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LYAPUNOV CONSTRAINTS AND GLOBAL ASYMPTOTIC STABILIZATION

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Abstract. In this paper, we develop a method for stabilizing underactuated mechanical systems by imposing kinematic constraints (more precisely *Lyapunov constraints*). If these constraints can be implemented by actuators, i.e., if there exists a related constraint force exerted by the actuators, then the existence of a Lyapunov function for the system under consideration is guaranteed. We establish necessary and sufficient conditions for the existence and uniqueness of constraint forces. These conditions give rise to a system of PDEs whose solution is the required Lyapunov function. To illustrate our results, we solve these PDEs for certain underactuated mechanical systems of interest such as the inertia wheel-pendulum, the inverted pendulum on a cart system and the ball and beam system.

1. **Introduction.** Consider a dynamical system on a smooth connected manifold P, defined by a smooth vector field $X \in \mathfrak{X}(P)$. It is well-known (see [16, 26]) that the system is *globally asymptotically stable* at a point $\alpha_o \in P$ (i.e. all trajectories converge to α_o) if there exists a non negative function $V : P \to \mathbb{R}$, called *Lyapunov function*, such that

P1: $V(\alpha) = 0$ only if $\alpha = \alpha_o$;
P2: V is a proper map;
P3: $\langle dV(\alpha), X(\alpha)\rangle < 0$ for all $\alpha \neq \alpha_o$.

This work is based on the following observation: in terms of the trajectories $\Gamma : I \subset \mathbb{R} \to P$ of the system, property 3 implies that, for all $t \in I$,

$$\langle dV(\Gamma(t)), X(\Gamma(t))\rangle = -\mu(\Gamma(t)), \qquad (*)$$

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* Sadly Prof. Jerrold Marsden passed away on September 21, 2010.

or equivalently

$$\langle dV(\Gamma(t)), \Gamma'(t)\rangle = -\mu(\Gamma(t)), \qquad (**)$$

where $\mu : P \to \mathbb{R}$ is a nonnegative function. In other words, property 3 can be seen as a "kinematic constraint" on the system. So, if we want to asymptotically stabilize a dynamical system, we can impose on the system constraints of the form $(**)$ for some appropriate[1] non negative functions V and μ. Constraints of the form $(**)$ will be called ***Lyapunov constraints***.[2] For instance, if we have an actuated Hamiltonian system with phase space $P = T^*Q$ and Hamiltonian function $H : T^*Q \to \mathbb{R}$, we need to find a vertical vector field $Y \in \mathfrak{X}(T^*Q)$ such that $(*)$ holds for $X = X_H + Y$. In this case, assuming that $\dim Q = n$ and (q^i, p_j) are local coordinates for T^*Q, Eq. $(**)$ takes the form

$$\sum_{i=1}^{n} \left(\frac{\partial V}{\partial q^i}(q,p)\, \dot{q}^i + \frac{\partial V}{\partial p_i}(q,p)\, \dot{p}_i \right) = -\mu(q,p). \qquad (1)$$

From (1), it is clear that $(**)$ defines a second order constraint in the configuration variable of the system. Also note that the role of Y is two-fold: as a *control signal* and as a *constraint force*. Thus, applying the idea of Lyapunov constraints to an actuated Hamiltonian system, in order to find a control law that asymptotically stabilizes the system we can try to find the required constraint force which the actuators must exert in order to implement the second order constraint $(**)$. The purpose of this paper is in answering the following questions.

- How do we guarantee the existence of such constraint forces for a given pair of functions V and μ?
- Assuming such a constraint force exists, is there a constructive method to find it?

More precisely, we shall develop, in the context of actuated Hamiltonian systems on a cotangent bundle T^*Q, a global non linear method based on Lyapunov constraints. The method consists of solving a certain PDE for V (see (63)) such that for each solution satisfying properties **P1** and **P2**, for some $\alpha_o \in T^*Q$, there exists a unique control law corresponding to the vertical field Y that globally asymptotically stabilizes the system at α_o. Moreover, the solution of the PDE is the required Lyapunov function for the system.

There are other global energy-based methods in the literature, for example the method of controlled Lagrangians [5, 4] (or IDA-PBC on the Hamiltonian side [21, 11]), also known as the energy shaping method. We shall show that these methods can be seen as a particular case of the technique we develop in this paper. As seen in [29], the method of controlled Lagrangians can not be easily extended to systems subjected to friction forces. We shall see that our method give rise to additional "degrees of freedom" that enable us to deal with friction forces in a more effective way.

It is worth mentioning that, as is the case with any energy shaping methods, solving the PDEs to construct a Lyapunov function is not an easy task in general. In the present work, we shall solve them for particular underactuated systems, but we shall not make a systematic study of existence and uniqueness of solutions. We expect to do that in the future, in the framework of the geometric theory of PDEs,

[1] We can consider weaker conditions than those appearing above and still arrive to useful results. For instance, if condition 2 does not hold, constraint $(*)$ ensure, unless, *local* asymptotic stability.

[2] The idea of Lyapunov constraints has already appeared in references [14, 15]. Here, we are going to study it in a systematic way.

following the same ideas as those developed in [12] for the energy shaping method's PDEs.

As it is clear from the first paragraph of this introduction, the idea of Lyapunov constraints can be applied to arbitrary dynamical systems. However, as a first step toward a more systematic study of this technique, in this paper we focus only on (constrained) Hamiltonian and Lagrangian systems. More precisely, throughout this paper, we work within the framework of *higher order constrained systems*, developed in [9, 10, 13, 14]. In this setting, it is possible to construct a closed-loop mechanical system (CLMS) (see Definition 2.4 for the definition of CLMS) from a system with second order constrains, in such a way that both systems are equivalent (i.e. their trajectories are in bijection) and the control law of the former coincides with the constraint force of the later. Moreover, it can be shown that every CLMS can be constructed in this way [15]. Some applications can be found in [23]. (See [8, 18, 22, 25] for the first order case.) We briefly review these ideas in Section 2. In Section 3, we focus on the case in which the contraints are given by Lyapunov constraints. We derive the equation that must be solved in order to find the forces which must be implemented for such constraints. In Section 4 we study, for a particular subclass of Lyapunov constraints (that we call *simple*), necessary and sufficient conditions for existence and uniqueness of constraint forces. This is done globally and in local coordinates and also for systems with friction. These conditions amount to solving a system of PDEs (see (63)) whose solution plays the role of a Lyapunov function for the dynamical system under investigation. At the end of this section we study how to use the solutions of these PDEs for asymptotically stabilizing underactuated systems, including an analysis of the required La'Salle surface. Then, in Section 5, we present several solutions of such PDEs for the inertia wheel pendulum, the inverted pendulum on a cart and the ball-beam system. Finally, in Section 6, we consider a larger class of Lyapunov constraints (called *quasi-simple*) than the ones considered in Section 4 which enable us to deal with system with friction forces.

We assume that the reader is familiar with basic concepts of Differential Geometry (see [6, 17, 20]) and with the ideas of Lagrangian and Hamiltonian systems in the context of Geometric Mechanics (see [1, 2, 19]). For an introduction to geometric control theory see [3] and [7].

2. **Constrained and closed-loop mechanical systems.**

In order to control a mechanical system, we can impose on it a set of constraints to make the system evolve in the desired way, and then obtain the control law as the related constraint force. In other words, to design a control strategy, we can *think of constraints*, i.e. we can impose on the system appropriate constraints to achieve a given behavior. In this way, we are constructing a closed-loop mechanical system from a constrained one. This idea originally appeared (to the best of our knowledge) in the papers [18] and [25], and it was further developed in [8, 9, 13, 14, 22, 23]. In this section we show how the technique of "constraints" works in the Hamiltonian case as in Ref. [15].

2.1. **Second order constrained systems.**

Throughout this paper, Q will denote a smooth n-manifold. By a **second order constrained (Hamiltonian) system** on Q we mean a triple $(H, \mathcal{P}, \mathcal{W})$, where $H : T^*Q \to \mathbb{R}$, $\mathcal{P}$ is a subset of TT^*Q,

and $\mathcal{W}$ is a vertical distribution on T^*Q, i.e. $\mathcal{W} \subset \ker \pi_*$, with $\pi : T^*Q \to Q$ the canonical projection. This is a special case of **higher order constrained systems (HOCS)** studied in [9] and [14]. The subset $\mathcal{P} \subset TT^*Q$ represents the kinematic constraints and the distribution $\mathcal{W}$ defines the directions along which constraint forces act.

Remark 1. From now on, given a manifold M, by a **distribution** on M we mean a map $\Delta : m \mapsto \Delta_m$, with Δ_m a subspace of T_mM, or equivalently, a subset $\Delta \subset TM$ such that for each $m \in M$ the intersection $\Delta_m \equiv \Delta \cap T_mM$ is a linear subspace. We say that the distribution is a **C^∞-distribution** if for each $v \in \Delta_m$ there exists a local C^∞ vector field X contained in the image of Δ satisfying $X(m) = v$. A **codistribution** and a **C^∞-codistribution** will be the obvious dual objects. The dimension of Δ_m need not be a constant for all m, i.e. Δ can have **varying rank** as m varies. Note that, if Δ is a C^∞-distribution with constant rank, then Δ is a linear subbundle of TM.

For the purposes of this paper, $\mathcal{P}$ will be given by the zero level subset of a set of smooth functions. In other words, in local coordinates, $\mathcal{P}$ will be given by

$$\mathcal{P} = \{(q, p, \dot{q}, \dot{p}) | F\ (q, p, \dot{q}, \dot{p}) = 0.\} \tag{2}$$

Note that (2) effectively imposes a second order derivative constraint on the system.

Definition 2.1. By a **trajectory** of $(H, \mathcal{P}, \mathcal{W})$ we mean an integral curve $\Gamma : I \subset \mathbb{R} \to T^*Q$ of a vector field $X \in \mathfrak{X}(T^*Q)$ satisfying

$$X \subset \mathcal{P} \quad \textit{and} \quad X - X_H \subset \mathcal{W}, \tag{3}$$

where X_H is the Hamiltonian vector field of H.[3] We say that $Y = X - X_H$ is the **constraint force** related to X and that the map $y : I \to TT^*Q$ given by

$$y(t) = Y(\Gamma(t)) = X(\Gamma(t)) - X_H(\Gamma(t)),$$

is the **constraint force** for the trajectory Γ.

Such a trajectory of $(H, \mathcal{P}, \mathcal{W})$ was called a *strong trajectory* in [15].

2.1.1. *Constraint forces and equations of motion.* Given a vector field $X \in \mathfrak{X}(T^*Q)$ satisfying (3), if Y is the constraint force related to X, it is clear that $Y \subset \mathcal{W}$, i.e.

$$Y(\alpha) \in \mathcal{W}_\alpha, \quad \forall \alpha \in T^*Q.$$

We require $\mathcal{W}$ to be vertical because, in the Hamiltonian formalism, external forces are given by vertical vectors.

Remark 2. Note that finding the trajectories of a triple $(H, \mathcal{P}, \mathcal{W})$ is the same as finding the vector fields $Y \in \mathfrak{X}(T^*Q)$ such that

$$Y \subset \mathcal{W} \quad \textit{and} \quad X_H + Y \subset \mathcal{P}. \tag{5}$$

In other words, there exist trajectories for $(H, \mathcal{P}, \mathcal{W})$ if and only if constraints defined by $\mathcal{P}$ can be implemented by a constraint force Y living inside $\mathcal{W}$.

[3] Recall that the Hamiltonian vector field X_F of a function $F : T^*Q \to \mathbb{R}$ is defined as $X_F = \omega^\sharp(dF)$, being ω the canonical symplectic 2-form of T^*Q. In usual coordinates,

$$X_F = \left(\frac{\partial F}{\partial p_1}, ..., \frac{\partial F}{\partial p_n}; -\frac{\partial F}{\partial q^1}, ..., -\frac{\partial F}{\partial q^n}\right). \tag{4}$$

Remark 3. Also note that, for a fixed solution X of (3), or equivalently, for a fixed Y satisfying (5), the corresponding trajectories Γ are the integral curves of $X = X_H + Y$, i.e. they satisfy

$$\Gamma'(t) = X_H(\Gamma(t)) + Y(\Gamma(t)). \tag{6}$$

In other words, they are the trajectories of an externally forced Hamiltonian system with Hamiltonian H and forcing Y.

After fixing an affine connection ∇ on Q, one can give an useful alternative description of constraint forces. The connection ∇ gives the diffeomorphism

$$\beta : TT^*Q \to T^*Q \oplus TQ \oplus T^*Q$$

such that given $v \in T_\alpha T^*Q$,

$$\beta(v) = \alpha \oplus \pi_*(v) \oplus \frac{Dw}{Dt}(0),$$

where $w : (-\epsilon, \epsilon) \to T^*Q$ is some curve passing through α at $t = 0$ with derivative v (see [9] and [14] for details) . Also, for each $\alpha \in T^*Q$ we have the linear isomorphism

$$\beta_\alpha : T_\alpha T^*Q \to T_{\pi(\alpha)}Q \oplus T^*_{\pi(\alpha)}Q,$$

such that $\beta(v) = \alpha \oplus \beta_\alpha(v)$. In terms of β, the vertical distribution at $\alpha \in T^*Q$ reads

$$\ker \pi_{*,\alpha} = \beta^{-1}\left(\alpha \oplus 0 \oplus T^*_{\pi(\alpha)}Q\right)$$

$$= \beta_\alpha^{-1}\left(0 \oplus T^*_{\pi(\alpha)}Q\right).$$

Thus, we can write

$$\mathcal{W}_\alpha = \beta^{-1}(\alpha \oplus 0 \oplus W_\alpha) = \beta_\alpha^{-1}(0 \oplus W_\alpha), \tag{7}$$

where W_α is a subspace of $T^*_{\pi(\alpha)}Q$. Moreover, if $Y \subset \mathcal{W}$, there exists a unique fiber preserving map

$$f : T^*Q \to T^*Q$$

such that

$$f \subset W, \qquad i.e. \quad f(\alpha) \in W_\alpha,$$

and

$$Y(\alpha) = \beta^{-1}(\alpha \oplus 0 \oplus f(\alpha)) = \beta_\alpha^{-1}(0 \oplus f(\alpha)), \tag{8}$$

$\forall \alpha \in T^*Q$. It can be shown that f is independent of the choice of the connection. In local coordinates, we have

$$Y = (0, ..., 0; f_1, ..., f_n), \tag{9}$$

where f_j is the j-th component of f in the given coordinate system and (see (4))

$$X = X_H + Y = \left(\frac{\partial H}{\partial p_1}, ..., \frac{\partial H}{\partial p_n}; -\frac{\partial H}{\partial q^1} + f_1, ..., -\frac{\partial H}{\partial q^n} + f_n\right). \tag{10}$$

Thus, constraint forces can be equivalently described as vertical vector fields Y on $\mathfrak{X}(T^*Q)$ or (by fixing an affine connection on Q) as fiber preserving maps f on T^*Q.

Also, using a connection, one can write Eq. (6), the *equations of motion*, in an alternative way which will be useful later. To do that, recall that given a function $F : T^*Q \to \mathbb{R}$, we have its *fiber derivative* and its *base derivative*

$$\mathbb{F}F : T^*Q \to TQ \quad \textit{and} \quad \mathbb{B}F : T^*Q \to T^*Q,$$

defined at $\alpha \in T^*Q$ by

$$\langle \mathbb{F}F(\alpha), \sigma \rangle = \left. \frac{dF(\alpha + s\,\sigma)}{ds} \right|_{s=0}, \quad \forall\, \sigma \in T^*Q, \tag{11}$$

and

$$\langle \mathbb{B}F(\alpha), X \rangle = \left. \frac{dF(w(s))}{ds} \right|_{s=0}, \quad \forall X \in TQ, \tag{12}$$

respectively, where $w : (-\epsilon, \epsilon) \to T^*Q$ is a horizontal curve such that $w(0) = \alpha$ and $(\pi \circ w)_* \left(d/dt|_0 \right) = X$. If $\Gamma : I \to T^*Q$ is a trajectory of $(H, \mathcal{P}, \mathcal{W})$, then Γ is a solution of (6) for some Y. Defining $q(t) = \pi(\Gamma(t))$, it can be shown that

$$\beta(\Gamma'(t)) = \Gamma(t) \oplus q'(t) \oplus \frac{D}{Dt}\Gamma(t)$$

and (see [14])

$$\beta(X_H(\alpha)) = \alpha \oplus \mathbb{F}H(\alpha) \oplus (-\mathbb{B}H(\alpha)), \quad \forall\, \alpha \in T^*Q. \tag{13}$$

Finally, applying β to both members of (6) and combining last two equations and (8), we get

$$q'(t) = \mathbb{F}H(\Gamma(t)), \qquad \frac{D}{Dt}\Gamma(t) = -\mathbb{B}H(\Gamma(t)) + f(\Gamma(t)). \tag{14}$$

2.1.2. *Affine constraints and* normality*: existence and uniqueness.* We say that $(H, \mathcal{P}, \mathcal{W})$ has **affine** (resp. **linear**) **constraints**, or that it is an **affine** (resp. **linear**) **HOCS**, if for each $\alpha \in T^*Q$ the subset $\mathcal{P}_\alpha \subset T_\alpha T^*Q$ is an affine (resp. linear) subspace. In such a case, for each $\alpha \in T^*Q$, we have

$$\mathcal{P}_\alpha = (\Delta_{\mathcal{P}})_\alpha + Z_{\mathcal{P}}(\alpha),$$

where $(\Delta_{\mathcal{P}})_\alpha \subset T_\alpha T^*Q$ is a linear subspace and $Z_{\mathcal{P}}(\alpha) \in T_\alpha T^*Q$ a vector. Note that $(\Delta_{\mathcal{P}})_\alpha$ defines a distribution $\Delta_{\mathcal{P}}$ on T^*Q (see Remark 1). In local coordinates, the function $\digamma$ in Eq. (2) is given by

$$\digamma(q, p, \dot{q}, \dot{p}) = \textstyle\sum_{i=1}^n \left[\varrho_i(q,p)\, \dot{q}^i + \vartheta^i(q,p)\, \dot{p}_i \right] - \gamma(q,p).$$

For affine HOCSs, assuming some regularity properties, we can give simple conditions that ensure existence and uniqueness of trajectories. Let $(\Delta_{\mathcal{P}})_\alpha$ be a C^∞-distribution and $Z_{\mathcal{P}}(\alpha)$ be a section $Z_{\mathcal{P}} \in \mathfrak{X}(T^*Q)$. Moreover, assume that $\Delta_{\mathcal{P}}$ and $\mathcal{W}$ are constant rank C^∞-distributions (see Remark 1 again) such that

$$TT^*Q = \Delta_{\mathcal{P}} \oplus \mathcal{W}. \tag{15}$$

These affine HOCSs were called **normal** in [14, 18]. For normal HOCSs, we have the following theorem (see [14] for a proof).

Theorem 2.2. *Let $(H, \mathcal{P}, \mathcal{W})$ be a normal HOCS. Denote by $\mathfrak{p} : TT^*Q \to TT^*Q$ the projection with range $\Delta_{\mathcal{P}}$ related to decomposition (15). The trajectories of $(H, \mathcal{P}, \mathcal{W})$ are the integral curves of*

$$X = \mathfrak{p} \circ (X_H - Z_{\mathcal{P}}) + Z_{\mathcal{P}},$$

and their related constraint forces are given by the field

$$Y = (id - \mathfrak{p}) \circ (Z_{\mathcal{P}} - X_H)\,. \tag{16}$$

By weakening the above conditions, we can still ensure uniqueness of trajectories. Suppose that for $(H, \mathcal{P}, \mathcal{W})$ there exists a dense subset $\mathfrak{A} \subset T^*Q$ such that

$$T_{\mathfrak{A}}T^*Q = \Delta_{\mathcal{P}}|_{\mathfrak{A}} \oplus \mathcal{W}|_{\mathfrak{A}}\,. \tag{17}$$

Then we have the following result.[4]

Theorem 2.3. *Given a triple $(H, \mathcal{P}, \mathcal{W})$ satisfying (17), if there exists a field X such that*

$$X \subset \mathcal{P} \quad \text{and} \quad X - X_H \subset \mathcal{W},$$

then this field is unique. In terms of constraint forces (see Remark 2), if there exists a field Y such that

$$Y \subset \mathcal{W} \quad \text{and} \quad X_H + Y \subset \mathcal{P},$$

then this field is unique.

(Again, see [14] for a proof.) Affine HOCSs satisfying (17) were called **almost normal** in [14]. In this paper, we will be using **almost normal** HOCSs.

2.2. **Closed-loop mechanical systems.** We first describe closed-loop mechanical systems (CLMS) in Hamiltonian form, and then show how to construct a CLMS from a HOCS such that

- they are equivalent dynamical systems, i.e., their trajectories are in bijection;
- the control law of the former coincides with the constraint force of the latter.

Consider an actuated Hamiltonian system on a manifold Q, defined by a Hamiltonian function $H : T^*Q \to \mathbb{R}$. In general, because of underactuation, the control forces are applied only along certain directions. For Hamiltonian systems, since these directions are described by vertical vectors, the control directions form a vertical distribution

$$\mathcal{W} \subset \ker \pi_* \subset TT^*Q$$

on T^*Q. And a *control law* will then be defined by a vector field $Y : T^*Q \to TT^*Q$ satisfying

$$Y(\alpha) \in \mathcal{W}_\alpha, \quad \forall \alpha \in T^*Q.$$

Note that, in general, the pair $(H, \mathcal{W})$ defines an *underactuated system*. The system is *fully actuated* only if $\mathcal{W} = \ker \pi_*$. Also, each pair (H, Y) defines a *closed-loop system*. If we want to emphasize the underactuated nature of the system, i.e. if $Y \subset \mathcal{W}$, we will denote it by $(H, Y)_{\mathcal{W}}$. On the other hand, we know that the trajectories of a forced Hamiltonian system, with Hamiltonian H and external force given by a vector field Y are the integral curves of $X_H + Y$. (Compare to Remark 3.) Based upon these observations, we consider the following definition.

Definition 2.4. By a **closed-loop mechanical system (CLMS)**, denoted by $(H, Y)_{\mathcal{W}}$, we mean a dynamical system defined by

1. a function H on a cotangent bundle T^*Q,
2. a vertical distribution $\mathcal{W}$ on T^*Q,
3. and a vector field Y such that $Y \subset \mathcal{W}$,

and trajectories given by the integral curves of $X_H + Y$.

[4] Note that, we are not asking $\mathcal{W}$ and $\Delta_{\mathcal{P}}$ to be (constant rank) C^∞-distributions.

Remark 4. Of course, a CLMS is nothing else but an externally forced Hamiltonian system with external force $Y \subset \mathcal{W}$.

