## Trigonometric Functions

The derivative of $\sin x$ is $\cos x$ and of $\cos x$ is $-\sin x$; everything else follows from this.

Many problems involving angles, circles, and periodic motion lead to trigonometric functions. In this chapter, we study the calculus of these functions, and we apply our knowledge to solve new problems.

The chapter begins with a review of trigonometry. Well-prepared students may skim this material and move on quickly to the second section. Students who do not feel prepared or who failed Orientation Quiz C at the beginning of the book should study this review material carefully.

### 5.1 Polar Coordinates and Trigonometry

Trigonometric functions provide the link between polar and cartesian coordinates.


Figure 5.1.1. The circumference and area of a circle.

This section contains a review of trigonometry, with an emphasis on the topics which are most important for calculus. The derivatives of the trigonometric functions will be calculated in the next section.

The circumference $C$ and area $A$ of a circle of radius $r$ are given by

$$
C=2 \pi r, \quad A=\pi r^{2}
$$

(see Fig. 5.1.1), where $\pi$ is an irrational number whose value is approximately 3.14159 ... ${ }^{1}$
${ }^{1}$ For details on the fascinating history of $\pi$, see P. Beckman, A History of $\pi$, Golem Press, 1970. To establish deeper properties of $\pi$ such as its irrationality (discovered by Lambert and Legendre around 1780), a careful and critical examination of the definition of $\pi$ is needed. The first explicit expression for $\pi$ was given by Viete $(1540-1603)$ as

$$
\frac{2}{\pi}=\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}}}} \cdots
$$

which is obtained by inscribing regular polygons in a circle. Euler's famous expression $\pi / 4=1-\frac{1}{3}+\frac{1}{5} \cdots \cdots$ is discussed in Example 3, Section 12.5. For an elementary proof of the irrationality of $\pi$, see M. Spivak, Calculus, Benjamin, 1967.


Figure 5.1.2. The length $C_{\theta}$ and area $A_{\theta}$ are proportional to $\theta$.

If two rays are drawn from the center of the circle, both the length and area of the part of the circle between the rays are proportional to the angle between the rays. Thus, if we measure angles in degrees, the length $C_{\theta}$ and area $A_{\theta}$ between rays making an angle $\theta$ (see Fig. 5.1.2) are determined by the relations

$$
\frac{C_{\theta}}{2 \pi r}=\frac{\theta}{360}, \quad \frac{A_{\theta}}{\pi r^{2}}=\frac{\theta}{360} \quad(\theta \text { in degrees })
$$

since a full circle corresponds to an angle of 360 degrees.
These formulas become simpler if we adopt the radian unit of measure, in which the total angular measure of a circle is defined to be $2 \pi$. Then our previous formulas become

$$
\frac{C_{\theta}}{2 \pi r}=\frac{\theta}{2 \pi}, \quad \frac{A_{\theta}}{\pi r^{2}}=\frac{\theta}{2 \pi} \quad(\theta \text { in radians })
$$

or simply

$$
C_{\theta}=r \theta, \quad A_{\theta}=\frac{1}{2} r^{2} \theta
$$

The formulas of calculus are also simpler when angles are measured in radians rather than degrees. Unless explicit mention is made of degrees, all angles in this book will be expressed in radians. If you use a calculator to do computations with angles measured in radians, be sure that it is operating in the radian mode.

Example 1 An arc of length 10 meters on a circle of radius 4 meters subtends what angle at the center of the circle? How much area is enclosed in this part of the circle?

Solution In the formula $C_{\theta}=r \theta$, we have $C_{\theta}=10$ and $r=4$, so $\theta=\frac{10}{4}=2 \frac{1}{2}$ (radians). The area enclosed is $A_{\theta}=\frac{1}{2} r^{2} \theta=\frac{1}{2} \cdot 16 \cdot \frac{5}{2}=20$ square meters.
Conversions between degrees and radians are made by multiplying or dividing by the factor $360 / 2 \pi=180 / \pi \approx 57.296$ degrees per radian.

## Degrees and Radians

To convert from radians to degrees, multiply by $\frac{180^{\circ}}{\pi} \approx 57^{\circ} 18^{\prime} \approx 57.296^{\circ}$.
To convert from degrees to radians, multiply by $\frac{\pi}{180^{\circ}} \approx 0.01745$.

The following table gives some important angles in degrees and radians:

| Degrees | $0^{\circ}$ | $30^{\circ}$ | $45^{\circ}$ | $60^{\circ}$ | $90^{\circ}$ | $120^{\circ}$ | $135^{\circ}$ | $150^{\circ}$ | $180^{\circ}$ | $270^{\circ}$ | $360^{\circ}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Radians | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ | $\frac{3 \pi}{2}$ | $2 \pi$ |

The measures of right angles and straight angles are shown in Fig. 5.1.3. circle, a right angle, and a straight angle in degrees and radians.



Figure 5.1.4. $\theta, \theta-2 \pi$, and $\theta+2 \pi$ measure the same geometric angle.


Figure 5.1.7. The polar coordinates $(r, \theta)$ of a point $P$.

Negative numbers and numbers larger than $2 \pi$ (or $360^{\circ}$ ) can also be used to represent angles. The convention is that $\theta$ and $\theta+2 \pi$ represent the same geometric angle; hence so do $\theta+4 \pi, \theta+6 \pi, \ldots$ as well as $\theta-2 \pi$, $\theta-4 \pi, \ldots$ (see Fig. 5.1.4). The angle $-\theta$ equals $2 \pi-\theta$ and is thus the "mirror image" of $\theta$ (see Fig. 5.1.5). Note, also, that rays making angles of $\theta$ and $\theta+\pi$ with a given ray point in opposite directions along the same straight line (see Fig. 5.1.6).


Figure 5.1.5. The angle $-\theta$, or $2 \pi-\theta$, is the mirror image of $\theta$.


Figure 5.1.6. The rays making angles of $\theta$ and $\theta+\pi$ with $O P$ point in opposite directions.

Example 2 (a) Convert to radians: $36^{\circ}, 160^{\circ}, 280^{\circ},-300^{\circ}, 460^{\circ}$.
(b) Convert to degrees: $5 \pi / 18,2.6,6.27,0.2,-9.23$.

Solution (a) $36^{\circ} \rightarrow 36 \times 0.01745=0.6282$ radian;
$160^{\circ} \rightarrow 160 \times 0.01745:=2.792$ radians;
$280^{\circ} \rightarrow 280 \times 0.01745=4.886$ radians;
$-300^{\circ} \rightarrow-300 \times 0.01745=-5.235$ radians, or
$-300 \times \pi / 180=-5 \pi / 3$ radians;
$460^{\circ} \rightarrow 460^{\circ}-360^{\circ}=100^{\circ} \rightarrow 100 \times 0.01745=1.745$ radians.
(b) $5 \pi / 18 \rightarrow 5 \pi / 18 \times 180 / \pi=50^{\circ}$;
$2.6 \rightarrow 2.6 \times 57.296=148.97^{\circ}$;
$6.27 \rightarrow 6.27 \times 57.296=359.25^{\circ}$;
$0.2 \rightarrow 0.2 \times 57.296=11.46^{\circ}$.

$$
-9.23 \rightarrow-9.23 \times 180 / \pi=-528.84^{\circ} \rightarrow 720^{\circ}-528.84^{\circ}=191.16^{\circ}
$$

Cartesian coordinates $(x, y)$ represent points in the plane by their distances from two perpendicular lines. In the polar coordinate representation, a point $P$ is associated with each pair $(r, \theta)$ of numbers in the following way. ${ }^{2}$ First, a ray is drawn through the origin making an angle of $\theta$ with the positive $x$ axis. Then one travels a distance $r$ along this ray, if $r$ is positive. (See Fig. 5.1.7.) If $r$ is negative, one travels a distance $-r$ along the ray traced in the opposite


Figure 5.1.8. Plotting $(r, \theta)$ for negative $r$.

[^0]direction. One arrives at the point $P$; we call $(r, \theta)$ its polar coordinates. Notice that the resulting point is the same as the one with polar coordinates $(-r, \theta+\pi)$ (see Fig. 5.1.8) and that the pair $(r, \theta+2 \pi n)$ represents the same point as $(r, \theta)$, for any integer $n$.

Example 3 Plot the points $P_{1}, P_{2}, P_{3}$, and $P_{4}$ whose polar coordinates are $(5, \pi / 6)$, $(-5, \pi / 6),(5,-\pi / 6)$, and $(-5,-\pi / 6)$, respectively.
Solution (See Fig. 5.1.9.) The point $(-5,-\pi / 6)$ is obtained by rotating $\pi / 6=30^{\circ}$ clockwise to give an angle of $-\pi / 6$ and then moving 5 units backwards on

Figure 5.1.9. Some points in polar coordinates.

this line to the point $P_{4}$ shown. The other points are ploted in a similar way.

Example 4 Describe the set of points $P$ whose polar coordinates $(r, \theta)$ satisfy $0 \leqslant r \leqslant 2$ and $0 \leqslant \theta<\pi$.
Solution


Figure 5.1.10. The region $0 \leqslant r \leqslant 2,0 \leqslant \theta<\pi$.


Figure 5.1.11. The cosine and sine of $\theta$ are the $x$ and $y$ coordinates of the point $P$.

Since $0 \leqslant r \leqslant 2$, we can range from the origin to 2 units from the origin. Our angle with the $x$ axis varies from 0 to $\pi$, but not including $\pi$. Thus we are confined to the region in Fig. 5.1.10. The negative $x$ axis is dashed since it is not included in the region.
If $\theta$ is a real number, we define $\cos \theta$ to be $x$ and $\sin \theta$ to be $y$, where $(x, y)$ are the cartesian coordinates of the point $P$ on the circle of cadius one whose polar coordinates are ( $1, \theta$ ). (See Fig. 5.1.11.) If an angle $\phi^{\circ}$ is given in degrees, $\sin \phi^{\circ}$ or $\cos \phi^{\circ}$ means $\sin \theta$ or $\cos \theta$, where $\theta$ is the same angle measured in radians. Thus $\sin 45^{\circ}=\sin (\pi / 4), \cos 60^{\circ}=\cos (\pi / 3)$, and so on.

The sine and cosine functions can also be defined in terms of ratios of sides of right triangles. (See Fig. 5.1.12.) By definition, $\cos \theta=\left|O A^{\prime}\right|$, and by similar triangles,

$$
\cos \theta=\left|O A^{\prime}\right|=\left|\frac{O A^{\prime}}{1}\right|=\frac{\left|O A^{\prime}\right|}{\left|O B^{\prime}\right|}=\frac{|O A|}{|O B|}=\frac{\text { side adjacent to } \theta}{\text { hypotenuse }} .
$$

Figure 5.1.12. The triangles $O A B$ and $O A^{\prime} B^{\prime}$ are similar; $\cos \theta=|O A| /|O B|$ and $\sin \theta=|A B| /|O B|$.



Figure 5.1.13. Converting polar to cartesian coordinates.

In the same way, we see that

$$
\sin \theta=\frac{|A B|}{|O B|}=\frac{\text { side opposite to } \theta}{\text { hypotenuse }}
$$

It follows (see Fig. 5.1.13) that if the point $B$ has cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$, then $\cos \theta=|O A| /|O B|=x / r$ and $\sin \theta$ $=|A B| /|O B|=y / r$, so we obtain the following relations.

## Cartesian and Polar Coordinates

$$
x=r \cos \theta \quad \text { and } \quad y=r \sin \theta
$$

where $(x, y)$ are cartesian coordinates and $(r, \theta)$ are polar coordinates.

Example 5 Convert from cartesian to polar coordinates: (2, -4 ); and from polar to cartesian coordinates: $(6,-\pi / 8)$.
Solution We plot $(2,-4)$ as in Fig. 5.1.14. Then $r=\sqrt{2^{2}+(-4)^{2}}=\sqrt{20}=2 \sqrt{5}$ and


Figure 5.1.14. Find the polar coordinates of $(2,-4)$.

Example 6

Solution


Figure 5.1.15. If $\theta$
$=\angle B O A$, then $\angle O B A$
$=\pi / 2-\theta$. $\cos \theta=2 /(2 \sqrt{5})=1 / \sqrt{5} \approx 0.447214$, so from tables or a calculator ${ }^{3}, \theta=$ 1.107; but we must take $\theta=-1.107$ (or 5.176 ) since we are in the fourth quadrant. Thus the polar coordinates of $(2,-4)$ are $(2 \sqrt{5},-1.107)$.

The cartesian coordinates of the point with polar coordinates $(6,-\pi / 8)$ are

$$
x=r \cos \theta=6 \cos (-\pi / 8)=(6)(0.92388)=5.5433
$$

and

$$
y=r \sin \theta=6 \sin (-\pi / 8)=(6)(-0.38268)=-2.2961
$$

That is, $(5.5433,-2.2961)$. This point is also in the fourth quadrant as it should be.
(a) Show that $\sin \theta=\cos (\pi / 2-\theta)$ for $0 \leqslant \theta \leqslant \pi / 2$. (b) Show that $\sin$ is an odd function: $\sin (-\theta)=-\sin \theta$ (assume that $0 \leqslant \theta \leqslant \pi / 2$ ).
(a) In Fig. 5.1.15, the angle $O B A$ is $\pi / 2-\theta$ since the three angles must add up to $\pi$ by plane geometry. Therefore $\sin \theta=$ (opposite/hypotenuse) $=$ $|A B| /|O B|$ and $\cos (\pi / 2-\theta)=($ adjacent $/$ hypotenuse $)=|A B| /|O B|=$ $\sin \theta$.
(b) Referring to Fig. 5.1.16, we see that if $\theta$ is switched to $-\theta$, this changes the $\operatorname{sign}$ of $y=\sin \theta$. Hence $\sin (-\theta)=-\sin \theta$.

Figure 5.1.16. $\sin \theta$ is an odd function.

${ }^{3}$ Many calculators are equipped with a $\cos ^{-1}$ (arccos) function which computes the angle whose cosine is given. If your calculator does not have such a function, you can use the cosine function together with the method of bisection (see Example 7, Section 3.1). The inverse cosine (and other trigonometric) functions are discussed in Section 5.4.


Figure 5.1.17. Two basic examples.


Figure 5.1.19. Illustrating the sine and cosine of $2 \pi / 3$.

The other trigonometric functions can be defined in terms of the sine and cosine:

$$
\begin{array}{ll}
\text { Tangent: } & \tan \theta=\frac{\sin \theta}{\cos \theta}=\frac{|A B|}{|O A|}=\frac{\text { side opposite }}{\text { side adjacent }} . \\
\text { Cotangent: } & \cot \theta=\frac{\cos \theta}{\sin \theta}=\frac{1}{\tan \theta}=\frac{|O A|}{|A B|} . \\
\text { Secant: } & \sec \theta=\frac{1}{\cos \theta}=\frac{|O B|}{|O A|} . \\
\text { Cosecant: } & \csc \theta=\frac{1}{\sin \theta}=\frac{|O B|}{|A B|} .
\end{array}
$$

Some frequently used values of the trigonometric functions can be read off the right triangles shown in Fig. 5.1.17. For example, $\cos (\pi / 4)=1 / \sqrt{2}$, $\sin (\pi / 4)=1 / \sqrt{2}, \tan (\pi / 4)=1, \cos (\pi / 6)=\sqrt{3} / 2$, and $\tan (\pi / 3)=\sqrt{3}$. (The proof that the $1,2, \sqrt{3}$ triangle has angles $\pi / 3, \pi / 6, \pi / 2$ is an exercise in euclidean geometry; see Fig. 5.1.18.)

Figure 5.1.18. The angles of an equilateral triangle are all equal to $\pi / 3$.


Special care should be taken with functions of angles which are not between 0 and $\pi / 2$-that is, angles not in the first quadrant-to ensure that their signs are correct. For instance, we notice in Fig. 5.1.19 that $\sin (2 \pi / 3)=\sqrt{3} / 2$ and $\cos (2 \pi / 3)=-\frac{1}{2}$.

The following table gives some commonly used values of $\sin , \cos$, and $\tan$ :

| $\theta^{\circ}$ | $0^{\circ}$ | $30^{\circ}$ | $45^{\circ}$ | $60^{\circ}$ | $90^{\circ}$ | $120^{\circ}$ | $135^{\circ}$ | $150^{\circ}$ | $180^{\circ}$ | $270^{\circ}$ | $360^{\circ}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ | $\frac{3 \pi}{2}$ | $2 \pi$ |
| $\cos \theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | 0 | 1 |
| $\sin \theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | -1 | 0 |
| $\tan \theta$ | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | $\pm \infty$ | $-\sqrt{3}$ | -1 | $-\frac{\sqrt{3}}{3}$ | 0 | $\pm \infty$ | 0 |

Over the centuries, large tables of values of the trigonometric functions have been complied. The first such table, compiled by Hipparchus and Ptolemy, appeared in Ptolemy's Almagest. Today these values are also on many pocket calculators. Since angles as well as some lengths can be directly measured (as in surveying), the trigonometric relations can then enable us to compute lengths which may be inaccessible (see Example 7).

## 匋 Calculator Discussion

You may be curious about how pocket calculators compute their values of $\sin \theta$ and $\cos \theta$. Some analytic expressions are available, such as

$$
\sin \theta=\theta-\frac{\theta^{3}}{3 \cdot 2}+\frac{\theta^{5}}{5 \cdot 4 \cdot 3 \cdot 2}-\frac{\theta^{7}}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}+\cdots
$$

(as will be proved in Section 12.5), but using these is inefficient and inaccurate. Instead a rational function of $\theta$ is fitted to many known values of $\sin \theta$ (or $\cos \theta, \tan \theta$, and so on) and this rational function is used to calculate approximate values at the remaining points. Thus when $\theta$ is entered and $\sin \theta$ pressed, a program in the calculator calculates the value of this rational function.

If you experiment with your calculator-for example, by calculating $\tan \theta$ for $\theta$ near $\pi / 2-$ you might discover some inaccuracies in this method. ${ }^{4}$

Example 7 (a) In Fig. 5.1.20, find $x$. (b) A tree 50 meters away subtends an angle of $53^{\circ}$ as seen by an observer. How tall is the tree?

Figure 5.1.20. Find $x$.


Solution (a) We find that $\tan 22^{\circ}=10.3 / x$, so $x=10.3 / \tan 22^{\circ}$. From tables or a calculator, $\tan 22^{\circ} \approx 0.404026$, so $x \approx 25.4934$. (b) Refer to Fig. 5.1.21. $|A B|=|O A| \tan 53^{\circ}=50 \tan 53^{\circ}$. Using tables or a calculator, this becomes $50(1.3270)=66.35$ meters .

Figure 5.1.21. Trigonometry used to find the height of a tree.


From the definition of $\sin$ and $\cos$, the point $P$ with cartesian coordinates $x=\cos \theta$ and $y=\sin \theta$ lies on the unit circle $x^{2}+y^{2}=1$. Therefore, for any value of $\theta$,

$$
\begin{equation*}
\cos ^{2} \theta+\sin ^{2} \theta=1 \tag{1}
\end{equation*}
$$

This is an example of trigonometric identity-a relationship among the trigonometric functions which is valid for all $\theta$.

[^1]

Figure 5.1.23. Data for the law of cosines: $c^{2}=a^{2}+$ $b^{2}-2 a b \cos \theta$.


Figure 5.1.24. Find $x$.


Figure 5.1.25. Geometry for the proof of the addition formulas.

