alpha, \theta^\alpha)$ and $(\bar{p}_\alpha, \bar{\theta}^\alpha)$. In the $\hat{c} = 3$ model on the Calabi-Yau space, basic observables are associated to cohomology elements of the Calabi-Yau space. For example, for $\omega \in H^{1,1}$, we have $\sigma = \omega_{i\bar{j}} \psi^i_L \psi^{\bar{j}}_R$. In the $\hat{c} = 5$ formalism, it can be multiplied by any function of $\theta, \bar{\theta}$ as $\Phi(\theta, \bar{\theta})\, \sigma$, giving rise to a vertex operator for the $\mathcal{N} = 2$ vector multiplet in four dimensions associated to σ. We can turn on the auxiliary fields in this multiplet to break the $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$.[2] For example, we can turn on the perturbation,

$$\int d^2 z G^- \bar{G}^- \left[N(\theta-\bar{\theta})^2 \sigma \right].$$

This corresponds to turning on R-R flux through the cycle dual to ω, represented by $\epsilon_{\alpha\beta}\theta^\alpha\bar{\theta}^\beta\ \sigma$ [6], combined with an appropriate amount of NS-NS flux, represented by $\epsilon_{\alpha\beta}(\theta^\alpha\theta^\beta+\bar{\theta}^\alpha\bar{\theta}^\beta)\ \sigma$, through the dual cycle. The strength of the NS-NS flux (related to the coupling constant τ of the dual gauge theory) is dictated by the condition of extremization of the glueball superpotential [6], leading to preservation of $\mathcal{N}=1$ supersymmetry. Note that this term reduces the supersymmetry to $\mathcal{N}=1$ given by simultaneous shift of $\theta, \bar{\theta}$.

With these fluxes turned on and the supersymmetry reduced, the $\mathcal{N}=2$ vector multiplet is decomposed into an $\mathcal{N}=1$ vector multiplet v_α and the chiral multiplet t. These couple to the worldsheet as

$$\int d^2 z G^- \bar{G}^- \left[\left(t + v_\alpha(\theta-\bar{\theta})^\alpha + N(\theta-\bar{\theta})^2\right) \sigma \right] \tag{4.1}$$

where we included the effect of the fluxes. In section 2, we saw that, in the $\hat{c}=3$ model, the Kähler moduli appears as a coefficient of the linear superpotential (2.3). The coupling (4.1) in the $\hat{c}=5$ model can also be written in term of a superpotential given by

$$W = \left(t + v_\alpha(\Theta-\bar{\Theta}) + N(\Theta-\bar{\Theta})^2\right) \Sigma, \tag{4.2}$$

where Σ is the superfield in the $\hat{c}=3$ model with σ as the lowest component, and $\Theta, \bar{\Theta}$ are fermionic superfields whose lowest components are θ and $\bar{\theta}$. The contribution of W to the worldsheet action is

$$S_{int} = \int d^2 z G^- \bar{G}^- W. \tag{4.3}$$

[2] Normally one does not consider “turning on” auxiliary fields since their values are fixed by equations of motion. However, in Wick-rotated signatures (2,2) or (4,0), there may be supersymmetric backgrounds which violate equations of motion. For example, the auxiliary fields D_{ij} in an $\mathcal{N}=2$ vector multiplet transform as a triplet under the R-symmetry group which gets Wick-rotated from $SU(2)$ to $SL(2)$. For a free $\mathcal{N}=2$ multiplet, the potential is $D_{++}D_{--}+(D_{+-})^2$ and one has an $\mathcal{N}=1$ supersymmetric background when $D_{++}=-D_{--}=D_{+-}=N$ for any value of N. After Wick-rotation back to Minkowski space, the value of N is uniquely determined by the reality conditions on D_{ij}. For example, for a free multiplet in Minkowski space, $N=0$ is the unique supersymmetric background consistent with the reality condition $D_{++}=(D_{--})^*$. However, in a non-trivial background such as that of [21] or [22], the reality conditions together with supersymmetry may imply a non-zero value for N.

Noting that G^- is a linear combination of operators acting on the 4d part and the Calabi-Yau part, we can express it as

$$\begin{aligned}\int d^2z G^-\bar{G}^- W = \int d^2z \Big[& \left(t + v_\alpha(\theta-\bar\theta) + N(\theta-\bar\theta)^2\right) G^-_{CY}\bar{G}^-_{CY}\sigma \\ & + \left(v_\alpha \partial x^{\alpha\dot{-}} + 2N(\theta-\bar\theta)_\alpha \partial x^{\alpha\dot{-}}\right)\bar{G}^-_{CY}\sigma \\ & + \left(v_\alpha \bar\partial x^{\alpha\dot{-}} + 2N(\theta-\bar\theta)_\alpha \bar\partial x^{\alpha\dot{-}}\right) G^-_{CY}\sigma - 2N\epsilon_{\alpha\beta}\partial x^{\beta\dot{-}}\bar\partial x^{\alpha\dot{-}}\sigma\Big] .\end{aligned}$$