Suppose we have an underactuated system $(H, \mathcal{W})$ on a manifold Q with Hamiltonian $H : T^*Q \to \mathbb{R}$ and a vertical distribution $\mathcal{W}$ on T^*Q which determines that actuation directions. As we noted earlier at the beginning of this section, in order to control such a system, we can impose on it a set of constraints to make the system follow a desired behavior and then obtain the control law as the related constraint force. Assume that

- we have constraints $\mathcal{P}$,
- the related triple $(H, \mathcal{P}, \mathcal{W})$ defines a HOCS with existence and uniqueness of trajectories.

Then, the trajectories of $(H, \mathcal{P}, \mathcal{W})$, which by hypothesis have the desired behavior, are the integral curves of a (unique) vector field $X = X_H + Y$, with $Y \subset \mathcal{W}$ defining the corresponding constraint forces (which in the normal case is given by Eq. (16)). Since such trajectories coincide with those of the CLMS $(H, Y)_{\mathcal{W}}$ (see Definition 2.4) and as a consequence $(H, Y)_{\mathcal{W}}$ has the desired behavior, Y determines the required control law. We are in effect controlling an underactuated system $(H, \mathcal{W})$ by applying a control signal Y defined as a constraint force implementing the constraints $\mathcal{P}$.

Remark 5. In other words, by thinking of constraints, we are replacing the problem of finding the control law for $(H, \mathcal{W})$ to that of finding the constraint force for $(H, \mathcal{P}, \mathcal{W})$ for a given $\mathcal{P}$.

In particular, we have constructed a CLMS $(H, Y)_{\mathcal{W}}$ from a HOCS $(H, \mathcal{P}, \mathcal{W})$ such that both dynamical systems are equivalent: their trajectories coincide. It was shown in Ref. [15] that every CLMS can be constructed in this way, i.e. every control law can be seen as the constraint force implementing a given set of constraints. This means that one does not loose generality by *thinking of constraints*.

3. **Lyapunov constraints.**

In this section we show how to use "constraints" for asymptotic stabilization. More precisely, given an underactuated system $(H, \mathcal{W})$, we will impose on it an appropriate affine constraint $\mathcal{P}$ in such a way that the related affine HOCS $(H, \mathcal{P}, \mathcal{W})$ is asymptotically stable at some point. Moreover, asymptotic stabilization property can be ensured by exhibiting a Lyapunov function. The problem to be solved is that of finding the corresponding constraint force.

3.1. **The constraint equation.**

Let Q be a smooth connected manifold and $V : T^*Q \to \mathbb{R}$ a non negative function such that:

P1: $V(\alpha) = 0$ implies $\alpha = \alpha_o \in T^*Q$.
P2: V is proper [i.e. if $I \subset \mathbb{R}$ is compact, then $V^{-1}(I)$ is compact too].

Note that, as a consequence, $dV(\alpha) = 0$ if $\alpha = \alpha_o$. Consider an underactuated system defined by a pair $(H, \mathcal{W})$. If we impose on the trajectories $\Gamma : I \subset \mathbb{R} \to T^*Q$ of the system the constraint

$$\langle dV(\Gamma(t)), \Gamma'(t) \rangle = -\mu(\Gamma(t)) \tag{18}$$

for some non negative function $\mu : T^*Q \to \mathbb{R}$ satisfying **P1**, it is clear that V would be a Lyapunov function for this system and global asymptotic stabilization at α_o follows (see [16, 26]).

Of course, we can consider weaker conditions on V and μ and still have (under additional hypothesis) useful results on stabilization, i.e. *Lyapunov-like theorems*. For instance, condition **P1** for μ says that $\mu^{-1}(0) = \{\alpha_o\}$. If $\mu^{-1}(0)$ were larger than the singleton $\{\alpha_o\}$, we could still ensure global stability of α_o if, in addition, the only invariant subset inside $\mu^{-1}(0)$ $\{\alpha_o\}$. In this case, *La'Salle invariance principle* would ensure global asymptotic stability for such a point. Here, $\mu^{-1}(0)$ is the *La'Salle surface*. If V does not satisfy condition **P2** , then one is only guaranteed local asymptotic stabilization. (For a proof of these results, see [16].)

This is why we consider constraints of the form (18) for general functions V and μ. We shall call them **Lyapunov constraints**, as in Refs. [14, 15]. In most cases, we assume the corresponding Lyapunov function is non negative (although non negativity for V is not always needed).

Let us study (18) in a more general situation. If we fix a point $\alpha \in T^*Q$, then Eq. (18) defines a linear nonhomogeneous equation

$$\langle dV(\alpha), v\rangle = -\mu(\alpha) \tag{19}$$

for vectors $v \in T_\alpha T^*Q$. Since Eq. (19) has no solutions for those α such that $dV(\alpha) = 0$ and $\mu(\alpha) \neq 0$, we require V and μ to satisfy

M1: $\mu(\alpha) = 0$ if $dV(\alpha) = 0$.

If $dV(\alpha) = 0$ then v can take any value in $T_\alpha T^*Q$. So, let us focus on the $dV(\alpha) \neq 0$ cases. The homogeneous solution of (19) is given by the subspace $\langle dV(\alpha)\rangle^o$, the annihilator of the covector $dV(\alpha)$. To find an expression for the set of all solutions, consider a Riemannian metric Φ on T^*Q. Then

$$z(\alpha) = \frac{-\mu(\alpha)\ \Phi^\sharp(dV(\alpha))}{\langle dV(\alpha), \Phi^\sharp(dV(\alpha))\rangle}$$

is a particular solution. Therefore, for each $\alpha \in T^*Q$, the space of solutions of (19) is an affine subset given by

$$\mathcal{P}_\alpha = (\Delta_\mathcal{P})_\alpha + Z_\mathcal{P}(\alpha), \tag{20}$$

with

$$(\Delta_\mathcal{P})_\alpha = \langle dV(\alpha)\rangle^o \tag{21}$$

and

$$Z_\mathcal{P}(\alpha) = \begin{cases} z(\alpha), & if \quad dV(\alpha) \neq 0, \\ 0, & otherwise. \end{cases} \tag{22}$$

Note that

$$\dim(\Delta_\mathcal{P})_\alpha = \begin{cases} 2n-1, & if \quad dV(\alpha) \neq 0, \\ 2n, & otherwise. \end{cases} \tag{23}$$

Thus, the underactuated system $(H, \mathcal{W})$ together with the Lyapunov constraint $\mathcal{P}$ defines an affine HOCS $(H, \mathcal{P}, \mathcal{W})$.

Since we are mainly interested in asymptotic stabilization of some point α_o, we will also assume that (beside non negativity)

M2: $\mu^{-1}(0) \subset T^*Q$ is a (closed) nowhere dense subset containing α_o, or equivalently, $\mu^{-1}(0)$ has empty interior and contains α_o.

Remark 6. Recall that a closed set is nowhere dense if and only if its complement is dense.

Assumption **M2** says that the set of points α such that $\mu(\alpha) = 0$ does not contain any open set. If the zero level set of μ contained an open set O, then V will be constant on O. In this case, asymptotic stability would be much harder to show.

Remark 7. Assumptions **M1** and **M2** imply that V and dV vanish at most along a (closed) nowhere dense subset. As a consequence, it follows from (23) that $\dim(\Delta_{\mathcal{P}})_\alpha = 2n-1$ along a (open) dense subset (see Remark 6).

3.2. **Space of actuators and constraint forces.** The triple $(H, \mathcal{P}, \mathcal{W})$ with $\mathcal{P}$ given by (20), (21) and (22) is not a normal HOCS in general. For instance, $\Delta_{\mathcal{P}}$ is not a constant rank C^∞-distribution (dV typically vanishes somewhere). Since $\dim(\Delta_{\mathcal{P}})_\alpha = 2n-1$ on an open dense subset (see Remark 7), in order to guarantee existence and uniqueness of trajectories and taking into account the results mentioned in §2.1.2, it is natural to require $\dim \mathcal{W}_\alpha = 1$ along an open dense subset. For simplicity we will assume that

W1: $\mathcal{W} = \langle \Omega \rangle$, with Ω a vertical vector field vanishing at most on a closed nowhere dense subset of T^*Q.

Informally, in this case we have a single actuation force at almost all the points of the phase space.

Remark 8. If we have more than one actuator, e.g. $\mathcal{W}$ is generated by k vector fields $\{\Omega_1, ..., \Omega_k\}$, we can study each 1-dimensional (sub)distribution $\langle \Omega \rangle$ of $\mathcal{W}$ given by

$$\Omega = \sum\nolimits_{i=1}^{k} f_i \, \Omega_i, \qquad f_i \in C^\infty(T^*Q). \tag{24}$$

In most applications of interest, the actuation directions are fixed and does not depend upon the current velocity or momenta. In other words, after fixing a connection on Q, the distribution $\mathcal{W}$ is given by (see (7))

$$\mathcal{W}_\alpha = \beta^{-1}\left(\alpha \oplus 0 \oplus W_{\pi(\alpha)}\right) = \beta_\alpha^{-1}\left(0 \oplus W_{\pi(\alpha)}\right),$$

where $W \subset T^*Q$ is a codistribution on Q. Given this, we will also assume that

W2: $\Omega(\alpha) = \beta_\alpha^{-1}(0 \oplus \xi(\pi(\alpha)))$, with $\xi : Q \to T^*Q$.

Note that ξ does not depend upon the choice of a connection. From assumption **W1**, it follows that ξ vanishes at most on a closed nowhere dense subset of Q.

To find the trajectories of $(H, \mathcal{P}, \mathcal{W})$, we must look for a vector field $Y \subset \mathcal{W}$ such that $X_H + Y \subset \mathcal{P}$ (see Remark 2), i.e.,

$$X_H(\alpha) + Y(\alpha) \in (\Delta_{\mathcal{P}})_\alpha + Z_{\mathcal{P}}(\alpha),$$

or equivalently

$$\langle dV(\alpha), X_H(\alpha) + Y(\alpha) \rangle = -\mu(\alpha),$$

for all $\alpha \in T^*Q$. In other words, using the identity

$$\langle dV(\alpha), X_H(\alpha) \rangle = \{V, H\}(\alpha),$$

where $\{\cdot,\cdot\}$ is the canonical Poisson bracket on T^*Q, we need to solve for a field $Y \subset \mathcal{W}$ satisfying

$$\langle dV(\alpha), Y(\alpha) \rangle = -\mu(\alpha) - \{V, H\}(\alpha), \quad \forall \alpha \in T^*Q. \tag{25}$$

As mentioned in Remark 5, the problem of finding the control law for the underactuated system $(H, \mathcal{W})$ is replaced by that of finding the constraint force for the HOCS $(H, \mathcal{P}, \mathcal{W})$. In the present case, this means solving Eq. (25) for $Y \subset \mathcal{W}$.

Each solution Y defines the control law of a CLMS $(H,Y)_{\mathcal{W}}$ with the desired behavior. Therefore, Eq. (25) can be seen as a *global method*, based on Lyapunov constraints, *for stabilization of underactuated systems*. We will dwelve into more details in the last two sections.

Consider assumption **W1**, i.e. $\mathcal{W} = \langle \Omega \rangle$. If we write $Y = \lambda\Omega$, with $\lambda \in C^{\infty}(T^*Q)$, we get

$$\lambda(\alpha) = -\frac{\mu(\alpha) + \{V,H\}(\alpha)}{\langle dV(\alpha), \Omega(\alpha)\rangle} \tag{26}$$

for all α such that $\langle dV(\alpha), \Omega(\alpha)\rangle \neq 0$. (Note that λ plays the role of a *Lagrange multiplier*.)

Remark 9. Again, if we have more than one actuator, e.g. $\mathcal{W} = \langle \Omega_1, ..., \Omega_k \rangle$, as in Remark 8, we can study Eq. (25) for each 1-dimensional distribution $\langle \Omega \rangle$ contained in $\mathcal{W}$ (see (24)). The conditions that Ω must satisfy in order to ensure existence of solutions of (25) in $\langle \Omega \rangle$ will be dealt with in a future work. In this paper, the vector field Ω is given in advance, so we will look for conditions on μ and V that ensure existence of solutions of Eq. (25) inside a given 1-dimensional distribution $\langle \Omega \rangle$.

If, in addition, we also use assumption **W2**, i.e. $\Omega(\alpha) = \beta_{\alpha}^{-1}(0 \oplus \xi(\pi(\alpha)))$, and note that

$$\langle dV(\alpha), \Omega(\alpha)\rangle = \langle \xi(\pi(\alpha)), \mathbb{F}V(\alpha)\rangle,$$

where $\mathbb{F}V : T^*Q \to TQ$ is the fiber derivative of V (see (11)), Eq. (26) translates to

$$\lambda(\alpha) = -\frac{\mu(\alpha) + \{V,H\}(\alpha)}{\langle \xi(\pi(\alpha)), \mathbb{F}V(\alpha)\rangle}. \tag{27}$$

Thus, existence and uniqueness of solutions of (25) is related to the zeros of the denominator in the r.h.s. of (26) and (27), i.e. it is related to the subset

$$\mathfrak{C} = \{\alpha \in T^*Q \ : \ \langle dV(\alpha), \Omega(\alpha)\rangle = 0\}, \tag{28}$$

or equivalently

$$\mathfrak{C} = \{\alpha \in T^*Q \ : \ \langle \xi(\pi(\alpha)), \mathbb{F}V(\alpha)\rangle = 0\}. \tag{29}$$

In other words, the main problem of (25) is that the function

$$\frac{\mu + \{V,H\}}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \tag{30}$$

may be not well-defined on the whole of T^*Q.

Remark 10. $\alpha \in \mathfrak{C}$ if and only if $\Omega(\alpha) \in \langle dV(\alpha)\rangle^o = (\Delta_{\mathcal{P}})_{\alpha}$, i.e.,

$$\mathcal{W}_{\alpha} \cap (\Delta_{\mathcal{P}})_{\alpha} \neq 0 \quad or \quad \Omega(\alpha) = 0,$$

or in other words, $\alpha \notin \mathfrak{C}$ if and only if (since $\dim(\Delta_{\mathcal{P}})_{\alpha} \geq 2n-1$)

$$\mathcal{W}_{\alpha} \cap (\Delta_{\mathcal{P}})_{\alpha} = 0, \quad dim(\Delta_{\mathcal{P}})_{\alpha} = 2n-1, \quad and \quad dim\mathcal{W}_{\alpha} = 1,$$

i.e.,

$$T_{\alpha}T^*Q = (\Delta_{\mathcal{P}})_{\alpha} \oplus \mathcal{W}_{\alpha}. \tag{31}$$

Therefore, if the complement of $\mathfrak{C}$ is dense, then $(H, \mathcal{P}, \mathcal{W})$ is almost normal.

A necessary condition to ensure existence is, of course (see (26) and (27)),

$$\mu(\alpha) + \{V, H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}. \tag{32}$$

In fact, this is needed for expression (30) to belong to $C^\infty(T^*Q)$. We will show in the next section that, for certain classes of function H, V and μ, Equation (32) is not only a necessary, but also a sufficient condition. For such functions V, since the subset $\mathfrak{C}$ is nowhere dense, it follows from Eq. (31) that $(H, \mathcal{P}, \mathcal{W})$ is almost normal (recall (17) and Remark 6) and as a consequence, uniqueness is also guaranteed.

3.3. **Controlled Lagrangians and Lyapunov constraints.** The method of *controlled Lagrangian* [5, 4] and its equivalent Hamiltonian counterpart IDA-PBC [21, 11] is an energy shaping based procedure to stabilize underactuated mechanical systems $(H, \mathcal{W})$ at a point $\alpha_o = (x, 0) \in T^*Q$. The method consists of constructing a controller

$$Y = Y^{cons} + Y^{diss}$$

with $Y^{cons}, Y^{diss} \subset \mathcal{W}$ such that:[5]

CL1: The dynamical system defined by $X_H + Y^{cons}$ is Lyapunov stable and locally exponentially stable at α_o.

CL2: $X_H + Y^{cons}$ is equivalent to a Hamiltonian vector field plus a gyroscopic term. More precisely, Y^{cons} is such that

$$X_H + Y^{cons} = \left(\mathbb{F}H^{-1} \circ \mathbb{F}H_c\right)_* (X_{H_c} + Y^{gyr})$$

for some Hamiltonian function $H_c : T^*Q \to \mathbb{R}$, called the *controlled Hamiltonian*, and some vertical vector field Y^{gyr} fulfilling

$$\langle dH_c, Y^{gyr} \rangle = 0.$$

CL3: H_c is positive definite w.r.t. α_o.

CL4: Y^{diss} satisfies

$$\left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* Y^{diss} \right\rangle \leq 0. \tag{33}$$

Defining

$$V = H_c \circ \mathbb{F}H_c^{-1} \circ \mathbb{F}H, \tag{34}$$

we have

$$\begin{aligned}
&\left\langle dV, X_H + Y^{cons} + Y^{diss} \right\rangle = \\
&= \left\langle dV, \left(\mathbb{F}H^{-1} \circ \mathbb{F}H_c\right)_* (X_{H_c} + Y^{gyr}) + Y^{diss} \right\rangle \\
&= \left\langle dV, \left(\mathbb{F}H^{-1} \circ \mathbb{F}H_c\right)_* (X_{H_c} + Y^{gyr}) \right\rangle + \left\langle dV, Y^{diss} \right\rangle \\
&= \langle dH_c, X_{H_c} + Y^{gyr} \rangle + \left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* Y^{diss} \right\rangle \\
&= \left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* Y^{diss} \right\rangle \leq 0.
\end{aligned}$$

Therefore, V is a Lyapunov function for the system defined by $X = X_H + Y$ and, defining

$$\mu(\alpha) = -\left\langle dH_c(\alpha), \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* Y^{diss}(\alpha) \right\rangle = -\left\langle dV(\alpha), Y^{diss}(\alpha) \right\rangle, \tag{35}$$

$\mu^{-1}(0)$ is its La'Salle surface. In particular,

$$\left\langle dV(\alpha), Y^{cons}(\alpha) + Y^{diss}(\alpha) \right\rangle = -\mu(\alpha) - \{V, H\}(\alpha),$$

[5] Here, we are actually describing the Hamiltonian side of the *controlled Lagrangian method*.

i.e. this method gives a solution $Y = Y^{cons} + Y^{diss}$ of Eq. (25) for V and μ given by (34) and (35), respectively. Thus, the method of Controlled Lagrangians (and that of Controlled Hamiltonians) can be seen as a special case of our method based on Lyapunov constraints, defined by Equation (25).

To end this subsection, let us see what happen when $\mathcal{W}$ is generated by a vector field Ω. In this case

$$Y^{cons} = \lambda^{cons}\,\Omega \quad \textit{and} \quad Y^{diss} = \lambda^{diss}\,\Omega,$$

and the method of controlled Lagrangians makes the following natural choice for the dissipative control:

$$\lambda^{diss} = -\left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* \Omega\right\rangle = -\left\langle dV, \Omega\right\rangle.$$

As a consequence,

$$\begin{aligned}\left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* Y^{diss}\right\rangle &= -\left\langle dH_c, \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* \Omega\right\rangle^2 \\ &= -\left(\lambda^{diss}\right)^2 \leq 0,\end{aligned}$$

and (33) follows. So, if we find Y^{cons} and H_c satisfying **CL1, CL2, CL3**, then **CL4** can be easily achieved. From (35), we also have

$$\mu(\alpha) = \left\langle dH_c(\alpha), \left(\mathbb{F}H_c^{-1} \circ \mathbb{F}H\right)_* \Omega(\alpha)\right\rangle^2 = \left\langle dV(\alpha), \Omega(\alpha)\right\rangle^2,$$

and accordingly

$$\mathfrak{C} = \left\{\alpha \in T^*Q \ : \ \left\langle dV(\alpha), \Omega(\alpha)\right\rangle = 0\right\} = \mu^{-1}(0). \tag{36}$$

That is to say, the La'Salle surface coincides with $\mathfrak{C}$.

3.4. **Hamiltonian systems with friction.** Suppose that, instead of a Hamiltonian system, we want to stabilize a Hamiltonian system subjected to friction forces. We assume that these dissipative forces are derived from a *Rayleigh dissipation function* $F : TQ \to \mathbb{R}$ [7, 24, 28] defined by

$$F(v) = \frac{1}{2} R(v, v), \tag{37}$$

where $R : TQ \times TQ \to \mathbb{R}$ is a positive-semidefinite tensor (i.e. $R(v, v) \geq 0$ for all $v \in TQ$). Note that F satisfies

$$F(v) = \frac{1}{2}\left\langle \mathbb{F}F(v), v\right\rangle, \quad \forall v \in TQ,$$

or equivalently,

$$\mathbb{F}F = R^\flat. \tag{38}$$

After fixing a connection, the trajectories of a Hamiltonian system with friction forces are given by (compare with (14))

$$q'(t) = \mathbb{F}H(\Gamma(t)) \quad \textit{and} \quad \frac{D}{Dt}\Gamma(t) = -\mathbb{B}H(\Gamma(t)) - \mathbb{F}F(q'(t)),$$

where $q(t) = \pi(\Gamma(t))$. Equivalently, they can be described as the integral curves of $X = X_H + Y_F$, with

$$Y_F(\alpha) = -\beta_\alpha^{-1}\left(0 \oplus \mathbb{F}F(\mathbb{F}H(\alpha))\right), \quad \forall \alpha \in T^*Q. \tag{39}$$

In canonical coordinates (see (9))

$$Y_F = \left(0, ..., 0; -R_{1j}\frac{\partial H}{\partial p_j}, ..., -R_{nj}\frac{\partial H}{\partial p_j}\right)$$

and (see (10))

$$X=\left(\frac{\partial H}{\partial p_1},...,\frac{\partial H}{\partial p_n};-\frac{\partial H}{\partial q^1}-R_{1j}\frac{\partial H}{\partial p_j},...,-\frac{\partial H}{\partial q^n}-R_{nj}\frac{\partial H}{\partial p_j}\right).$$

Suppose we have an underactuated system with friction defined by H, $\mathcal{W}$ and F (or R). It is clear that for functions V and μ the Lyapunov constraints can be implemented on the system if and only if there exists a vertical section $Y\subset\mathcal{W}$ satisfying

$$\langle dV(\alpha),X_H(\alpha)+Y_F(\alpha)+Y(\alpha)\rangle=-\mu(\alpha),\qquad\forall\alpha\in T^*Q.$$

Since (recall (37) and (38))

$$\begin{aligned}\langle dV(\alpha),Y_F(\alpha)\rangle&=-\langle\mathbb{F}F(\mathbb{F}H(\alpha)),\mathbb{F}V(\alpha)\rangle\\&=-R(\mathbb{F}H(\alpha),\mathbb{F}V(\alpha)),\end{aligned}$$

Equation (25) modifies to

$$\langle dV(\alpha),Y(\alpha)\rangle=-\mu(\alpha)-\{V,H\}(\alpha)+R(\mathbb{F}H(\alpha),\mathbb{F}V(\alpha)),\tag{40}$$

$\forall\alpha\in T^*Q$. Equation (40) is the analogue of (25) for systems with friction. In other words, it can be seen as a *method for stabilizing underactuated systems with friction*. If $\mathcal{W}$ is generated by a vector field Ω, following the same steps as in §3.2, we have $Y=\lambda\Omega$ with $\lambda\in C^\infty(T^*Q)$ given by

$$\lambda(\alpha)=-\frac{\mu(\alpha)+\{V,H\}(\alpha)-R(\mathbb{F}H(\alpha),\mathbb{F}V(\alpha))}{\langle dV(\alpha),\Omega(\alpha)\rangle}\tag{41}$$

for all $\alpha\notin\mathfrak{C}$. Here, $\mathfrak{C}$ is defined in §3.2 (see Eq. (28)). Of course, if we have more than one actuation, we can study (40) for each 1-dimensional distribution $\langle\Omega\rangle$ contained in $\mathcal{W}$ (see Remarks 8 and 9). We shall come back to Eq. (40) in §4.1.2 and §6.1.