Relationship (1) is, in essence, a statement of Pythagoras' theorem for a right triangle $\left(O A^{\prime} B^{\prime}\right.$ in Fig. 5.1.12). For a general triangle, the correct relationship between the three sides is given by the law of cosines: with notation as in Fig. 5.1.22, we have

$$
\begin{equation*}
c^{2}=a^{2}+b^{2}-2 a b \cos \theta \tag{2}
\end{equation*}
$$


(a) 0 acute

(b) 0 obtuse

Figure 5.1.22. Proving the law of cosines.

To prove equation (2), note that the $(x, y)$ coordinates of $B$ are $x$ $=b \cos \theta$ and $y=b \sin \theta$; those of $A$ are $x=a, y=0$. By the distance formula and equation (1),

$$
\begin{aligned}
c^{2} & =(b \cos \theta-a)^{2}+(b \sin \theta)^{2}=b^{2} \cos ^{2} \theta-2 a b \cos \theta+a^{2}+b^{2} \sin ^{2} \theta \\
& =b^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right)+a^{2}-2 a b \cos \theta=b^{2}+a^{2}-2 a b \cos \theta,
\end{aligned}
$$

so equation (2) is proved.
In Fig. 5.1.22 we situated the triangles in a particular way, but this was just a device to prove equation (2); since any triangle can be moved into this special position, the formula holds in the general situation of Fig. 5.1.23.

Note that when $\theta=\pi / 2, \cos \theta=0$ and so equation (2) reduces to Pythagoras' theorem: $c^{2}=a^{2}+b^{2}$.

Example 8 In Fig. 5.1.24, find $x$.
Solution By the law of cosines

$$
\begin{aligned}
x^{2} & =(20.2)^{2}+(13.4)^{2}-2(20.2)(13.4) \cos \left(10.3^{\circ}\right) \\
& =(408.04)+(179.56)-532.64=54.96 .
\end{aligned}
$$

Taking square roots, we find $x \approx 7.41$.
Now consider the situation in Fig. 5.1.25. By the distance formula,

$$
\begin{aligned}
|P Q|^{2} & =(\cos \phi-\cos \theta)^{2}+(\sin \phi-\sin \theta)^{2} \\
& =\cos ^{2} \phi-2 \cos \phi \cos \theta+\cos ^{2} \theta+\sin ^{2} \phi-2 \sin \phi \sin \theta+\sin ^{2} \theta \\
& =2-2 \cos \phi \cos \theta-2 \sin \phi \sin \theta
\end{aligned}
$$

On the other hand, by the law of cosines (2) applied to $\triangle O P Q$,

$$
|P Q|^{2}=1^{2}+1^{2}-2 \cos (\phi-\theta),
$$

since $\phi-\theta$ is the angle at the vertex $O$. Comparing our two expressions for $|P Q|^{2}$ gives the identity

$$
\cos (\phi-\theta)=\cos \phi \cos \theta+\sin \phi \sin \theta
$$

which is valid for all $\phi$ and $\theta$. If we replace $\theta$ by $-\theta$ and recall that $\cos (-\theta)=\cos \theta$ and $\sin (-\theta)=-\sin \theta$, this identity yields

$$
\begin{equation*}
\cos (\theta+\phi)=\cos \theta \cos \phi-\sin \theta \sin \phi . \tag{3}
\end{equation*}
$$

Now if we write $\pi / 2-\phi$ for $\phi$ and recall that $\cos (\pi / 2-\phi)=\sin \phi$ and $\sin (\pi / 2-\phi)=\cos \phi$, the same identity gives

$$
\begin{equation*}
\sin (\theta+\phi)=\sin \theta \cos \phi+\cos \theta \sin \phi . \tag{4}
\end{equation*}
$$

Identities (3) and (4), called the addition formulas for sine and cosine, will be essential for calculus. From these basic identities, we can also derive many others by algebraic manipulation.

For integral calculus, two of the most important consequences of (3) are the double-angle formulas. Setting $\theta=\phi$ in (3) gives

$$
\cos 2 \theta=\cos ^{2} \theta-\sin ^{2} \theta=\left(1-\sin ^{2} \theta\right)-\sin ^{2} \theta=1-2 \sin ^{2} \theta .
$$

Thus

$$
\begin{equation*}
\sin ^{2} \theta=\frac{1}{2}(1-\cos 2 \theta) \tag{5}
\end{equation*}
$$

Similarly,

$$
\cos 2 \theta=\cos ^{2} \theta-\sin ^{2} \theta=2 \cos ^{2} \theta-1
$$

gives

$$
\begin{equation*}
\cos ^{2} \theta=\frac{1}{2}(\cos 2 \theta+1) . \tag{6}
\end{equation*}
$$

Example 9 (a) Prove the product formula: $\cos \theta \cos \phi=\frac{1}{2}[\cos (\theta-\phi)+\cos (\theta+\phi)]$.
(b) Prove that $1+\tan ^{2} \theta=\sec ^{2} \theta$.

Solution (a) Add the identity for $\cos (\theta-\phi)$ to that for $\cos (\theta+\phi)$ :

$$
\begin{aligned}
\cos (\theta-\phi)+\cos (\theta+\phi) & =\cos \phi \cos \theta+\sin \phi \sin \theta+\cos \theta \cos \phi-\sin \theta \sin \phi \\
& =2 \cos \theta \cos \phi .
\end{aligned}
$$

Dividing by 2 gives the product formula.
(b) Divide both sides of $\sin ^{2} \theta+\cos ^{2} \theta=1$ by $\cos ^{2} \theta$; then, using $\tan \theta=$ $\sin \theta / \cos \theta$ and $1 / \cos \theta=\sec \theta$, we get $\tan ^{2} \theta+1=\sec ^{2} \theta$ as required.
Some of the most important trigonometric identities are listed on the inside front cover of the book for handy reference. They are all useful, but you can get by quite well by memorizing only (1) through (4) above and deriving the rest when you need them.

From the available values of the trigonometric functions, one can accurately draw their graphs. The calculus of these functions, studied in the next section, confirms that these graphs are correct, so there are no maxima, minima, or inflection points other than those in plain view in Fig. 5.1.26 on the following page.

Perhaps the most important fact about these functions is their periodicity: When a function $f$ satisfies $f(\theta+\tau)=f(\theta)$ for all $\theta$ and a given positive $\tau, f$ is said to be periodic with period $\tau$. The reciprocal $1 / \tau$ is called the frequency. All the functions in Fig. 5.1.26 are periodic with period $2 \pi$; this enables us to draw the entire graph by repeating the segment over an interval of length $2 \pi$. The trigonometric functions also have $4 \pi, 6 \pi, 8 \pi, \ldots$ as additional periods, but the least period of a periodic function is always unique. Note that the functions $\tan$ and $\cot$ have $\pi$ as their least period, while $2 \pi$ is the least period of the other four trigonometric functions.



Figure 5.1.26. Graphs of the trigonometric functions.

(c) $y=\tan \theta$

(d) $y=\sec \theta$

(f) $y=\cot \theta$

Solution (a) We obtain $y=\cos 2 x$ by taking the graph of $y=\cos x$ and compressing

Example 10

Figure 5.1.27. The graph of $y=\cos 2 x$.
(a) Sketch the graph $y=\cos 2 x$. What is the (least) period of this function?
(b) Sketch the graph of $y=3 \cos 5 \theta$. the graph horizontally by a factor of 2 (see Fig. 5.1.27). The function repeats every $\pi$ units on the $x$ axis, so it is periodic with (least) period $\pi$.

(b) We obtain $y=3 \cos 5 \theta$ by compressing the graph of $y=\cos \theta$ horizontally by a factor of 5 and stretching it vertically by a factor of 3 (see Fig. 5.1.28).

Figure 5.1.28. The graph of $y=3 \cos 5 \theta$.


Example 11 Where are the inflection points of $\tan \theta$ ? For which values of $\theta$ do you expect $\tan \theta$ to be a differentiable function?

Solution Recall that an inflection point is a point where the second derivative changes sign-that is, a point between different types of concavity. On the graph of $\tan \theta$, notice that the graph is concave upward on ( $0, \pi / 2$ ) and downward on $(-\pi / 2,0)$. Hence 0 is an inflection point, as are $\pi,-\pi$, and so forth. The general inflection point is $n \pi$, where $n=0, \pm 1, \pm 2, \ldots$ (But $\pi / 2$ is not an inflection point because $\tan \theta$ is not defined there.)

From the graph, we expect that $\tan \theta$ will be a differentiable function of $\theta$ except at $\pm \pi / 2, \pm 3 \pi / 2, \ldots$.

## Exercises for Section 5.1

1. If an arc of a circle with radius 10 meters subtends an angle of $22^{\circ}$, how long is the arc? How much area is enclosed in this part of the circle?
2. An arc of radius 15 feet subtends an angle of 2.1 radians. How long is the arc? How much area is enclosed in this part of the circle?
3. An arc of radius 18 meters has length 5 meters. What angle does it subtend? How much area is enclosed in this part of the circle?
4. An arc of length 110 meters subtends an angle of
$24^{\circ}$. What is the radius of the arc? How much area is enclosed in this part of the circle?
5. Convert to radians: $29^{\circ}, 54^{\circ}, 255^{\circ}, 130^{\circ}, 320^{\circ}$.
6. Convert to degrees: $5, \pi / 7,3.2,2 \pi / 9, \frac{1}{2}, 0.7$.
7. Simplify so that $0 \leqslant \theta<2 \pi$ or $0 \leqslant \theta^{\circ}<360^{\circ}$.
(a) Radians: $7 \pi / 3,16 \pi / 5,15 \pi$.
(b) Degrees: $520^{\circ}, 1745^{\circ}, 385^{\circ}$.
8. Simplify so that $0 \leqslant \theta<2 \pi$ or $0 \leqslant \theta^{\circ}<360^{\circ}$.
(a) Radians: $5 \pi / 2,48 \pi / 11,13 \pi+1$.
(b) Degrees: $470^{\circ}, 604^{\circ}, 75^{\circ}, 999^{\circ}$.
9. Plot the following points given in polar coordinates: $(3, \pi / 2),(5,-\pi / 4),(1,2 \pi / 3),(-3, \pi / 2)$.
10. Plot the following points given in polar coordinates: $(6,3 \pi / 2),(-2, \pi / 6),(7,-2 \pi / 3),(1, \pi / 2)$, (4, $-\pi / 6$ ).
In Exercises 11-14, sketch the set of points whose polar coordinates $(r, \theta)$ satisfy the given conditions.
11. $-1 \leqslant r \leqslant 2 ; \pi / 3 \leqslant \theta<\pi / 2$.
12. $0<r<4 ;-\pi / 6<\theta<\pi / 6$.
13. $2 \leqslant r<3 ;-\pi / 2 \leqslant \theta \leqslant \pi$.
14. $-2 \leqslant r \leqslant-1 ;-\pi / 4<\theta<0$.
15. Find the polar coordinates of $(x, y)=$ $(5,-2)$.
16. Find the cartesian coordinates of $(r, \theta)=$ (2, $\pi / 6$ ).
17. Convert from cartesian to polar coordinates:
(a) $(1,0),(b)(3,4)$,
(c) $(\sqrt{3}, 1)$,
(d) $(\sqrt{3},-1)$,
(e) $(-\sqrt{3}, 1)$.
18. Convert from polar to cartesian coordinates:
(a) $(0, \pi / 8),(b)(1,0),(c)(2, \pi / 4),(d)(8,3 \pi / 2)$, (e) $(2, \pi)$.
19. Convert from cartesian to polar coordinates:
(a) $(1,-1)$,
(b) $(0,2),(c)\left(\frac{1}{2}, 7\right)$,
(d) $(-12,-5)$,
(e) $(-3,8)$, (f) $\left(\frac{3}{4}, \frac{3}{4}\right)$.
20. Convert from cartesian to polar coordinates:
(a) $(-4,4),(b)(1,15),(c)(19,-3),(d)(-5,-6)$, (e) $(0.3,0.9)$, (f) $\left(-\frac{3}{2}, \frac{1}{2}\right)$.
21. Convert from polar to cartesian coordinates:
(a) $(6, \pi / 2)$,
(b) $(-12,3 \pi / 4)$,
(c) $(4,-\pi)$,
(d) $(2,13 \pi / 2)$, (e) $(8,-2 \pi / 3),(f)(-1,2)$.
22. Convert from polar to cartesian coordinates:
(a) $(-1,-1),(b)(1, \pi)$, (c) $(10,2.7)$,
(d) $(5,7 \pi / 2),(e)(8,7 \pi)$, (f) $(4,-3 \pi)$.
23. Show that $\tan \theta=\cot (\pi / 2-\theta)$ assuming that $0 \leqslant \theta \leqslant \pi / 2$.
24. Show that $\sec \theta=\csc (\pi / 2-\theta)$ for $0 \leqslant \theta \leqslant \pi / 2$.
25. Show that $\cos \theta=\cos (-\theta)$ for $0 \leqslant \theta \leqslant \pi$.
26. Show that $\tan \theta=-\tan (-\theta)$ for $0 \leqslant \theta \leqslant \pi$, $\theta \neq \pi / 2$.
Refer to Fig. 5.1.29 for Exercises 27-30.
27. Find $a$.
28. Find $b$.

29. Find $d$.


Figure 5.1.29. Find $a, b, c, d$.
31. An airplane flying at 5000 feet has an angle of elevation of $25^{\circ}$ from observer $A$. Observer $B$ sees that airplane directly overhead. How far apart are $A$ and $B$ ?
32. A leaning tower tilts at $9^{\circ}$ from the vertical directly away from an observer who is 500 meters away from its base. If the observer sees the top of the tower at an angle of elevation of $22^{\circ}$, how high is the tower?
33. A mountain 3000 meters away subtends an angle of $17^{\circ}$ at an observer. How tall is the mountain?
34. A pedestrian 100 meters from the outdoor elevator at the Fairhill Hotel at noon sees the elevator at an angle of $10^{\circ}$. The elevator, steadily rising, makes an angle of $20^{\circ}$ after 30 seconds has elapsed. How fast is the elevator rising? When will it make an angle of $30^{\circ}$ ?
Refer to Fig. 5.1.30 for Exercises 35-38.
35. Find $p$.
36. Find $q$.

37. Find $r$.

38. Find $s$.


Figure 5.1.30. Find $p, q, r$, and $s$.
39. Prove that $\cos ^{2} \frac{\theta}{2}=\frac{1+\cos \theta}{2}$.
40. Prove that $\tan \frac{\theta}{2}=\frac{\sin \theta}{1+\cos \theta}$.
41. Prove that $\sin \theta \sin \phi=\frac{\cos (\theta-\phi)-\cos (\theta+\phi)}{2}$.
42. Express $\sin (3 \theta)$ in terms of $\sin \theta$ and $\cos \theta$. Simplify the expressions in Exercises 43-50.
43. $\sin \left(\theta+\frac{\pi}{2}\right)$
44. $\sin \left(\frac{3 \pi}{2}+\theta\right)$
45. $\cos \left(\frac{3 \pi}{2}-\theta\right)$
46. $\tan \left(\theta+\frac{7}{2} \pi\right)$
47. $\sec (6 \pi+\theta)$
48. $\sin \left(\theta-\frac{9}{2} \pi\right)$
49. $\cos \left(\theta+\frac{\pi}{2}\right) \sin \left(\phi-\frac{3 \pi}{2}\right)$.
50. $\frac{\sin \left(\theta+\frac{5}{2} \pi\right)}{\cos \left(\frac{\pi}{2}-\theta\right)}$.

Compute the quantities in Exercises 51-54 by using trigonometric identities, without using tables or a calculator.
51. $\cos 7 \frac{1}{2}^{\circ}$
52. $\tan 22 \frac{1}{2}^{\circ}$
53. $\sec \frac{\pi}{12}$
54. $\sin \frac{\pi}{12}$

Derive the identities in Exercises 55-60, making use of the table of identities on the inside front cover of the book.
55. $\sec \theta+\tan \theta=\frac{1+\sin \theta}{1-2 \sin ^{2}(\theta / 2)}$.
56. $8 \cos \theta+8 \cos 2 \theta=-9+16\left(\cos \theta+\frac{1}{4}\right)^{2}$.
57. $\sec ^{2} \frac{\theta}{2}=\frac{2 \sec \theta}{\sec \theta+1}$.
58. $\csc ^{2} \frac{\theta}{2}=\frac{2 \sec \theta}{\sec \theta-1}$.
59. $\csc \theta \csc \phi=\frac{2 \sec (\theta+\phi) \sec (\theta-\phi)}{\sec (\theta+\phi)-\sec (\theta-\phi)}$.
60. $\tan \theta \tan \phi=\frac{\sec (\theta+\phi)-\sec (\theta-\phi)}{\sec (\theta+\phi)+\sec (\theta-\phi)}$.

Sketch the graph of the functions in Exercises 61-68.
61. $2 \cos 3 \theta$
62. $\cos \left(3 \theta+\frac{\pi}{6}\right)$
63. $\tan \frac{3 \theta}{2}$
64. $\tan \frac{\theta}{2}$
65. $4 \sin 2 x \cos 2 x$
66. $\sin x+\cos x$
67. $\sin 3 \theta+1$
68. $\csc 2 \theta$
69. Locate the inflection points of $\cot \theta$ by inspecting the graph.
70. Locate the maximum and minimum points of $\sin \theta$ by inspecting the graph.
71. For what $\theta$ do you expect $\sec \theta$ and $\cot \theta$ to be differentiable?
72. Where is $\sec \theta$ concave upward?
73. Light travels at velocity $v_{1}$ in a certain medium, enters a second medium at angle of incidence $\theta_{1}$ (measured from the normal to the surface), and refracts at angle $\theta_{2}$ while travelling a different velocity $v_{2}$ (in the second medium). According to Snell's law, $v_{1} / v_{2}=\sin \theta_{1} / \sin \theta_{2}$.
(a) Light enters at $60^{\circ}$ and refracts at $30^{\circ}$. The first medium is air ( $v_{1}=3 \times 10^{10}$ centimeters per second). Find the velocity in the second medium.
(b) Show that if $v_{1}=v_{2}$, then the light travels in a straight line.
(c) The speed halves in passing from one medium to another, the angle of incidence being $45^{\circ}$. Calculate the angle of refraction.
 tions $\sin \theta=\theta$ and $\cos \theta=1$, valid for $\theta$ near zero. Experiment with your hand calculator to determine a region of validity for $\theta$ that guarantees eight-place accuracy for these approximations.
Exercises 75-78 concern the law of sines.
75. Using the notation in Fig. 5.1.31, prove that

$$
\frac{\sin \alpha}{a}=\frac{\sin \beta}{b}=\frac{\sin \gamma}{c} .
$$



Figure 5.1.31. Law of sines.
76. Find $a$ in Fig. 5.1.32.