Since W is annihilated by $\oint G^+$ and $\oint \bar{G}^+$ of (3.1), W is a chiral superpotential which implies that (4.3) is in the BRST cohomology. Actually, annihilation by G^+ and $\bar{G}^+$ of (3.1) implies that W is chiral using the worldsheet equations of motion of the undeformed theory. In principle, one still needs to check that W is chiral after including any possible back-reaction to the worldsheet equations of motion. Fortunately, there is no back-reaction to the worldsheet equations of motion for the $d=4$ fields $(x^{\alpha\dot{-}}, \theta^\alpha, \bar\theta^\alpha)$ which appear in W. This is clear since the equations of motion for these $d=4$ fields come from varying $(x^{\alpha\dot{+}}, p_\alpha, \bar{p}_\alpha)$, which are absent from (4.3).

On the other hand, since the vertex operator for the spacetime curvature and the graviphoton field strength contain p_α and $\bar{p}_\alpha$, in the $\hat{c}=5$ formalism formulated in the last section, there may be a subtlety in simultaneously turning on the gravity fields and the R-R flux. Since it is clear from the target space point of view that supersymmetry is still preserved with both of them turned on, there should be a manifestly supersymmetric description of such a background on the worldsheet. It would be interesting to understand how to apply the $\hat{c}=5$ formalism in this case. On the open string side, turning on the spacetime curvature and the graviphoton field strength generates the C-deformation of the gluino field [12,13]. Thus it is reasonable to expect a phenomenon dual to it in the closed string side. In the following, we will consider the large N duality in the absence of the gravity field strengths.

As in the $\hat{c}=3$ model for the conifold, the $\hat{c}=5$ model has two branches, the H branch with $\sigma=0$ and the C branch with $\sigma\neq 0$. We identify each C domain as a hole on the worldsheet. Whereas the C branch of the $\hat{c}=3$ model is described as the Landau-Ginzburg model with the superpotential (2.3) (and with the path integral measure $d\sigma/\sigma^2$), the C branch in the $\hat{c}=5$ model is the Landau-Ginzburg model with (4.2). In particular, its target space is the supermanifold with coordinates $(\Sigma, \Theta^\alpha, \bar\Theta^\alpha)$. As in the $\hat{c}=3$ case, the C branch does not contribute to a string amplitude unless its domain has the topology of the disk. This statement just follows from the functional integral over Σ and the operation of $\oint d\sigma \partial/\partial\sigma$ and is independent of whether there are extra degrees of freedom.

The functional integral over the disk C domain indeed gives the correct boundary condition for the N D branes extended in the $\mathbf{R}^4$ direction with the gluino field $\mathcal{W}_\alpha$ turned on. To see this, let us integrate over Σ first. As in the case of the $\hat{c}=3$ model [3], it gives

$$\oint \frac{d\sigma}{\sigma^2} \exp\left[-\left(t + v_\alpha(\theta-\bar{\theta})^\alpha + N(\theta-\bar{\theta})^2\right)\sigma\right] = t + v_\alpha(\theta-\bar{\theta})^\alpha + N(\theta-\bar{\theta})^2. \tag{4.4}$$

According to the large N duality [6], t and v_α are related to the open string variable $\mathcal{W}_\alpha$ as

$$\begin{aligned} t &= \mathrm{tr}\mathcal{W}_\alpha \mathcal{W}^\alpha \\ v_\alpha &= \mathrm{tr}\mathcal{W}_\alpha \\ N &= \mathrm{tr}1. \end{aligned} \tag{4.5}$$

Using this, the right-hand side of (4.4) can be written as

$$t + v_\alpha(\theta-\bar{\theta})^\alpha + N(\theta-\bar{\theta})^2 = \mathrm{tr}\left[\exp\left(\mathcal{W}^\alpha \frac{\partial}{\partial\theta^\alpha}\right)(\theta-\bar{\theta})^2\right]. \tag{4.6}$$

We can then identify $(\theta-\bar{\theta})^2$ as the boundary state for the D brane extended in the $\mathbf{R}^4$ direction. As in any state which is invariant under the topological BRST symmetry, the boundary state can be decomposed into a chiral primary state and a BRST trivial part. It was shown in [23] that the chiral primary part is determined by the (quantum) period of the cycle on which the D brane is wrapped. For the D brane extended in the $\mathbf{R}^4$ direction, the chiral primary part is $(\theta-\bar{\theta})^2$; indeed it imposes the correct boundary condition $\theta^\alpha = \bar{\theta}^\alpha$, which is associated with Neumann boundary conditions for x^m. We can then identify the action of the differential operator $\exp\left(\mathcal{W}^\alpha \frac{\partial}{\partial\theta^\alpha}\right)$ as an insertion of $\oint \mathcal{W}^\alpha(p_\alpha + \bar{p}_\alpha)$ on the boundary of the disk, giving rise to the correct coupling of the gluino on the boundary. This shows that the superpotential for t and the kinetic term for v_α computed in the closed string theory agree with those for the glueball superfield and the $U(1)$ part of $\mathcal{W}_\alpha$ in the open string theory according to the correspondence (4.5). This is what we wanted to show.