4. ***Simple* Lyapunov constraints.** In this section, we study Eq. (25) for particular classes of functions H, V and μ. We consider H and V to be *simple Hamiltonian functions*, i.e., with "kinetic plus potential terms" form. More precisely, writing $q=\pi(\alpha)$,

$$H(\alpha)=\frac{1}{2}\rho\left(\rho^\sharp(\alpha),\rho^\sharp(\alpha)\right)+h(q)\tag{42}$$

and

$$V(\alpha)=\frac{1}{2}\phi\left(\phi^\sharp(\alpha),\phi^\sharp(\alpha)\right)+v(q),\tag{43}$$

where ρ and ϕ are Riemannian metrics on Q corresponding to the kinetic energy terms and h,v are functions on Q corresponding to the potential energy terms. It is easy to show that (recall (11))

$$\mathbb{F}H=\rho^\sharp\quad\textit{and}\quad\mathbb{F}V=\phi^\sharp.\tag{44}$$

For now, we will only assume μ to be non negative.

4.1. **The existence and uniqueness problem.** As in Section 3.2, assume that $\mathcal{W}$ satisfies conditions **W1** and **W2**. The following theorem gives a necessary condition for existence of solutions $Y \subset \mathcal{W}$ of Equation (25).

Theorem 4.1. *Given a pair $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ defined by a map $\xi : Q \to T^*Q$, if there exists a solution $Y \subset \mathcal{W}$ of (25), for V simple and μ non negative, then*

$$\mu(\alpha) = \{V, H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C};$$

where (see (29))

$$\mathfrak{C} = \{\alpha \in T^*Q \ : \ \langle \xi(q), \mathbb{F}V(\alpha)\rangle = 0\}. \tag{45}$$

The solution Y is unique and given by

$$Y(\alpha) = \beta_\alpha^{-1}\left(0 \oplus \frac{-\mu(\alpha) - \{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q)\right). \tag{46}$$

Proof. If there exists a solution $Y \subset \mathcal{W}$ of (25), we know from Section 3.2 that Y is given by

$$Y(\alpha) = \beta_\alpha^{-1}(0 \oplus \lambda(\alpha)\,\xi(q)),$$

with $\lambda \in C^\infty(T^*Q)$ and (see Eqs. (27) and (45))

$$\lambda(\alpha) = \frac{-\mu(\alpha) - \{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}, \qquad \forall \alpha \notin \mathfrak{C}, \tag{47}$$

i.e. for all α in the complement of $\mathfrak{C}$. This has two consequences. First, using $\mathbb{F}V(\alpha) = \phi^\sharp(\alpha)$, it follows that (see (45) again)

$$\mathfrak{C} = \left\{\alpha \in T^*Q \ : \ \left\langle \xi(q), \phi^\sharp(\alpha)\right\rangle = 0\right\}. \tag{48}$$

So, $\mathfrak{C}$ is a codistribution on an open dense subset of Q (since ξ vanishes at most on a closed nowhere dense subset of Q, by assumptions **W1** and **W2**) with $\mathfrak{C}_q = \langle \xi(q)\rangle^{\perp_\phi}$. Therefore, $\mathfrak{C}$ is a closed nowhere dense subset of T^*Q and, accordingly, its complement is a (open) dense subset (see Remark 6). Then, the values of λ along $\mathfrak{C}$ can be obtained from Eq. (47) by continuity and we have[6]

$$\frac{\mu + \{V, H\}}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q). \tag{49}$$

Therefore, if Y exists, it is uniquely given by the formula (46). This proves the last statement of the theorem. On the other hand, since $\lambda \in C^\infty$ the equation

$$\mu(\alpha) + \{V, H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}, \tag{50}$$

is satisfied. But $\mu(\alpha) \geq 0$ for all α, so $\{V, H\}(\alpha) \leq 0$ for all $\alpha \in \mathfrak{C}$. Since $\mathfrak{C}$ is a codistribution, if $\alpha \in \mathfrak{C}$ then $-\alpha \in \mathfrak{C}$. Therefore, we have

$$\{V, H\}(\alpha) \leq 0 \quad \textit{and} \quad \{V, H\}(-\alpha) \leq 0, \qquad \forall \alpha \in \mathfrak{C}. \tag{51}$$

Using a connection on Q, the Poisson bracket between V and H can be written as (see [14] for details)

$$\{V, H\}(\alpha) = \langle \mathbb{B}V(\alpha), \mathbb{F}H(\alpha)\rangle - \langle \mathbb{B}H(\alpha), \mathbb{F}V(\alpha)\rangle,$$

[6] With this notation we are saying that formula

$$\frac{\mu(\alpha) + \{V, H\}}{\langle \xi(\pi(\alpha)), \mathbb{F}V(\alpha)\rangle},$$

which is defined in a dense subset, can be uniquely extended to a C^∞ function on T^*Q.

and, using in addition the kinetic plus potential form of H and V (see (42), (43) and (44)),

$$\begin{aligned}\{V,H\}(\alpha) &= \left\langle \mathbb{B}V(\alpha), \rho^\sharp(\alpha)\right\rangle - \left\langle \mathbb{B}H(\alpha), \phi^\sharp(\alpha)\right\rangle \\ &= \left\langle \alpha, \rho^\sharp(\mathbb{B}V(\alpha)) - \phi^\sharp(\mathbb{B}H(\alpha))\right\rangle. \end{aligned} \tag{52}$$

Here, $\mathbb{B}H$ and $\mathbb{B}V$ denote the derivatives of H and V w.r.t. configuration variables, respectively, i.e., their base derivatives (see (12)). It can also be shown that

$$\phi^\sharp(\mathbb{B}H(\alpha)) = \Gamma_H(\alpha,\alpha) + \phi^\sharp[dh(q)] \tag{53}$$

and

$$\rho^\sharp(\mathbb{B}V(\alpha)) = \Gamma_V(\alpha,\alpha) + \rho^\sharp[dv(q)], \tag{54}$$

where $\Gamma_{V,H} : T^*Q \times_Q T^*Q \to TQ$ are bilinear maps (see Eqs. (72) and (73) for local expressions). So $\{V,H\}(\alpha)$ is a odd function, and Eq. (51) holds if and only if

$$\{V,H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}. \tag{55}$$

Using (55) and (50), we get

$$\mu(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}.$$

□

Remark 11. Note that, if there exists a solution of (25) for a simple V, then $\mathfrak{C} \subset \mu^{-1}(0)$ [i.e. $\mu(\alpha) = 0, \forall \alpha \in \mathfrak{C}$]. This implies, in particular, assumption **M1** of §3.1: $\mu(\alpha) = 0$ if $dV(\alpha) = 0$.

Remark 12. If we write $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ (see (8)), Theorem 4.1 gives

$$f(\alpha) = -\frac{\mu(\alpha) + \{V,H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q). \tag{56}$$

Remark 13. As remarked in Section 3.2, the uniqueness of constraint force Y can be seen as a consequence of the fact that the triple $(H, \mathcal{P}, \mathcal{W})$, under the conditions of Theorem 4.1, is almost normal. In fact, as seen in the proof of Theorem 4.1, $\mathfrak{C}$ is a closed nowhere dense subset. Therefore, its complement $\mathfrak{A}$ is a (open) dense subset [see Remarks 6 and 10, and Eq. (31)] such that

$$T_\alpha T^*Q = (\Delta_{\mathcal{P}})_\alpha \oplus \mathcal{W}_\alpha, \qquad \forall \alpha \in \mathfrak{A}.$$

This, according to Eq. (17), is exactly the condition that defines an almost normal HOCS.

We now derive a sufficient condition for existence. In addition to **W1** and **W2**, we will assume

W3: $\xi(q) \neq 0$ for all $q \in Q$, i.e. ξ is a nowhere vanishing map.

This assumption is satisfied by all the example applications we consider in this paper.

Theorem 4.2. *Given a pair $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ defined by nowhere vanishing map $\xi : Q \to T^*Q$, there exists a solution $Y \subset \mathcal{W}$ of (25), for V simple and μ non negative, if and only if*

$$\{V,H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}, \tag{57}$$

and[7]

$$\frac{\mu}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q). \tag{58}$$

Proof. If there exists a solution $Y \subset \mathcal{W}$ of (25), Theorem 4.1 says that the quotient

$$\frac{\mu(\alpha) + \{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle},$$

which is well-defined outside $\mathfrak{C}$, can be extended to all of T^*Q as a C^∞ function (see (49)). We also proved that this implies $\{V, H\}(\alpha) = 0$ for all α in $\mathfrak{C}$, which gives Eq. (57). To prove Eq. (58), we will show that if (57) holds true, then

$$\frac{\{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle} \tag{59}$$

can be extended to all of T^*Q as a C^∞ function. From (52), (53) and (54),

$$\{V, H\}(\alpha) = \langle \alpha, \Gamma(\alpha, \alpha)\rangle + \langle \alpha, g(q)\rangle,$$

where

$$\Gamma = \Gamma_V - \Gamma_H \quad \textit{and} \quad g = \rho^\sharp \circ dv \circ \pi - \phi^\sharp \circ dh \circ \pi. \tag{60}$$

Let us write $\alpha = \alpha_1 + \alpha_2$, with

$$\alpha_1 = h(\alpha)\,\xi(q) \qquad \textit{and} \qquad \alpha_2 = \alpha - h(\alpha)\,\xi(q),$$

where

$$h(\alpha) = \left\langle \xi(q), \phi^\sharp(\alpha)\right\rangle = \langle \xi(q), \mathbb{F}V(\alpha)\rangle.$$

Taking $\xi(q)$ such that

$$\phi(\xi(q), \xi(q)) = 1, \qquad \forall q \in Q,$$

which is possible since $\xi(q) \neq 0$ for all q (see assumption **W3**), it is clear that $\alpha_2 \in \mathfrak{C}$. Then

$$\{V, H\}(\alpha) = \langle \alpha_1, \Gamma(\alpha, \alpha)\rangle + \langle \alpha_1, g(\alpha)\rangle + \langle \alpha_2, \Gamma(\alpha_1, \alpha_1)\rangle$$

$$+ \langle \alpha_2, \Gamma(\alpha_1, \alpha_2)\rangle + \langle \alpha_2, \Gamma(\alpha_2, \alpha_1)\rangle,$$

where we have used (57), i.e.

$$\{V, H\}(\alpha_2) = \langle \alpha_2, \Gamma(\alpha_2, \alpha_2)\rangle + \langle \alpha_2, g(\alpha_2)\rangle = 0.$$

Recalling that $\alpha_1 = h(\alpha)\,\xi(q)$, we get

$$\begin{aligned}\frac{\{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle} = \frac{\{V, H\}(\alpha)}{h(\alpha)} &= \langle \xi(q), \Gamma(\alpha, \alpha)\rangle + \langle \xi(q), g(\alpha)\rangle + \\ &+ h(\alpha)\,\langle \alpha_2, \Gamma(\xi(q), \xi(q))\rangle + \langle \alpha_2, \Gamma(\xi(q), \alpha_2)\rangle \\ &+ \langle \alpha_2, \Gamma(\alpha_2, \xi(q))\rangle,\end{aligned} \tag{61}$$

which shows that equation (59) defines a C^∞ function. We now prove the converse.

If (57) holds, the calculations for the necessary conditions show that

$$\frac{\{V, H\}}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q).$$

[7]Note that condition (58) implies that $\mu(\alpha) = 0$, $\forall \alpha \in \mathfrak{C}$.

Since, by hypothesis, Eq. (58) also holds, it is clear that

$$\frac{\mu + \{V, H\}}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q).$$

So,

$$Y(\alpha) = \beta_\alpha^{-1}\left(0 \oplus \frac{-\mu(\alpha) - \{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q)\right)$$

gives a solution of (25) in $\mathcal{W}$. □

Under the conditions of Theorem 4.2, with μ satisfying (58), it is easy to show that Eq. (57) is equivalent to Eq. (32). In other words, if μ is non negative and satisfies (58), we have existence and uniqueness for (25) if and only if (32) holds, as we have anticipated at the end of Section 3.2. In this paper, we prefer to work with Eq. (57).

In order to satisfy (58), we can choose

$$\mu(\alpha) = \zeta\,\langle \xi(q), \mathbb{F}V(\alpha)\rangle^2, \quad \textit{with} \;\; \zeta > 0. \tag{62}$$

Remark 14. Note that, in general, existence implies that $\mathfrak{C} \subset \mu^{-1}(0)$, but in the last case $\mathfrak{C} = \mu^{-1}(0)$. This is what we have by using the method of controlled Lagrangians (see Eqs. (35) and (36)). As a consequence, assumptions **M1** and **M2** of Section 3.1 are automatically fulfilled for Controlled Lagrangians.

To summarize, for an underactuated system $(H, \mathcal{W})$ satisfying the conditions of Theorem 4.2 and choosing μ as in Eq. (62), the Theorem 4.2 says that, in order to implement a Lyapunov constraint defined by V and μ, it is necessary and sufficient for V to satisfy (57). In this case, the equation

$$\{V, H\}(\alpha) = 0, \qquad \forall \alpha \in \mathfrak{C}, \tag{63}$$

which defines a system of PDEs, can be seen as a global method for stabilization of mechanical systems *via* a simple Lyapunov function. We will call it **simple Lyapunov constraint based method**. In fact, as discussed at the beginning of §3.1, if we find a solution V of (57) satisfying property **P1** for some $\alpha_o \in T^*Q$, then the related constrained system (and its equivalent closed-loop system) is locally stable at α_o, and it is globally stable if **P2** also holds. For asymptotic stabilization, see §4.2.

4.1.1. *Coordinate expressions.* Let us study Eq. (57) in local coordinates. Using canonical coordinates systems for T^*Q, the functions H and V given by Eqs. (42) and (43) respectively, take the form

$$H(q,p) = \frac{1}{2}p^t\,\mathbb{H}(q)\,p + h(q) = \frac{1}{2}p_i^t\,\mathbb{H}^{ij}(q)\,p_j + h(q) \tag{64}$$

and

$$V(q,p) = \frac{1}{2}p^t\,\mathbb{V}(q)\,p + v(q) = \frac{1}{2}p_i^t\,\mathbb{V}^{ij}(q)\,p_j + v(q), \tag{65}$$

where $\mathbb{H}(q)$ and $\mathbb{V}(q)$ are positive definite matrices for all q. (Summation over repeated index convention is assumed from now on.) In these coordinates, Eq. (58) says that

$$\frac{\mu(q,p)}{p^t\,\mathbb{V}(q)\,\xi(q)} = \frac{\mu(q,p)}{p_i^t\,\mathbb{V}^{ij}(q)\,\xi_j(q)} \in C^\infty(T^*Q), \tag{66}$$

i.e., the quotient is well-defined for all (q,p). Here, ξ_k is the k-th component of ξ in the chosen coordinates.

Theorem 4.3. *Consider an underactuated system $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ satisfying **W1**, **W2** and **W3**. There exists a solution $Y \subset \mathcal{W}$ of (25), for V given by (65) and μ non negative and satisfying (66), if and only if (see Eqs. (64) and (65))*

$$\left(\frac{\partial \mathbb{V}^{ij}(q)}{\partial q^k}\,\mathbb{H}^{kl}(q) - \frac{\partial \mathbb{H}^{ij}(q)}{\partial q^k}\,\mathbb{V}^{kl}(q)\right) p_i\, p_j\, p_l = 0 \tag{67}$$

and

$$\left(\frac{\partial v(q)}{\partial q^k}\,\mathbb{H}^{kl}(q) - \frac{\partial h(q)}{\partial q^k}\,\mathbb{V}^{kl}(q)\right) p_l = 0 \tag{68}$$

for all (q,p) such that $p^t\,\mathbb{V}(q)\,\xi(q) = 0$. In such a case (recall (56)), Y is uniquely given by $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ with[8]

$$f_k(q,p) = -\frac{\mu(q,p) + \{V,H\}(q,p)}{p^t\,\mathbb{V}(q)\,\xi(q)}\,\xi_k(q)\,. \tag{69}$$

Proof. From last theorem, it is enough to show that (67) and (68) are the local expressions of (57). We shall first show that Eq. (57) splits into two independent equations. More precisely, recalling that

$$\begin{aligned}\{V,H\}(\alpha) = {} & \langle \alpha, \Gamma_V(\alpha,\alpha) - \Gamma_H(\alpha,\alpha)\rangle \\ & + \left\langle \alpha, \rho^\sharp[dv(q)] - \phi^\sharp[dh(q)]\right\rangle,\end{aligned}$$

(see the proof of Theorem 4.2) we shall show that $\{V,H\}(\alpha) = 0$ for all $\alpha \in \mathfrak{C}$ if and only if

$$\langle \alpha, \Gamma_V(\alpha,\alpha) - \Gamma_H(\alpha,\alpha)\rangle = 0 \tag{70}$$

and

$$\left\langle \alpha, \rho^\sharp[dv(q)] - \phi^\sharp[dh(q)]\right\rangle = 0 \tag{71}$$

for all $\alpha \in \mathfrak{C}$. To do that, given $\varsigma \in T^*Q$ and $\epsilon \in \mathbb{R}$, note that

$$\begin{aligned}\{V,H\}(\epsilon\,\varsigma) = {} & \epsilon^3\,\langle \varsigma, \Gamma_V(\varsigma,\varsigma) - \Gamma_H(\varsigma,\varsigma)\rangle \\ & + \epsilon\,\left\langle \varsigma, \rho^\sharp[dv(\pi(\varsigma))] - \phi^\sharp[dh(\pi(\varsigma))]\right\rangle,\end{aligned}$$

where we have used the bilinearity of the maps Γ_V and Γ_H, and the fact that $\pi(\epsilon\,\varsigma) = \pi(\varsigma)$. Fixing $\varsigma \in \mathfrak{C}$, since $\epsilon\,\varsigma \in \mathfrak{C}$ (recall that $\mathfrak{C}$ is a codistribution), Equation (57) says that the cubic polynomial $A_\varsigma\,\epsilon^3 + C_\varsigma\,\epsilon$, with

$$A_\varsigma = \langle \varsigma, \Gamma_V(\varsigma,\varsigma) - \Gamma_H(\varsigma,\varsigma)\rangle$$

and

$$C_\varsigma = \left\langle \varsigma, \rho^\sharp[dv(\pi(\varsigma))] - \phi^\sharp[dh(\pi(\varsigma))]\right\rangle,$$

must be identically zero. This is possible only if $A_\varsigma = C_\varsigma = 0$ and we see that (70) and (71) holds true. The maps $\Gamma_{V,H} : T^*Q \times_Q T^*Q \to TQ$ (see Eqs. (53) and (54)) are given by

$$[\Gamma_V(q,p,\widetilde{p})]^l = \frac{\partial \mathbb{V}^{ij}}{\partial q^k}\,\mathbb{H}^{kl}\,p_i\,\widetilde{p}_j \tag{72}$$

and

$$[\Gamma_H(q,p,\widetilde{p})]^l = \frac{\partial \mathbb{H}^{ij}}{\partial q^k}\,\mathbb{V}^{kl}\,p_i\,\widetilde{p}_j \tag{73}$$

in local coordinates. We now see that Eq. (70) modifies to (67). We leave it to the reader to verify that Eq. (71) is Eq. (68) in local coordinates. □

[8] f_k is the k-th component of f in the chosen coordinates.

4.1.2. *Systems with friction.* Lets consider again the case of systems with friction discussed in Section 3.4. Suppose we have an underactuated system $(H, \mathcal{W})$ subjected to Raleigh type friction forces given by R. Let us impose on it a Lyapunov constraint related to functions V and μ. We continue with the assumptions we have made so far. For instance, H and V are simple and given by (42) and (43), μ is non negative, and $\mathcal{W}$ satisfies **W1** and **W2** with the map $\xi : Q \to T^*Q$. We have seen in Section 3.4 that, if there exists $Y \subset \mathcal{W}$ implementing $\mathcal{P}$, i.e. solving (40), then $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ with (see (41))

$$f(\alpha) = -\frac{\mu(\alpha) + \{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q),$$

for all $\alpha \notin \mathfrak{C}$. Accordingly,

$$\mu(\alpha) + \{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)) = 0, \quad \forall \alpha \in \mathfrak{C}, \tag{74}$$

is a necessary condition for existence. In order to find a sufficient condition, we shall again assume that $\mathcal{W}$ satisfies **W3**.

We saw in Theorem 4.1 that $\mathfrak{C}$ is a codistribution such that $\mathfrak{C}_q = \langle \xi(q)\rangle^{\perp_\phi}$(see Eq. (48)), ϕ being the metric defining the kinetic term of V. The assumption **W3** ensures that orthogonal projection $\mathfrak{p} : T^*Q \to T^*Q$, w.r.t. ϕ and with range $\mathfrak{C}$, is a C^∞ function.

The following theorem is the analogue of Theorem 4.2 for systems with friction.