Figure 5.1.32. Find $a$.
77. Show that the common value of $(\sin \alpha) / a$, $(\sin \beta) / b$, and $(\sin \gamma) / c$ is the reciprocal of twice the radius of the circumscribed circle.
78. (a) A parallelogram is formed with acute angle $\theta$ and sides $l, L$. Find a formula for its area. (b) A parallelogram is formed with an acute angle $\theta$, a base of length $L$, and a diagonal opposite $\theta$ of length $d$. Find a formula for its area.
79. Show that $A \cos \omega t+B \sin \omega t=\alpha \cos (\omega t-\theta)$, where $(\alpha, \theta)$, are the polar coordinates of $(A, B)$.
80. Use Exercise 79 to write $f(t)=\cos t+\sqrt{3} \sin t$ as $\alpha \cos (t-\theta)$ for some $\alpha$ and $\theta$. Use this to graph $f$.
81. Light of wavelength $\lambda$ is diffracted through a single slit of width $a$. This light then passes through a lens and falls on a screen. A point $P$ is on the screen, making an angle $\theta$ with the lens axis. The intensity $I$ at the point $P$ on the screen is

$$
I=I_{0}\left[\frac{\sin ([\pi a \sin \theta] / \lambda)}{[\pi a \sin \theta] / \lambda}\right]^{2}, \quad 0<\theta<\pi,
$$

where $I_{0}$ is the intensity when $P$ is on the lens axis.
(a) Show that the intensity is zero for $\sin \theta$ $=\lambda / a$.
(b) Find all values of $\theta$ for which $\theta>0$ and $I=0$.
(c) Verify that $I$ is approximately $I_{0}$ when $[\pi a \sin \theta] / \lambda$ is close enough to zero. (Use $\sin \theta \approx \theta$; see Exercise 74.)
(d) In practice, $\lambda=5 \times 10^{-5}$ centimeters, and $a=10^{-2}$ centimeters. Check by means of a calculator or table that $\sin \theta=\lambda / a$ is approximately $\theta=\lambda / a$.
82. The current $I$ in a circuit is given by the formula $I(t)=20 \sin (311 t)+40 \cos (311 t)$. Let $r=\left(20^{2}+\right.$ $\left.40^{2}\right)^{1 / 2}$ and define the angle $\theta$ by $\cos \theta=20 / r$, $\sin \theta=40 / r$.
(a) Verify by use of the sum formula for the sine function that $I(t)=r \sin (311 t+\theta)$.
(b) Show that the peak current is $r$ (the maximum value of $I$ ).
(c) Find the period and frequency.
(d) Determine the phase shift (in radians), that is, the value of $t$ which makes $311 t+\theta=0$.
83. The instantaneous power input to an AC circuit is $p=v i$, where $v$ is the instantaneous potential difference between the circuit terminals and $i$ is the instantaneous current. If the circuit is a pure resistor, then $v=V \sin (\omega t)$ and $i=I \sin (\omega t)$.
(a) Verify by means of trigonometric identities that

$$
p=\frac{V I}{2}-\frac{V I}{2} \cos (2 \omega t)
$$

(b) Draw a graph of $p$ as a function of $t$.
84. Two points are located on one bank of the Colorado River, 500 meters apart. A point on the opposite bank makes angles of $88^{\circ}$ and $80^{\circ}$ with the line joining the two points. Find the lengths of two cables to be stretched across the river connecting the points.

### 5.2 Differentiation of the Trigonometric Functions



Figure 5.2.1. The point $P$ moves at unit speed around the circle.


Figure 5.2.2. Geometry used to determine $\cos ^{\prime} 0$ and $\sin ^{\prime} 0$.

Differentiation rules for sine and cosine follow from arguments using limits and the addition formulas.

In this section, we will derive differentiation formulas for the trigonometric functions. In the course of doing so, we will use many of the basic properties of limits and derivatives introduced in the first two chapters.

The unit circle $x^{2}+y^{2}=1$ can be described by the parametric equations $x=\cos \theta, y=\sin \theta$. As $\theta$ increases, the point $(x, y)=(\cos \theta, \sin \theta)$ moves along the circle in a counterclockwise direction (see Fig. 5.2.1).

The length of arc on the circle between the point $(1,0)$ (corresponding to $\theta=0$ ) and the point $(\cos \theta, \sin \theta)$ equals the angle $\theta$ subtended by the arc. If we think of $\theta$ as time, the point $(\cos \theta, \sin \theta)$ travels a distance $\theta$ in time $\theta$, so it is moving with unit speed around the circle. At $\theta=0$, the tangent line to the circle is vertical, so the velocity of the point is 1 in the vertical direction; thus, we expect that

$$
\left.\frac{d x}{d \theta}\right|_{\theta=0}=0 \quad \text { and }\left.\quad \frac{d y}{d \theta}\right|_{\theta=0}=1 .
$$

That is,

$$
\begin{align*}
& \left.\frac{d \cos \theta}{d \theta}\right|_{\theta=0}=0  \tag{1}\\
& \left.\frac{d \sin \theta}{d \theta}\right|_{\theta=0}=1 \tag{2}
\end{align*}
$$

We will now derive (1) and (2) using limits.
According to the definition of the derivative, formulas (1) and (2) amount to the following statements about limits:

$$
\begin{equation*}
\lim _{\Delta \theta \rightarrow 0} \frac{\cos \Delta \theta-1}{\Delta \theta}=0 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\Delta \theta \rightarrow 0} \frac{\sin \Delta \theta}{\Delta \theta}=1 \tag{4}
\end{equation*}
$$

To prove (3) and (4), we use the geometry in Fig. 5.2 .2 and shall denote $\Delta \theta$ by the letter $\phi$ for simplicity of notation. When $0<\phi<\pi / 2$, we have
area triangle $O C B=\frac{1}{2}|O C| \cdot|A B|=\frac{1}{2} \sin \phi$,
area triangle $O C B<$ area sector $O C B=\frac{1}{2} \phi$,
and

$$
\text { area sector } \begin{aligned}
O C B<\text { area triangle } \begin{aligned}
O C D & =\frac{1}{2}|O C| \cdot|C D| \\
& =\frac{1}{2} \tan \phi=\frac{1}{2} \frac{\sin \phi}{\cos \phi}
\end{aligned} .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\sin \phi<\phi \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi<\frac{\sin \phi}{\cos \phi} \tag{6}
\end{equation*}
$$

For $-\pi / 2<\phi<0, \sin \phi$ is negative, and from (5) we have $-\sin \phi=\sin (-\phi)$ $<-\phi$, so $|\sin \phi|<|\phi|$ for all $\phi \neq 0$. Thus, if $\phi$ approaches zero, so must $\sin \phi$ :

$$
\begin{equation*}
\lim _{\phi \rightarrow 0} \sin \phi=0 \tag{7}
\end{equation*}
$$

Now, $\lim _{\phi \rightarrow 0} \cos \phi=\lim _{\phi \rightarrow 0} \sqrt{1-\sin ^{2} \phi}=\sqrt{1-\left(\lim _{\phi \rightarrow 0} \sin \phi\right)^{2}}=1$. (By continuity of the square root function, we can take the limit under the root sign.) Thus we have

$$
\begin{equation*}
\lim _{\phi \rightarrow 0} \cos \phi=1 \tag{8}
\end{equation*}
$$

Next, we use (5) and (6) to get

$$
\begin{equation*}
\cos \phi<\frac{\sin \phi}{\phi}<1 \tag{9}
\end{equation*}
$$

for $0<\phi<\pi / 2$. However, the expressions in (9) are unchanged if $\phi$ is replaced by $-\phi$, so (9) holds for $-\pi / 2<\phi<0$ as well. Now (9) and (8) imply

$$
\lim _{\phi \rightarrow 0} \frac{\sin \phi}{\phi}=1
$$

since $\sin \phi / \phi$ is squeezed between 1 and a function which approaches 1 . Thus, the limit statement (4) is proved.

To prove (3), we again use the Pythagorean identity:

$$
\sin ^{2} \phi=1-\cos ^{2} \phi=(1-\cos \phi)(1+\cos \phi)
$$

which implies $(1-\cos \phi) / \phi=\sin ^{2} \phi / \phi(1+\cos \phi)$; using the product and quotient rules for limits,

$$
\lim _{\phi \rightarrow 0} \frac{1-\cos \phi}{\phi}=\lim _{\phi \rightarrow 0} \frac{\sin \phi}{\phi} \frac{\lim _{\phi \rightarrow 0} \sin \phi}{1+\lim _{\phi \rightarrow 0} \cos \phi}=1 \cdot \frac{0}{1+1}=0
$$

## Calculator Discussion

We can confirm (3) and (4) by some numerical experiments. For instance, on our HP-15C calculator, we compute

$$
\frac{1-\cos (\Delta \theta)}{\Delta \theta} \text { and } \frac{\sin (\Delta \theta)}{\Delta \theta}
$$

for $\Delta \theta=0.02$ and 0.001 to be

$$
\begin{aligned}
& \frac{1-\cos \Delta \theta}{\Delta \theta}=0.009999665 \quad \text { for } \quad \Delta \theta=0.02 \\
& \frac{1-\cos \Delta \theta}{\Delta \theta}=0.000500000 \quad \text { for } \quad \Delta \theta=0.001
\end{aligned}
$$

and

$$
\begin{array}{ll}
\frac{\sin \Delta \theta}{\Delta \theta}=0.999933335 & \text { for } \quad \Delta \theta=0.02 \\
\frac{\sin \Delta \theta}{\Delta \theta}=0.999999833 & \text { for } \quad \Delta \theta=0.001
\end{array}
$$

Your answer may differ because of calculator inaccuracies. However, these numbers confirm that $(1-\cos \Delta \theta) / \Delta \theta$ is near zero for $\Delta \theta$ small and that $(\sin \Delta \theta) / \Delta \theta$ is near 1 for $\Delta \theta$ small.

Now we are ready to compute the derivatives of $\sin \theta$ and $\cos \theta$ at all values of $\theta$. According to the definition of the derivative,

$$
\frac{d}{d \theta} \sin \theta=\lim _{\Delta \theta \rightarrow 0}\left[\frac{\sin (\theta+\Delta \theta)-\sin \theta}{\Delta \theta}\right]
$$

From the addition formula for $\sin$, the right-hand side equals

$$
\begin{aligned}
\lim _{\Delta \theta \rightarrow 0} & {\left[\frac{\sin \theta \cos \Delta \theta+\cos \theta \sin \Delta \theta-\sin \theta}{\Delta \theta}\right] } \\
& =\lim _{\Delta \theta \rightarrow 0}\left[\frac{\sin \theta(\cos (\Delta \theta)-1)}{\Delta \theta}+\frac{\cos \theta \sin (\Delta \theta)}{\Delta \theta}\right] \\
& =\sin \theta \lim _{\Delta \theta \rightarrow 0}\left[\frac{\cos (\Delta \theta)-1}{\Delta \theta}\right]+\cos \theta \lim _{\Delta \theta \rightarrow 0}\left[\frac{\sin (\Delta \theta)}{\Delta \theta}\right]
\end{aligned}
$$

by the sum rule and constant multiple rule for limits. Substituting from (3) and (4) gives $(\sin \theta) \cdot 0+(\cos \theta) \cdot 1=\cos \theta$. Thus

$$
\frac{d}{d \theta} \sin \theta=\cos \theta
$$

We compute the derivative of $\cos \theta$ in a similar way:

$$
\begin{aligned}
\frac{d}{d \theta} \cos \theta & =\lim _{\Delta \theta \rightarrow 0}\left[\frac{\cos (\theta+\Delta \theta)-\cos \theta}{\Delta \theta}\right] \\
& =\lim _{\Delta \theta \rightarrow 0}\left[\frac{\cos \theta \cos (\Delta \theta)-\sin \theta \sin (\Delta \theta)-\cos \theta}{\Delta \theta}\right] \\
& =\lim _{\Delta \theta \rightarrow 0}\left[\cos \theta\left(\frac{\cos (\Delta \theta)-1}{\Delta \theta}\right)-\frac{\sin \theta \sin (\Delta \theta)}{\Delta \theta}\right] \\
& =-\sin \theta
\end{aligned}
$$

## Derivative of Sine and Cosine

$$
\frac{d}{d \theta} \sin \theta=\cos \theta \quad \text { and } \quad \frac{d}{d \theta} \cos \theta=-\sin \theta
$$

In words, the derivative of the sine function is the cosine and the derivative of the cosine is minus the sine. These formulas are worth memorizing. Study Fig. 5.2 .3 to check that they are consistent with the graphs of sine and cosine. For example, notice that on the interval $(0, \pi / 2), \sin \theta$ is increasing and its derivative $\cos \theta$ is positive.

Figure 5.2.3. Graphs of $\sin$, $\cos$, and their derivatives.



## Example 1 Differentiate

(a) $(\sin \theta)(\cos \theta)$,
(b) $\sin ^{2} \theta$
(c) $\sin 5 \theta$,
(d) $\frac{\sin 3 \theta}{\cos \theta+\theta^{4}}$.

Solution (a) By the product rule,

$$
\begin{aligned}
\frac{d}{d \theta}(\sin \theta)(\cos \theta) & =\left(\frac{d}{d \theta} \sin \theta\right) \cos \theta+\sin \theta\left(\frac{d}{d \theta} \cos \theta\right) \\
& =\cos \theta \cos \theta-\sin \theta \sin \theta=\cos ^{2} \theta-\sin ^{2} \theta
\end{aligned}
$$

(b) By the power of a function rule,

$$
\frac{d}{d \theta} \sin ^{2} \theta=\frac{d}{d \theta}(\sin \theta)^{2}=2 \sin \theta \frac{d}{d \theta} \sin \theta=2 \sin \theta \cos \theta
$$

(c) By the chain rule,

$$
\begin{aligned}
\frac{d}{d \theta} \sin 5 \theta & =\frac{d}{d u} \sin u \frac{d u}{d \theta} \quad(\text { where } u=5 \theta) \\
& =(\cos u)(5)=5 \cos 5 \theta
\end{aligned}
$$

(d) By the quotient rule and chain rule,

$$
\begin{aligned}
\frac{d}{d \theta} \frac{\sin 3 \theta}{\cos \theta+\theta^{4}} & =\frac{\left(\cos \theta+\theta^{4}\right)(d / d \theta) \sin 3 \theta-\sin 3 \theta(d / d \theta)\left(\cos \theta+\theta^{4}\right)}{\left(\cos \theta+\theta^{4}\right)^{2}} \\
& =\frac{\left(\cos \theta+\theta^{4}\right) 3 \cos 3 \theta-\sin 3 \theta\left(-\sin \theta+4 \theta^{3}\right)}{\left(\cos \theta+\theta^{4}\right)^{2}} \\
& =\frac{3 \cos \theta \cos 3 \theta+3 \theta^{4} \cos 3 \theta+\sin \theta \sin 3 \theta-4 \theta^{3} \sin 3 \theta}{\left(\cos \theta+\theta^{4}\right)^{2}}
\end{aligned}
$$

Example 2 Differentiate (a) $\cos \theta \sin ^{2} \theta$ and (b) $(\sin 3 x) /\left(1+\cos ^{2} x\right)$.
Solution (a) By the product rule and the power rule,

$$
\begin{aligned}
\frac{d}{d \theta}\left(\cos \theta \sin ^{2} \theta\right) & =\left(\frac{d}{d \theta} \cos \theta\right) \sin ^{2} \theta+\cos \theta\left(\frac{d}{d \theta} \sin ^{2} \theta\right) \\
& =(-\sin \theta) \sin ^{2} \theta+\cos \theta \cdot 2 \sin \theta \cos \theta \\
& =2 \cos ^{2} \theta \sin \theta-\sin ^{3} \theta
\end{aligned}
$$

(b) Here the independent variable is called $x$ rather than $\theta$. By the chain rule,

$$
\frac{d}{d x} \sin 3 x=3 \cos 3 x
$$

so, by the quotient rule,

$$
\begin{aligned}
\frac{d}{d x} \frac{\sin 3 x}{1+\cos ^{2} x} & =\frac{\left(1+\cos ^{2} x\right) \cdot 3 \cos 3 x-\sin 3 x \cdot 2 \cos x(-\sin x)}{\left(1+\cos ^{2} x\right)^{2}} \\
& =\frac{3 \cos 3 x\left(1+\cos ^{2} x\right)+2 \cos x \sin x \cdot \sin 3 x}{\left(1+\cos ^{2} x\right)^{2}} .
\end{aligned}
$$

Now that we know how to differentiate the sine and cosine functions, we can differentiate the remaining trigonometric functions by using the rules of calculus. For example, consider $\tan \theta=\sin \theta / \cos \theta$. The quotient rule gives:

$$
\begin{aligned}
\frac{d}{d \theta} \tan \theta & =\frac{\cos \theta(d / d \theta) \sin \theta-\sin \theta(d / d \theta) \cos \theta}{\cos ^{2} \theta} \\
& =\frac{\cos \theta \cdot \cos \theta+\sin \theta \cdot \sin \theta}{\cos ^{2} \theta}=\frac{1}{\cos ^{2} \theta}=\sec ^{2} \theta .
\end{aligned}
$$

In a similar way, we see that

$$
\frac{d}{d \theta} \cot \theta=-\csc ^{2} \theta
$$

Writing $\csc \theta=1 / \sin \theta$, we get $\csc ^{\prime} \theta=\left(-\sin ^{\prime} \theta\right) /\left(\sin ^{2} \theta\right)=(-\cos \theta) /\left(\sin ^{2} \theta\right)$ $=-\cot \theta \csc \theta$ and, similarly, $\sec ^{\prime} \theta=\tan \theta \sec \theta$.

The results we have obtained are summarized in the following box.

## Differentiation of Trigonometric Functions

| Function | Derivative | Leibniz notation |
| :--- | :--- | :--- |
| $\sin \theta$ | $\cos \theta$ | $\frac{d(\sin \theta)}{d \theta}=\cos \theta$ |
| $\cos \theta$ | $-\sin \theta$ | $\frac{d(\cos \theta)}{d \theta}=-\sin \theta$ |
| $\tan \theta$ | $\sec ^{2} \theta$ | $\frac{d(\tan \theta)}{d \theta}=\sec ^{2} \theta$ |
| $\cot \theta$ | $-\csc ^{2} \theta$ | $\frac{d(\cot \theta)}{d \theta}=-\csc ^{2} \theta$ |
| $\sec \theta$ | $\tan \theta \sec \theta$ | $\frac{d(\sec \theta)}{d \theta}=\tan \theta \sec \theta$ |
| $\csc \theta$ | $-\cot \theta \csc \theta$ | $\frac{d(\csc \theta)}{d \theta}=-\cot \theta \csc \theta$ |

## Example 3 Differentiate $\csc x \tan 2 x$

Solution Using the product rule and chain rule,

$$
\begin{aligned}
\frac{d}{d x} \csc x \tan 2 x & =\left(\frac{d}{d x} \csc x\right)(\tan 2 x)+\csc x\left(\frac{d}{d x} \tan 2 x\right) \\
& =-\cot x \cdot \csc x \cdot \tan 2 x+\csc x \cdot 2 \cdot \sec ^{2} 2 x \\
& =2 \csc x \sec ^{2} 2 x-\cot x \csc x \tan 2 x .
\end{aligned}
$$

Example 4 Differentiate: (a) $(\tan 3 x) /\left(1+\sin ^{2} x\right)$, (b) $1-\csc ^{2} 5 x$, (c) $\sin \left(\sqrt{3 \theta^{2}+1}\right)$.
Solution (a) By the quotient and chain rules,

$$
\frac{d}{d x} \frac{\tan 3 x}{1+\sin ^{2} x}=\frac{\left(3 \sec ^{2} 3 x\right)\left(1+\sin ^{2} x\right)-(\tan 3 x)(2 \sin x \cos x)}{\left(1+\sin ^{2} x\right)^{2}} .
$$

(b) $\frac{d}{d x}\left(1-\csc ^{2} 5 x\right)=-2 \csc 5 x \frac{d}{d x}(\csc 5 x)=10 \csc ^{2} 5 x \cot 5 x$.
(c) By the chain rule,

$$
\begin{aligned}
\frac{d}{d \theta} \sin \left(\sqrt{3 \theta^{2}+1}\right) & =\cos \left(\sqrt{3 \theta^{2}+1}\right) \frac{d}{d \theta} \sqrt{3 \theta^{2}+1} \\
& =\cos \left(\sqrt{3 \theta^{2}+1}\right) \cdot \frac{1}{2} \frac{3 \cdot 2 \theta}{\sqrt{3 \theta^{2}+1}} \\
& =\frac{3 \theta}{\sqrt{3 \theta^{2}+1}} \cos \sqrt{3 \theta^{2}+1} \cdot
\end{aligned}
$$

Example 5 Differentiate: (a) $\tan (\cos \sqrt{x})$, (b) $\csc \sqrt{x}$.
Solution (a) By the chain rule,

$$
\begin{aligned}
\frac{d}{d x} \tan (\cos \sqrt{x}) & =\frac{d}{d u} \tan u \frac{d u}{d x} \quad(u=\cos \sqrt{x}) \\
& =\sec ^{2} u \cdot \frac{d}{d x} \cos \sqrt{x} \\
& =\sec ^{2}(\cos \sqrt{x})(-\sin \sqrt{x}) \frac{d}{d x} \sqrt{x} \\
& =\frac{-\sec ^{2}(\cos \sqrt{x}) \sin \sqrt{x}}{2 \sqrt{x}} .
\end{aligned}
$$

(b) By the chain rule,

$$
\frac{d}{d x} \csc \sqrt{x}=-\frac{\csc \sqrt{x} \cot \sqrt{x}}{2 \sqrt{x}} .
$$

By reversing the formulas for derivatives of trigonometric functions and multiplying through by -1 where necessary, we obtain the following indefinite integrals (antiderivatives).