We note that one can start with a different combination of fluxes, for example,

$$\int d^2z G^- \bar{G}^- \left[N(\theta^1 \pm \bar{\theta}^1)(\theta^2 \pm \bar{\theta}^2)\sigma\right], \tag{4.7}$$

and repeat the derivation. (We can also consider more general quadratic combinations of θ and $\bar{\theta}$ that preserve 4 supercharges. Here we are presenting simple ones for an illustration.) One will then find the boundary state whose chiral primary part is represented by $(\theta^1 \pm \bar{\theta}^1)(\theta^2 \pm \bar{\theta}^2)$. We can interpret it as the boundary state for a D_{2n+2} brane wrapping on the $\mathbf{S}^3$ of the deformed conifold and extending in a $2n$-dimensional plane in $\mathbf{R}^4$, where n is the number of minus signs in (4.7). This is consistent with what one expects from T-dual of the open/closed string duality that we discussed in this paper.

The original argument [24] for the existence of the large N dualities of the type discussed in this paper starts with the conjectured equivalence of the D brane description involving open strings and the closed string description motivated by the computation of the R-R charges [25]. The result of this paper provides the worldsheet explanation for the equivalence of the two descriptions, at the level of F terms. For the closed string, the vertex operator $N(\theta - \bar{\theta})^2\sigma$ represents the closed string background with N units of R-R flux turned on. We have found that turning on this worldsheet interaction generates the open string sector whose boundary state for the $4d$ part of the target space is represented by $N(\theta - \bar{\theta})^2$. This boundary state indeed carries the correct amount of R-R charge expected from the duality. We hope that our result in this paper will turn out to be a useful step toward deriving the full large N duality in the superstring.

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References

[1] R. Gopakumar and C. Vafa, “On the gauge theory/geometry correspondence,” Adv. Theor. Math. Phys. **3**, 1415 (1999) [arXiv:hep-th/9811131].

[2] E. Witten, “Chern-Simons gauge theory as a string theory,” Prog. Math. **133**, 637 (1995) [arXiv:hep-th/9207094].

[3] H. Ooguri and C. Vafa, “Worldsheet derivation of a large N duality,” Nucl. Phys. B **641**, 3 (2002) [arXiv:hep-th/0205297].

[4] E. Witten, “Phases of $\mathcal{N}=2$ theories in two dimensions,” Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[6] C. Vafa, “Superstrings and topological strings at large N,” J. Math. Phys. **42**, 2798 (2001) [arXiv:hep-th/0008142].

[7] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, “A geometric unification of dualities,” Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[8] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” arXiv:hep-th/0208048.

[9] N. Berkovits, “Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background,” Nucl. Phys. **B431**, 258 (1994), [arXiv: hep-th/9404162].

[10] N. Berkovits and C. Vafa, “$\mathcal{N}=4$ topological strings,” Nucl. Phys. **B433**, 123 (1995, [arXiv: hep-th/9407190].

[11] N. Berkovits, “A New Description of the Superstring,” [arXiv: hep-th/9604123].

[12] H. Ooguri and C. Vafa, “The C-deformation of gluino and non-planar diagrams,” arXiv:hep-th/0302109.

[13] H. Ooguri and C. Vafa, “Gravity induced C-deformation,” arXiv:hep-th/0303063.

[14] R. Dijkgraaf, M.T. Grisaru, H. Ooguri, C. Vafa, and D. Zanon, “Planar gravitational corrections for supersymmetric gauge theories,” arXiv:hep-th/0310061.

[15] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” Nucl. Phys. B **577**, 419 (2000) [arXiv:hep-th/9912123].

[16] J. M. Labastida and M. Marino, “Polynomial invariants for torus knots and topological strings,” Commun. Math. Phys. **217**, 423 (2001) [arXiv:hep-th/0004196].

[17] P. Ramadevi and T. Sarkar, “On link invariants and topological string amplitudes,” Nucl. Phys. B **600**, 487 (2001) [arXiv:hep-th/0009188].

[18] J. M. Labastida, M. Marino and C. Vafa, “Knots, links and branes at large N,” JHEP **0011**, 007 (2000) [arXiv:hep-th/0010102].

[19] J. M. Labastida and M. Marino, “A new point of view in the theory of knot and link invariants,” arXiv:math.qa/0104180.

[20] M. Marino and C. Vafa, "Framed knots at large N," arXiv:hep-th/0108064.

[21] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and χSB-resolution of naked singularities," JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].

[22] J. M. Maldacena and C. Nunez, "Towards the large N limit of pure $\mathcal{N} = 1$ super Yang Mills," Phys. Rev. Lett. **86**, 588 (2001) [arXiv:hep-th/0008001].

[23] H. Ooguri, Y. Oz and Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. B **477**, 407 (1996) [arXiv:hep-th/9606112].

[24] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200].

[25] J. Polchinski, "Dirichlet-branes and Ramond-Ramond charges," Phys. Rev. Lett. **75**, 4724 (1995) [arXiv:hep-th/9510017].