Theorem 4.4. *Suppose H and V are simple Hamiltonian functions, μ is non negative, and $\mathcal{W}$ satisfies **W1**-**W3** with the map $\xi : Q \to T^*Q$. Then, there exists a solution $Y \subset \mathcal{W}$ of (40), for a dissipation tensor R, if and only if*

$$\{V, H\}(\alpha) = 0, \quad \forall \alpha \in \mathfrak{C}, \tag{75}$$

and

$$\frac{\mu - R(\mathbb{F}H(\mathfrak{p}(\cdot)), \mathbb{F}V(\mathfrak{p}(\cdot)))}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q). \tag{76}$$

Such a solution is uniquely given by

$$f(\alpha) = -\frac{\mu(\alpha) + \{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q).$$

Proof. If a solution $Y \subset \mathcal{W}$ of (40) exists, then it can be shown, as in Theorem 4.1, that formula above can be extended to all of T^*Q by continuity in a unique way. (To do this, the fact that $\mathfrak{C}$ is nowhere dense is crucial.) In other words,

$$\frac{\mu + \{V, H\} - R(\mathbb{F}H(\cdot), \mathbb{F}V(\cdot))}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q). \tag{77}$$

This gives the expression for f. On the other hand, Eq. (77) implies (74) and, since $\mu \geq 0$,

$$\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)) \leq 0, \quad \forall \alpha \in \mathfrak{C}. \tag{78}$$

Now, from the proof of Theorem 4.3, given $\varsigma \in T^*Q$ and $\epsilon \in \mathbb{R}$,

$$\{V, H\}(\epsilon\varsigma) = A_\varsigma \epsilon^3 + C_\varsigma \epsilon,$$

with

$$A_\varsigma = \langle \varsigma, \Gamma_V(\varsigma, \varsigma) - \Gamma_H(\varsigma, \varsigma)\rangle$$

and

$$C_\varsigma = \left\langle \varsigma, \rho^\sharp[dv(\pi(\varsigma))] - \phi^\sharp[dh(\pi(\varsigma))]\right\rangle.$$

Also, from the linearity of $\mathbb{F}H = \rho^\sharp$ and $\mathbb{F}V = \phi^\sharp$ and the bilinearity of R,

$$-R\left(\mathbb{F}H\left(\epsilon\,\varsigma\right), \mathbb{F}V\left(\epsilon\,\varsigma\right)\right) = B_\varsigma\,\epsilon^2,$$

with

$$B_\varsigma = -R\left(\mathbb{F}H\left(\varsigma\right), \mathbb{F}V\left(\varsigma\right)\right).$$

Therefore, if $\varsigma \in \mathfrak{C}$, Eq. (78) implies that

$$A_\varsigma\,\epsilon^3 + B_\varsigma\,\epsilon^2 + C_\varsigma\,\epsilon \leq 0, \qquad \forall \epsilon \in \mathbb{R}.$$

But this is possible only if $A_\varsigma = C_\varsigma = 0$ and $B_\varsigma \leq 0$. The first two conditions give Eq. (75). This equation ensures that (see Theorem 4.2)

$$\frac{\{V, H\}}{\left\langle \xi\left(\pi\left(\cdot\right)\right), \mathbb{F}V\left(\cdot\right)\right\rangle} \in C^\infty\left(T^*Q\right).$$

If we show that

$$\frac{R\left(\mathbb{F}H\left(\mathfrak{p}\left(\cdot\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\cdot\right)\right)\right) - R\left(\mathbb{F}H\left(\cdot\right), \mathbb{F}V\left(\cdot\right)\right)}{\left\langle \xi\left(\pi\left(\cdot\right)\right), \mathbb{F}V\left(\cdot\right)\right\rangle} \in C^\infty\left(T^*Q\right), \tag{79}$$

then using (77) we see that (76) holds. Lets consider (79). As in Theorem 4.2, write $\alpha = \alpha_1 + \alpha_2$, with

$$\alpha_1 = h\left(\alpha\right)\,\xi\left(q\right) \qquad \textit{and} \qquad \alpha_2 = \alpha - h\left(\alpha\right)\,\xi\left(q\right),$$

where

$$h\left(\alpha\right) = \left\langle \xi\left(q\right), \phi^\sharp\left(\alpha\right)\right\rangle = \left\langle \xi\left(q\right), \mathbb{F}V\left(\alpha\right)\right\rangle.$$

Choosing $\xi\left(q\right)$ such that

$$\phi\left(\xi\left(q\right), \xi\left(q\right)\right) = 1, \qquad \forall q \in Q,$$

it is clear that $\alpha_2 \in \mathfrak{C}$. Then

$$R\left(\mathbb{F}H\left(\mathfrak{p}\left(\alpha\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\alpha\right)\right)\right) = R\left(\mathbb{F}H\left(\alpha_2\right), \mathbb{F}V\left(\alpha_2\right)\right) \tag{80}$$

and

$$\begin{aligned} R\left(\mathbb{F}H\left(\alpha\right), \mathbb{F}V\left(\alpha\right)\right) &= R\left(\mathbb{F}H\left(\alpha_1\right), \mathbb{F}V\left(\alpha_1\right)\right) + R\left(\mathbb{F}H\left(\alpha_1\right), \mathbb{F}V\left(\alpha_2\right)\right) \\ &\quad + R\left(\mathbb{F}H\left(\alpha_2\right), \mathbb{F}V\left(\alpha_2\right)\right) + R\left(\mathbb{F}H\left(\alpha_2\right), \mathbb{F}V\left(\alpha_2\right)\right) \\ &= h^2\left(\alpha\right)\; R\left(\mathbb{F}H\left(\xi\left(q\right)\right), \mathbb{F}V\left(\xi\left(q\right)\right)\right) \\ &\quad + h\left(\alpha\right)\; R\left(\mathbb{F}H\left(\xi\left(q\right)\right), \mathbb{F}V\left(\alpha_2\right)\right) \\ &\quad + h\left(\alpha\right)\; R\left(\mathbb{F}H\left(\alpha_2\right), \mathbb{F}V\left(\xi\left(q\right)\right)\right) \\ &\quad + R\left(\mathbb{F}H\left(\alpha_2\right), \mathbb{F}V\left(\alpha_2\right)\right). \end{aligned} \tag{81}$$

Substracting (80) from (81) and quotienting by $\left\langle \xi\left(q\right), \mathbb{F}V\left(\alpha\right)\right\rangle = h\left(\alpha\right)$, condition (79) follows. The converse of the theorem is immediate. □

In the proof of Theorem 4.4, we saw that existence of solutions of (40) implies that $B_\varsigma \leq 0$ for all $\varsigma \in \mathfrak{C}$, i.e.

$$R\left(\mathbb{F}H\left(\varsigma\right), \mathbb{F}V\left(\varsigma\right)\right) \geq 0, \qquad \forall \varsigma \in \mathfrak{C}.$$

Let us state that as a corollary.

Corollary 1. *Under the conditions of theorem above, if there exists a solution $Y \subset \mathcal{W}$ of (40), for a dissipation tensor R, then*

$$R\left(\mathbb{F}H\left(\alpha\right), \mathbb{F}V\left(\alpha\right)\right) \geq 0, \qquad \forall \alpha \in \mathfrak{C}, \tag{82}$$

or equivalently

$$R\left(\mathbb{F}H\left(\mathfrak{p}\left(\alpha\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\alpha\right)\right)\right) \geq 0, \qquad \forall \alpha \in T^*Q. \tag{83}$$

Remark 15. Another way of proving last corollary is by noting that condition (76) implies

$$\mu\left(\alpha\right) - R\left(\mathbb{F}H\left(\mathfrak{p}\left(\alpha\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\alpha\right)\right)\right) = 0, \qquad \forall \alpha \in \mathfrak{C}.$$

Then, since $\mu \geq 0$, Eqs. (82) and (83) immediately follow.

The following corollary is an alternative (and redundant) presentation of Theorem 4.4, but, we think, it is more useful when considering applications.

Corollary 2. *Under the conditions of theorem above, there exists a solution $Y \subset \mathcal{W}$ of (40), for V simple and*

$$\mu\left(\alpha\right) = \eta\left(\alpha\right) + R\left(\mathbb{F}H\left(\mathfrak{p}\left(\alpha\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\alpha\right)\right)\right),$$

with η such that

$$\frac{\eta}{\left\langle \xi\left(\pi\left(\cdot\right)\right), \mathbb{F}V\left(\cdot\right)\right\rangle} \in C^\infty\left(T^*Q\right), \tag{84}$$

if and only if

$$\left\{V, H\right\}\left(\alpha\right) = 0 \quad \textit{and} \quad R\left(\mathbb{F}H\left(\alpha\right), \mathbb{F}V\left(\alpha\right)\right) \geq 0, \qquad \forall \alpha \in \mathfrak{C}. \tag{85}$$

Remark 16. If (82) holds, in order to fulfill condition (84) for η (or equivalently, (76) for μ), we can take (compare to Eq. (62))

$$\mu\left(\alpha\right) = \zeta\ \left\langle \xi\left(q\right), \mathbb{F}V\left(\alpha\right)\right\rangle^2 + R\left(\mathbb{F}H\left(\mathfrak{p}\left(\alpha\right)\right), \mathbb{F}V\left(\mathfrak{p}\left(\alpha\right)\right)\right), \quad \textit{with} \ \ \zeta > 0. \tag{86}$$

Eq. (82) ensures that $\mu \geq 0$.

In summary, consider an underactuated system $(H, \mathcal{W})$ with a Raleigh dissipation tensor R satisfying the conditions of Theorem 4.4, and fix μ, for instance, as in Eq. (86). If we find V satisfying (85), we can implement a Lyapunov constraint, defined by V and μ, with a unique constraint force $Y \subset \mathcal{W}$. Moreover, as in Section 4.1, if V satisfies property **P1** for some $\alpha_o \in T^*Q$, then the related constrained system (and its equivalent closed-loop system) is locally stable at α_o; and it is globally stable if **P2** also holds. Thus, we can say that Equations (85) constitutes the **simple Lyapunov constraint based method for systems with friction**. The main problem with this method is solving condition

$$R\left(\mathbb{F}H\left(\alpha\right), \mathbb{F}V\left(\alpha\right)\right) \geq 0,$$

which in coordinates reads

$$R_{ij}\left(q\right)\ \mathbb{H}^{ik}\left(q\right)\ p_k\ \mathbb{V}^{jl}\left(q\right)\ p_l \geq 0. \tag{87}$$

This condition is equivalent to Eq. 40 of Ref. [29]. The problem with (87) is that it is not satisfied by most underactuated mechanical systems (even if it is restricted to $\mathfrak{C}$). For example, an inverted cart-pendulum (see [29]). We believe it is hard to avoid such a condition if we work with simple Lyapunov functions. This is our main motivation for considering a bigger class of functions V (see Section 6).

4.1.3. *Fully actuated systems.* In this subsection, we are going to discuss a completely different situation: that in which $\mathcal{W}$, instead of being generated by one vector field, coincides with the whole vertical distribution $\ker \pi_*$. In this case, of course, we can not ensure uniqueness of solutions for (25), but we can always ensure existence.

Theorem 4.5. *For every pair $(H, \mathcal{W})$, with H simple and $\mathcal{W} = \ker \pi_*$, and for every simple function V, if we take μ such that $\mu(\alpha) = \langle \alpha, \mathbb{F}V(\alpha)\rangle$, Eq. (25) has at least one solution $Y \subset \mathcal{W}$. Expressing H and V as in (42) and (43), respectively, one of these solutions is $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ with*

$$f(\alpha) = -\alpha - \phi^\flat \circ \rho^\sharp \circ \mathbb{B}V(\alpha) + \mathbb{B}H(\alpha). \tag{88}$$

Proof. For f given by (88), it follows that

$$\langle f(\alpha), \mathbb{F}V(\alpha)\rangle = -\left\langle \alpha + \phi^\flat \circ \rho^\sharp \circ \mathbb{B}V(\alpha) - \mathbb{B}H(\alpha), \phi^\sharp(\alpha)\right\rangle$$

$$= -\left\langle \alpha, \phi^\sharp(\alpha)\right\rangle - \left\langle \alpha, \rho^\sharp \circ \mathbb{B}V(\alpha) - \phi^\sharp \circ \mathbb{B}H(\alpha)\right\rangle.$$

Since (see (52))

$$\{V, H\}(\alpha) = \left\langle \alpha, \rho^\sharp(\mathbb{B}V(\alpha)) - \phi^\sharp(\mathbb{B}H(\alpha))\right\rangle,$$

then

$$\langle f(\alpha), \mathbb{F}V(\alpha)\rangle = -\left\langle \alpha, \phi^\sharp(\alpha)\right\rangle - \{V, H\}(\alpha).$$

Thus, taking $\mu(\alpha) = \left\langle \alpha, \phi^\sharp(\alpha)\right\rangle$, our statement follows. □

Remark 17. For a system with Raleigh dissipation function F, if

$$Y(\alpha) = \beta_\alpha^{-1}(0 \oplus g(\alpha))$$

is a solution of (25), it is easy to see that (recall Eq. (39))

$$f(\alpha) = g(\alpha) + \mathbb{F}F(\mathbb{F}H(\alpha))$$

defines a solution of (40).

4.2. **Stabilization and the La'Salle surface.** Let us consider again the problem of (asymptotic) stabilization. To do that, let us recall our main results. Fix an underactuated system $(H, \mathcal{W})$ on Q, with H simple and $\mathcal{W}$ satisfying assumptions **W1-W3**. Impose on the system the affine constraint $\mathcal{P}$ given by

$$\langle dV(\Gamma(t)), \Gamma'(t)\rangle = -\mu(\Gamma(t)),$$

where V is simple and μ is non negative and satisfies (58).

Remark 18. For systems with a dissipation tensor R, we require (76) instead of (58), and replace μ by $\mu - R(\mathbb{F}H(\cdot), \mathbb{F}V(\cdot))$ in the expressions of f and Y bellow.

From the results in the paper so far, we get that the following statements are equivalent:

- trajectories of the triple $(H, \mathcal{P}, \mathcal{W})$ exists and are unique,
- there exists a unique constraint force $Y \subset \mathcal{W}$ implementing $\mathcal{P}$,
- there exists a unique $Y \subset \mathcal{W}$ such that $X_H + Y \subset \mathcal{P}$,
- there exists a unique $Y \subset \mathcal{W}$ fulfilling (25),
- $\{V, H\}(\alpha) = 0$, $\forall \alpha \in \mathfrak{C}$ (recall (45)).

In any case, the trajectories Γ of the system are the integral curves of $X_H + Y$, with $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ and

$$f(\alpha) = -\frac{\mu(\alpha) + \{V, H\}(\alpha)}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q). \tag{89}$$

Moreover, they satisfy

$$\frac{dV(\Gamma(t))}{dt} \leq 0,$$

and

$$\frac{dV(\Gamma(t))}{dt} = 0 \quad \Longleftrightarrow \quad \Gamma(t) \in \mu^{-1}(0).$$

In this situation one says that V is a Lyapunov function for the system defined by $X_H + Y$, and $\mu^{-1}(0)$ is its La'Salle surface.

Remark 19. Note that, from (58), $\mathfrak{C} \subset \mu^{-1}(0)$. Thus, if a forced Hamiltonian system, defined by a simple Hamiltonian and with one position-dependent actuator, has a simple Lyapunov function V, the related La'Salle surface $\mu^{-1}(0)$ will always be larger than a single point.

Suppose there exists a point $\alpha_o \in T^*Q$ such that V is positive definite w.r.t. α_o, i.e. V is non negative and $V(\alpha) = 0$ only for $\alpha = \alpha_o$ (see property **P1** of Section 3.1). In order to ensure asymptotic stability, taking into account Remark 19, the *La'Salle invariance principle* (see Ref. [16]) tells us that we need to study the invariant submanifolds of $\mu^{-1}(0)$. To do that, we can apply the following algorithm. Given a pair $(X, \mathfrak{M})$, where $\mathfrak{M}$ is a submanifold of T^*Q and $X \in \mathfrak{X}(T^*Q)$, let us define

$$\mathfrak{M}_k = \{\alpha \in \mathfrak{M}_{k-1} : X(\alpha) \in T\mathfrak{M}_{k-1}\}, \qquad for \;\; k \geq 1, \tag{90}$$

being $\mathfrak{M}_0 = \mathfrak{M}$, and assume that each $\mathfrak{M}_k$ is a submanifold. Note that each $\mathfrak{M}_k$ represents the initial conditions inside $\mathfrak{M}_{k-1}$ whose trajectories, given by integral curves of X, remain inside $\mathfrak{M}_{k-1}$. If the process stops for some k, we denote $\mathfrak{M}_\infty$ the last submanifold. It is clear that $\mathfrak{M}_\infty$ is an invariant submanifold for the vector field X. Applying this construction to the pair $(X, \mathfrak{M})$, with $X = X_H + Y$ and $\mathfrak{M} = \mu^{-1}(0)$, according to the La'Salle invariance principle, if $\mathfrak{M}_\infty = \{\alpha_o\}$, then α_o is an asymptotically stable point of the system. In addition, if Q is connected and V is proper (see property **P2** of Section 3.1), α_o is globally asymptotically stable. Thus, we can see condition (57) and algorithm (90) as a **simple Lyapunov constraint based method for (global) asymptotic stabilization**. In the case of systems with friction, we just replace (57) by (85).

Remark 20. If α_o is an equilibrium point for X, it is easy to show that $\alpha_o = (x, 0)$, with $x \in Q$. Accordingly, $\alpha_o \in \mathfrak{C} \subset \mathfrak{M}$ (because $\mathfrak{C}$ is a codistribution) and moreover, $\alpha_o \in \mathfrak{M}_k$ for all $k \geq 0$ (since $X(\alpha_o) = 0 \in T\mathfrak{M}_{k-1}$ for all $k \geq 1$).

In the following, we present some results about submanifolds $\mathfrak{M}_k$. For simplicity, lets assume

$$\mu(\alpha) = \kappa \, \langle \xi(q), \mathbb{F}V(\alpha)\rangle^2, \quad with \;\; \kappa > 0. \tag{91}$$

According to Remark 14, $\mu^{-1}(0) = \mathfrak{C}$, so $\mathfrak{M}_0 = \mathfrak{C}$.

Theorem 4.6. *Let $(H, \mathcal{W})$ be an underactuated system on Q, with H simple and $\mathcal{W}$ fulfilling **W1-W3** with map $\xi : Q \to T^*Q$. Let V be simple and such that*

$$\{V, H\}(\alpha) = 0, \quad \forall \alpha \in \mathfrak{C},$$

and suppose μ is given by (91). Then, after choosing a connection ∇ on Q, the submanifold $\mathfrak{M}_1$ is given by the points $\alpha \in \mathfrak{C}$ such that

$$\langle \mathbb{B}H(\alpha) - f(\alpha), \mathbb{F}V[\xi(q)]\rangle = \left\langle \alpha, \nabla_{\mathbb{F}H(\alpha)}[\mathbb{F}V \circ \xi](q)\right\rangle, \tag{92}$$

where f is given by (89).

Proof. Consider a curve $\alpha(t)$ satisfying

$$\left\langle \alpha(t), \phi^\sharp[\xi(q(t))]\right\rangle = 0, \tag{93}$$

where $q(t) = \pi(\alpha(t))$. Writing $\vartheta = \phi^\sharp \circ \xi$, differentiating equation (93) and using a connection on Q, we have

$$\begin{aligned} 0 &= \left\langle \alpha(t), \frac{D}{Dt}\vartheta(q(t))\right\rangle + \left\langle \frac{D}{Dt}\alpha(t), \vartheta(q(t))\right\rangle \\ &= \left\langle \alpha(t), \nabla_{q'(t)}\vartheta(q(t))\right\rangle + \left\langle \frac{D}{Dt}\alpha(t), \vartheta(q(t))\right\rangle. \end{aligned} \tag{94}$$

If $\alpha(t)$ is, in addition, an integral curve of $X_H + Y$, i.e.

$$\alpha'(t) = X_H(\alpha(t)) + Y(\alpha(t)),$$

then (see Eq. (14)),

$$q'(t) = \mathbb{F}H(\alpha(t)), \qquad \frac{D}{Dt}\alpha(t) = -\mathbb{B}H(\alpha(t)) + f(\alpha(t)). \tag{95}$$

Combining (94) and (95), we have the required result. □

The following corollary will be used later, in a particular example.

Corollary 3. *Under the conditions of Theorem 4.6, if Q has a trivializable tangent bundle, the map ξ is constant and the metrics ρ and ϕ, corresponding to H and V respectively are also constant, then*

$$\mathfrak{M}_1 = T^*_{Q_1}Q \cap \mathfrak{C}, \quad \text{with} \quad Q_1 = \left\{q \in Q \ : \ \left\langle dv(q), \rho^\sharp(\xi)\right\rangle = 0\right\}. \tag{96}$$

Here, ξ denotes the constant value of $\xi : Q \to T^*Q$.

Proof. Since Q has a trivializable tangent bundle, we can consider a trivial connection to express Eq. (92). Also, since ξ is constant, (92) reduces to

$$\left\langle \mathbb{B}H(\alpha), \phi^\sharp(\xi)\right\rangle - \left\langle f(\alpha), \phi^\sharp(\xi)\right\rangle = 0, \qquad \forall \alpha \in \mathfrak{C}, \tag{97}$$

where we are using that $\mathbb{F}V = \phi^\sharp$. When the metrics corresponding to H and V are constant, and the connection is trivial, it is easy to show that

$$\mathbb{B}H(\alpha) = dh(q), \qquad \mathbb{B}V(\alpha) = dv(q), \tag{98}$$

and (see (72) and (73))

$$\Gamma_V = \Gamma_H = 0. \tag{99}$$

Therefore, from Eqs. (60), (61) and (99),

$$\frac{\{V, H\}(\alpha)}{\langle \xi, \mathbb{F}V(\alpha)\rangle} = \langle \xi, g(\alpha)\rangle = \left\langle \xi, \rho^\sharp(dv(q)) - \phi^\sharp(dh(q))\right\rangle.$$

Then, using (89) and (91),

$$\begin{aligned} f(\alpha) &= -\frac{\mu(\alpha) + \{V, H\}(\alpha)}{\langle \xi, \mathbb{F}V(\alpha)\rangle}\xi \\ &= -\left[\kappa\langle \xi, \mathbb{F}V(\alpha)\rangle + \left\langle \xi, \rho^\sharp(dv(q)) - \phi^\sharp(dh(q))\right\rangle\right]\xi \end{aligned}$$

in general and

$$f(\alpha) = -\left\langle \xi, \rho^\sharp(dv(q)) - \phi^\sharp(dh(q)) \right\rangle \xi$$

along $\mathfrak{C}$. Also, since $\mathbb{B}H(\alpha) = dh(q)$ (see (98)),

$$\begin{aligned}
&\left\langle \mathbb{B}H(\alpha), \phi^\sharp(\xi) \right\rangle - \left\langle f(\alpha), \phi^\sharp(\xi) \right\rangle = \\
&= \left\langle dh(q), \phi^\sharp(\xi) \right\rangle + \left\langle \xi, \rho^\sharp(dv(q)) - \phi^\sharp(dh(q)) \right\rangle \left\langle \xi, \phi^\sharp(\xi) \right\rangle \\
&= \left\langle \xi, \rho^\sharp(dv(q)) \right\rangle \left\langle \xi, \phi^\sharp(\xi) \right\rangle.
\end{aligned}$$

Therefore, Eq. (97) modifies to

$$\left\langle \xi, \rho^\sharp(dv(q)) \right\rangle = 0,$$

as required. □

Now, lets consider a fully actuated system (see Section 4.1.3). Assume again that H and V are simple Hamiltonian functions. Theorem 4.5 tells us that if we choose $\mu(\alpha) = \langle \alpha, \mathbb{F}V(\alpha) \rangle$, then the following statements are equivalent:

- the triple $(H, \mathcal{P}, \ker \pi_*)$ has existence of trajectories,
- there exists a constraint force Y implementing $\mathcal{P}$,
- there exists $Y \subset \ker \pi_*$ such that $X_H + Y \subset \mathcal{P}$,
- there exists $Y \subset \ker \pi_*$ satisfying (25),

where $\mathcal{P}$ is the Lyapunov constraint defined by V and μ. The trajectories Γ of the system are the integral curves of $X = X_H + Y$, with $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ and

$$f(\alpha) = -\alpha - \phi^\flat \circ \rho^\sharp \circ \mathbb{B}V(\alpha) + \mathbb{B}H(\alpha). \tag{100}$$

Again, V is a Lyapunov function for the system defined by X, and $\mu^{-1}(0) = \mathbb{O}_Q$ (the null codistribution) is its La'Salle surface. If for a point $\alpha_o = (x, 0)$, with $x \in Q$, i.e. $\alpha_o \in \mathbb{O}_Q$, we choose V to be positive definite w.r.t. α_o, which is equivalent to requiring T to be positive definite w.r.t. x and if

$$dv(q) = 0 \quad \text{only if} \quad q = x, \tag{101}$$

then the following theorem shows that the system defined by X is globally asymptotically stable at α_o.