## Antiderivatives of Some Trigonometric Functions

$$
\begin{array}{ll}
\int \cos \theta d \theta=\sin \theta+C & \int \sin \theta d \theta=-\cos \theta+C \\
\int \sec ^{2} \theta d \theta=\tan \theta+C & \int \csc ^{2} \theta d \theta=-\cot \theta+C \\
\int \tan \theta \sec \theta d \theta=\sec \theta+C & \int \cot \theta \csc \theta d \theta=-\csc \theta+C
\end{array}
$$

For instance, to check that

$$
\int \sec ^{2} \theta d \theta=\tan \theta+C
$$

we simply recall from the preceding display that $(d / d \theta)(\tan \theta)=\sec ^{2} \theta$.
Example 6 Find $\int \sec \theta(\sec \theta+3 \tan \theta) d \theta$.
Solution Multiplying out, we have

$$
\begin{aligned}
\int\left(\sec ^{2} \theta+3 \sec \theta \tan \theta\right) d \theta & =\int \sec ^{2} \theta d \theta+3 \int \sec \theta \tan \theta d \theta \\
& =\tan \theta+3 \sec \theta+C .
\end{aligned}
$$

Example 7 Find $\int \sin 4 u d u$.
Solution If we guess $-\cos 4 u$ as the antiderivative, we find $(d / d u)(-\cos 4 u)=4 \sin 4 u$, which is four times too big, so

$$
\int \sin 4 u d u=-\frac{1}{4} \cos 4 u+C
$$

Example 8 Find the following antiderivatives: (a) $\int 2 \cos 4 s d s$, (b) $\int\left(1+\sec ^{2} \theta\right) d \theta$, (c) $\int \tan 2 x \sec 2 x d x$, (d) $\int(\sin x+\sqrt{x}) d x$.

Solution (a) Since $(d / d s) \sin 4 s=4 \cos 4 s,(d / d s) \frac{1}{2} \sin 4 s=\frac{1}{2} 4 \cos 4 s=2 \cos 4 s$. Thus $\int 2 \cos 4 s d s=\frac{1}{2} \sin 4 s+C$.
(b) By the sum rule for integrals, $\int\left(1+\sec ^{2} \theta\right) d \theta=\theta+\tan \theta+C$.
(c) Since $(d / d x) \sec (2 x)=(\sec 2 x \tan 2 x) \cdot 2$, we have

$$
\int \tan 2 x \sec 2 x d x=\frac{1}{2} \sec (2 x)+C
$$

(d) $\int(\sin x+\sqrt{x}) d x=-\cos x+\left(x^{3 / 2} / \frac{3}{2}\right)+C=-\cos x+\frac{2}{3} x^{3 / 2}+C$.

We can use these methods for indefinite integrals to calculate some definite integrals as well.

Example 9 Calculate: (a) $\int_{0}^{1}\left(2 \sin x+x^{3}\right) d x$, (b) $\int_{\pi / 4}^{\pi / 2} \cos 2 x d x$.
Solution (a) By the sum and power rules,

$$
\int\left(2 \sin x+x^{3}\right) d x=2 \int \sin x d x+\int x^{3} d x=-2 \cos x+\frac{x^{4}}{4}+C
$$

Thus, by the fundamental theorem of calculus,

$$
\begin{aligned}
\int_{0}^{1}\left(2 \sin x+x^{3}\right) d x & =\left.\left(-2 \cos x+\frac{x^{4}}{4}\right)\right|_{0} ^{1}=-2(\cos 1-\cos 0)+\frac{1}{4} \\
& =\frac{9}{4}-2 \cos 1
\end{aligned}
$$

(b) As Example 7, we find that an antiderivative for $\cos 2 x$ is $(1 / 2) \sin 2 x$. Therefore, by the fundamental theorem,

$$
\int_{\pi / 4}^{\pi / 2} \cos 2 x d x=\left.\frac{1}{2} \sin 2 x\right|_{\pi / 4} ^{\pi / 2}=\frac{1}{2}(\sin \pi-\sin \pi / 2)=-\frac{1}{2}
$$

Our list of antiderivatives still leaves much to be desired. Where, for instance, is $\int \tan \theta d \theta$ ? The absence of this and other entries in the previous display is related to the missing antiderivative for $1 / x$ (see Exercise 65). The gap will be filled in the next chapter.

## Exercises for Section 5.2

Differentiate the functions of $\theta$ in Exercises 1-12.

1. $\cos \theta+\sin \theta$
2. $8 \cos \theta-10 \sin \theta$
3. $5 \cos 3 \theta+10 \sin 2 \theta$
4. $8 \sin 10 \theta-10 \cos 8 \theta$
5. $(\cos \theta)(\sin \theta+\theta)$
6. $\cos ^{2} \theta-3 \sin ^{3} \theta$
7. $\cos ^{3} 3 \theta$
8. $\sin ^{4} 5 \theta$
9. $\frac{\cos \theta}{\cos \theta-1}$
10. $\frac{\sin \theta}{\cos \theta-1}$
11. $\frac{\cos \theta+\sin \theta}{\sin \theta+1}$
12. $\frac{\theta+\cos \theta}{\cos \theta \sin \theta}$

Differentiate the functions of $x$ in Exercises 13-24.
13. $(\cos x)^{3}$
14. $\sin \left(20 x^{2}\right)$
15. $(\sqrt{x}+\cos x)^{4}$
16. $\frac{\cos \sqrt{x}}{1+\sqrt{x}}$
17. $\sin (x+\sqrt{x})$
18. $\csc \left(x^{2}+\sqrt{x}\right)$
19. $\frac{x}{\cos x+\sin \left(x^{2}\right)}$
20. $\frac{\cos x}{\tan x+\sqrt{x}}$
21. $\tan x+2 \cos x$
22. $\sec x+8 \csc x$
23. $\sec 3 x$
24. $\tan 10 x$

Differentiate each of the functions in Exercises 25-36.
25. $f(x)=\sqrt{x}+\cos 3 x$
26. $f(x)=\left[(\sin 2 x)^{2}+x^{2}\right]$
27. $f(x)=\sqrt{\cos x}$
28. $f(x)=\sin ^{2} x$
29. $f(t)=\left(4 t^{3}+1\right) \sin \sqrt{t}$
30. $f(t)=\csc t \cdot \sec ^{2} 3 t$
31. $f(x)=\sin \left(\sqrt{1-x^{3}}\right)+\tan \left(\frac{x}{x^{4}+1}\right)$.
32. $f(\theta)=\tan \left(\theta+\frac{1}{\theta}\right)$.
33. $f(\theta)=\left[\csc \left(\frac{\theta}{\sqrt{\theta^{2}+1}}\right)+1\right]^{3 / 2}$.
34. $f(r)=\frac{r^{2}+\sqrt{1-r^{2}}}{r \sin r}$.
35. $f(v)=\left(\tan \sqrt{v^{2}+1}\right)\left(\sec \left[\frac{1}{\left(v^{2}+1\right)}\right]\right)$.
36. $f(s)=\frac{\sin \left(s \sqrt[3]{1+s^{2}}\right)}{\cos s}$.

Find the antiderivatives in Exercises 37-44.
37. $\int\left(x^{3}+\sin x\right) d x$
38. $\int\left(\cos 3 x+5 x^{3 / 2}\right) d x$
39. $\int\left(x^{4}+\sec 2 x \tan 2 x\right) d x$
40. $\int(\sin 2 x+\sqrt{x}) d x$
41. $\int \sin \left(\frac{u}{2}\right) d u$
42. $\int \cos \left(\frac{3 v}{5}\right) d v$
43. $\int \cos \left(\theta^{2}\right) \theta d \theta$. (Hint: Compute the derivative of $\sin \left(\theta^{2}\right)$.)
44. $\int \sin \left(\phi^{3}\right) \phi^{2} d \phi$.
45. Find $\int \cos \theta \sin \theta d \theta$ by using a trigonometric identity.
46. Find $\int \cos ^{2} \theta d \theta$ by using (a) a half-angle formula; (b) a double-angle formula.
47. Find $\int \sin ^{2} \theta d \theta+\int \cos ^{2} \theta d \theta$.
48. Find $\int \sin ^{2} \theta d \theta$ by using a trigonometric identity.

Evaluate the definite integrals in Exercises 49-56
49. $\int_{0}^{\pi / 2} \sin \left(\frac{\theta}{4}\right) d \theta$
50. $\int_{0}^{2 \pi} \sin \left(\frac{2 \theta}{3}\right) d \theta$
51. $\int_{-7}^{7}\left(\sin t+\sin ^{3} t\right) d t$. (Hint: No calculation is necessary.)
52. $\int_{-5}^{5}\left(2 \cos t+\sin ^{9} t\right) d t$
53. $\int_{0}^{1} \cos (3 \pi t) d t$
54. $\int_{1 / 4}^{1 / 2} \sin \left(\frac{\pi}{3} s\right) d s$
55. $\int_{0}^{\pi} \cos ^{2} \theta d \theta$ (see Exercise 46).
56. $\int_{0}^{10 \pi} \sin ^{2} \theta d \theta$ (see Exercise 48).
57. Prove the following inequalities by using trigonometric identities and the inequalities established at the beginning of this section:
(a) $\phi<\frac{\sin 2 \phi}{1+\cos 2 \phi}$ for $0<\phi<\pi / 2$;
(b) $\sec \phi>\phi$ for $0<\phi<\frac{\pi}{2}$.
58. Find $\lim _{\phi \rightarrow 0} \frac{\tan 2 \phi}{3 \phi}$.
59. Find $\lim _{\Delta \theta \rightarrow 0} \frac{\sin a \Delta \theta}{\Delta \theta}$ where $a$ is any constant.
60. Find $\lim _{\phi \rightarrow 0} \frac{\sin a \phi}{\sin b \phi}$, where $a$ and $b$ are constants.
61. Show that $f(x)=\sin x$ and $f(x)=\cos x$ satisfy $f^{\prime \prime}(x)+f(x)=0$.
62. Find a function $f(x)$ which satisfies $f^{\prime \prime}(x)+4 f(x)$ $=0$.
63. Show that $f(x)=\tan x$ satisfies the equation $f^{\prime}(x)=1+[f(x)]^{2}$ and $f(x)=\cot x$ satisfies $f^{\prime}(x)$ $=-\left(1+[f(x)]^{2}\right)$.
64. Show that $f(x)=\sec x$ satisfies $f^{\prime \prime}+f-2 f^{3}=0$.
65. Suppose that $f^{\prime}(x)=1 / x$. (We do not yet have such a function at our disposal, but we will see in Chapter 6 that there is one.) Show that $\int \tan \theta d \theta$ $=-f(\cos \theta)+C$.
66. Show that $\int \tan \theta d \theta=f(\sec \theta)+C$, where $f$ is a function such that $f^{\prime}(x)=1 / x$. (The apparent conflict with Exercise 65 will be resolved in the next chapter.)
67. Show that $(d / d \theta) \cos \theta=-\sin \theta$ can be derived from $(d / d \theta) \sin \theta=\cos \theta$ by using the identity $\cos \theta=\sin (\pi / 2-\theta)$ and the chain rule.
68. (a) Evaluate $\lim _{\Delta \theta \rightarrow 0}(\tan \Delta \theta) / \Delta \theta$ using the methods of the beginning of this section.
(b) Use part (a) and the addition formula for the tangent (see the inside front cover) to prove that $\tan ^{\prime} \theta=\sec ^{2} \theta,-\pi / 2<\theta<\pi / 2$, that is, prove that

$$
\lim _{\Delta \theta \rightarrow 0} \frac{\tan (\theta+\Delta \theta)-\tan \theta}{\Delta \theta}=\sec ^{2} \theta .
$$

*69. Suppose that $\phi(x)$ is a function "appearing from the blue" with the property that

$$
\frac{d \phi}{d x}=\frac{1}{\cos x}
$$

Calculate: (a) $\frac{d}{d x}(\phi(3 x) \cos x)$, (b) $\int_{0}^{1} \frac{1}{\cos x} d x$,
(c) $\frac{d^{2}}{d x^{2}}(\phi(2 x) \sin 2 x)$.
$\star 70$. Let $\psi$ be a function such that

$$
\frac{d \psi}{d x}=\phi
$$

where $\phi$ is described in Exercise 69. Prove that

$$
\begin{aligned}
& \frac{d}{d x}\left(\psi(x) \sin x+\frac{d}{d x}\left(\psi(x) \cos x-\frac{x^{2}}{2}\right)\right) \\
& \quad=-\phi(x) \sin x
\end{aligned}
$$

*71. Give a geometric "proof" that $\sin ^{\prime}=\cos$ and $\cos ^{\prime}=-\sin$ by these steps:
(a) Consider the parametric curve $(\cos \theta, \sin \theta)$ as in the beginning of this section. Argue that $\left(\cos ^{\prime} \theta\right)^{2}+\left(\sin ^{\prime} \theta\right)^{2}=1$, since the point moves at unit speed around the circle (see Exercise 34, Section 2.4).
(b) Use the relation $\cos ^{2} \theta+\sin ^{2} \theta=1$ to show that $\sin \theta \sin ^{\prime} \theta+\cos \theta \cos ^{\prime} \theta=0$.
(c) Conclude from (a) and (b) that $\left(\sin ^{\prime} \theta\right)^{2}$ $=\cos ^{2} \theta$ and $\left(\cos ^{\prime} \theta\right)^{2}=\sin ^{2} \theta$.
(d) Give a geometric argument to get the correct signs in the square roots of the relations in (c).

### 5.3 Inverse Functions

The derivative of an inverse function is the reciprocal of its derivative.
Sometimes two variable quantities are related in such a way that either one may be considered as a function of the other. The relationship between the quantities may be expressed by either of two functions, which are called inverses of one another. In this section, we will learn when a given function has an inverse, and we will see a useful relationship between the derivative of a function and the derivative of its inverse.

A simple example of a function with an inverse is the linear function $y=f(x)=m x+b$ with $m \neq 0$. We can solve for $x$ in terms of $y$ to get $x=(1 / m) y-b / m$. Considering the expression $(1 / m) y-b / m$ as a function $g(y)$ of $y$, we find that $y=f(x)$ whenever $x=g(y)$, and vice versa.

The graphs of $f$ and $g$ are very simply related. We interchange the role of the $x$ and $y$ axes by flipping the graph over the diagonal line $y=x$ (which bisects the right angle between the axes) or by viewing the graph through the back of the page held so that the $x$ axis is vertical. (See Fig. 5.3.1 for a specific example.) Notice that the (constant) slopes of the two graphs are reciprocals of one another.

Whenever two functions, $f$ and $g$, have the property that $y=f(x)$ whenever $x=g(y)$, and vice versa, we say that $f$ and $g$ are inverses of one another; the graphs of $f$ and $g$ are then related by flipping the $x$ and $y$ axes, as in Fig. 5.3.1. If we are given a formula for $y=f(x)$, we can try to find the inverse function by solving for $x$ in terms of $y$.

Figure 5.3.1. The functions $f(x)=-2 x+4$ and $g(y)=-\frac{1}{2} y+2$ are inverses to one another.



Example 1 Find an inverse function for $f(x)=x^{3}$. Graph $f$ and its inverse.
Solution Solving the relation $y=x^{3}$ for $x$ in terms of $y$, we find $x=\sqrt[3]{y}$, so the cuberoot function $g(y)=\sqrt[3]{y}$ is the inverse function to the cubing function $f(x)=x^{3}$. The graphs of $f$ and $g$ are shown in parts (a) and (b) of Fig. 5.3.2. In part (c), we have illustrated the important fact that the names of variables

used with a function are arbitrary; since we like to have $y$ as a function of $x$, we may write $g(x)=\sqrt[3]{x}$, and so the graph $y=\sqrt[3]{x}$ is another acceptable picture of the cuberoot function.

Not every function has an inverse. For instance, if $y=f(x)=x^{2}$, we may solve for $x$ in terms of $y$ to get $x= \pm \sqrt{y}$, but this does not give $x$ as a function defined for all $y$, for two reasons:
(1) if $y<0$, the square root $\sqrt{y}$ is not defined;
(2) if $y>0$, the two choices of positive or negative sign give two different values for $x$.

We can also see the difficulty geometrically. If we interchange the axes by flipping the parabola $y=x^{2}$ (Fig. 5.3.3), the resulting "horizontal parabola" is


not the graph of a function defined on the real numbers, since it intersects some vertical lines twice and others not at all.

This situation is similar to one we encountered in Section 2.3, where a

Figure 5.3.4. The function $f(x)=x^{2}$ and its inverse $g(y)=\sqrt{y}$.
curve like $x^{2}+y^{2}=1$ defined $y$ as an implicit function of $x$ only if we looked at part of the curve. In the present case, we can obtain an invertible function if we restrict $f(x)=x^{2}$ to the domain $[0, \infty)$. (The choice $(-\infty, 0]$ would do as well.) Then the inverse function $g(y)=\sqrt{y}$ is well defined with domain $[0, \infty)$ (see Fig. 5.3.4). The domain $(-\infty, 0]$ for $f$ would have led to the other square root sign $-\sqrt{y}$ for $g$.


The inverse to a function $f$, when it exists, is sometimes denoted by $f^{-1}$ and read " $f$ inverse". A function with an inverse is said to be invertible.

Warning Notice from this example that the inverse $f^{-1}$ is not in general the same as $1 / f$.

We summarize our work to this point in the following box.

## Inverse Functions

The inverse function to a function $f$ is a function $g$ for which $g(y)=x$ when $y=f(x)$, and vice versa.

The inverse function to $f$ is denoted by $f^{-1}$.
To find a formula for $f^{-1}$, try to solve the equation $y=f(x)$ for $x$ in terms of $y$. If the solution is unique, set $f^{-1}(y)=x$.

The graph of $f^{-1}$ is obtained from that of $f$ by flipping the figure to interchange the horizontal and vertical axes.

It may be necessary to restrict the domain of $f$ before there is an inverse function.