Theorem 4.7. *Algorithm (90) applied to $\mathfrak{M} = \mathbb{O}_Q$ and $X = X_H + Y$, with Y given by (100) and v satisfying (101), gives*

$$\mathfrak{M}_1 = \mathfrak{M}_\infty = \{\alpha_o\}.$$

Proof. Fixing a connection on Q, we have for X that (see (8) and (13))

$$\begin{aligned}
\beta(X(\alpha)) &= \alpha \oplus \mathbb{F}H(\alpha) \oplus (-\mathbb{B}H(\alpha) + f(\alpha)) \\
&= \alpha \oplus \mathbb{F}H(\alpha) \oplus \left(-\alpha - \phi^\flat \circ \rho^\sharp \circ \mathbb{B}V(\alpha)\right)
\end{aligned}$$

and, if $\alpha \in \mathbb{O}_Q$, i.e. $\alpha = (q, 0)$ with $q \in Q$,

$$\beta(X(\alpha)) = 0 \oplus 0 \oplus \left(-\phi^\flat \circ \rho^\sharp \circ dv\right)(q).$$

Since

$$\beta(T\mathbb{O}_Q) = 0 \oplus TQ \oplus 0,$$

then, for $\mathfrak{M} = \mathbb{O}_Q$, submanifold $\mathfrak{M}_1 = \{\alpha \in \mathfrak{M} : X(\alpha) \in T\mathfrak{M}\}$ is given by the points $q \in Q$ such that $dv(q) = 0$. But this only happens at $q = x$ (recall (101)), so

$$\mathfrak{M}_1 = \mathfrak{M}_\infty = \{(x, 0)\}.$$

□

Locally, we can always find a positive definite function v w.r.t. any point x, and satisfying (101). Thus, last theorem implies the following fact: *every fully actuated Hamiltonian system on Q can always be (locally) asymptotic stabilized at any point of $\mathbb{O}_Q$.*

5. **Some examples.**

5.1. **Inertia wheel pendulum.**

Consider a Hamiltonian system on $Q = S^1 \times S^1$, with Hamiltonian function

$$H(\theta, \psi, p_\theta, p_\psi) = \frac{1}{2}\,(p_\theta, p_\psi) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} p_\theta \\ p_\psi \end{pmatrix} + M\,(1 + \cos\theta)\,,$$

where M, a, b and c are constants and $a, b, M, ac - b^2$ are each strictly greater than zero. Following the notation of previous sections, we identify (see Eq. (64))

$$\rho^\sharp = \mathbb{H} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

For the choice $\xi(\pi(\alpha)) = \xi = (0, 1)$, the underactuated system corresponding to H and ξ is called *the inertia wheel pendulum* provided $a = b$ and $c > a$.

Let us look for a simple function V satisfying (57), or, in coordinates, Eqs. (67) and (68). We shall focus on V of the form

$$V(\theta, \psi, p_\theta, p_\psi) = \frac{1}{2}\,(p_\theta, p_\psi) \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix} \begin{pmatrix} p_\theta \\ p_\psi \end{pmatrix} + v(\theta, \psi)\,,$$

where f, g and $\hbar$ are constants satisfying $f\hbar - g^2 > 0$ and $\hbar > 0$. Therefore, (see Eq. (65)),

$$\phi^\sharp = \mathbb{V} = \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}.$$

Remark 21. Since $\mathbb{H}$ and $\mathbb{V}$ are constants, Eq. (67) holds true trivially and Eq. (57) reduces to Eq. (68).

We have

$$\begin{aligned} \{V, H\} &= \frac{\partial V}{\partial \theta}\frac{\partial H}{\partial p_\theta} + \frac{\partial V}{\partial \psi}\frac{\partial H}{\partial p_\psi} - \frac{\partial H}{\partial \theta}\frac{\partial V}{\partial p_\theta} - \frac{\partial H}{\partial \psi}\frac{\partial V}{\partial p_\psi} \\ &= \frac{\partial v}{\partial \theta}\,(a\,p_\theta + b\,p_\psi) + \frac{\partial v}{\partial \psi}\,(b\,p_\theta + c\,p_\psi) \\ &\quad + M\,\sin\theta\,(f\,p_\theta + g\,p_\psi)\,. \end{aligned} \tag{102}$$

and

$$\left\langle \xi, \phi^\sharp(\alpha) \right\rangle = g\,p_\theta + \hbar\,p_\psi, \tag{103}$$

Therefore, the codistribution $\mathfrak{C}$ is given by pairs (p_θ, p_ψ) such that $g\,p_\theta + \hbar\,p_\psi = 0$. Thus, using (57), (102) and (103), we get

$$\frac{\partial v}{\partial \theta}\,(-a\,\hbar + b\,g) + \frac{\partial v}{\partial \psi}\,(-b\,\hbar + c\,g) + M\,\sin\theta\,\left(-f\,\hbar + g^2\right) = 0. \tag{104}$$

Therefore, (57) gives rise to a first order linear nonhomogeneous PDE for v.

The equilibrim point of interest to us is $\alpha_o = (0, 0, 0, 0)$. We want to find a solution v of (104) which is positive definite at $(\theta, \psi) = (0, 0)$. For a particular solution, we choose

$$v_p(\theta, \psi) = N\,(1 - \cos\theta)\,,$$

with

$$N = M \frac{f\,\hbar - g^2}{-a\,\hbar + b\,g},$$

and require that $N > 0$.

Remark 22. The condition $N > 0$ can not be satisfied if $\rho^\sharp = \phi^\sharp$. In fact, in such a case $N = -M$.

The related homogeneous equation

$$\frac{\partial v}{\partial \theta}\,(-a\,\hbar + b\,g) + \frac{\partial v}{\partial \psi}\,(-b\,\hbar + c\,g) = 0$$

has a general solution of the form $K\,(\psi - n\,\theta)$, with

$$n = \frac{-b\,\hbar + c\,g}{-a\,\hbar + b\,g}. \tag{105}$$

We can choose n to be a natural number and choose $K\,(y) = \chi\,(1 - \cos\,y)$. Then,

$$v_h\,(\theta, \psi) = \chi\,(1 - \cos\,(\psi - n\,\theta))$$

is a solution of the homogeneous equation and it is well-defined on the whole torus $S^1 \times S^1$. Choosing $\chi > 0$, we have that

$$v\,(\theta, \psi) = v_h\,(\theta, \psi) + v_p\,(\theta, \psi) = \chi\,(1 - \cos\,(\psi - n\,\theta)) + N\,(1 - \cos\theta)$$

is positive and only vanishes at $(\theta, \psi) = (0, 0)$ as required. In summary, the requirements on f, $\hbar$ and g are

$$f\hbar - g^2 > 0, \qquad \hbar > 0, \quad -a\,\hbar + b\,g > 0 \quad \textit{and} \quad \frac{-b\,\hbar + c\,g}{-a\,\hbar + b\,g} \in \mathbb{N}.$$

(Note that, to have a solution of last equations, b must be non zero.) Then, f, $\hbar$ and g are given by

$$\hbar = d\,\frac{n\,b - c}{a\,c - b^2}, \quad g = d\,\frac{n\,a - b}{a\,c - b^2} \quad \textit{and} \quad f = \frac{g^2 + e}{\hbar},$$

with $d, e > 0$, and n some natural number such that $n\,b > c$. In particular, this implies that $n\,a > b$ if $b > 0$ and $n\,a < b$ if $b < 0$. For fixed values of f, g and $\hbar$, we have defined a function

$$\begin{aligned} V\,(\theta, \psi, p_\theta, p_\psi) = \frac{1}{2}\,(p_\theta, p_\psi) \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix} \begin{pmatrix} p_\theta \\ p_\psi \end{pmatrix} + N\,(1 - \cos\theta) \\ + \chi\,(1 - \cos\,(\psi - n\,\theta)) \end{aligned}$$

that satisfies **P1** and **P2**.

To write down the control law, let us calculate

$$\frac{\{V, H\}}{\langle \xi, \phi^\sharp\,(\cdot)\rangle} = \frac{\{V, H\}}{g\,p_\theta + \hbar\,p_\psi}.$$

Replacing p_ψ by $p_\psi + \frac{g}{\hbar}\, p_\theta - \frac{g}{\hbar}\, p_\theta$ in Eq. (102), we have

$$\begin{aligned}\frac{\{V,H\}}{g\, p_\theta + \hbar\, p_\psi} &= \left[\frac{\partial v}{\partial \theta}\left(a - \frac{g\, b}{\hbar}\right) + \frac{\partial v}{\partial \psi}\left(b - \frac{g\, c}{\hbar}\right)\right] \frac{p_\theta}{g\, p_\theta + \hbar\, p_\psi} \\ &\quad + M\, \sin\theta \left(f - \frac{g^2}{\hbar}\right) \frac{p_\theta}{g\, p_\theta + \hbar\, p_\psi} \\ &\quad + \left[\frac{\partial v}{\partial \theta}\, b + \frac{\partial v}{\partial \psi}\, c + M\, \sin\theta\, g\right] \frac{1}{\hbar} \\ &= \frac{\partial v}{\partial \theta}\, \frac{b}{\hbar} + \frac{\partial v}{\partial \psi}\, \frac{c}{\hbar} + M\, \sin\theta\, \frac{g}{\hbar}.\end{aligned}$$

Since

$$\frac{\partial v}{\partial \theta} = N \sin\theta - \chi\, n\, \sin(\psi - n\,\theta)$$

and

$$\frac{\partial v}{\partial \psi} = \chi\, \sin(\psi - n\,\theta)\,,$$

it follows that

$$\frac{\{V,H\}}{g\, p_\theta + \hbar\, p_\psi} = \frac{M\, (f\, b - g\, a)}{-a\,\hbar + b\, g}\, \sin\theta + \chi\, \frac{b^2 - a\, c}{-a\,\hbar + b\, g}\, \sin(\psi - n\,\theta)\,,$$

and in terms of n, d and e,

$$\begin{aligned}\frac{\{V,H\}}{g\, p_\theta + \hbar\, p_\psi} &= -\frac{M\, \left[(n\, a - b)\, d^2 + e\, \left(a\, c - b^2\right)^2\, b\right]}{(n\, b - c)\, d^2}\, \sin\theta \\ &\quad + \chi\, \frac{b^2 - a\, c}{d}\, \sin(\psi - n\,\theta)\,.\end{aligned}$$

Now, choosing

$$\mu(\theta, \psi, p_\theta, p_\psi) = \kappa\, (g\, p_\theta + \hbar\, p_\psi)^2\,, \qquad \kappa > 0,$$

which is non negative and makes

$$\frac{\mu}{\langle \xi(\pi(\cdot)), \phi^\sharp(\cdot)\rangle}$$

a C^∞ function, the corresponding control law (see (56) and (69)) will be a function $f = (f_\theta, f_\psi)$ with $f_\theta = 0$ and

$$\begin{aligned}f_\psi(\theta, \psi, p_\theta, p_\psi) &= -\zeta\, \left(d\, \frac{(n\, a - b)\, p_\theta + (n\, b - c)\, p_\psi}{a\, c - b^2}\right) + \\ &\quad + \frac{M\, \left[(n\, a - b)\, d^2 + e\, \left(a\, c - b^2\right)^2\, b\right]}{(n\, b - c)\, d^2}\, \sin\theta \\ &\quad + \chi\, \frac{a\, c - b^2}{d}\, \sin(\psi - n\,\theta)\,.\end{aligned}$$

So far, we have been able to ensure global Lyapunov stability of α_o. We now demonstrate asymptotic stability. The conditions of Corollary 3 are satisfied, so for $\mathfrak{M}_1$ we just need to use Eq. (96). In this case, Q_1 is given by

$$0 = \left\langle dv(q)\,, \rho^\sharp(\xi)\right\rangle = b\, \frac{\partial v}{\partial \theta} + c\, \frac{\partial v}{\partial \psi} = N\, b\, \sin\theta + \chi\, (c - b\, n) \sin(\psi - n\,\theta)\,. \quad (106)$$

Let us calculate $\mathfrak{M}_2$. Consider a trajectory of the system $(\theta(t), \psi(t))$ inside Q_1. Differentiating equation (106) and replacing the time derivatives of θ and ψ by their corresponding momenta (using Legendre transformation), we have

$$\begin{aligned} &N\, b\, \cos\theta\, (a\, p_\theta + b\, p_\psi) - n\, \chi\, (c - b\, n)\, \cos(\psi - n\, \theta)\, (a\, p_\theta + b\, p_\psi) + \\ &+\chi\, (c - b\, n)\, \cos(\psi - n\, \theta)\, (b\, p_\theta + c\, p_\psi) = 0. \end{aligned}$$

Since $g\, p_\theta + \hbar\, p_\psi = 0$, it follows that

$$N\, b\, \cos\theta\, (a\, \hbar - b\, g) + \chi\, (c - b\, n)\, [-n\, (a\, \hbar - b\, g) + (b\, \hbar - c\, g)] \cos(\psi - n\, \theta) = 0.$$

But $-n\,(a\,\hbar - b\,g) + (b\,\hbar - c\,g) = 0$ (recall (105)) and $N, b, (a\,\hbar - b\,g) \neq 0$. Therefore,

$$p_\theta\, \cos\theta = 0.$$

Differentiating again we have

$$\dot{p}_\theta\, \hbar\, \cos\theta - \sin\theta\, (a\, \hbar - b\, g)\, p_\theta = 0.$$

From these equations it follows that $p_\theta = p_\psi = 0$. Thus, $\mathfrak{M}_2$ is the null codistribution along Q_1. Let us calculate $\mathfrak{M}_3$. For all $\alpha \in \mathfrak{M}_2$ we must have

$$X_H(\alpha) + Y(\alpha) \in T\mathfrak{M}_2,$$

which means that $dh(\alpha) + f(\alpha) = 0$. Therefore,

$$\frac{\partial H}{\partial \theta} = 0 \quad \textit{and} \quad \frac{\partial H}{\partial \psi} + f_\psi = 0,$$

or equivalently, $M\, \sin\theta = 0$ and

$$\frac{M\, (g\, a - f\, b)}{-a\, \hbar + b\, g}\, \sin\theta + \chi\, \frac{a\, c - b^2}{-a\, \hbar + b\, g}\, \sin(\psi - n\, \theta) = 0.$$

This implies $\theta = 0, \pi$ and $\psi = 0, \pi$. Therefore,

$$\mathfrak{M}_3 = \mathfrak{M}_\infty = \{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}.$$

and this gives us that from almost all initial conditions the trajectories of the system converge to $\theta = 0$. In this case, the system is *quasi-globally* asymptotically stable at $(0,0)$.

We now present some simulation plots in Figure (5.1). The system parameters, which are taken from [27] are $a = 0.0032$, $b = 0.0032$, $c = 0.4846$, $M = 37.98$. The control parameters are chosen to be $n = 154$, $\xi = 100$, $\kappa = 0.3$, $d = 1$, $e = 1000$ and $N = 37985$.

5.2. **Inverted pendulum on a cart.** Now, consider a Hamiltonian system on $Q = S^1 \times \mathbb{R}$, with

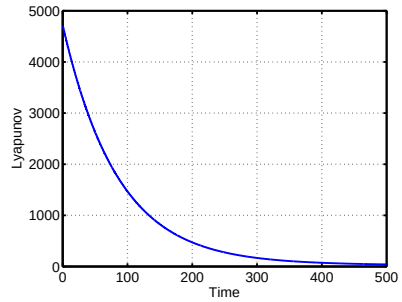
$$\begin{aligned} H(\theta, x, p_\theta, p_x) = \frac{1}{2}\, \frac{(p_\theta, p_x)}{ac - b^2\, \cos^2\theta} \begin{bmatrix} a & -b\, \cos\theta \\ -b\, \cos\theta & c \end{bmatrix} \begin{pmatrix} p_\theta \\ p_x \end{pmatrix} \\ + d\, (1 + \cos\theta), \end{aligned}$$

where a, b, c, d and $ac - b^2$ are strictly positive. In this case,

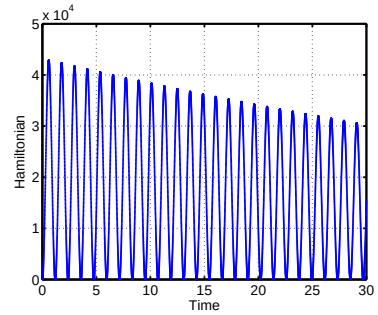
$$\rho^\sharp = \frac{1}{ac - b^2\, \cos^2\theta} \begin{bmatrix} a & -b\, \cos\theta \\ -b\, \cos\theta & c \end{bmatrix}.$$

Again, $\xi(\pi(\alpha)) = \xi = (0,1)$. The mechanical system corresponding to H and ξ is the underactuated *inverted pendulum on a cart* system provided

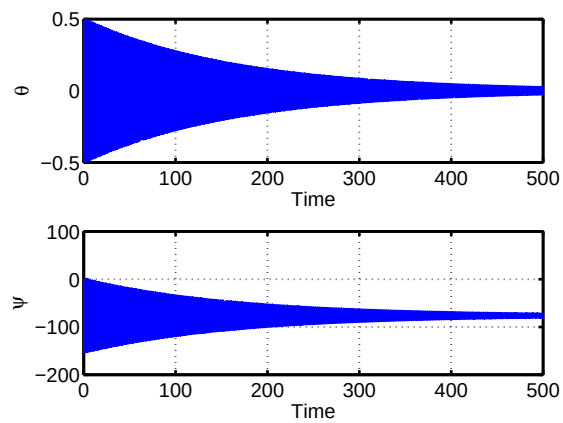
$$a = M + m, \quad b = ml, \quad c = ml^2 \quad \textit{and} \quad d = mgl,$$



(a) Decay of Lyapunov function for the inertia wheel pendulum



(b) Evolution of the Hamiltonian for the inertia wheel pendulum



(c) Plot of θ, ψ versus time for the inertia wheel pendulum

where M is the mass of the cart, m the mass of the pendulum, l the length of the pendulum and g the acceleration of gravity. Let us look for a simple Lyapunov function

$$V(\theta, x, p_\theta, p_x) = \frac{1}{2}\,(p_\theta, p_x)\begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}\begin{pmatrix} p_\theta \\ p_x \end{pmatrix} + v(\theta, x)$$

satisfying (57), such that f, g and $\hbar$ are functions of θ, and $f\hbar - g^2, \hbar, f > 0$. Of course,

$$\phi^\sharp = \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}.$$

We want to find a solution with v positive definite at $(\theta, x) = (0, 0)$.

$\mathfrak{C}$ is given by p_θ, p_x such that $g\,p_\theta + \hbar\,p_x = 0$. As we know from Theorem 4.3, the condition $\{V, H\} = 0$ on $\mathfrak{C}$ splits into two equations

$$\begin{aligned} &\left(\hbar^2 f' - 2\,g\,g'\,\hbar + g^2\,\hbar'\right)\,(a\,\hbar + b\,g\,\cos\theta)\,\left(ac - b^2\,\cos^2\theta\right) = \\ &= -2\,b\,\sin\theta\,\left[b\,\cos\theta\,\left((a\,\hbar + b\,g\,\cos\theta)\,\hbar + (g\,c + b\,\hbar\,\cos\theta)\,g\right)\right]\,\left(f\,\hbar - g^2\right) \\ &\quad -2\,b\,\sin\theta\,g\,\hbar\,\left(ac - b^2\,\cos^2\theta\right)\,\left(f\,\hbar - g^2\right) \end{aligned} \tag{107}$$

and

$$\begin{aligned}&(a\,\hbar + b\,g\,\cos\theta)\,\tfrac{\partial v}{\partial\theta} - (g\,c + b\,\hbar\,\cos\theta)\,\tfrac{\partial v}{\partial x} = \\ &-\left(f\,\hbar - g^2\right)\,\left(ac - b^2\,\cos^2\theta\right)\,d\,\sin\theta,\end{aligned} \tag{108}$$

corresponding to Eqs. (67) and (68) respectively. In (107), the primes f', g' and $\hbar'$ denote derivatives of f, g and $\hbar$ w.r.t. θ. For Eq. (108) we have the particular solution (depending only on θ) given by

$$v_p\,(\theta) = -\int_0^\theta \frac{\left(f\,\hbar - g^2\right)\,\left(ac - b^2\,\cos^2 y\right)\,d\,\sin y}{a\,\hbar + b\,g\,\cos y}dy.$$

To simplify (108), we choose two options for $\hbar$ and g:

Option 1:

$$a\,\hbar + b\,g\,\cos\theta = \alpha\,\left(a\,c - b^2\,\cos^2\theta\right)$$

and

$$g\,c + b\,\hbar\,\cos\theta = \beta\,\left(a\,c - b^2\,\cos^2\theta\right),$$

where α and β are constants.

Option 2:

$$a\,\hbar + b\,g\,\cos\theta = \alpha\,\left(a\,c - b^2\,\cos^2\theta\right)\,\left(f\,\hbar - g^2\right)$$

and

$$g\,c + b\,\hbar\,\cos\theta = \beta\,\left(a\,c - b^2\,\cos^2\theta\right)\,\left(f\,\hbar - g^2\right),$$

where α is a constant and β is a function of θ.