Example 2 Let $f(x)=x^{2}+2 x+3$. Restrict $f$ to a suitable interval so that it has an inverse. Find the inverse function and sketch its graph.
Solution We may solve the equation $y=x^{2}+2 x+3$ for $x$ in terms of $y$ by the quadratic formula:

$$
x^{2}+2 x+3-y=0
$$

gives $x=[-2 \pm \sqrt{4-4(3-y)}] / 2=-1 \pm \sqrt{y-2}$. If we choose $y \geqslant 2$ and the + sign, we get $x=-1+\sqrt{y-2}$ for the inverse function. The restriction $y \geqslant 2$ corresponds to $x \geqslant-1$ (see Fig. 5.3.5). (The answer $x=-1-$ $\sqrt{y-2}$ for $y \geqslant 2$ and $x \leqslant-1$ is also acceptable-this is represented by the dashed portion of the graph.

Figure 5.3.5. Restricting the domain of $f(x)$ to $[-1, \infty)$ gives a function with an inverse defined on $[2, \infty)$.

Example 3 Sketch the graph of the inverse function for each function in Fig. 5.3.6.

Figure 5.3.6. Sketch the graph of the inverse.

Solution

Figure 5.3.7. Graphs of the inverse functions (compare Fig. 5.3.6).

(a)

(b)

(c)

The graphs which we obtain by viewing the graphs in Fig. 5.3.6 from the reverse side of the page, are shown in Fig. 5.3.7.

(a)

(b)

(c)


There is a simple geometric test for invertibility; a function is invertible if each horizontal line meets the graph in at most one point.

Example 4 Determine whether or not each function in Fig. 5.3.8 is invertible on its domain.


Solution Applying the test just mentioned, we find that the functions (a) and (c) are invertible while (b) is not.
A function may be invertible even though we cannot find an explicit formula for the inverse function. This fact gives us a way of obtaining "new func-
tions." The following is a useful calculus test for finding intervals on which a function is invertible. In the next section, we shall use it to obtain inverses for the trigonometric functions.

## Inverse Function Test

Suppose that $f$ is continuous on $[a, b]$ and that $f$ is increasing at each point of $(a, b)$. (For instance, this holds if $f^{\prime}(x)>0$ for each $x$ in $(a, b)$.) Then $f$ is invertible on $[a, b]$, and the inverse $f^{-1}$ is defined on the interval $[f(a), f(b)]$.

If $f$ is decreasing rather than increasing at each point of $(a, b)$, then $f$ is still invertible; in this case, the domain of $f^{-1}$ is $[f(b), f(a)]$.

To justify this test, we apply the increasing function test from Section 3.2 to conclude that $f$ is increasing on $[a, b]$; that is, if $a \leqslant x_{1}<x_{2} \leqslant b$, then $f\left(x_{1}\right)<f\left(x_{2}\right)$. In particular, $f(a)<f(b)$. If $y$ is any number in $(f(a), f(b))$, then by the intermediate value theorem (first version, Section 3.1), there is an $x$ in $(a, b)$ such that $f(x)=y$. If $y=f(a)$ or $f(b)$, we can choose $x=a$ or $x=b$. Since $f$ is increasing on $[a, b]$, for any $y$ in $[f(a), f(b)]$ there can only be one $x$ such that $y=f(x)$. Thus, by definition, $f$ is invertible on $[a, b]$ and the domain of $f^{-1}$ is the range $[f(a), f(b)]$ of values of $f$ on $[a, b]$. The proof of the second assertion in the inverse function test is similar.

We can allow open or infinite intervals in the inverse function test. For instance, if $f$ is continuous and increasing on $[a, \infty)$, and if $\lim _{x \rightarrow \infty} f(x)=\infty$, then $f$ has an inverse defined on $[f(a), \infty)$.

Example 5 Verify that $f(x)=x^{2}+x$ has an inverse if $f$ is defined on $\left[-\frac{1}{2}, \infty\right)$.
Solution Since $f$ is differentiable on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$ and hence on $\left[-\frac{1}{2}, \infty\right)$. But $f^{\prime}(x)=2 x+1>0$ for $x>-\frac{1}{2}$. Thus $f$ is increasing. Also, $\lim _{x \rightarrow \infty} f(x)=\infty$. Hence the inverse function test guarantees that $f$ has an inverse defined on $\left[-\frac{1}{2}, \infty\right)$.

Example 6 Let $f(x)=x^{5}+x$.
(a) Show that $f$ has an inverse on $[-2,2]$. What is the domain of this inverse?
(b) Show that $f$ has an inverse on $(-\infty, \infty)$.
(c) What is $f^{-1}(2)$ ?

畨(d) Numerically calculate $f^{-1}(3)$ to two decimal places of accuracy.
Solution (a) $f^{\prime}(x)=5 x^{4}+1>0$, so by the inverse function test, $f$ is invertible on $[-2,2]$. The domain of the inverse is $[f(-2), f(2)]$, which is $[-34,34]$.
(b) Since $f^{\prime}(x)>0$ for all $x$ in $(-\infty, \infty), f$ is increasing on $(-\infty, \infty)$. Now $f$ takes arbitrarily large positive and negative values as $x$ varies over $(-\infty, \infty)$; it takes all values in between by the intermediate value theorem, so the domain of $f^{-1}$ is $(-\infty, \infty)$. There is no simple formula for $f^{-1}(y)$, the solution of $x^{5}+x=y$, but we can calculate $f^{-1}(y)$ for any specific values of $y$ to any desired degree of accuracy. (This is really no worse than the situation for $\sqrt{x}$. If the inverse function to $x^{5}+x$ had as many applications as the square-root function, we would learn about it in high school, tables would be readily available for it, calculators would calculate it at the touch of a key, and there would be a standard notation like $\mathscr{f} y$ for the solution of $x^{5}+x$ $=y$, just as $\sqrt[5]{y}$ is the standard notation for the solution of $x^{5}=y$.)
(c) Since $f(1)=1^{5}+1=2, f^{-1}(2)$ must equal 1 .
(d) To calculate $f^{-1}(3)-$ that is, to seek an $x$ such that $x^{5}+x=3$-we use the method of bisection described in Example 7, Section 3.1. Since $f(1)=2<3$ and $f(2)=34>3, x$ must lie between 1 and 2 . We can squeeze toward the correct answer by calculating:

$$
\begin{array}{rll}
f(1.5)=9.09375 & \text { so } & 1<x<1.5 \\
f(1.25)=4.30176 & \text { so } & 1<x<1.25 \\
f(1.1)=2.71051 & \text { so } & 1.1<x<1.25 \\
f(1.15)=3.16135 & \text { so } & 1.1<x<1.15 \\
f(1.14)=3.06541 & \text { so } & 1.1<x<1.14 \\
f(1.13)=2.97244 & \text { so } & 1.13<x<1.14 \\
f(1.135)=3.01856 & \text { so } & 1.13<x<1.135
\end{array}
$$

Thus, to two decimal places, $x \approx 1.13$. (About 10 minutes of further experimentation gave $f(1.132997566)=3.000000002$ and $f(1.132997565)=$ 2.999999991 . What does this tell you about $f^{-1}(3) ?$ )

## 圈 Calculator Discussion

Recall (see Section R.6) that a function $f$ may be thought of as an operation key on a calculator. The inverse function should be another key, which we can label $f^{-1}$. According to the definition, if we feed in any $x$, then push $f$ to get $y=f(x)$, then push $f^{-1}$, we get back $x=f^{-1}(y)$, the number we started with. Likewise if we feed in a number $y$ and first push $f^{-1}$ and then $f$, we get $y$ back again. By Fig. 5.3.4, $y=x^{2}$ and $x=\sqrt{y}$ (for $x \geqslant 0, y \geqslant 0$ ) are inverse functions. Try it out numerically, by pushing $x=3.0248759$, then the $x^{2}$ key, then the $\sqrt{x}$ key. Try it also in the reverse order. (The answer may not come out exactly right because of roundoff errors.)

As the preceding calculator discussion suggests, there is a close relation between inverse functions and composition of functions as discussed in connection with the chain rule in Section 2.2. If $f$ and $g$ are inverse functions, then $g(y)$ is that number $x$ for which $f(x)=y$, so $f(g(y))=y$; i.e., $f \circ g$ is the identity function which takes each $y$ to itself: $(f \circ g)(y)=y$. Similarly, $g(f(x))$ $=x$ for all $x$, so $g \circ f$ is the identity function as well.

If we assume that $f^{-1}$ and $f$ are differentiable, we can apply the chain rule to the equation

$$
f^{-1}(f(x))=x
$$

to obtain $\left(f^{-1}\right)^{\prime}(f(x)) \cdot f^{\prime}(x)=1$, which gives

$$
\left(f^{-1}\right)^{\prime}(f(x))=\frac{1}{f^{\prime}(x)}
$$

Writing $y$ for $f(x)$, so that $x=f^{-1}(y)$, we obtain the formula

$$
\left(f^{-1}\right)^{\prime}(y)=\frac{1}{f^{\prime}\left(f^{-1}(y)\right)} .
$$

Since the expression $\left(f^{-1}\right)^{\prime}$ is awkward, we sometimes revert to the notation $g(y)$ for the inverse function and write

$$
g^{\prime}(y)=\frac{1}{f^{\prime}(x)}
$$




Figure 5.3.9. The inverse of the tangent line is the tangent line of the inverse.

Notice that although $d y / d x$ is not an ordinary fraction, the rule

$$
\frac{d x}{d y}=\frac{1}{d y / d x}
$$

is valid as long as $d y / d x \neq 0$. (Maybe the "reciprocal" notation $f^{-1}$ is not so bad after all!)

## Inverse Function Rule

To differentiate the inverse function $g=f^{-1}$ at $y$, take the reciprocal of the derivative of the given function at $x=f^{-1}(y)$ :

$$
\begin{aligned}
& g^{\prime}(y)=\frac{1}{f^{\prime}(g(y))} \quad \text { if } \quad g=f^{-1} \\
& \frac{d x}{d y}=\frac{1}{d y / d x}
\end{aligned}
$$

Notice that our earlier observation that the slopes of inverse linear functions are reciprocals is just a special case of the inverse function rule. Figure 5.3.9 illustrates how the general rule is related to this special case.

Assuming that the inverse $f^{-1}$ is continuous (see Exercise 41), we can prove that $f^{-1}$ is differentiable whenever $f^{\prime} \neq 0$ in the following way. Recall that $f^{\prime}\left(x_{0}\right)=d y / d x=\lim _{\Delta x \rightarrow 0}(\Delta y / \Delta x)$, where $\Delta x$ and $\Delta y$ denote changes in $x$ and $y$. On the other hand,

$$
g^{\prime}\left(y_{0}\right)=\frac{d x}{d y}=\lim _{\Delta y \rightarrow 0} \frac{\Delta x}{\Delta y}=\frac{1}{\lim _{\Delta y \rightarrow 0}(\Delta y / \Delta x)}
$$

by the reciprocal rule for limits. But $\Delta x \rightarrow 0$ when $\Delta y \rightarrow 0$, since $g$ is continuous, so

$$
g^{\prime}\left(y_{0}\right)=\frac{1}{\lim _{\Delta y \rightarrow 0}(\Delta y / \Delta x)}=\frac{1}{d y / d x}
$$

Example 7 Use the inverse function rule to compute the derivative of $\sqrt{x}$. Evaluate the derivative at $x=2$.
Solution Let us write $g(y)=\sqrt{y}$. This is the inverse function of $f(x)=x^{2}$. Since $f^{\prime}(x)=2 x$,

$$
g^{\prime}(y)=\frac{1}{f^{\prime}(g(y))}=\frac{1}{2 g(y)}=\frac{1}{2 \sqrt{y}}
$$

so $(d / d y)(\sqrt{y})=1 /(2 \sqrt{y})$. We may substitute any letter for $y$ in this result, including $x$, so we get the formula

$$
\frac{d}{d x} \sqrt{x}=\frac{1}{2 \sqrt{x}}
$$

When $x=2$, the derivative is $1 /(2 \sqrt{2})$.
Example 7 reproduces the rule for differentiating $x^{1 / 2}$ that we learned in Section 2.3. In fact, one can similarly use inverse functions to obtain an alternative proof of the rule for differentiating fractional powers: $(d / d x) x^{p / q}$ $=(p / q) x^{(p / q)-1}$.

Example 8
Solution

Figure 5.3.10. The sourness of yogurt as a function of fermentation temperature.


Figure 5.3.11. The graph of $T$ as a function of $S$.

Find $\left(f^{-1}\right)^{\prime}(2)$, where $f$ is the function in Example 6.
We know that $f^{-1}(2)=1$, so $\left(f^{-1}\right)^{\prime}(2)=1 / f^{\prime}(1)$. But $f^{\prime}(x)=5 x^{4}+1$, so $f^{\prime}(1)=6$, and hence $\left(f^{-1}\right)^{\prime}(2)=\frac{1}{6}$.
The important point in Example 8 is that we differentiated $f^{-1}$ without having an explicit formula for it. We will exploit this idea in the next section.

## Supplement to Section 5.3 Inverse Functions and Yogurt

If a yogurt culture is added to a quart of boiled milk and set aside for 4 hours, then the sourness of the resulting yogurt depends upon the temperature at which the mixture was kept. By performing a series of experiments, we can plot the graph of a function $S=f(T)$, where $T$ is temperature and $S$ is the sourness measured by the amount of lactic acid in grams in the completed yogurt. (See Fig. 5.3.10.) If $T$ is too low, the culture is dormant; if $T$ is too high, the culture is killed.


In making yogurt to suit one's taste, one might desire a certain degree of sourness and wish to know what temperature to use. (Remember that we are fixing all other variables, including the time of fermentation.) To find the temperature which gives $S=2$, for instance, one may draw the horizontal line $S=2$, see if it intersects the graph of $f$, and read off the value of $T$ (Fig. 5.3.10).

From the graph, we see that there are two possible values of $T: 38^{\circ} \mathrm{C}$ and $52^{\circ} \mathrm{C}$. Similarly, there are two possible temperatures to achieve any value of $S$ strictly between zero and the maximum value 3.8 of $f$. If, however, we restrict the allowable temperatures to the interval [20,47], then we will get a unique value of $T$ for each $S$ in $[0,3.8]$. The new function $T=g(S)$, which assigns to each the sourness value the proper temperature for producing it, is the inverse function to $f$ (Fig. 5.3.11).

## Exercises for Section 5.3

Find the inverse for each of the functions in Exercises $1-6$ on the given interval.

1. $f(x)=2 x+5$ on $[-4,4]$.
2. $f(x)=-\frac{1}{3} x+2$ on $(-\infty, \infty)$.
3. $f(x)=x^{5}$ on $(-\infty, \infty)$.
4. $f(x)=x^{8}$ on $(0,1]$.
5. $h(t)=\frac{t-10}{t+3}$ on $[-1,1]$.
6. $a(s)=\frac{2 s+5}{-s+1}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.
7. Find the inverse function for the function $f(x)$ $=(a x+b) /(c x+d)$ on its domain. What must you assume about $a, b, c$, and $d$ ?
8. Find an inverse function $g$ for $f(x)=x^{2}+2 x+1$ on some interval containing zero. What is $g(9)$ ? What is $g(x)$ ?
9. Sketch the graph of the inverse of each of the functions in Fig. 5.3.12.


(a)

(c)

Figure 5.3.12. Sketch the graph of the inverse of each of these functions.
10. Determine whether each function in Fig. 5.3.13 has an inverse. Sketch the inverse if there is one.
 functions have inverses?
11. Draw a graph of $f(x)=(3 x+1) /(2 x-2)$ and a graph of its inverse function.
12. Draw a graph of $f(x)=(x-1) /(x+1)$ and a graph of its inverse function.
13. Find the largest possible intervals on which $f(x)$ $=1 /\left(x^{2}-1\right)$ is invertible. Sketch the graphs of the inverse functions.
14. Sketch a graph of $f(x)=x /\left(1+x^{2}\right)$ and find an interval on which $f$ is invertible.
15. Let $f(x)=x^{3}-4 x^{2}+1$.
(a) Find an interval containing 1 on which $f$ is invertible. Denote the inverse by $g$.
(b) Compute $g(-7)$ and $g^{\prime}(-7)$.
(c) What is the domain of $g$ ?

16. Let $f(x)=x^{5}+x$. (a) Find $f^{-1}(246)$.
it (b) Find $f^{-1}(4)$, correct to at least two decimal places.
17. Show that $f(x)=\frac{1}{3} x^{3}-x$ is not invertible on any open interval containing 1 .
18. Find intervals on which $f(x)=x^{5}-x$ is invertible.
19. Show that if $n$ is odd, $f(x)=x^{n}$ is invertible on $(-\infty, \infty)$. What is the domain of the inverse function?
20. Discuss the invertibility of $f(x)=x^{n}$ for $n$ even.
21. Show that $f(x)=-x^{3}-2 x+1$ is invertible on $[-1,2]$. What is the domain of the inverse?
22. (a) Show that $f(x)=x^{3}-2 x+1$ is invertible on $[2,4]$. What is the domain of the inverse function?
(b) Find the largest possible intervals on which $f$ is invertible.
23. Verify the inverse function rule for $y=x^{3}$.
24. Verify the inverse function rule for the function $y=(a x+b) /(c x+d)$ by finding $d y / d x$ and $d x / d y$ directly. (See Exercise 7.)
25. If $f(x)=x^{3}+2 x+1$, show that $f$ has an inverse on $[0,2]$. Find the derivative of the inverse function at $y=4$.
26. Find $g^{\prime}(0)$, where $g$ is the inverse function to $f(x)=x^{9}+x^{5}+x$.
27. Let $y=x^{3}+2$. Find $d x / d y$ when $y=3$.
28. If $f(x)=x^{5}+x$, find the derivative of the inverse function when $y=34$.
For each function $f$ in Exercises 29-32, find the derivative of the inverse function $g$ at the points indicated.
29. $f(x)=3 x+5$; find $g^{\prime}(2), g^{\prime}\left(\frac{3}{4}\right)$.
30. $f(x)=x^{5}+x^{3}+2 x$; find $g^{\prime}(0), g^{\prime}(4)$.
31. $f(x)=\frac{1}{12} x^{3}-x$ on $[-1,1]$; find $g^{\prime}(0), g^{\prime}\left(\frac{11}{12}\right)$.
32. $f(x)=\sqrt{x-3}$ on $[3, \infty)$; find $g^{\prime}(4), g^{\prime}(8)$.
[533. Enter the number 2.6 on your calculator, then push the $x^{2}$ key followed by the $\sqrt{x}$ key. Is there any roundoff error? Try the $\sqrt{x}$ key, then the $x^{2}$ key. Also try a sequence such as $x^{2}, \sqrt{x}, x^{2}$, $\sqrt{x}, x^{2}, \sqrt{x}$. Do you get the original number? Try these experiments with different starting numbers.
34. If we think of a French-English dictionary as defining a function from the set of French words to the set of English words (does it really?), how is the inverse function defined? Discuss.
35. Suppose that $f(x)$ is the number of pounds of beans you can buy for $x$ dollars. Let $g(y)$ be the inverse function. What does $g(y)$ represent?
36. Let $f_{1}(x)=x, f_{2}(x)=1 / x, f_{3}(x)=1-x, f_{4}(x)$ $=1 /(1-x), f_{5}(x)=(x-1) / x$, and $f_{6}(x)=$ $x /(x-1)$.
(a) Show that the composition of any two functions in this list is again in the list. Complete the "composition table" below. For example $\left(f_{2} \circ f_{3}\right)(x)=f_{2}(1-x)=1 /(1-x)=f_{4}(x)$.

| $\circ$ | $f_{1}$ | $f_{2}$ | $f_{3}$ | $f_{4}$ | $f_{5}$ | $f_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{1}$ |  |  | $f_{3}$ |  |  |  |
| $f_{2}$ |  |  | $f_{4}$ |  |  |  |
| $f_{3}$ |  | $f_{5}$ |  | $f_{5}$ |  |  |
| $f_{4}$ |  |  |  |  |  |  |
| $f_{5}$ |  |  |  |  |  |  |
| $f_{6}$ |  |  |  |  |  |  |

(b) Show that the inverse of any function in the preceding list is again in the list. Which of the functions equal their own inverses?
37. (a) Find the domain of

$$
f(x)=\sqrt{(2 x+5) /(3 x+7)}
$$

(b) By solving for the $x$ in the equation $y=$ $\sqrt{(2 x+5) /(3 x+7)}$, find a formula for the inverse function $g(y)$.
(c) Using the inverse function rule, find a formula for $f^{\prime}(x)$.
(d) Check your answer by using the chain rule.
38. Suppose that $f$ is concave upward and increasing on $[a, b]$.
(a) By drawing a graph, guess whether $f^{-1}$ is concave upward or downward on the inter$\operatorname{val}[f(a), f(b)]$.
(b) What if $f$ is concave upward and decreasing on $[a, b]$ ?
39. Show that if the inverse function to $f$ on $S$ is $g$, with domain $T$, then the inverse function to $g$ on $T$ is $f$, with domain $S$. Thus, the inverse of the inverse function is the original function, that is, $\left(f^{-1}\right)^{-1}=f$.
*40. Under what conditions on $a, b, c$, and $d$ is the function $f(x)=(a x+b) /(c x+d)$ equal to its own inverse function?
*41. Suppose that $f^{\prime}\left(x_{0}\right)>0$ and $f\left(x_{0}\right)=y_{0}$. If $g$ is the inverse of $f$ with $g\left(y_{0}\right)=x_{0}$, show that $g$ is continuous at $y_{0}$ by filling in the details of the following argument:
(a) For $\Delta x$ sufficiently small,

$$
\frac{3}{2} f^{\prime}\left(x_{0}\right)>\Delta y / \Delta x>\frac{1}{2} f^{\prime}\left(x_{0}\right)
$$

(b) As $\Delta y \rightarrow 0, \Delta x \rightarrow 0$ as well.
(c) Let $\Delta y=f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)$. Then $\Delta x=$ $g\left(y_{0}+\Delta y\right)-g\left(y_{0}\right)$.