5.2.1. *Option 1.* This option gives

$$\hbar\,(\theta) = \alpha\,c - \beta\,b\,\cos\theta \qquad \textit{and} \qquad g\,(\theta) = \beta\,a - \alpha\,b\,\cos\theta. \tag{109}$$

Therefore,

$$v_p\,(\theta) = -\frac{d}{\alpha}\int_0^\theta \left(f\,\hbar - g^2\right)\,\sin y\,dy. \tag{110}$$

Since we want $v_p \geq 0$ and $v_p = 0$ only at $\theta = 0$, we require that $\alpha < 0$ (recall that $d > 0$). On the other hand, since $\hbar$ is positive,

$$\alpha\,c - \beta\,b\,\cos\theta > 0. \tag{111}$$

Since $c > 0$ and $\alpha < 0$, this condition can not be satisfied for all θ. For this condition to hold in a neighborhood of $\theta = 0$, we require

$$\alpha\,c > \beta\,b.$$

In particular, $\beta < 0$ (since $b > 0$). The neighborhood where (111) holds turns is

$$-\cos^{-1}\left(\frac{\alpha\,c}{\beta\,b}\right) < \theta < \cos^{-1}\left(\frac{\alpha\,c}{\beta\,b}\right). \tag{112}$$

We need to solve (107) for f and the homogeneous part of (108). Under the choices made for $\hbar$ and g (see (109)), these equations reduce to

$$\left(\hbar^2\,f' - 2\,g\,g'\,\hbar + g^2\,\hbar'\right) = -2\,b\,\beta\,\sin\theta\,\left(f\,\hbar - g^2\right) \tag{113}$$

and

$$\alpha\,\frac{\partial v}{\partial\theta} - \beta\,\frac{\partial v}{\partial x} = 0, \tag{114}$$

respectively. Equation (114) has solutions of the form $v(\theta,x) = K\left(\theta + \frac{\alpha}{\beta}\,x\right)$ for any C^1 function K. Let us choose K such that $K \geq 0$, and $K = 0$ only at $\theta = 0$. For instance,

$$K(y) = M\,(1-\cos y)\,, \qquad with \quad M > 0. \tag{115}$$

Thus,

$$v(\theta,x) = v_p(\theta) + K\left(\theta + \frac{\alpha}{\beta}\,x\right) \tag{116}$$

will be a non negative function in the interval given by (112) vanishing only at $(\theta,x) = (0,0)$. To solve (113), we just need to integrate a first order ODE for $f(\theta)$. To get a simpler expression, note that

$$\hbar^2 f' - 2\,g\,g'\,\hbar + g^2\,\hbar' = \hbar\,\left(f\,\hbar - g^2\right)' - \left(f\,\hbar - g^2\right)\,\hbar'. \tag{117}$$

So, instead of an equation in f, we can consider (107) as an equation for $\Delta = f\,\hbar - g^2$. In (113) we now substitute the expression for g and $\hbar$ given by (109) and replace $f\,\hbar - g^2$ by Δ to get the ODE

$$\Delta' + \Delta\,\chi(\theta) = 0,$$

with

$$\chi(\theta) = \frac{\beta\,b\,\sin\theta}{\alpha\,c - \beta\,b\,\cos\theta}.$$

We need a positive solution Δ defined in some subset of (112). The general solution is

$$\Delta(\theta) = N\exp\left(-\int_0^\theta \chi(y)\;dy\right). \tag{118}$$

It is enough to choose N positive in order for Δ to be positive inside the interval (112). Then

$$f(\theta) = \frac{\Delta(\theta) + g^2(\theta)}{\hbar(\theta)} = \frac{\Delta(\theta) + (\beta\,a - \alpha\,b\,\cos\theta)^2}{\alpha\,c - \beta\,b\,\cos\theta}. \tag{119}$$

Therefore, we have constructed a function

$$V(\theta,x,p_\theta,p_x) = \frac{1}{2}\,(p_\theta,p_x)\begin{bmatrix} f(\theta) & g(\theta) \\ g(\theta) & \hbar(\theta)\end{bmatrix}\begin{pmatrix} p_\theta \\ p_x\end{pmatrix} + v(\theta,x)\,,$$

defined in (112) with (recall (110), (115) and (116))

$$v(\theta,x) = -\frac{N\,d}{\alpha}\int_0^\theta \exp\left(-\int_0^y \chi(z)\;dz\right)\sin y\,dy + M\left[1-\cos\left(\theta + \frac{\alpha}{\beta}\,x\right)\right],$$

$\chi(\theta)$ given by (118), $f(\theta)$ by (119), and $g(\theta)$ and $\hbar(\theta)$ by (109). The constants α, β, M, N must satisfy

$$\alpha < 0, \quad \beta < 0, \quad M > 0, \quad N > 0 \quad and \quad \alpha\,c > \beta\,b.$$

The function V now satisfies **P1** and **P2** with $\alpha_o = (0,0,0,0)$. To construct the control law, we can choose

$$\mu(\theta,x,p_\theta,p_x) = \kappa\,(g\,p_\theta + \hbar\,p_x)^2\,, \qquad \kappa > 0.$$

The controller is $f = (f_\theta, f_x)$ with $f_\theta = 0$ and

$$f_x(\theta,x,p_\theta,p_x) = -\frac{\mu(\theta,x,p_\theta,p_x) + \{V,H\}(\theta,x,p_\theta,p_x)}{g\,p_\theta + \hbar\,p_x}.$$

5.2.2. *Option 2.* The second option gives

$$\hbar(\theta) = (\alpha\, c - \beta\, b\, \cos\theta)\, \left(f\, \hbar - g^2\right) \qquad \textit{and} \qquad g(\theta) = (\beta\, a - \alpha\, b\, \cos\theta)\, \left(f\, \hbar - g^2\right).$$

Therefore,

$$v_p(\theta) = -\frac{d}{\alpha} \int_0^\theta \sin y\, dy = -\frac{d}{\alpha}\, (1 - \cos\theta).$$

As in Option 1, since we want that $v_p \geq 0$ and $v_p = 0$ only at $\theta = 0$, and since $d > 0$, we require $\alpha < 0$. On the other hand, since $\hbar$ must be positive, we need

$$\alpha\, c - \beta\, b\, \cos\theta > 0. \tag{120}$$

We now need to solve (107) for f, and the homogeneous part of (108). Under the choices made for $\hbar$ and g (see (109)), these equations reduce to

$$\left(\hbar^2\, f' - 2\, g\, g'\, \hbar + g^2\, \hbar'\right) = -2\, b\, \beta\, \sin\theta\, \left(f\, \hbar - g^2\right)^2 \tag{121}$$

and

$$\alpha\, \frac{\partial v}{\partial \theta} - \beta\, \frac{\partial v}{\partial x} = 0, \tag{122}$$

respectively. Let us solve (121). Writing $\Delta = f\, \hbar - g^2$ and using the fact that $\hbar = (\alpha\, c - \beta\, b\, \cos\theta)\, \Delta$, it follow that (recall (117))

$$\hbar\, \Delta' - \Delta\, \hbar' = -\Delta^2\, (-\beta'\, b\, \cos\theta + \beta\, b\, \sin\theta).$$

Therefore, Eq. (121) becomes

$$-\Delta^2\, (-\beta'\, b\, \cos\theta + \beta\, b\, \sin\theta) = -2\, b\, \beta\, \sin\theta\, \Delta^2,$$

i.e.

$$-\beta'\, \cos\theta = \beta\, \sin\theta.$$

with the most general solution being

$$\beta(\theta) = N\, \cos\theta.$$

Since we want (120) to hold around $\theta = 0$, we must choose $N < \alpha\, c/b$. Note that we have no conditions on Δ and on f. We shall choose f such that Δ is constant. With these choices, Eq. (122) transforms to

$$\alpha\, \frac{\partial v}{\partial \theta} - N\, \cos\theta\, \frac{\partial v}{\partial x} = 0.$$

It is easy to show that

$$v_h(\theta, x) = K\left(x + \frac{N}{\alpha}\, \sin\theta\right),$$

for any C^1 function K provides a solution. We choose $K(y) = M\, \mathrm{th}^2(y)$, with $M > 0$. In summary, we have a Lyapunov function

$$V(\theta, x, p_\theta, p_x) = \frac{1}{2}\, (p_\theta, p_x) \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix} \begin{pmatrix} p_\theta \\ p_x \end{pmatrix} + v(\theta, x)$$

with

$$\hbar(\theta) = \left(\alpha\, c - N\, b\, \cos^2\theta\right)\, \Delta,$$
$$g(\theta) = \cos\theta\, (N\, a - \alpha\, b)\, \Delta,$$
$$f(x) = \frac{\Delta + g^2(\theta)}{\hbar(\theta)},$$

and

$$v(\theta, x) = -\frac{d}{\alpha}\,(1-\cos\theta) + M\,K\left(x + \frac{N}{\alpha}\sin\theta\right).$$

For the constants involved we choose

$$\alpha < 0, \qquad N\,b < \alpha\,c \quad \textit{and} \quad \Delta, M > 0.$$

The function V is defined for all θ such that

$$-\cos^{-1}\sqrt{\frac{\alpha\,c}{N\,b}} < \theta < \cos^{-1}\sqrt{\frac{\alpha\,c}{N\,b}}.$$

The control law is given by the μ-term, and the following 5 terms:

$$(1): \frac{-\alpha\,\sin\theta\,(N\,a-\alpha\,b)\,\left(\alpha\,c + N\,b\,\cos^2\theta\right)\,p_\theta^2}{\left(\alpha\,c - N\,b\,\cos^2\theta\right)^3}$$

$$(2): \frac{b\,\sin\theta\,\left(4\,b\,N\,\cos^2\theta - \alpha\,c\right)\,p_\theta^2}{\left(\alpha\,c - N\,b\,\cos^2\theta\right)^3\,\Delta\,\left(a\,c - b^2\,\cos^2\theta\right)}$$

$$(3): -\frac{b\,\cos^2\theta\,\sin\theta\,(N\,a-\alpha\,b)\,p_\theta^2}{\left(\alpha\,c - N\,b\,\cos^2\theta\right)^3\,\Delta^2\,\left(a\,c - b^2\,\cos^2\theta\right)}\left[\alpha\,c\,\left(2\,b\,\Delta^2\,\alpha + 3\,a\,N\right)\right]$$
$$-\frac{b\,\cos^2\theta\,\sin\theta\,(N\,a-\alpha\,b)\,p_\theta^2}{\left(\alpha\,c - N\,b\,\cos^2\theta\right)^3\,\Delta^2\,\left(a\,c - b^2\,\cos^2\theta\right)}\left[b\,N\,\cos^2\theta\,\left(2\,N\,a\,\Delta^2 + 3\,b\,\alpha\right)\right]$$

$$(4): \frac{\left(d\,b\cos\,\theta\,\sin\theta + \left(\alpha\,c - N\,b\,\cos^2\theta\right)\,K\prime\left(x + \frac{N}{\alpha}\sin\theta\right)\right)}{\alpha\,\left(\alpha\,c - N\,b\,\cos^2\theta\right)\,\Delta\,\left(a\,c - b^2\,\cos^2\theta\right)}$$

$$(5): \frac{d\,\sin\theta\,\cos\theta\,(N\,a-\alpha\,b)}{\alpha\,c - N\,b\,\cos^2\theta}$$

We shall make a simulation for this second option only. In Figure (5.2.2) the system parameters are chosen to be $a = 4$, $b = 1$, $c = 1$, $d = 9.8$ and the control parameters are chosen to be $M = 1$, $\alpha = -1$, $N = -2$, $\Delta = 1$ and $\kappa = 2$.

5.3. **The ball and the beam.** Consider a homogeneous ball of radius R and mass m rolling without sliding on a road of length l. The road is contained in a vertical plane with its center O fixed. So, the configuration space of the system can be described as the open rectangle $Q = (-l/2, l/2) \times (-\pi/2, \pi/2)$. The Hamiltonian for the system is

$$H(x, \theta, p_x, p_\theta) = \frac{1}{2}\,\frac{(p_x, p_\theta)}{(a + m\,x^2)\,c - b^2}\begin{bmatrix} a + m\,x^2 & b \\ b & c \end{bmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$$
$$+\,d\,(x\,\sin\theta + R\,\cos\theta).$$

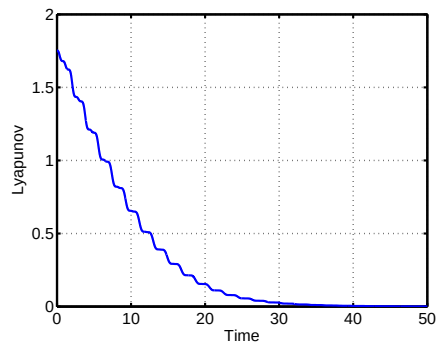
The x coordinate is the distance from O to the center of mass of the ball, and θ is the angle between the beam and the horizontal reference line. The constants a, b, c and d are

$$a = I + m\,R^2, \quad b = m\,R, \quad c = J/R^2 + m \quad \textit{and} \quad d = m\,g,$$

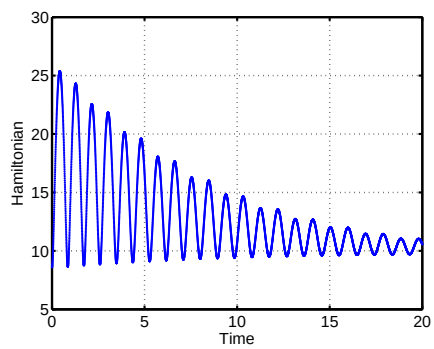
where I and J are the moments of inertia of the road and the ball respectively. Note that a, b, c and d, and $a\,c - b^2$, are strictly positive.

Assume $\xi(\pi(\alpha)) = \xi = (0, 1)$ and $\alpha_o = (0, 0, 0, 0)$. We need to look for a solution V of (57) of the form

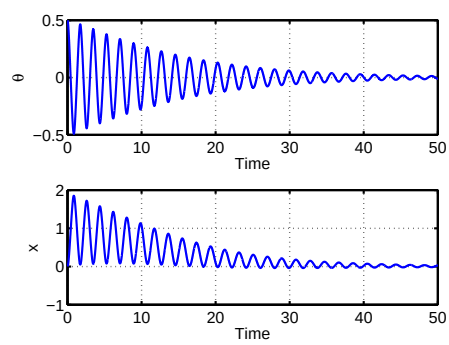
$$V(x, \theta, p_x, p_\theta) = \frac{1}{2}\,(p_x, p_\theta)\begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + v(x, \theta),$$



(d) Decay of Lyapunov function for the pendulum on a cart system



(e) Evolution of the Hamiltonian for the pendulum on a cart system



(f) Plot of x, θ versus time for the pendulum on a cart system

where f, g and $\hbar$ are functions of x such that $f\hbar - g^2 > 0$ and $f, \hbar > 0$ such that v is positive definite at $(x, \theta) = (0, 0)$. Of course, $\mathfrak{C}$ is given by

$$g\, p_x + \hbar\, p_\theta = 0.$$

Equation (57) splits into (see (67) and (68))

$$\begin{aligned}&\left(\hbar^2 f' - 2\,g\,g'\,\hbar + g^2\,\hbar'\right)\,\left(\left(a + m\,x^2\right)\,\hbar - b\,g\right)\,\left(\left(a + m\,x^2\right)\,c - b^2\right) = \\ &= -2\,m\,x\,c\,\left[\left(\left(a + m\,x^2\right)\,\hbar - b\,g\right)\,\hbar\right]\,\left(f\,\hbar - g^2\right) \\ &\quad -2\,m\,x\,c\,\left[(g\,c - b\,\hbar)\,g - \hbar^2/c\,\left(\left(a + m\,x^2\right)\,c - b^2\right)\right]\,\left(f\,\hbar - g^2\right)\end{aligned} \tag{123}$$

and

$$\begin{aligned}&\left(\left(a + m\,x^2\right)\,\hbar - b\,g\right)\,\tfrac{\partial v}{\partial x} - (g\,c - b\,\hbar)\,\tfrac{\partial v}{\partial \theta} = \\ &= \left(f\,\hbar - g^2\right)\,\left(\left(a + m\,x^2\right)\,c - b^2\right)\,d\,\sin\theta.\end{aligned} \tag{124}$$

In (123), the primes f', g' and $\hbar'$ denote derivatives of f, g and $\hbar$ w.r.t. x. As in the pendulum on a cart system from previous section, we choose $\hbar$ and g such that

$$\left(a + m\,x^2\right)\,\hbar - b\,g = \alpha\,\left(f\,\hbar - g^2\right)\,\left(\left(a + m\,x^2\right)\,c - b^2\right)$$

and

$$g\,c - b\,\hbar = \beta\,\left(f\,\hbar - g^2\right)\,\left(\left(a + m\,x^2\right)\,c - b^2\right),$$

where α is a function of x and β a constant. This gives

$$\hbar\,(x) = (c\,\alpha + b\,\beta)\,\left(f\,\hbar - g^2\right) \qquad \textit{and} \qquad g\,(x) = \left(b\,\alpha + \left(a + m\,x^2\right)\,\beta\right)\,\left(f\,\hbar - g^2\right).$$

With this choice, (123) and (124) becomes

$$\left(\hbar^2 f' - 2\,g\,g'\,\hbar + g^2\,\hbar'\right)\,\alpha = -2\,m\,x\,\beta^2\,\left(f\,\hbar - g^2\right)^2$$

and

$$\alpha\,\frac{\partial v}{\partial x} - \beta\,\frac{\partial v}{\partial \theta} = d\,\sin\theta.$$

Substituting $\Delta = f\,\hbar - g^2$ and combining Eq. (117) and the fact that $\hbar = (c\,\alpha + b\,\beta)\,\Delta$, we have

$$\begin{aligned}\hbar^2 f' - 2\,g\,g'\,\hbar + g^2\,\hbar' = \hbar\,\Delta' - \Delta\,\hbar' &= (c\,\alpha + b\,\beta)\,DD' \\ &\quad - \Delta\,\left(c\,\alpha'\,\Delta + (c\,\alpha + b\,\beta)\,\Delta'\right) \\ &= -\Delta^2\,c\,\alpha'.\end{aligned}$$

Therefore, (123) gives

$$-\Delta^2\,c\,\alpha\,\alpha' = -2\,m\,x\,\beta^2\,\Delta^2$$

i.e.

$$c\,\alpha\,\alpha' = 2\,m\,x\,\beta^2.$$

The general solution for this equation is

$$\alpha\,(x) = \pm\sqrt{\frac{2\,m\,\beta^2}{c}\,x^2\,+\delta},$$

where δ is a non negative constant (we need α to be defined at $x = 0$). Note that we have no conditions on Δ, i.e. on f. For simplicity, we shall take f such that Δ is a positive constant Δ. Now, let us solve (124). Note first that

$$v_p\,(\theta) = -\frac{d}{\beta}\,(1 - \cos\theta)$$

is a particular solution. We shall take $\beta < 0$ to ensure that v_p is positive and only vanishes at 0. Since $\hbar$ must be positive and taking into account the negativity of β, the condition $c\,\alpha + b\,\beta > 0$ implies that

$$\alpha\,(x) = \sqrt{\frac{2\,m\,\beta^2}{c}\,x^2\,+\delta},$$

i.e. α is positive and

$$\delta > \frac{b^2\,\beta^2}{c^2}.$$

For the homogenous equation

$$\alpha\,(x)\,\frac{\partial v}{\partial x} - \beta\,\frac{\partial v}{\partial \theta} = 0, \tag{125}$$

consider a primitive ϱ of $1/\alpha$, say

$$\varrho\,(x) = \int_0^x \left(\frac{2\,m\,\beta^2}{c}\,y^2\ + \delta\right)^{-1/2} dy = \frac{\ln(\mu x + \sqrt{\mu^2 x^2 + \delta})}{\mu}$$

where $\mu^2 = \frac{2m\beta^2}{c}$. It is easy to see that

$$v_h\,(x,\theta) = K\,(\theta + \beta\,\varrho\,(x))\,,$$

where K is any C^1 function is a solution of (125). We choose

$$K\,(y) = M\ (1 - \cos\,(y))\,, \qquad N > 0.$$

Then $v\,(x,\theta) = v_h\,(x,\theta) + v_p\,(\theta) \geq 0$ and $v\,(x,\theta) = 0$ if and only if $(x,\theta) = (0,0)$. Therefore, we have a Lyapunov function

$$V\,(x,\theta,p_x,p_\theta) = \frac{1}{2}\ (p_x,p_\theta)\ \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}\ \begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + v\,(x,\theta)$$

with

$$\hbar\,(x) = \left(c\sqrt{\frac{2\,m\,\beta^2}{c}\,x^2\ + \delta} + b\,\beta\right)\,\Delta,$$

$$g\,(x) = \left(b\sqrt{\frac{2\,m\,\beta^2}{c}\,x^2\ + \delta} + \left(a + m\,x^2\right)\,\beta\right)\,\Delta,$$

$$f\,(x) = \frac{\Delta + g^2\,(x)}{\hbar\,(x)},$$

and

$$v\,(x,\theta) = -\frac{d}{\beta}\ (1 - \cos\theta) + M\ (1 - \cos\,(\theta + \beta\,\varrho\,(x)))\,,$$

where $\varrho\,(x)$ is given by (125). For the implicit constants, we have

$$\beta < 0, \quad \Delta, M > 0 \quad \textit{and} \quad \delta > \frac{b^2\,\beta^2}{c^2}.$$

For our simulation in Figure (5.3), we choose the system parameters to be $a = 0.01$, $b = 0.0016$, $c = 0.1544$, $d = 1.078$, $m = 0.11$, $R = 0.015$, $J = 10^{-5}$ and the control parameters to be $\Delta = 10$, $M = 0.27$, $\beta = -2$, $\delta = 0.1$, $\kappa = 4$ and $\mu = 2.39$.

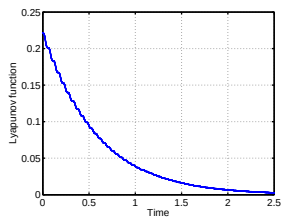
6. ***Quasi-simple*** **Lyapunov constraints.** In this section, we will consider **quasi-simple** Lyapunov functions where V takes the form

$$V\,(\alpha) = \frac{1}{2}\,\phi\left(\phi^\sharp\,(\alpha - \kappa\,(q))\,,\phi^\sharp\,(\alpha - \kappa\,(q))\right) + v\,(q)\,,$$

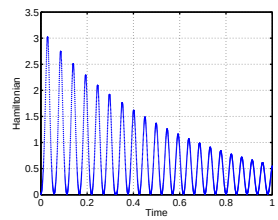
with $\kappa : Q \to T^*Q$, $v : Q \to \mathbb{R}$, ϕ a Riemannian metric on Q, and $q = \pi\,(\alpha)$. Note that, in this case,

$$\mathbb{F}V\,(\alpha) = \phi^\sharp\,(\alpha - \kappa\,(q))\,. \tag{126}$$

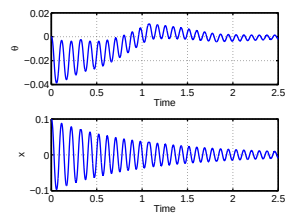
Again, μ will be non negative.



(g) Decay of Lyapunov function for the ball on a beam system



(h) Evolution of the Hamiltonian for the ball on a beam system



(i) Plot of x, θ versus time for the ball on a beam system

We will concentrate on systems with friction given by a Rayleigh dissipation tensor R (and obtain friction-free ones as the $R \equiv 0$ case). In fact, as we said before, stabilization of systems with friction is our main motivation for considering Lyapunov functions which are different from the simple ones.

6.1. **The existence and uniqueness problem.** Suppose we have, as in Sections 3.4 and 4.1.2, an underactuated system subjected to friction forces, defined by H, $\mathcal{W}$ and R, with $\mathcal{W}$ satisfying **W1-W3** and defined by $\xi : Q \to T^*Q$. Regarding $\mathfrak{C}$, since we are assuming that V is quasi-simple, each subset $\mathfrak{C}_q = \mathfrak{C} \cap T_q Q$ is no longer a linear subspace, but an affine one. In fact, it follows from (126) that

$$\langle \xi(q), \mathbb{F}V(\alpha) \rangle = \left\langle \xi(q), \phi^\sharp(\alpha - \kappa(q)) \right\rangle = \phi(\xi(q), \alpha - \kappa(q)).$$

So, $\mathfrak{C}_q = \mathfrak{C}_q^{vec} + \kappa(q)$ with

$$\mathfrak{C}_q^{vec} = \langle \xi(q) \rangle^{\perp_\phi}.$$

We will state the main result in this section. Let $\mathfrak{p} : T^*Q \to T^*Q$ be the orthogonal projection w.r.t. ϕ with range $\mathfrak{C}^{vec}$, and define $\mathfrak{P} : T^*Q \to T^*Q$ as

$$\mathfrak{P}(\alpha) = \mathfrak{p}(\alpha - \kappa(q)) + \kappa(q).$$

(Of course, if $\alpha \in \mathfrak{C}$, then $\mathfrak{P}(\alpha) = \alpha$.)

Theorem 6.1. *Given a pair $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ defined by a nowhere vanishing map $\xi : Q \to T^*Q$ and given a Rayleigh dissipation tensor R, there exists a solution $Y \subset \mathcal{W}$ of (40) for a quasi-simple V and μ non negative, if and only if the function*

$$\eta(\alpha) = \mu(\alpha) + \{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha)))$$

satisfies

$$\frac{\eta}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot) \rangle} \in C^\infty(T^*Q). \tag{127}$$

The solution is unique and given by $Y(\alpha) = \beta_\alpha^{-1}(0 \oplus f(\alpha))$ *with*

$$f(\alpha) = -\frac{\mu(\alpha) + \{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle}\xi(q). \tag{128}$$

In order to prove this theorem, lets consider local coordinate expressions. In canonical coordinates,

$$H(q,p) = \frac{1}{2}p^t\,\mathbb{H}(q)\,p + h(q) = \frac{1}{2}p_i^t\,\mathbb{H}^{ij}(q)\,p_j + h(q)$$

and

$$\begin{aligned} V(q,p) &= \frac{1}{2}\,(p-\kappa(q))^t\,\mathbb{V}(q)\,(p-\kappa(q)) + v(q) \\ &= \frac{1}{2}\,(p_i-\kappa_i(q))^t\,\mathbb{V}^{ij}(q)\,(p_j-\kappa_j(q)) + v(q). \end{aligned}$$

So,

$$\begin{aligned} \{V,H\}(\alpha) = &-(p_i-\kappa_i(q))\,\mathbb{V}^{ij}(q)\,\frac{\partial\kappa_j(q)}{\partial q_k}\,\mathbb{H}^{kl}(q)\,p_l \\ &+\frac{1}{2}\,(p_i-\kappa_i(q))\,\frac{\partial\mathbb{V}^{ij}(q)}{\partial q_k}\,(p_j-\kappa_j(q))\,\mathbb{H}^{kl}(q)\,p_l \\ &+\frac{\partial v(q)}{\partial q_k}\,\mathbb{H}^{kl}(q)\,p_l \\ &-\left[\frac{1}{2}p_i\,\frac{\partial\mathbb{H}^{ij}(q)}{\partial q_k}\,p_j + \frac{\partial h(q)}{\partial q_k}\right]\mathbb{V}^{kl}(q)\,(p_l-\kappa_l(q)) \end{aligned}$$

and

$$R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)) = R_{ij}(q)\,\mathbb{H}^{ik}(q)\,p_k\,\mathbb{V}^{jl}(q)\,(p_l-\kappa_l(q)).$$

Let $\lambda_l = p_l - \kappa_l(q)$. Then, after omitting the dependence on q, we have

$$\begin{aligned} \{V,H\}(\alpha) = &\left[-\lambda_i\,\mathbb{V}^{ij}\,\frac{\partial\kappa_j}{\partial q_k} + \frac{1}{2}\lambda_i\,\frac{\partial\mathbb{V}^{ij}}{\partial q_k}\,\lambda_j + \frac{\partial v}{\partial q_k}\right]\mathbb{H}^{kl}\,(\lambda_l+\kappa_l) \\ &-\left[\frac{1}{2}\,(\lambda_i+\kappa_i)\,\frac{\partial\mathbb{H}^{ij}}{\partial q_k}\,(\lambda_j+\kappa_j) + \frac{\partial h}{\partial q_k}\right]\mathbb{V}^{kl}\,\lambda_l \end{aligned}$$

and

$$R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)) = R_{ij}\,\mathbb{H}^{ik}\,(\lambda_k+\kappa_k)\,\mathbb{V}^{jl}\,\lambda_l.$$

We want to study the difference

$$\{V,H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)).$$

This difference can be decomposed in terms homogeneous in λ (ordered form 0 to 3)

$$\begin{aligned} D =& \frac{\partial v}{\partial q_k}\,\mathbb{H}^{kl}\,\kappa_l, \\ C_\lambda =& -\lambda_i\,\mathbb{V}^{ij}\,\frac{\partial \kappa_j}{\partial q_k}\,\mathbb{H}^{kl}\kappa_l + \frac{\partial v}{\partial q_k}\,\mathbb{H}^{kl}\,\lambda_l - \frac{1}{2}\,\kappa_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q_k}\,\kappa_j\,\mathbb{V}^{kl}\,\lambda_l \\ & - \frac{\partial h}{\partial q_k}\,\mathbb{V}^{kl}\,\lambda_l - R_{ij}\,\mathbb{H}^{ik}\,\kappa_k\,\mathbb{V}^{jl}\,\lambda_l, \\ B_\lambda =& -\lambda_i\,\mathbb{V}^{ij}\,\frac{\partial \kappa_j}{\partial q_k}\,\mathbb{H}^{kl}\,\lambda_l + \frac{1}{2}\,\lambda_i\,\frac{\partial \mathbb{V}^{ij}}{\partial q_k}\,\lambda_j\,\mathbb{H}^{kl}\,\kappa_l \\ & - \kappa_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q_k}\,\lambda_j\,\mathbb{V}^{kl}\,\lambda_l - R_{ij}\,\mathbb{H}^{ik}\,\lambda_k\,\mathbb{V}^{jl}\,\lambda_l, \\ A_\lambda =& \frac{1}{2}\,\lambda_i\,\frac{\partial \mathbb{V}^{ij}}{\partial q_k}\,\lambda_j\,\mathbb{H}^{kl}\,\lambda_l - \frac{1}{2}\,\lambda_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q_k}\,\lambda_j\,\mathbb{V}^{kl}\,\lambda_l. \end{aligned} \tag{129}$$

In particular, after fixing $\varsigma \in T_q^*Q$ and $\epsilon \in \mathbb{R}$, for $\lambda = \epsilon\,\varsigma$ we have that

$$\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))$$

is equal to the cubic polynomial

$$A_\varsigma\,\epsilon^3 + B_\varsigma\,\epsilon^2 + C_\varsigma\,\epsilon + D, \tag{130}$$

with coefficients given by (129). Of course, we can give global expressions for all these coefficients, but we prefer to work with their local versions.

We are now in a position to prove the theorem.

Proof of Theorem 6.1. According to Sections 3.4 and 4.1.2, there exists a solution $Y \subset \mathcal{W}$ of (40) if and only if

$$\frac{\mu(\alpha) + \{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle} \tag{131}$$

can be extended to all of T^*Q as a unique C^∞ function. In other words, if and only if

$$\frac{\mu + \{V, H\} - R(\mathbb{F}H(\cdot), \mathbb{F}V(\cdot))}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot)\rangle} \in C^\infty(T^*Q).$$

This prove the last part of the theorem. If we show that

$$\begin{aligned} & \frac{\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle} - \\ & - \frac{\{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha)))}{\langle \xi(q), \mathbb{F}V(\alpha)\rangle} \end{aligned}$$

can be be extended to all of T^*Q as a C^∞ function, then the same is true for (131) if and only if (127) holds. To show this, let $\alpha = \alpha_1 + \alpha_2 + \kappa(q)$ with

$$\alpha_1 = h(\alpha)\,\xi(q) \quad \textit{and} \quad \alpha_2 = \alpha - \kappa(q) - h(\alpha)\,\xi(q),$$

where $h(\alpha) = \langle \xi(q), \phi^\sharp(\alpha - \kappa(q))\rangle = \langle \xi(q), \mathbb{F}V(\alpha)\rangle$. Choose $\xi(q)$ such that

$$\phi(\xi(q), \xi(q)) = 1, \quad \forall q \in Q,$$

which is possible since $\xi(q) \neq 0$ for all q. It is clear that $\alpha_2 \in \mathfrak{C}_q^{vec}$ (compare with Theorem 4.2). If we write

$$\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))$$

with $\alpha = \alpha_1 + \alpha_2 + \kappa$, we get a cubic polynomial in $h(\alpha)$, with coefficients depending on α_2. (To see this, use Eqs. (129) and (130)). The 0-th order term is

$$\{V, H\}(\alpha_2 + \kappa(q)) - R(\mathbb{F}H(\alpha_2 + \kappa(q)), \mathbb{F}V(\alpha_2 + \kappa(q))),$$

which coincides with

$$\{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha))).$$

Therefore, the difference

$$\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha))$$

$$- [\{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha)))]$$

is a cubic polynomial in $h(\alpha)$, with coefficients depending on α_2, and **no** constant term. So, this difference divided by $h(\alpha) = \langle \xi(q), \mathbb{F}V(\alpha) \rangle$ defines a C^∞ function. □

If (127) is satisfied, it is clear that $\eta(\alpha) = 0$ for all $\alpha \in \mathfrak{C}$. Since $\mu \geq 0$, then, using the definition of η,

$$\{V, H\}(\alpha) - R(\mathbb{F}H(\alpha), \mathbb{F}V(\alpha)) \leq 0, \quad \forall \alpha \in \mathfrak{C}, \tag{132}$$

or equivalently,

$$\{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha))) \leq 0, \quad \forall \alpha \in T^*Q.$$

This enable us to choose (say)

$$\mu(\alpha) = \eta(\alpha) - [\{V, H\}(\mathfrak{P}(\alpha)) - R(\mathbb{F}H(\mathfrak{P}(\alpha)), \mathbb{F}V(\mathfrak{P}(\alpha)))] \tag{133}$$

where

$$\eta(\alpha) = \zeta \, \langle \xi(q), \mathbb{F}V(\alpha) \rangle^2, \quad \text{with} \quad \zeta > 0. \tag{134}$$

(Recall Eqs. (62) and (86).) Therefore, with μ defined as in (133), equation (132) is a necessary and a sufficient condition for existence of solutions of (40). One can in fact say a little bit more. Lets choose $\varsigma \in \mathfrak{C}_q^{vec}$ and consider the element $\epsilon\,\varsigma$, with $\epsilon \in \mathbb{R}$, which also belongs to $\mathfrak{C}_q^{vec}$. For $\lambda = \epsilon\,\varsigma$, equation (132) is (see (129) and (130))

$$A_\varsigma\, \epsilon^3 + B_\varsigma\, \epsilon^2 + C_\varsigma\, \epsilon + D \leq 0,$$

for all $\epsilon \in \mathbb{R}$, $q \in Q$ and $\varsigma \in \mathfrak{C}_q^{vec}$, i.e. for all ς such that $\xi_l\, \mathbb{V}^{kl}\, \varsigma_l = 0$. But a cubic polynomial $p(x) = a\,x^3 + b\,x^2 + c\,x + d$ satisfies $p(x) \leq 0$ for all x if and only if $a = 0$, $b, d \leq 0$ and $c^2 \leq 4\,b\,d$. Therefore, we have the following result.

Theorem 6.2. *Given a pair $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ defined by $\xi : Q \to T^*Q$ and given a Rayleigh dissipation tensor R, there exists a solution $Y \subset \mathcal{W}$ of (40), for V quasi-simple and μ given by (133) and (134), if and only if*

$$A_\varsigma = 0, \quad B_\varsigma, D \leq 0 \quad \textit{and} \quad C_\varsigma^2 \leq 4 B_\varsigma\, D \tag{135}$$

for all $q \in Q$ and $\varsigma \in \mathfrak{C}_q^{vec}$.

Remark 23. If $\kappa \equiv 0$, then $D = 0$. Therefore, condition (135) reduces to

$$A_\varsigma = C_\varsigma = 0 \quad \textit{and} \quad B_\varsigma \leq 0.$$

Since in this case

$$B_\varsigma = -R_{ij}\, \mathbb{H}^{ik}\, \varsigma_k\, \mathbb{V}^{jl}\, \varsigma_l, \tag{136}$$

(136) is exactly the same as (87) with $p = \varsigma$. On the other hand, if in addition $R \equiv 0$, we only have the first two conditions in (135) which correspond to Eqs. (26) and (27) with $p = \varsigma$.

In short, if (135) is satisfied for a quasi-simple function V, choosing μ as in Eqs. (133) and (134), the related Lyapunov constraint can be implemented by a unique constraint force $Y \subset \mathcal{W}$ (given by (128)). Of course, if V is a solution of (135) satisfying property **P1** for some $\alpha_o \in T^*Q$, then the related constrained system (and its equivalent closed-loop system) is locally stable at α_o; and it is globally stable if **P2** also holds. Thus, we can say that Eq. (135) defines the **quasi-simple Lyapunov constraint based method for stabilization of systems with friction**. The main different with the method of Section 4.1.2 is the new *degree of freedom* introduced by function κ. It transforms Eq. (87), i.e.

$$-R_{ij}\, \mathbb{H}^{ik}\, \varsigma_k\, \mathbb{V}^{jl}\, \varsigma_l \leq 0$$

(see also (136)) into

$$\begin{aligned} &-\varsigma_i\, \mathbb{V}^{ij}\, \frac{\partial \kappa_j}{\partial q_k}\, \mathbb{H}^{kl}\, \varsigma_l + \frac{1}{2}\, \varsigma_i\, \frac{\partial \mathbb{V}^{ij}}{\partial q_k}\, \varsigma_j\, \mathbb{H}^{kl}\, \kappa_l \\ &-\kappa_i\, \frac{\partial \mathbb{H}^{ij}}{\partial q_k}\, \varsigma_j\, \mathbb{V}^{kl}\, \varsigma_l - R_{ij}\, \mathbb{H}^{ik}\, \varsigma_k\, \mathbb{V}^{jl}\, \varsigma_l \leq 0. \end{aligned} \tag{137}$$

In the next section we will find solutions to (137) and (135). (See the comments at the end of Section 4.1.2.) Also, because of this new degree of freedom, the condition $\mathfrak{C} \subset \mu^{-1}(0)$ is no longer necessary. That is, the La'Salle surface can be smaller than $\mathfrak{C}$ or even a singleton set (see the comments at the end of Sections 6.2.1 and 6.2.2). In any case, asymptotic stabilization can be easier to establish. Otherwise, we need to apply the algorithm described in Section 4.2.

6.2. **Some examples.**

6.2.1. *Inertia wheel pendulum with friction.* Lets add a friction tensor to the example in §5.1 given by

$$R = \begin{pmatrix} \chi & y \\ y & z \end{pmatrix}.$$

Recall that

$$\mathbb{V} = \begin{pmatrix} f & g \\ g & \hbar \end{pmatrix} \quad \textit{and} \quad \mathbb{H} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

and that the scalars $\chi, z, f, \hbar, a, c, f\,\hbar - g^2$ and $a\,c - b^2$ are positive. The vectors λ such that $\xi_l\, \mathbb{V}^{kl}\, \lambda_l = 0$ are of the form

$$\lambda = \epsilon \begin{pmatrix} \hbar \\ -g \end{pmatrix}.$$

So, we can choose

$$\varsigma = \begin{pmatrix} \hbar \\ -g \end{pmatrix}.$$

We need to find v and κ satisfying (see (135))

$$B = -\left(\frac{\partial \kappa_i}{\partial q_j} + R_{ij}\right) \mathbb{V}_{il}\, \mathbb{H}_{jk}\, \varsigma_k\, \varsigma_l \leq 0,$$

$$D = \frac{\partial v}{\partial q_k}\, \mathbb{H}^{kl}\, \kappa_l \leq 0,$$

such that

$$C = \left(\frac{\partial v}{\partial q_k}\, \mathbb{H}^{kl}\, \varsigma_l - \frac{\partial h}{\partial q_k}\, \mathbb{V}^{kl}\, \varsigma_l\right) - \left(\frac{\partial \kappa_i}{\partial q_j} + R_{ij}\right) \mathbb{V}_{il}\, \mathbb{H}_{jk}\, \kappa_k\, \varsigma_l$$

satisfies $C^2 \leq 4\, B\, D$. The condition $A = 0$ is automatically satisfied in this case. For simplicity, we have omitted suffix ς. We will consider two different kinds of solutions.

Case 1.

$$v = N\, (1 - \cos\theta)\,, \quad \kappa_1 = e_1\, \sin\theta \quad \text{and} \quad \kappa_2 = e_2\, \sin\theta.$$

We will show that the above choice is a solution. For simplicity, set $y = 0$.

For our choice of v and κ, we have

$$B = -\left(f\, \hbar - g^2\right)\, (e_1\, \cos\theta + \chi)\, (a\, \hbar - b\, g)\,,$$

$$D = N\, \sin^2\theta\, (a\, e_1 + b\, e_2)$$

and

$$C = N\, \sin\theta\, (a\, \hbar - b\, g) + \left(f\, \hbar - g^2\right)\, M\, \sin\theta$$

$$- \left(f\, \hbar - g^2\right)\, (e_1\, \cos\theta + \chi)\, \sin\theta\, (a\, e_1 + b\, e_2)\,.$$

Now, B and D will be non positive (for a neighbourhood of θ around zero) if

$$a\, \hbar - b\, g < 0, \quad e_1 < -\chi, \quad N > 0 \quad \textit{and} \quad a\, e_1 + b\, e_2 < 0.$$

Note that $D \leq 0$ for all θ, but $B \leq 0$ only for $|\theta| \leq \theta_o \equiv \cos^{-1}\left|\frac{\chi}{e_1}\right|$. The larger $|e_1|$ is, the closer θ_o is to $\pi/2$. Let us define $e = -e_1$ and $\widetilde{e} = -\,(a\, e_1 + b\, e_2)$. If in addition,

$$N = -\frac{\left(f\, \hbar - g^2\right)\, M}{a\, \hbar - b\, g}$$

and

$$e \leq \frac{4\, M}{\widetilde{e}} + \chi,$$

then $C^2 \leq 4\, B\, D$ for all θ. So, if

$$N > 0, \quad a\, \hbar - b\, g < 0, \quad N\, (a\, \hbar - b\, g) = -\left(f\, \hbar - g^2\right)\, M$$

and

$$\widetilde{e} > 0, \quad \chi < e \leq \chi + \frac{4\, M}{\widetilde{e}},$$

then there exist a control law (or a constraint force) f such that $dV/dt = -\mu$ for all $|\theta| \leq \theta_o \equiv \cos^{-1}\left|\frac{\chi}{e_1}\right|$, where

$$V = \frac{1}{2}\, (p_\theta - e_1\, \sin\theta, p_\psi - e_2\, \sin\theta)\, \mathbb{V} \begin{pmatrix} p_\theta - e_1\, \sin\theta \\ p_\psi - e_2\, \sin\theta \end{pmatrix} + N\, (1 - \cos\theta)$$

and (recall (133) and (134))

$$\begin{aligned}\mu =& \eta\left(p_\theta, p_\psi\right) - N \sin^2\theta \left(a\, e_1 + b\, e_2\right)\\ &+ \frac{1}{\hbar}\left(f\,\hbar - g^2\right)\left(e_1 \cos\theta + \chi\right)\sin\theta \left(a\, e_1 + b\, e_2\right)\left(p_\theta - e_1 \sin\theta\right)\\ &+ \frac{1}{\hbar^2}\left(f\,\hbar - g^2\right)\left(e_1 \cos\theta + \chi\right)\left(a\,\hbar - b\, g\right)\left(p_\theta - e_1 \sin\theta\right)^2,\end{aligned}$$

with

$$\eta\left(p_\theta, p_\psi\right) = \zeta\left[g\left(p_\theta - e_1 \sin\theta\right) + \hbar\left(p_\psi - e_2 \sin\theta\right)\right]^2.$$

Case 2.

$$v = N\left(1 - \cos\frac{\theta}{2}\right), \quad \kappa_1 = e_1 \sin\frac{\theta}{2} \quad \text{and} \quad \kappa_2 = e_2 \sin\frac{\theta}{2},$$

for $-\pi < \theta < \pi$. In this case

$$B = -\left(f\,\hbar - g^2\right)\left(\frac{e_1}{2}\cos\frac{\theta}{2} + \chi\right)\left(a\,\hbar - b\, g\right),$$

$$D = \frac{N}{2}\sin^2\frac{\theta}{2}\left(a\, e_1 + b\, e_2\right)$$

and

$$\begin{aligned}C =& \frac{N}{2}\sin\frac{\theta}{2}\left(a\,\hbar - b\, g\right) + \left(f\,\hbar - g^2\right) M \sin\theta\\ &- \left(f\,\hbar - g^2\right)\left(\frac{e_1}{2}\cos\frac{\theta}{2} + \chi\right)\sin\frac{\theta}{2}\left(a\, e_1 + b\, e_2\right).\end{aligned}$$

Now, B and D are non positive if

$$a\,\hbar - b\, g < 0, \quad \frac{e_1}{2} < -\chi, \quad N > 0 \quad \textit{and} \quad a\, e_1 + b\, e_2 < 0.$$

$D \le 0$ for all θ and $B \le 0$ only for $|\theta| \le \theta_o \equiv 2\cos^{-1}\left|\frac{2\chi}{e_1}\right|$. The larger $|e_1|$ is, the closer θ_o is to π. If

$$\frac{N}{2}\left(a\,\hbar - b\, g\right) = \left(f\,\hbar - g^2\right)\chi\left(a\, e_1 + b\, e_2\right)$$

and

$$M = \frac{e_1}{4}\left(a\, e_1 + b\, e_2\right),$$

then $C = 0$ (in particular $C^2 \le 4\, B\, D$) for all θ. So, for the choice of constant above, there exist a control law f, for functions

$$\begin{aligned}V =& \frac{1}{2}\left(p_\theta - e_1 \sin\frac{\theta}{2}, p_\psi - e_2 \sin\frac{\theta}{2}\right)\mathbb{V}\begin{pmatrix} p_\theta - e_1 \sin\frac{\theta}{2}\\ p_\psi - e_2 \sin\frac{\theta}{2}\end{pmatrix}\\ &+ N\left(1 - \cos\frac{\theta}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\mu =& \eta - \frac{N}{2}\sin^2\frac{\theta}{2}\left(a\, e_1 + b\, e_2\right)\\ &+ \frac{1}{\hbar^2}\left(f\,\hbar - g^2\right)\left(\frac{e_1}{2}\cos\frac{\theta}{2} + \chi\right)\left(a\,\hbar - b\, g\right)\left(p_\theta - e_1 \sin\frac{\theta}{2}\right)^2,\end{aligned}$$

with

$$\eta\left(p_\theta, p_\psi\right) = \zeta\left[g\left(p_\theta - e_1 \sin\frac{\theta}{2}\right) + \hbar\left(p_\psi - e_2 \sin\frac{\theta}{2}\right)\right]^2,$$

such that the controlled system satisfies $dV/dt = -\mu$ if

$$|\theta| \leq \theta_o \equiv 2\cos^{-1}\left|\frac{2\chi}{e_1}\right|.$$

In this case V is not defined for all θ, but the La'Salle surface $\mu^{-1}(0)$ is very simple. It is given by the compact subset $\theta = p_\theta = p_\psi = 0$. Therefore, each invariant submanifold is a point of the form

$$(\theta, \psi, p_\theta, p_\psi) = (0, \psi_o, 0, 0).$$

The control law. Let

$$\kappa_1(\theta) = e_1\,\kappa(\theta) \qquad \textit{and} \qquad \kappa_2(\theta) = e_2\,\kappa(\theta),$$

then the control law in both cases is given by

$$\begin{aligned} f_\psi = &-\zeta\left[g\left(p_\theta - e_1\,\sin\frac{\theta}{2}\right) + \hbar\left(p_\psi - e_2\,\sin\frac{\theta}{2}\right)\right] \\ &-\frac{1}{\hbar}\,(a\,e_1 + b\,e_2)\,(g\,e_1 + \hbar\,e_2)\,\kappa\prime \\ &-\frac{b}{\hbar}\,v\prime - \frac{g}{\hbar}\,M\,\sin\theta + \frac{b}{\hbar^2}\,[g\,(p_\theta - e_1\,\kappa) + \hbar\,(p_\psi - e_2\,\kappa)]\,(g\,e_1 + \hbar\,e_2)\,\kappa\prime \\ &+\frac{(p_\theta - e_1\,\kappa)\,\kappa\prime}{\hbar^2}\,\left[(g\,e_1 + \hbar\,e_2)\,(a\,\hbar - b\,g) + e_1\,\left(f\,\hbar - g^2\right)\right]. \end{aligned}$$

Here, $\kappa\prime$ and $v\prime$ are the derivatives of κ and v w.r.t. θ.