### 5.4 The Inverse Trigonometric Functions

The derivatives of the inverse trigonometric functions have algebraic formulas.
In the previous section, we discussed the general concept of inverse function and developed a formula for differentiating the inverse. Now we will apply this formula to study the inverses of the sine, cosine, and the other trigonometric functions.

We begin with the function $y=\sin x$, using the inverse function test to locate an interval on which $\sin x$ has an inverse. Since $\sin ^{\prime} x=\cos x>0$ on $(-\pi / 2, \pi / 2), \sin x$ is increasing on this interval, so $\sin x$ has an inverse on the interval $[-\pi / 2, \pi / 2]$. The inverse is denoted $\sin ^{-1} y .{ }^{4}$ We obtain the graph of $\sin ^{-1} y$ by interchanging the $x$ and $y$ coordinates. (See Fig. 5.4.1.)

The values of $\sin ^{-1} y$ may be obtained from a table for $\sin x$. (Many pocket calculators can evaluate the inverse trigonometric functions as well as the trigonometric functions.)

## The Inverse Sine Function

$x=\sin ^{-1} y$ means that $\sin x=y$ and $-\pi / 2 \leqslant x \leqslant \pi / 2$. The number $\sin ^{-1} y$ is expressed in radians unless a degree sign is explicitly shown.

[^2]Figure 5.4.1. The graph of $\sin x$ on $[-\pi / 2, \pi / 2]$ together with its inverse.



Example 1 Calculate $\sin ^{-1} 1, \sin ^{-1} 0, \sin ^{-1}(-1), \sin ^{-1}\left(-\frac{1}{2}\right)$, and $\sin ^{-1}(0.342)$.
Solution Since $\sin (\pi / 2)=1, \sin ^{-1} 1=\pi / 2$. Similarly, $\sin ^{-1} 0=0, \sin ^{-1}(-1)=-\pi / 2$. Also $\sin (-\pi / 6)=-\frac{1}{2}$, so $\sin ^{-1}\left(-\frac{1}{2}\right)=-\pi / 6$. Using a calculator, or tables, we find $\sin ^{-1}(0.342)=0.349$ (or $20^{\circ}$ ).

We could have used any other interval on which $\sin x$ has an inverse, such as $[\pi / 2,3 \pi / 2]$, to define an inverse sine function; had we done so, the function obtained would have been different. The choice $[-\pi / 2, \pi / 2]$ is standard and is usually the most convenient.

Example 2 (a) Calculate $\sin ^{-1}\left(\frac{1}{2}\right), \sin ^{-1}(-\sqrt{3} / 2)$, and $\sin ^{-1}(2)$. (b) Simplify $\tan \left(\sin ^{-1} y\right)$.
Solutilon (a) Since $\sin (\pi / 6)=\frac{1}{2}, \sin ^{-1}\left(\frac{1}{2}\right)=\pi / 6$. Similarly, $\sin ^{-1}(-\sqrt{3} / 2)=-\pi / 3$. Finally, $\sin ^{-1}(2)$ is not defined $\operatorname{since} \sin x$ always lies between -1 and 1 .
(b) From Fig. 5.4.2 we see that $\theta=\sin ^{-1} y$ (that is, $\sin \theta=|A B| /|O B|$


Figure 5.4.2. $\tan \left(\sin ^{-1} y\right)$ $=y / \sqrt{1-y^{2}}$.
$=y$ ) and $\tan \theta=y / \sqrt{1-y^{2}}$, so $\tan \left(\sin ^{-1} y\right)=y / \sqrt{1-y^{2}}$.
Let us now calculate the derivative of $\sin ^{-1} y$. By the formula for the derivative of an inverse function from page 278 ,

$$
\frac{d}{d y} \sin ^{-1} y=\frac{1}{(d / d x) \sin x}=\frac{1}{\cos x}
$$

where $y=\sin x$. However, $\cos ^{2} x+\sin ^{2} x=1$, so $\cos x=\sqrt{1-y^{2}}$. (The negative root does not occur since $\cos x$ is positive on $(-\pi / 2, \pi / 2)$.)

Thus,

$$
\begin{equation*}
\frac{d}{d y} \sin ^{-1} y=\frac{1}{\sqrt{1-y^{2}}}=\left(1-y^{2}\right)^{-1 / 2}, \quad-1<y<1 \tag{1}
\end{equation*}
$$

Notice that the derivative of $\sin ^{-1} y$ is not defined at $y= \pm 1$ but is "infinite" there. This is consistent with the appearance of the graph in Fig. 5.4.1.

Example 3 (a) Differentiate $h(y)=\sin ^{-1}\left(3 y^{2}\right)$. (b) Differentiate $f(x)=x \sin ^{-1} 2 x$.
(c) Calculate $(d / d x)\left(\sin ^{-1} 2 x\right)^{3 / 2}$.

Solution (a) From (1) and the chain rule, with $u=3 y^{2}$,

$$
h^{\prime}(y)=\left(1-u^{2}\right)^{-1 / 2} \frac{d u}{d y}=6 y\left(1-9 y^{4}\right)^{-1 / 2}
$$

(b) Here we are using $x$ for the variable name. Of course we can use any letter we please. By the product and chain rules, and equation (1),

$$
\begin{aligned}
f^{\prime}(x) & =\left(\frac{d x}{d x}\right)\left(\sin ^{-1} 2 x\right)+x \frac{d}{d x}\left(\sin ^{-1} 2 x\right) \\
& =\sin ^{-1} 2 x+2 x\left(1-4 x^{2}\right)^{-1 / 2}
\end{aligned}
$$


(c) By the power rule, chain rule, and (1),

$$
\begin{aligned}
\frac{d}{d x}\left(\sin ^{-1} 2 x\right)^{3 / 2} & =\frac{3}{2}\left(\sin ^{-1} 2 x\right)^{1 / 2} \frac{d}{d x} \sin ^{-1} 2 x \\
& =\frac{3}{2}\left(\sin ^{-1} 2 x\right)^{1 / 2} \cdot 2 \cdot \frac{1}{\sqrt{1-(2 x)^{2}}} \\
& =3\left(\frac{\sin ^{-1} 2 x}{1-4 x^{2}}\right)^{1 / 2} \cdot
\end{aligned}
$$



Figure 5.4.3. The graph of cos and its inverse.

It is interesting that while $\sin ^{-1} y$ is defined in terms of trigonometric functions, its derivative is an algebraic function, even though the derivatives of the trigonometric functions themselves are still trigonometric.

The rest of the inverse trigonometric functions can be introduced in the same way as $\sin ^{-1} y$. The derivative of $\cos x,-\sin x$, is negative on $(0, \pi)$, so $\cos x$ on $(0, \pi)$ has an inverse $\cos ^{-1} y$. Thus for $-1 \leqslant y \leqslant 1, \cos ^{-1} y$ is that number (expressed in radians) in $[0, \pi]$ whose cosine is $y$. The graph of $\cos ^{-1} y$ is shown in Fig. 5.4.3.

The derivative of $\cos ^{-1} y$ can be calculated in the same manner as we calculated $(d / d y) \sin ^{-1} y$ :

$$
\begin{align*}
\frac{d}{d y} \cos ^{-1} y & =\frac{1}{(d / d x) \cos x}=\frac{1}{-\sin x} \\
& =\frac{-1}{\sqrt{1-y^{2}}} \tag{2}
\end{align*}
$$

Example 4 Differentiate $\tan \left(\cos ^{-1} x\right)$.
Solution By the chain rule and equation (2) (with $x$ in place of $y$ ),

$$
\frac{d}{d x} \tan \left(\cos ^{-1} x\right)=\sec ^{2}\left(\cos ^{-1} x\right) \cdot\left(\frac{-1}{\sqrt{1-x^{2}}}\right)
$$

From Fig. 5.4 .4 we see that $\sec \left(\cos ^{-1} x\right)=1 / x$, so

$$
\frac{d}{d x}\left[\tan \left(\cos ^{-1} x\right)\right]=\frac{-1}{x^{2} \sqrt{1-x^{2}}}
$$

Another method is to use Fig. 5.4.4 directly to obtain

$$
\tan \left(\cos ^{-1} x\right)=\frac{\sqrt{1-x^{2}}}{x}
$$

Differentiating by the quotient and chain rules,

$$
\begin{aligned}
\frac{d}{d x} \frac{\sqrt{1-x^{2}}}{x} & =\frac{x \cdot(-2 x) \cdot \frac{1}{2}\left(1-x^{2}\right)^{-1 / 2}-\left(1-x^{2}\right)^{1 / 2}}{x^{2}} \\
& =-\frac{1}{x^{2}}\left(\frac{x^{2}}{\sqrt{1-x^{2}}}+\sqrt{1-x^{2}}\right) \\
& =\frac{-1}{x^{2} \sqrt{1-x^{2}}},
\end{aligned}
$$

which agrees with our previous answer.

Figure 5.4.5. $\tan x$ and its inverse.

Next, we construct the inverse tangent. Since $\tan ^{\prime} x=\sec ^{2} x, \tan x$ is increasing at every point of its domain. It is continuous on $(-\pi / 2, \pi / 2)$ and has range $(-\infty, \infty)$, so $\tan ^{-1} y$ is defined on this domain; see Fig. 5.4.5. Thus, for $-\infty<y<\infty, \tan ^{-1} y$ is the unique number in $(-\pi / 2, \pi / 2)$ whose tangent is $y$.


The derivative of $\tan ^{-1} y$ can be calculated as in (1) and (2):

$$
\begin{equation*}
\frac{d}{d y} \tan ^{-1} y=\frac{1}{(d / d x) \tan x}=\frac{1}{\sec ^{2} x}=\frac{1}{1+\tan ^{2} x}=\frac{1}{1+y^{2}} \tag{3}
\end{equation*}
$$

Thus $\frac{d}{d y} \tan ^{-1} y=\frac{1}{1+y^{2}}$.

Example 5 Differentiate $f(x)=\left(\tan ^{-1} \sqrt{x}\right) /\left(\cos ^{-1} x\right)$. Find the domain of $f$ and $f^{\prime}$.
Solution By the quotient rule, the chain rule, and (3),

$$
\begin{aligned}
\frac{d}{d x} & \frac{\tan ^{-1} \sqrt{x}}{\cos ^{-1} x} \\
& =\left\{\cos ^{-1} x \cdot\left[\frac{1}{1+(\sqrt{x})^{2}} \cdot \frac{1}{2 \sqrt{x}}\right]-\tan ^{-1} \sqrt{x} \cdot\left(-\frac{1}{\sqrt{1-x^{2}}}\right)\right\}\left(\cos ^{-1} x\right)^{-2} \\
& =\frac{\sqrt{1-x^{2}} \cos ^{-1} x+2 \sqrt{x}(1+x) \tan ^{-1} \sqrt{x}}{2 \sqrt{x}(1+x) \sqrt{1-x^{2}}\left(\cos ^{-1} x\right)^{2}}
\end{aligned}
$$

The domain of $f$ consists of those $x$ for which $x \geqslant 0$ (so that $\sqrt{x}$ is defined) and $-1<x<1$ (so that $\cos ^{-1} x$ is defined and not zero)-that is, the domain of $f$ is $[0,1)$. For $f^{\prime}$ to be defined, the denominator in the derivative must be nonzero. This requires $x$ to belong to the interval ( 0,1 ). Thus, the domain of $f^{\prime}$ is $(0,1)$.
The remaining inverse trigonometric functions can be treated in the same way. Their graphs are shown in Fig. 5.4.6 and their properties are summarized in the box on the next page.

Remembering formulas such as those on the next page is an unpleasant chore for most students (and professional mathematicians as well). Many people prefer to memorize only a few basic formulas and to derive the rest as needed. It is also useful to develop a short mental checklist: Is the sign right? Is the sign consistent with the appearance of the graph? Is the derivative undefined at the proper points?


Figure 5.4.6. cot, sec, csc, and their inverses.

## Inverse Trigonometric Functions

| Function | Domain on which function has an inverse | Derivative of function | Inverse | Domain of inverse | Derivative of inverse |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sin x$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $\cos x$ | $\sin ^{-1} y$ | $[-1,1]$ | $\frac{1}{\sqrt{1-y^{2}}}$, | $-1<y<1$ |
| $\cos x$ | $[0, \pi]$ | $-\sin x$ | $\cos ^{-1} y$ | $[-1,1]$ | $-\frac{1}{\sqrt{1-y^{2}}}$ | $-1<y<1$ |
| $\tan x$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ | $\sec ^{2} x$ | $\tan ^{-1} y$ | $(-\infty, \infty)$ | $\frac{1}{1+y^{2}}$, | $-\infty<y<\infty$ |
| $\cot x$ | $(0, \pi)$ | $-\csc ^{2} x$ | $\cot ^{-1} y$ | $(-\infty, \infty)$ | $-\frac{1}{1+y^{2}}$, | $-\infty<y<\infty$ |
| $\sec x$ | $\left[0, \frac{\pi}{2}\right) \text { and }\left(\frac{\pi}{2}, \pi\right]$ | $\tan x \sec x$ | $\sec ^{-1} y$ | $\begin{aligned} & (-\infty,-1] \\ & \text { and }[1, \infty) \end{aligned}$ | $\frac{1}{\sqrt{y^{2}\left(y^{2}-1\right)}}$ | $\begin{aligned} -\infty & <y<-1, \\ 1 & <y<\infty \end{aligned}$ |
| $\csc x$ | $\left[-\frac{\pi}{2}, 0\right)$, and $\left(0, \frac{\pi}{2}\right]$ | $-\cot x \csc x$ | $\csc ^{-1} y$ | $\begin{aligned} & (-\infty,-1] \\ & \text { and }[1, \infty) \end{aligned}$ | $-\frac{1}{\sqrt{y^{2}\left(y^{2}-1\right)}},$ | $\begin{gathered} -\infty<y<-1, \\ 1<y<\infty \end{gathered}$ |

Example 6 Calculate $\cos ^{-1}\left(-\frac{1}{2}\right), \tan ^{-1}(1)$, and $\csc ^{-1}(2 / \sqrt{3})$.
Solution We find that $\cos ^{-1}\left(-\frac{1}{2}\right)=2 \pi / 3$ since $\cos (2 \pi / 3)=-\frac{1}{2}$, as is seen from Fig. 5.4.7. Similarly, $\tan ^{-1}(1)=\pi / 4$ since $\tan (\pi / 4)=1$. Finally, $\csc ^{-1}(2 / \sqrt{3})$ $=\pi / 3$ since $\csc (\pi / 3)=2 / \sqrt{3}$.


Figure 5.4.7. Evaluating some inverse trigonometric functions.

Example 7 Differentiate: (a) $\sec ^{-1}\left(y^{2}\right), y>0$; (b) $\cot ^{-1}\left[\left(x^{3}+1\right) /\left(x^{3}-1\right)\right]$.
Solution (a) By the chain rule,

$$
\frac{d}{d y} \sec ^{-1}\left(y^{2}\right)=\frac{1}{\sqrt{y^{4}\left(y^{4}-1\right)}} \cdot 2 y=\frac{2}{y \sqrt{y^{4}-1}} .
$$

(b) By the chain rule,

$$
\begin{aligned}
\frac{d}{d x} \cot ^{-1}\left(\frac{x^{3}+1}{x^{3}-1}\right) & =\frac{-1}{1+\left[\left(x^{3}+1\right) /\left(x^{3}-1\right)\right]^{2}} \frac{d}{d x}\left(\frac{x^{3}+1}{x^{3}-1}\right) \\
& =\frac{-\left(x^{3}-1\right)^{2}}{\left(x^{3}-1\right)^{2}+\left(x^{3}+1\right)^{2}}\left[\frac{\left(x^{3}-1\right) \cdot 3 x^{2}-\left(x^{3}+1\right) 3 x^{2}}{\left(x^{3}-1\right)^{2}}\right] \\
& =\frac{6 x^{2}}{\left(x^{3}-1\right)^{2}+\left(x^{3}+1\right)^{2}}=\frac{3 x^{2}}{x^{6}+1} \cdot
\end{aligned}
$$

Example 8 Differentiate $f(x)=\left(\csc ^{-1} 3 x\right)^{2}$. Find the domain of $f$ and $f^{\prime}$.
Solution By the chain rule,

$$
\begin{aligned}
\frac{d}{d x}\left(\csc ^{-1} 3 x\right)^{2} & =2\left(\csc ^{-1} 3 x\right) \frac{d}{d x} \csc ^{-1} 3 x \\
& =2 \csc ^{-1} 3 x \cdot 3 \cdot \frac{-1}{\sqrt{(3 x)^{2}\left[(3 x)^{2}-1\right]}} \\
& =\frac{-2 \csc ^{-1} 3 x}{\sqrt{x^{2}\left(9 x^{2}-1\right)}}=\frac{-2 \csc ^{-1} 3 x}{|x| \sqrt{9 x^{2}-1}}
\end{aligned}
$$

For $f(x)$ to be defined, $3 x$ should lie in $[1, \infty)$ or $(-\infty-1]$; that is, $x$ should lie in $\left[\frac{1}{3}, \infty\right)$ or $\left(-\infty,-\frac{1}{3}\right]$. The domain of $f^{\prime}$ is $\left(\frac{1}{3}, \infty\right)$, together with $\left(-\infty,-\frac{1}{3}\right)$.