6.2.2. *The inverted cart-pendulum with friction.* Consider again the system considered in Section 5.2. This time we will write the Hamiltonian function as

$$H = \frac{1}{2\Delta}\,(p_\theta, p_x)\begin{bmatrix} 1+\beta & -\cos\theta \\ -\cos\theta & 1 \end{bmatrix}\begin{pmatrix} p_\theta \\ p_x \end{pmatrix} + \cos\theta$$

where $\Delta = 1 + \beta - \cos^2\theta = \beta + \sin^2\theta$ and $\beta > 0$. Let the friction forces be given by the Raleigh matrix

$$R = \begin{pmatrix} \chi & 0 \\ 0 & z \end{pmatrix},$$

where χ and z are positive scalars. We need to find a solution

$$\begin{aligned} V(\theta, x, p_\theta, p_x) = &\frac{1}{2}\,(p_\theta - \kappa_\theta(\theta, x), p_x - \kappa_x(\theta, x))\begin{bmatrix} f & g \\ g & \hbar \end{bmatrix}\begin{pmatrix} p_\theta - \kappa_\theta(\theta, x) \\ p_x - \kappa_x(\theta, x) \end{pmatrix} \\ &+ v(\theta, x) \end{aligned}$$

of (135), where $f\hbar - g^2$, h and f are strictly positive. In this case

$$\mathbb{V} = \begin{pmatrix} f & g \\ g & \hbar \end{pmatrix}, \qquad \mathbb{H} = \frac{1}{\Delta}\begin{pmatrix} 1+\beta & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix} \qquad \textit{and} \qquad h = \cos\theta.$$

As in the previous example, the vectors λ such that $\xi_l\,\mathbb{V}^{kl}\,\lambda_l = 0$ are of the form

$$\lambda = \epsilon\begin{pmatrix} \hbar \\ -g \end{pmatrix}.$$

So, we choose

$$\varsigma = \begin{pmatrix} \hbar \\ -g \end{pmatrix}.$$

Let us assume that f, g and $\hbar$ are functions of θ only. Then, see (129),

$$D = \frac{\partial v}{\partial q^k}\,\mathbb{H}^{kl}\,\kappa_l = \frac{1}{\Delta}\left[\frac{\partial v}{\partial \theta}\,((1+\beta)\,\kappa_\theta - \kappa_x\,\cos\theta) + \frac{\partial v}{\partial x}\,(\kappa_x - \kappa_\theta\,\cos\theta)\right],$$

$$\begin{aligned}
C = &\left[\frac{\partial v}{\partial q^k}\,\mathbb{H}^{kl}\,\varsigma_l - \frac{\partial h}{\partial q^k}\,\mathbb{V}^{kl}\,\varsigma_l\right] - \left(\frac{\partial \kappa_i}{\partial q^j} + R_{ij}\right)\,\mathbb{V}^{il}\,\mathbb{H}^{jk}\,\kappa_k\,\varsigma_l \\
&- \frac{1}{2}\,\kappa_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q^k}\,\kappa_j\,\mathbb{V}^{kl}\,\varsigma_l \\
= &\frac{1}{\Delta}\left[\frac{\partial v}{\partial \theta}\,((1+\beta)\,\hbar + g\,\cos\theta) - \frac{\partial v}{\partial x}\,(g + \hbar\,\cos\theta)\right] + \sin\theta\,\delta \\
&- \frac{\delta}{\Delta}\left[\left(\frac{\partial \kappa_\theta}{\partial \theta} + \chi\right)\,((1+\beta)\,\kappa_\theta - \kappa_x\,\cos\theta) + \frac{\partial \kappa_\theta}{\partial x}\,(\kappa_x - \kappa_\theta\,\cos\theta)\right] \\
&+ \frac{\delta\,\sin\theta}{\Delta^2}\,\left(\cos\theta\,\left((1+\beta)\,\kappa_\theta^2 + \kappa_x^2\right) - \left(1+\beta+\cos^2\theta\right)\,\kappa_\theta\,\kappa_x\right),
\end{aligned}$$

$$\begin{aligned}
B = &\frac{1}{2}\,\varsigma_i\,\frac{\partial \mathbb{V}^{ij}}{\partial q^k}\,\varsigma_j\,\mathbb{H}^{kl}\,\kappa_l - \kappa_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q^k}\,\varsigma_j\,\mathbb{V}^{kl}\,\varsigma_l - \left(\frac{\partial \kappa_i}{\partial q^j} + R_{ij}\right)\,\mathbb{V}^{il}\,\mathbb{H}^{jk}\,\varsigma_k\,\varsigma_l \\
= &\frac{1}{2\,\Delta}\,(\hbar\,\delta' - \delta\,\hbar')\,((1+\beta)\,\kappa_\theta - \kappa_x\,\cos\theta) \\
&+ \frac{\delta\,\sin\theta}{\Delta^2}\,\left[2\cos\theta\,((1+\beta)\,\hbar\,\kappa_\theta - g\,\kappa_x) + \left(1+\beta+\cos^2\theta\right)\,(g\,\kappa_\theta - \hbar\,\kappa_x)\right] \\
&- \frac{\delta}{\Delta}\left[\left(\frac{\partial \kappa_\theta}{\partial \theta} + \chi\right)\,((1+\beta)\,\hbar + g\,\cos\theta) - \frac{\partial \kappa_\theta}{\partial x}\,(\hbar\,\cos\theta + g)\right]
\end{aligned}$$

and

$$\begin{aligned}
A = &\frac{1}{2}\,\varsigma_i\,\frac{\partial \mathbb{V}^{ij}}{\partial q^k}\,\varsigma_j\,\mathbb{H}^{kl}\,\varsigma_l - \frac{1}{2}\,\varsigma_i\,\frac{\partial \mathbb{H}^{ij}}{\partial q^k}\,\varsigma_j\,\mathbb{V}^{kl}\,\varsigma_l \\
= &\frac{((1+\beta)\,\hbar + g\,\cos\theta)}{\Delta}\,\left[\frac{1}{2}\,(\hbar\,\delta' - \delta\,\hbar') + (\hbar\,\cos\theta + g)\,\sin\theta\,\frac{\delta}{\Delta}\right].
\end{aligned}$$

where $\delta = f\,\hbar - g^2$. Here, δ' and $\hbar'$ denote the derivatives of δ and $\hbar$ w.r.t. θ. If we substitute $A = 0$ in B, assuming $(1+\beta)\,\hbar + g\,\cos\theta \neq 0$, we get

$$\begin{aligned}
B = &\frac{\delta\,((1+\beta)\,\hbar + \cos\theta\,g)}{\Delta}\,\left[(\cos\theta\,\kappa_\theta - \kappa_x)\,\frac{\sin\theta}{\Delta} - \left(\frac{\partial \kappa_\theta}{\partial \theta} + \chi\right)\right] \\
&+ \frac{\delta}{\Delta}\,\frac{\partial \kappa_\theta}{\partial x}\,(g + \hbar\,\cos\theta).
\end{aligned}$$

Thus,

$$D = \frac{1}{\Delta}\left[\frac{\partial v}{\partial \theta}\,((1+\beta)\,\kappa_\theta - \kappa_x\,\cos\theta) + \frac{\partial v}{\partial x}\,(\kappa_x - \kappa_\theta\,\cos\theta)\right],$$

$$\begin{aligned}
C = &\frac{1}{\Delta}\left[\frac{\partial v}{\partial \theta}\,((1+\beta)\,\hbar + g\,\cos\theta) - \frac{\partial v}{\partial x}\,(g + \hbar\,\cos\theta)\right] + \sin\theta\,\delta \\
&- \frac{\delta}{\Delta}\left[\left(\frac{\partial \kappa_\theta}{\partial \theta} + \chi\right)\,((1+\beta)\,\kappa_\theta - \kappa_x\,\cos\theta) + \frac{\partial \kappa_\theta}{\partial x}\,(\kappa_x - \kappa_\theta\,\cos\theta)\right] \\
&+ \frac{\delta\,\sin\theta}{\Delta^2}\,\left(\cos\theta\,\left((1+\beta)\,\kappa_\theta^2 + \kappa_x^2\right) - \left(1+\beta+\cos^2\theta\right)\,\kappa_\theta\,\kappa_x\right),
\end{aligned}$$

$$B = \frac{\delta\ ((1+\beta)\ \hbar + \cos\theta\, g)}{\Delta}\ \left[(\cos\theta\,\kappa_\theta - \kappa_x)\ \frac{\sin\theta}{\Delta} - \left(\frac{\partial\kappa_\theta}{\partial\theta} + \chi\right)\right]$$
$$+ \frac{\delta}{\Delta}\ \frac{\partial\kappa_\theta}{\partial x}\ (g + \hbar\ \cos\theta)\,.$$

and

$$0 = \frac{1}{2}\ (\hbar\,\delta' - \delta\,\hbar')\ +\ (\hbar\ \cos\theta + g)\ \sin\theta\,\frac{\delta}{\Delta}.$$

If

$$(1+\beta)\ \hbar + \cos\theta\, g = \Omega\,\Delta\,\delta$$

and

$$g + \hbar\ \cos\theta = \Theta\,\Delta\,\delta,$$

or equivalently

$$\hbar = \delta\ (\Omega - \Theta\ \cos\theta) \qquad \textit{and} \quad g = \delta\ (\Theta\ (1+\beta) - \Omega\ \cos\theta)\,,$$

then

$$D = \frac{1}{\Delta}\ \left[\frac{\partial v}{\partial\theta}\ ((1+\beta)\ \kappa_\theta - \kappa_x\ \cos\theta) + \frac{\partial v}{\partial x}\ (\kappa_x - \kappa_\theta\ \cos\theta)\right],$$
$$C = \left[\frac{\partial v}{\partial\theta}\,\Omega - \frac{\partial v}{\partial x}\ \Theta + \sin\theta\right]\ \delta$$
$$- \frac{\delta}{\Delta}\ \left[\left(\frac{\partial\kappa_\theta}{\partial\theta} + \chi\right)\ ((1+\beta)\ \kappa_\theta - \kappa_x\ \cos\theta) + \frac{\partial\kappa_\theta}{\partial x}\ (\kappa_x - \kappa_\theta\ \cos\theta)\right]$$
$$+ \frac{\delta\ \sin\theta}{\Delta^2}\ \left(\cos\theta\ \left((1+\beta)\ \kappa_\theta^2 + \kappa_x^2\right) - \left(1 + \beta + \cos^2\theta\right)\ \kappa_\theta\,\kappa_x\right),$$
$$B = -\,\delta^2\ \left[\frac{\partial\kappa_\theta}{\partial\theta}\,\Omega - \frac{\partial\kappa_\theta}{\partial x}\,\Theta\right]$$
$$+ \delta^2\,\Omega\ \left[(\cos\theta\,\kappa_\theta - \kappa_x)\ \frac{\sin\theta}{\Delta} - \chi\right],$$
$$\Omega' = \Theta'\ \cos\theta + \sin\theta\,\Theta. \tag{138}$$

Also, defining

$$\Phi = \frac{(1+\beta)\ \kappa_\theta - \kappa_x\ \cos\theta}{\Delta}$$

and

$$\Psi = \frac{\cos\theta\,\kappa_\theta - \kappa_x}{\Delta},$$

or equivalently

$$\kappa_\theta = \Phi - \cos\theta\,\Psi \qquad \textit{and} \qquad \kappa_x = \cos\theta\,\Phi - (1+\beta)\ \Psi,$$

we have

$$D = \frac{\partial v}{\partial\theta}\,\Phi - \frac{\partial v}{\partial x}\,\Psi,$$
$$C = \left[\left(\frac{\partial v}{\partial\theta}\,\Omega - \frac{\partial v}{\partial x}\ \Theta + \sin\theta\right) - \left(\frac{\partial\kappa_\theta}{\partial\theta}\,\Phi - \frac{\partial\kappa_\theta}{\partial x}\,\Psi - \sin\theta\,\Phi\,\Psi + \chi\,\Phi\right)\right]\ \delta,$$
$$B = -\delta^2\ \left(\frac{\partial\kappa_\theta}{\partial\theta}\,\Omega - \frac{\partial\kappa_\theta}{\partial x}\,\Theta - \sin\theta\,\Omega\,\Psi + \chi\Omega\right).$$

A solution to (138) is

$$\Omega = L + K\ \cos\theta, \quad \Theta = N\ \cos\theta - K,$$

i.e.

$$\hbar = \delta\,\left(L + 2K\,\cos\theta - N\,\cos^2\theta\right),$$
$$g = \delta\,\left((N\,\cos\theta - K)\,(1+\beta) - (L + K\,\cos\theta)\,\cos\theta\right),$$

with L, K and N constants and δ any positive function of θ. Choosing Ω and Θ as above and

$$\Phi = a\,\sin\theta, \qquad \Psi = -b\,x,$$

we get

$$D = \frac{\partial v}{\partial\theta}\,a\,\sin\theta + \frac{\partial v}{\partial x}\,b\,x,$$

$$C = \left(\frac{\partial v}{\partial\theta}\,(L + K\,\cos\theta) - \frac{\partial v}{\partial x}\,(N\,\cos\theta - K) + \sin\theta\right)\delta$$
$$- \left(\frac{\partial\kappa_\theta}{\partial\theta}a\,\sin\theta + \frac{\partial\kappa_\theta}{\partial x}\,b\,x + a\,b\,\sin^2\theta\,x + \chi\,a\,\sin\theta\right)\delta,$$

$$B = -\,\delta^2\,\frac{\partial\kappa_\theta}{\partial\theta}\,(L + K\,\cos\theta) - \frac{\partial\kappa_\theta}{\partial x}\,(N\,\cos\theta - K)$$
$$-\,\delta^2\,b\,\sin\theta\,(L + K\,\cos\theta)\,x + \chi\,(L + K\,\cos\theta).$$

If, in addition we choose,

$$v = M\,(1 - \cos\theta) + \frac{\rho}{2}\,x^2, \qquad M, \rho > 0,$$

and taking into account

$$\kappa_\theta = a\,\sin\theta + b\,x\,\cos\theta, \qquad \kappa_x = a\,\sin\theta\,\cos\theta + b\,x\,(1+\beta),$$

then we get

$$D = M\,a\,\sin^2\theta + \rho\,b\,x^2,$$

$$C = \left(M\,L + M\,K\,\cos\theta - a^2\,\cos\theta - a\,\chi + 1\right)\sin\theta\,\delta$$
$$-\left[\left(\rho\,N + b^2\right)\cos\theta - \rho\,K\right]x\,\delta,$$

and

$$B = -\delta^2\,\left[(a\,\cos\theta + \chi)\,(L + K\,\cos\theta) - b\,N\,\cos^2\theta + b\,K\,\cos\theta\right].$$

Note that

$$\hbar = \delta\,\left(L + 2\,K\,\cos\theta - N\,\cos^2\theta\right)$$

Choosing $K = 0$, we have

$$C = \left(M\,L - a^2\,\cos\theta - a\,\chi + 1\right)\sin\theta\,\delta - \left[\left(\rho\,N + b^2\right)\cos\theta\right]x\,\delta,$$

$$B = -\delta^2\,\left[(a\,\cos\theta + \chi)\,L - b\,N\,\cos^2\theta\right]$$

and

$$\hbar = \delta\,\left(L - N\,\cos^2\theta\right).$$

To get $D \le 0$ and $\hbar > 0$ (for θ in a neighbourhood of 0), we need

$$a, b \le 0 \tag{139}$$

and $L - N > 0$. Suppose $L < 0$. Then, $N < 0$ and

$$|N| > |L|. \tag{140}$$

In this case, $\hbar > 0$ for $\theta \in (-\theta_o, \theta_o)$, where

$$\theta_o = \cos^{-1} \sqrt{\frac{|L|}{|N|}}.$$

To have $B \leq 0$ near $\theta = 0$, lets impose the condition

$$0 < (a + \chi)\, L - b\, N = |L|\, (|a| - \chi) - |b\, N|\,. \tag{141a}$$

Then, $|a| > \chi$. To satisfy the condition $C^2 \leq 4\, B\, D$, choose

$$\rho = \frac{b^2}{|N|}. \tag{142}$$

If

$$\left(|a|\, \chi + 1 - M\, |L| - a^2\right)^2 < 4\, M\, |a|\, (|L|\, (|a| - \chi) - |b\, N|)\,, \tag{143}$$

then, $C^2 \leq 4\, B\, D$ holds near $\theta = 0$ and for all x. Choosing

$$M = \frac{1}{|L|} \quad \textit{and} \quad |b\, N| = \frac{|L|\, (|a| - \chi)}{n}, \qquad n > 1,$$

along with the condition $|a| > \chi$, we get

$$a^2 - \chi\, |a| < 4\, \left(1 - \frac{1}{n}\right).$$

If $a = -\,(\chi + \varepsilon)$ with $\varepsilon > 0$, we get

$$\chi < \frac{4\, \left(1 - \frac{1}{n}\right)}{\varepsilon} - \varepsilon.$$

If we choose $N = n\, L$, this implies $b = -\varepsilon / n^2$. Then, a solution to Eqs. (139)-(143) is given by the following choice

$$a = -\,(\chi + \varepsilon)\ , \quad b = -\frac{\varepsilon}{n^2}, \qquad N = n\, L, \qquad M = \frac{1}{|L|}, \qquad \rho = \frac{\varepsilon^2}{n^5\, |L|},$$

with

$$L < 0, \qquad \varepsilon > 0, \qquad \chi < \frac{4\, \left(1 - \frac{1}{n}\right)}{\varepsilon} - \varepsilon, \qquad n > 1. \tag{144}$$

Note that all the terms above depend on ε, n and L. So

$$\begin{aligned} V\,(\theta, x, p_\theta, p_x) = & \frac{1}{2}\, (p_\theta - \kappa_\theta\,(\theta, x)\,, p_x - \kappa_x\,(\theta, x)) \begin{bmatrix} f & g \\ g & \hbar \end{bmatrix} \begin{pmatrix} p_\theta - \kappa_\theta\,(\theta, x) \\ p_x - \kappa_x\,(\theta, x) \end{pmatrix} \\ & + v\,(\theta, x) \end{aligned}$$

and

$$\begin{aligned} \mu\,(\theta, x, p_\theta, p_x) = & \eta\, \{g\, [p_\theta - \kappa_\theta\,(\theta, x)] + \hbar\, [p_x - \kappa_x\,(\theta, x)]\}^2 \\ & + \delta^2 \left[\varepsilon\, \cos\theta\, \left(1 - \frac{1}{n} \cos\theta\right) - \chi\,(1 - \cos\theta)\right] |L| \left(\frac{p_\theta - \kappa_\theta\,(\theta, x)}{\hbar}\right)^2 \\ & + ((\chi + \varepsilon)\, \cos\theta - \chi)\, (\chi + \varepsilon)\, \sin\theta\, \delta\, \left(\frac{p_\theta - \kappa_\theta\,(\theta, x)}{\hbar}\right) \\ & + \frac{\chi + \varepsilon}{|L|}\, \sin^2\theta + \frac{\varepsilon^3}{n^7\, |L|}\, x^2 \end{aligned}$$

provides a solution to our problem with

$$v\,(\theta, x) = \frac{1}{|L|}\, (1 - \cos\theta) + \frac{\varepsilon^2}{2\, n^5\, |L|}\, x^2,$$

$$\kappa_\theta(\theta, x) = -(\chi + \varepsilon)\,\sin\theta - \frac{\varepsilon}{n^2}\,x\,\cos\theta,$$
$$\kappa_x(\theta, x) = -(\chi + \varepsilon)\,\sin\theta\,\cos\theta - \frac{\varepsilon}{n^2}\,x\,(1+\beta),$$

and

$$\hbar = \delta\,L\,\left(1 - n\,\cos^2\theta\right),$$
$$g = \delta\,L\,\cos\theta\,(n + n\,\beta - 1),$$
$$f = \frac{\delta + g^2}{\hbar},$$

where δ and η are constant positive scalars. It can be shown that near $\theta = 0$ where the function V has required Lyapunov properties, $\mu(\theta, x, p_\theta, p_x) = 0$ only if

$$(\theta, x, p_\theta, p_x) = (0, 0, 0, 0).$$

In this case, the La'Salle surface is the point that we are interested in stabilizing! Therefore, the inverted cart-pendulum with force $f = f(\theta, x, p_\theta, p_x)$ given by (128) is asymptotically stable near $\theta = 0$. The region of attraction (see the equation above (141a)) is the interval $(-\theta_o, \theta_o)$ where

$$\theta_o = \cos^{-1}\sqrt{\frac{1}{n}}.$$

If n is large and χ is small (which requires $\varepsilon < 2$ to satisfy (144)), the region of attraction is approximately given by the interval $(-\theta_1, \theta_1)$, where

$$\theta_1 = \cos^{-1}\frac{\chi}{\chi + \varepsilon}.$$

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