Example 9 Explain why the derivative of every inverse cofunction in the preceding box is the negative of that of the inverse function.
Solution Let $f(x)$ be one of the functions $\sin x, \tan x$, or $\sec x$, and let $g(x)$ be the corresponding cofunction $\cos x, \cot x$ or $\csc x$. Then we know that

$$
f\left(\frac{\pi}{2}-x\right)=g(x)
$$

If we let $y$ denote $g(x)$, then we get

$$
\frac{\pi}{2}-x=f^{-1}(y) \quad \text { and } \quad x=g^{-1}(y)
$$

so

$$
\frac{\pi}{2}-f^{-1}(y)=g^{-1}(y)
$$

It follows by differentation in $y$ that

$$
-\frac{d}{d y} f^{-1}(y)=\frac{d}{d y} g^{-1}(y)
$$

Hence, the derivatives of $f^{-1}(y)$ and $g^{-1}(y)$ are negatives, which is the general reason why this same phenomenon occurred three times in the box.
The differentiation formulas for the inverse trigonometric functions may be read backwards to yield some interesting antidifferentiation formulas. For example, since $(d / d x) \tan ^{-1} x=1 /\left(1+x^{2}\right)$, we get

$$
\begin{equation*}
\int \frac{d x}{1+x^{2}}=\tan ^{-1} x+C \tag{4}
\end{equation*}
$$

Formulas like this will play an important role in the techniques of integration.
Example 10 Find $\int \frac{x^{2}}{1+x^{2}} d x$ [Hint: Divide first.]
Solution Using long division, $x^{2} /\left(1+x^{2}\right)=1-1 /\left(1+x^{2}\right)$. Thus, by (4),

$$
\begin{aligned}
\int \frac{x^{2}}{1+x^{2}} d x & =\int 1 d x-\int \frac{1}{1+x^{2}} d x \\
& =x-\tan ^{-1} x+C
\end{aligned}
$$

## Exercises for Section 5.4

Calculate the quantities in Exercises 1-10.

1. $\sin ^{-1}\left(\frac{1}{\sqrt{3}}\right)$
2. $\sin ^{-1}\left(\frac{2}{\sqrt{3}}\right)$
3. $\sin ^{-1}\left(-\frac{\pi}{4}\right)$
4. $\sin ^{-1}(0.4)$
5. $\cos ^{-1}(1)$
6. $\cos ^{-1}(0.3)$
7. $\tan ^{-1}(\sqrt{3})$
8. $\cot ^{-1}(2.3)$
9. $\sec ^{-1}\left(\frac{2}{\sqrt{3}}\right)$
10. $\csc ^{-1}(-5)$

Differentiate the functions in Exercises 11-28.
11. $\sin ^{-1}(8 x)$
12. $\sin ^{-1}\left(\sqrt{1-x^{2}}\right)$
13. $x^{2} \sin ^{-1} x$
14. $\left(x^{2}-1\right) \sin ^{-1}\left(x^{2}\right)$
15. $\left(\sin ^{-1} x\right)^{2}$
16. $\cos ^{-1}\left(\frac{t}{t+1}\right)$
17. $\tan ^{-1}\left(\frac{2 x^{5}+x}{1-x^{2}}\right)$
18. $\cot ^{-1}\left(1-y^{2}\right)$
19. $\sec ^{-1}\left(y-\frac{1}{y^{2}}\right)$
20. $\left(\frac{1}{x}\right) \csc ^{-1}\left(\frac{1}{x}\right)$
21. $\frac{\sin ^{-1} 3 x}{x^{2}+2}$
22. $\frac{\sin ^{-1}(2 t)}{3 t^{2}-t+1}$
23. $\sin ^{-1}\left(\frac{t^{7}+t^{4}+1}{t^{5}+2 t}\right)$.
24. $\frac{u^{2}}{4}+2 \sin ^{-1}\left[\frac{1}{\left(u^{2}+1\right)^{3 / 2}}\right]$.
25. $\frac{\cos ^{-1} x}{1-\sin ^{-1} x}$.
26. $\left[\sin ^{-1}(3 x)+\cos ^{-1}(5 x)\right]\left[\tan ^{-1}(3 x)\right]$.
27. $\left(x^{2} \cos ^{-1} x+\tan x\right)^{3 / 2}$.
28. $\left(x^{3} \sin ^{-1} x+\cot x\right)^{5 / 3}$.

Find the antiderivatives in Exercises 29-36.
29. $\int\left(\frac{3}{1+x^{2}}+x\right) d x$
30. $\int \frac{1}{\sqrt{1-y^{2}}} d y$
31. $\int \frac{4}{\sqrt{1-x^{2}}} d x$
32. $\int \frac{1}{\sqrt{4-x^{2}}} d x$
33. $\int\left(\frac{3}{1+4 x^{2}}\right) d x$
34. $\int\left(\frac{1}{\sqrt{1-4 x^{2}}}\right) d x$
35. $\int\left(\frac{2}{\sqrt{y^{2}\left(y^{2}-1\right)}}\right) d y$
36. $\int\left(\frac{3}{\sqrt{4 u^{2}\left(u^{2}-1\right)}}\right) d u$
37. Prove: $\tan \left(\sin ^{-1} x\right)=x / \sqrt{1-x^{2}}$.
38. Prove $\csc ^{-1}(1 / x)=\sin ^{-1} x=\pi / 2-\cos ^{-1} x$.
39. Is $\cot ^{-1} y=1 /\left(\tan ^{-1} y\right)$ ? Explain.
40. Is the following correct:
$(d / d x) \cos ^{-1} x=(-1)\left(\cos ^{-2} x\right)[(d / d x) \cos x] ?$
Explain.
Calculate the quantities in Exercises 41-46.
41. $\frac{d^{2}}{d \theta^{2}} \cos ^{-1} \theta$
42. $\frac{d}{d \theta} \cot ^{-1}\left(\theta^{2}+1\right)$
43. $\frac{d}{d x}\left[\left(\sin ^{-1} 2 x\right)^{2}+x^{2}\right]$
44. $\frac{d}{d x}\left[\sin ^{-1}\left(\frac{1}{2} \sec \frac{2 x}{3}\right)\right]$
45. The rate of change of $\cos ^{-1}\left(8 s^{2}+2\right)$ with respect to $s$ at $s=0$.
46. The rate of change of $h(t)=\sin ^{-1}\left(3 t+\frac{1}{2}\right)$ with respect to $t$ at $t=0$.
47. What are the maxima, minima, and inflection points of $f(x)=\sin ^{-1} x$ ?
48. Prove that $y=\tan ^{-1} x$ has an inflection point at $x=0$.
49. Derive the formula $(d / d y) \cot ^{-1} y=-\frac{1}{1+y^{2}}$, $-\infty<y<\infty$.
50. Derive the formula

$$
\begin{aligned}
& \frac{d}{d y} \csc ^{-1} y=-\frac{1}{\sqrt{y^{2}\left(y^{2}-1\right)}}, \\
& -\infty<y<-1,1<y<\infty .
\end{aligned}
$$

51. (a) What is the domain of $\cos ^{-1}\left(x^{2}-3\right)$ ? Differentiate. (b) Sketch the graph of $\cos ^{-1}\left(x^{2}-3\right)$.
52. What is the equation of the line tangent to the graph of $\cos ^{-1}\left(x^{2}\right)$ at $x=0$ ?
53. Let $x$ and $y$ be related by the equation

$$
\frac{\sin (x+y)}{x y}=1
$$

and assume that $\cos (x+y)>0$.
(a) Find $d y / d x$.
(b) If $x=t /\left(1-t^{2}\right)$, find $d y / d t$.
(c) If $y=\sin ^{-1} t$, find $d x / d t$.
(d) If $x=t^{3}+2 t-1$, find $d y / d t$.
54. Find a function $f(x)$ which is differentiable and increasing for all $x$, yet $f(x)<\pi / 2$ for all $x$.
Calculate the definite integrals in Exercises 55-58.
55. $\int_{1}^{2} \frac{d x}{1+x^{2}}$.
56. $\int_{0}^{1} \frac{d x}{1+(2 x)^{2}}$.
57. $\int_{1 / 2}^{\sqrt{2} / 2}\left(\frac{x^{2}+1}{x^{2}}+\frac{1}{\sqrt{1-x^{2}}}+\cos x\right) d x$.
58. $\int_{2}^{10} \frac{d y}{\sqrt{y^{2}\left(y^{2}-1\right)}}$.
*59. Suppose that $\sin ^{-1}$ had been defined by inverting $\sin x$ on $[\pi / 2,3 \pi / 2]$ instead of on $[0, \pi]$. What would the derivative of $\sin ^{-1}$ have been?
$\star 60$. It is possible to approach the trigonometric functions without using geometry by defining the function $a(x)$ to be $\int_{0}^{x} d u / \sqrt{1-u^{2}}$ and the letting $\sin$ be the inverse function of $a$. Using this definition prove that:
(a) $(\sin \theta)^{2}+\left(\sin ^{\prime} \theta\right)^{2}$ is constant and equal to 1 ;
(b) $\sin ^{\prime \prime} \theta=-\sin \theta$.

### 5.5 Graphing and Word Problems

Many interesting word problems involve trigonometric functions.
The graphing and word problems in Chapter 3 were limited since, at that point, we could differentiate only algebraic functions. Now that we have more functions at our disposal, we can solve a wider variety of problems.

Example 1 A searchlight 10 kilometers from a straight coast makes one revolution every 30 seconds. How fast is the spot of light moving along a wall on the coast at the point $P$ in Fig. 5.5.1?


Solution Let $\theta$ denote the angle $P L Q$, and let $x$ denote the distance $|P Q|$. We know that $d \theta / d t=2 \pi / 30$, since $\theta$ changes by $2 \pi$ in 30 seconds. Now $x=10 \tan \theta$, so $d x / d \theta=10 \sec ^{2} \theta$, and the velocity of the $\operatorname{spot}$ is $d x / d t=(d x / d \theta)(d \theta / d t)$ $=10 \cdot \sec ^{2} \theta \cdot 2 \pi / 30=(2 \pi / 3) \sec ^{2} \theta$. A ${ }^{+} \quad x=8, \quad \sec \theta=|P L| /|L Q|=$ $\sqrt{8^{2}+10^{2}} / 10$, so the velocity is $d x / d t=(2 \pi / 3) \times\left(8^{2}+10^{2}\right) / 10^{2}=(82 \pi / 75)$ $\approx 3.4$ kilometers per second (this is very fast!).

Example 2 Find the point on the $x$ axis for which the sum of the distances from $(0,1)$ and ( $p, q$ ) is a minimum. (Assume that $p$ and $q$ are positive.)
Solution Let $(x, 0)$ be a point on the $x$ axis. The distance from $(0,1)$ is $\sqrt{1+x^{2}}$ and the distance from $(p, q)$ is $\sqrt{(x-p)^{2}+q^{2}}$. The sum of the distances is

$$
S=\sqrt{1+x^{2}}+\sqrt{(x-p)^{2}+q^{2}} .
$$

To minimize, we find:

$$
\begin{aligned}
\frac{d S}{d x} & =\frac{1}{2}\left(1+x^{2}\right)^{-1 / 2}(2 x)+\frac{1}{2}\left[(x-p)^{2}+q^{2}\right]^{-1 / 2} 2(x-p) \\
& =\frac{x}{\sqrt{1+x^{2}}}+\frac{x-p}{\sqrt{(x-p)^{2}+q^{2}}}
\end{aligned}
$$

Setting this equal to zero gives

$$
\frac{x}{\sqrt{1+x^{2}}}=\frac{p-x}{\sqrt{(x-p)^{2}+q^{2}}} .
$$

Instead of solving for $x$, we will interpret the preceding equation geometrically. Referring to Fig. 5.5.2, we find that $\sin \theta_{1}=x / \sqrt{1+x^{2}}$ and $\sin \theta_{2}=(p-x) / \sqrt{(x-p)^{2}+q^{2}}$; our equation says that these are equal, so $\theta_{1}=\theta_{2}$. Thus $(x, 0)$ is located at the point for which the lines from $(x, 0)$ to

Figure 5.5.2. The shortest path from $(0,1)$ to $(p, q)$ via the $x$ axis has $\theta_{1}=\theta_{2}$.

Figure 5.5.3. The pole in the corner.


Figure 5.5.4. $\phi=\tan ^{-1} \alpha$.

$(0,1)$ and $(p, q)$ make equal angles with a line parallel to the $y$ axis. This result is sometimes called the law of reflection.

Example 3 Two hallways, meeting at right angles, have widths $a$ and $b$. Find the length of the longest pole which will go around the corner; the pole must be in a horizontal position. (In Exercise 51, Section 3.5, you were asked to do the problem by minimizing the square of the length; redo the problem here by minimizing the length itself.)

Solution Refer to Fig. 5.5.3. The length of $P Q$ is

$$
f(\theta)=\frac{a}{\sin \theta}+\frac{b}{\cos \theta}
$$



The minimum of $f(\theta), 0 \leqslant \theta \leqslant \pi / 2$, will give the length of the longest pole which will fit around the corner. The derivative is

$$
f^{\prime}(\theta)=-\frac{a \cos \theta}{\sin ^{2} \theta}+\frac{b \sin \theta}{\cos ^{2} \theta}
$$

which is zero when $a \cos ^{3} \theta=b \sin ^{3} \theta$; that is, when $\tan ^{3} \theta=a / b$; hence $\theta$ $=\tan ^{-1}(\sqrt[3]{a / b})$. Since $f$ is large positive near 0 and $\pi / 2$, and there are no other critical points, this is a global minimum. (You can also use the second derivative test.) Thus, the answer is

$$
\frac{a}{\sin \theta}+\frac{b}{\cos \theta}
$$

where $\theta=\tan ^{-1}(\sqrt[3]{a / b})$. Using $\sin \left(\tan ^{-1} \alpha\right)=\alpha / \sqrt{1+\alpha^{2}}$ and $\cos \left(\tan ^{-1} \alpha\right)$ $=1 / \sqrt{1+\alpha^{2}}$ (Fig. 5.5.4), one can express the answer, after some simplification, as $\left(a^{2 / 3}+b^{2 / 3}\right)^{3 / 2}$.

One way to check the answer (which the authors actually used to catch an error) is to note its "dimension." The result must have the dimension of a length. Thus an answer like $a^{1 / 3}\left(a^{2 / 3}+b^{2 / 3}\right)^{3 / 2}$, which has dimension of (length) ${ }^{1 / 3} \times$ length, cannot be correct.

Example 4 (This problem was written on a train.) One normally chooses the window seat on a train to have the best view. Imagine the situation in Fig. 5.5.5 and see if this is really the best choice. (Ignore the extra advantage of the window seat which enables you to lean forward to see a special view.)

Figure 5.5.6. Which value of $x$ maximizes $\angle P O Q$ ?


Figure 5.5.5. Where should you sit to get the widest view?

## Solution

Is convenient to replace the diagram by a more abstract one (Fig. 5.5.6). We assume that the passenger's eye is located at a point $O$ on the line $A B$ when he is sitting upright and that he wishes to maximize the angle $\angle P O Q$ subtended by the window $P Q$. Denote by $x$ the distance from $O$ to $B$, which can be varied. Let the width of the window be $w$ and the distance from $A B$ to the window be $d$. Then we have

$$
\begin{aligned}
& \angle P O Q=\angle B O Q-\angle P O B \\
& \tan (\angle B O Q)=\frac{w+d}{x}
\end{aligned}
$$

and

$$
\tan (\angle P O B)=\frac{d}{x}
$$

So we wish to maximize

$$
f(x)=\angle P O Q=\tan ^{-1}\left(\frac{w+d}{x}\right)-\tan ^{-1}\left(\frac{d}{x}\right)
$$

Differentiating, we have

$$
\begin{aligned}
f^{\prime}(x) & =\frac{1}{1+[(w+d) / x]^{2}} \cdot\left(-\frac{w+d}{x^{2}}\right)-\frac{1}{1+(d / x)^{2}} \cdot\left(-\frac{d}{x^{2}}\right) \\
& =-\frac{w+d}{x^{2}+(w+d)^{2}}+\frac{d}{x^{2}+d^{2}}
\end{aligned}
$$

Setting $f^{\prime}(x)=0$ yields the equation

$$
\begin{aligned}
\left(x^{2}+d^{2}\right)(w+d) & =\left[x^{2}+(w+d)^{2}\right] d \\
w x^{2} & =(w+d)^{2} d-d^{2}(w+d) \\
x^{2} & =(w+d) \cdot d
\end{aligned}
$$

The solution, therefore, is $x=\sqrt{(w+d) d}$.
For example (all distances measured in feet), if $d=1$ and $w=5$, we should take $x=\sqrt{6} \approx 2.45$. Thus, it is probably better to take the second seat from the window, rather than the window seat.

There is a geometric interpretation for the solution of this problem. We may rewrite the solution as

$$
x^{2}=d^{2}+w d \quad \text { or } \quad x^{2}=\left(d+\frac{1}{2} w\right)^{2}-\left(\frac{1}{2} w\right)^{2}
$$

Figure 5.5.7. Geometric construction for the best seat.

This second formula leads to the following construction (which you may be able to carry out mentally before choosing your seat). Draw a line $R P$ through $P$ and parallel to $A B$. Now construct a circle with center at the midpoint $M$ of $P Q$ and with radius $M B$. Let $Z$ be the point where the circle intersects $R P$. Then $x=Z P$. (See Fig. 5.5.7.)


Let us turn from word problems to some problems in graphing, using the methods of Chapter 3.

Example 5 Discuss maxima, minima, concavity, and points of inflection for $f(x)=\sin ^{2} x$. Sketch its graph.
Solution If $f(x)=\sin ^{2} x, f^{\prime}(x)=2 \sin x \cos x$ and $f^{\prime \prime}(x)=2\left(\cos ^{2} x-\sin ^{2} x\right)$. The first derivative vanishes when either $\sin x=0$ or $\cos x=0$, at which points $f^{\prime \prime}$ is positive and negative, yielding minima and maxima. Thus, the minima of $f$ are at $0, \pm \pi, \pm 2 \pi, \ldots$, where $f=0$, and the maxima are at $\pm \pi / 2$, $\pm 3 \pi / 2, \ldots$, where $f=1$.

The function $f(x)$ is concave upward when $f^{\prime \prime}(x)>0$ (that is, $\cos ^{2} x$ $>\sin ^{2} x$ ) and downward when $f^{\prime \prime}(x)<0$ (that is, $\cos ^{2} x<\sin ^{2} x$ ). Also, $\cos x$ $= \pm \sin x$ exactly if $x= \pm \pi / 4, \pm \pi / 4 \pm \pi, \pm \pi / 4 \pm 2 \pi$, and so on (see the graphs of sine and cosine). These are then inflections points separating regions where $\sin ^{2} x$ is concave up and concave down. The graph is shown in Fig. 5.5.8.

Figure 5.5.8. The graph of $\sin ^{2} x$.


Now that we have done all this work, we observe that the graph could also have been found from the half-angle formula $\sin ^{2} \theta=\frac{1}{2}(1-\cos 2 \theta)$.

Example 6 Sketch the graph of the function $f(x)=\cos x+\cos 2 x$.
Solution The function is defined on $(-\infty, \infty)$; there are no asymptotes. We find that $f(-x)=\cos (-x)+\cos (-2 x)=\cos x+\cos 2 x=f(x)$, so $f$ is even. Furthermore, $f(x+2 \pi)=f(x)$, so the graph repeats itself every $2 \pi$ units (like that of $\cos x$ ). It follows that we need only look for features of the graph on $[0, \pi]$, because we can obtain $[-\pi, 0]$ by reflection across the $y$ axis and the rest of the graph by repetition of the part over $[-\pi, \pi]$.

We have

$$
f^{\prime}(x)=-\sin x-2 \sin 2 x \quad \text { and } f^{\prime \prime}(x)=-\cos x-4 \cos 2 x .
$$

To find roots of the first and second derivatives, it is best to factor using the formulas

$$
\sin 2 x=2 \sin x \cos x \quad \text { and } \quad \cos 2 x=2 \cos ^{2} x-1
$$

We obtain

$$
f^{\prime}(x)=-\sin x-4 \sin x \cos x=-\sin x(1+4 \cos x)
$$

and

$$
f^{\prime \prime}(x)=-\cos x-4\left(2 \cos ^{2} x-1\right)=-8 \cos ^{2} x-\cos x+4
$$

The critical points occur when $\sin x=0$ or $1+4 \cos x=0$-that is, when $x=0, \pi$, or $\cos ^{-1}\left(-\frac{1}{4}\right) \approx 1.82$ (radians).

We have

$$
\begin{aligned}
& f(0)=2, \quad f(\pi)=0, \quad f\left[\cos ^{-1}\left(-\frac{1}{4}\right)\right]=-1.125 \\
& f^{\prime \prime}(0)=-5, \quad f^{\prime \prime}(\pi)=-3, \quad f^{\prime \prime}\left[\cos ^{-1}\left(-\frac{1}{4}\right)\right]=3.75
\end{aligned}
$$

Hence 0 and $\pi$ are local maximum points, and $\cos ^{-1}\left(-\frac{1}{4}\right)$ is a local minimum point, by the second derivative test.

To find the points of inflection, we first find the roots of $f^{\prime \prime}(x)=0$; that is,

$$
-8 \cos ^{2} x-\cos x+4=0
$$

This is a quadratic equation in which $\cos x$ is the unknown, so

$$
\cos x=\frac{-1 \pm \sqrt{129}}{16} \approx 0.647 \text { and }-0.772
$$

Thus our candidates for points of inflection are

$$
x_{1}=\cos ^{-1}(0.647) \approx 0.87 \quad \text { and } \quad x_{2}=\cos ^{-1}(-0.772) \approx 2.45
$$

We can see from the previously calculated values for $f^{\prime \prime}(x)$ that $f^{\prime \prime}$ does change sign at these points, so they are inflection points. We calculate $f(x)$ and $f^{\prime}(x)$ at the inflection points:

$$
\begin{array}{ll}
f\left(x_{1}\right) \approx 0.48, & f^{\prime}\left(x_{1}\right) \approx-2.74 \\
f\left(x_{2}\right) \approx-0.58, & f^{\prime}\left(x_{2}\right) \approx 1.33
\end{array}
$$

Finally, the zeros of $f$ may be found by writing

$$
\begin{aligned}
f(x) & =\cos x+\cos 2 x \\
& =\cos x+2 \cos ^{2} x-1 \\
& =2(\cos x+1)\left(\cos x-\frac{1}{2}\right)
\end{aligned}
$$

Thus, $f(x)=0$ at $x=\pi$ and $x=\cos ^{-1}\left(\frac{1}{2}\right) \approx 1.047$. The graph on $[0, \pi]$ obtained from this information is shown in Fig. 5.5.9. Reflecting across the $y$ axis


Figure 5.5.10. The full graph of $\cos x+\cos 2 x$.
and then repeating the pattern, we obtain the graph shown in Fig. 5.5.10. Such graphs, with oscillations of varying amplitudes, are typical when sine and cosine functions with different frequencies are added.


## Exercises for Section 5.5

1. The height of an object thrown straight down from an initial altitude of 1000 feet is given by $h(t)=1000-40 t-16 t^{2}$. The object is being tracked by a searchlight 200 feet from where the object will hit. How fast is the angle of elevation of the searchlight changing after 4 seconds?
2. A searchlight 100 meters from a road is tracking a car moving at 100 kilometers per hour. At what rate (in degrees per second) is the searchlight turning when the car is 141 meters away?
3. A child is whirling a stone on a string 0.5 meter long in a vertical circle at 5 revolutions per second. The sun is shining directly overhead. What is the velocity of the stone's shadow when the stone is at the 10 o'clock position?
4. A bicycle is moving 10 feet per second. It has wheels of radius 16 inches and a reflector attached to the front spokes 12 inches from the center. If the reflector is at its lowest point at $t=0$, how fast is the reflector accelerating vertically at $t=5$ seconds?
5. Two weights $A$ and $B$ together on the ground are joined by a 20 -meter wire. The wire passes over a pulley 10 meters above the ground. Weight $A$ is slid along the ground at 2 meters per second. How fast is the distance between the weights changing after 3 seconds? (See Fig. 5.5.11.)


Figure 5.5.11. How fast are $A$ and $B$ separating?
6. Consider the two posts in Fig. 5.5.12. The light atop post $A$ moves vertically up and down the
post according to $h(t)=55+5 \sin t$ ( $t$ in seconds, height in meters). How fast is the length of the shadow of the 2 -meter statue changing at $t=20$ seconds?


Figure 5.5.12. How fast is the shadow's length changing when the light oscillates up and down?
7. Consider the situation sketched in Fig. 5.5.13. At what position on the road is the angle $\theta$ maximized?


Figure 5.5.13. Maximize $\theta$.
8. Two trains, each 50 meters long, are moving away from the intersection point of perpendicular tracks at the same speed. Where are the trains when $\operatorname{train} A$ subtends the largest angle as seen from the front of train $B$ ?
9. Particle $A$ is moving in the plane according to $x=3 \sin 3 t$ and $y=3 \cos 3 t$ and particle $B$ is moving according to $x=3 \cos 2 t$ and $y=3$ $\sin 2 t$. Find the maximum distance between $A$ and $B$.
10. Which points on the parametric curve $x=\cos t$, $y=4 \sin t$ are closest to $(0,1)$ ?
11. A slot racer travels at constant speed around a circular track, doing each lap in 3.1 seconds. The track is 3 feet in diameter.
(a) The position $(x, y)$ of the racer can be written as $x=r \cos (\omega t), y=r \sin (\omega t)$. Find the values of $r$ and $\omega$.
(b) The speed of the racer is the elapsed time divided into the distance traveled. Find its value and check that it is equal to

$$
\sqrt{(d x / d t)^{2}+(d y / d t)^{2}}
$$

12. The motion of a projectile (neglecting air friction and the curvature of the earth) is governed by the equations

$$
x=v_{0} t \cos \alpha \quad y=v_{0} t \sin \alpha-4.9 t^{2}
$$

where $v_{0}$ is the initial velocity and $\alpha$ is the initial angle of elevation. Distances are measured in meters and $t$ is the time from launch. (See Fig. 5.5.14.)


Figure 5.5.14. Path of a projectile near the surface of the earth.
(a) Find the maximum height of the projectile and the distance $R$ from launch to fall as a function of $\alpha$.
(b) Show that $R$ is maximized when $\alpha=\pi / 4$.
13. Molasses is smeared over the upper half $(y>0)$ of the $(x, y)$ plane. A bug crawls at 1 centimeter per minute where there is molasses and 3 centimeters per minute where there is none. Suppose that the bug is to travel from some point in the upper half-plane $(y>0)$ to some point in the lower half-plane $(y<0)$. The fastest route then consists of a broken line segment with a break on the $x$ axis. Find a relation between the sines of the angles made by the two parts of the segments with the $y$ axis.
14. Drywall sheets weighing 6000 lbs are moved across a level floor. The method is to attach a chain to the skids under the drywall stack, then pull it with a truck. The angle $\theta$ made by the chain and the floor is related to the force $F$ along the chain by $F=6000 k /(k \sin \theta+\cos \theta)$, where the number $k$ is the coefficient of friction.
(a) Compute $d F / d \theta$.
(b) Find $\theta$ for which $d F / d \theta=0$. (This is the angle that requires the least force.)
15. Discuss the maxima, minima, concavity, and points of inflection for $y=\sin 2 x-1$.
16. Find the maxima, minima, concavity, and inflection points of $f(x)=\cos ^{2} 3 x$.
17. Where is $f(x)=x \sin x+2 \cos x$ concave up? Concave down?
18. Where is $g(\theta)=\sin ^{3} \theta$ concave up? Concave down?
Sketch the graphs of the functions in Exercises 19-26.
19. $y=\cos ^{2} x$
20. $y=1+\sin 2 x$
21. $y=x+\cos x$
22. $y=x \sin x$
23. $y=2 \cos x+\cos 2 x$
24. $y=\cos 2 x+\cos 4 x$
25. $y=x^{2 / 3} \cos x$
26. $y=x^{1 / 2} \sin x$
27. Do Example 2 without using calculus. [Hint: Replace $(p, q)$ by $(p,-q)$.]
28. The displacement $x(t)$ from equilibrium of a mass $m$ undergoing harmonic motion on a spring of Hooke's constant $k$ is known to satisfy the equation $m x^{\prime \prime}(t)+k x(t)=0$. Check that $x(t)$ $=A \cos (\omega t+\theta)$ is a solution of this equation, where $\omega=\sqrt{k / m} ; A, \theta$ and $\omega$ are constants.
29. Determine the equations of the tangent and normal lines to the curve $y=\cos ^{-1} 2 x+\cos ^{-1} x$ at $(0, \pi)$.
30. Find the equation of the tangent line to the parametric curve $x=t^{2}, y=\cos ^{-1} t$ when $t=\frac{1}{2}$.
31. Sketch the graph of the function $(\sin x) /\left(1+x^{2}\right)$ for $0 \leqslant x<2 \pi$. (You may need to use a calculator to locate the critical points.)
$\star 32$. Let $f(x)=\sin ^{-1}\left[2 x /\left(x^{2}+1\right)\right]$.
(a) Show that $f(x)$ is defined for $x$ in $(-\infty, \infty)$.
(b) Compute $f^{\prime}(x)$. Where is it defined?
(c) Show that the maxima and minima occur at points where $f$ is not differentiable.
(d) Sketch the graph of $f$.
*33. Show that the function $(\sin x) / x$ has infinitely many local maxima and minima, and that they become approximately evenly spaced as $x \rightarrow \infty$.
$\star$ 34. Given two points outside a circle, find the shortest path between them which touches the circle. (Hint: First assume that the points are equidistant from the center of the circle, and put the figure in a standard position.)

### 5.6 Graphing in Polar Coordinates

Periodic functions are graphed as closed curves in polar coordinates.
The graph in polar coordinates of a function $f$ consists of all those points in the plane whose polar coordinates $(r, \theta)$ satisfy the relation $r=f(\theta)$. Such graphs are especially useful when $f$ is built up from trigonometric functions, since the entire graph is drawn when we let $\theta$ vary from 0 to $2 \pi$.

Various properties of $f$ may appear as symmetries of the graph. For instance, if $f(\theta)=f(-\theta)$, then its graph is symmetric in the $x$ axis; if $f(\pi-\theta)=f(\theta)$, it is symmetric in the $y$ axis; and if $f(\theta)=f(\pi+\theta)$, it is symmetric in the origin (see Fig. 5.6.1).


Figure 5.6.1. Symmetry in $x$ axis: $f(\theta)=f(-\theta)$;
symmetry in $y$ axis: $f(\pi-\theta)=f(\theta)$;
symmetry in origin: $f(\theta)=f(\pi+\theta)$.
Example 1 Plot the graph of $r=\cos 2 \theta$ in the $x y$ plane and discuss its symmetry.
Solution We know from the cartesian graph $y=\cos 2 x$ (Fig. 5.1.27) that as $\theta$ increases from 0 to $\pi / 4, r=\cos 2 \theta$ decreases from 1 to 0 . As $\theta$ continues from $\pi / 4$ to $\pi / 2, r=\cos 2 \theta$ becomes negative and decreases to -1 . Thus $(r, \theta)$ traces out the path in Fig. 5.6.2.


We can complete the path as $\theta$ goes through all values between 0 and $2 \pi$, sweeping out the four petals in Fig. 5.6.3, or else we may use symmetry in the $x$ and $y$ axes. In fact, $f(-\theta)=\cos (-2 \theta)=\cos 2 \theta=f(\theta)$, so we have symmetry in the $x$ axis. Also,


Figure 5.6.3. The full graph $r=\cos 2 \theta$, the "fourpetaled rose."

Example 2 Sketch the graph of $r=f(\theta)=\cos 3 \theta$.
Solution The graph is symmetric in the $x$ axis and, moreover, $f(\theta+\pi / 3)=-f(\theta)$. This means that we need only sketch the graph for $0 \leqslant \theta \leqslant \pi / 3$ and obtain the rest by reflection and rotations. Thus, we expect a three- or six-petaled rose. As $\theta$ varies from 0 to $\pi / 3,3 \theta$ increases from 0 to $\pi$, and $\cos 3 \theta$ decreases from 1 to - Hence, we get the graph in Fig. 5.6.4.


Reflect across the $x$ axis to complete the petal and then rotate by $\pi / 3$ and reflect through the origin (see Fig. 5.6.5).
 of $r=\cos 3 \theta$.

$$
\begin{aligned}
f(\pi-\theta) & =\cos [2(\pi-\theta)] \\
& =\cos 2 \pi \cos 2 \theta+\sin 2 \pi \sin 2 \theta \\
& =\cos 2 \theta=f(\theta)
\end{aligned}
$$

which gives symmetry in the $y$ axis. Finally,

$$
\begin{aligned}
f\left(\theta+\frac{\pi}{2}\right) & =\cos 2\left(\theta+\frac{\pi}{2}\right)=\cos (2 \theta+\pi) \\
& =\cos 2 \theta \cos \pi-\sin 2 \theta \sin \pi \\
& =-\cos 2 \theta=-f(\theta)
\end{aligned}
$$

Thus, the graph is unchanged when reflected in the $x$ axis and the $y$ axis. When we rotate by $90^{\circ}, r=f(\theta)$ reflects through the origin; that is, $r$ changes to $-r$.

Figure 5.6.4. Beginning the graph of $r=\cos 3 \theta$.

Figure 5.6.5. The graphing

Example 3 If $f(\theta+\pi / 2)=f(\theta)$, what does this tell you about the graph of $r=f(\theta)$ ?
Solution
This means that the graph will have the same appearance if it is rotated by $90^{\circ}$, since replacing $\theta$ by $\theta+\pi / 2$ means that we rotate through an angle $\pi / 2$.

Figure 5.6.6.
(a) $r=1+\cos \theta$
("cardioid");
(b) $r=1+2 \cos \theta$
("limaçon").

Figure 5.6 .6 shows two other graphs in polar coordinates with striking symmetry. (The curve in (b) is discussed in Example 7 below.)

(a)

(b)

Example 4 Convert the relation $r=1+2 \cos \theta$ to cartesian coordinates.
Solution We substitute $r=\sqrt{x^{2}+y^{2}}$ and $\cos \theta=x / r=x / \sqrt{x^{2}+y^{2}}$ to get

$$
\sqrt{x^{2}+y^{2}}=1+\frac{2 x}{\sqrt{x^{2}+y^{2}}}
$$

That is, $x^{2}+y^{2}-\sqrt{x^{2}+y^{2}}-2 x=0$.
Calculus can help us to draw graphs in polar coordinates by telling us the slope of tangent lines (see Fig. 5.6.7).


This slope at a point $(r, \theta)$ is not $f^{\prime}(\theta)$, since $f^{\prime}(\theta)$ is the rate of change of $r$ with respect to $\theta$, while the slope is the rate of change of $y$ with respect to $x$. To calculate $d y / d x$, we write

$$
x=r \cos \theta=f(\theta) \cos \theta \quad \text { and } \quad y=r \sin \theta=f(\theta) \sin \theta
$$

This is a parametric curve with $\theta$ as the parameter. According to the formula $d y / d x=(d y / d t) /(d x / d t)$ from Section 2.4, with $t$ replaced by $\theta$,

$$
\frac{d y}{d x}=\frac{d y / d \theta}{d x / d \theta}=\frac{f^{\prime}(\theta) \sin \theta+f(\theta) \cos \theta}{f^{\prime}(\theta) \cos \theta-f(\theta) \sin \theta}
$$

Dividing numerator and denominator by $\cos \theta$ gives

$$
\begin{equation*}
\frac{d y}{d x}=\frac{f^{\prime}(\theta) \tan \theta+f(\theta)}{f^{\prime}(\theta)-f(\theta) \tan \theta} \tag{1}
\end{equation*}
$$

## Tangents to Graphs in Polar Coordinates

The slope of the tangent to the graph of $r=f(\theta)$ at $(r, \theta)$ is

$$
\frac{(\tan \theta) d r / d \theta+r}{d r / d \theta-r \tan \theta}
$$

Example 5 (a) Find the slope of the line tangent to the graph of $r=3 \cos ^{2} 2 \theta$ at $\theta=\pi / 6$. (b) Find the slope of the line tangent to the graph of $r=\cos 3 \theta$ at $(r, \theta)=$ $(-1, \pi / 3)$.

Solution (a) Here, $f(\theta)=3 \cos ^{2} 2 \theta$, so $d r / d \theta=f^{\prime}(\theta)=-12 \cos 2 \theta \sin 2 \theta$ (by the chain rule). Now $f(\pi / 6)=3 \cos ^{2}(\pi / 3)=\frac{3}{4}$ and $f^{\prime}(\pi / 6)=-12 \cos (\pi / 3) \sin (\pi / 3)$ $=-12 \cdot \frac{1}{2} \cdot \sqrt{3} / 2=-3 \sqrt{3}$. Thus formula (1) gives

$$
\frac{d y}{d x}=\frac{-3 \sqrt{3}(1 / \sqrt{3})+3 / 4}{-3 \sqrt{3}-(3 / 4)(1 / \sqrt{3})}=\frac{3 \sqrt{3}}{13}
$$

so the slope of the tangent line is $3 \sqrt{3} / 13$.
(b) Here, $f(\theta)=\cos 3 \theta$, so the slope is, by formula (1),

$$
\begin{aligned}
\frac{f^{\prime}(\theta) \tan \theta+f(\theta)}{f^{\prime}(\theta)-f(\theta) \tan \theta} & =\frac{-3 \sin 3 \theta \tan \theta+\cos 3 \theta}{-3 \sin 3 \theta-\cos 3 \theta \tan \theta} \\
& =\frac{1-3 \tan 3 \theta \tan \theta}{-\tan \theta-3 \tan 3 \theta}
\end{aligned}
$$

Hence, at $\theta=\pi / 3$, the slope is $1 /-1.732 \approx-0.577$.
Calculus can aid us in other ways. A local maximum of $f(\theta)$ will be a point on the graph where the distance from the origin is a local maximum, as in Fig. 5.6.8. The methods of Chapter 3 can be used to locate these local maxima (as well as the local minima).

Figure 5.6.8. The point $P$ corresponds to a local maximum point of $f(\theta)$.


Example 6 Calculate the slope of the line tangent to $r=f(\theta)$ at $(r, \theta)$ if $f$ has a local maximum there. Interpret geometrically.

Solution At a local maximum, $f^{\prime}(\theta)=0$. Plugging this into formula (1) gives

$$
\begin{aligned}
\frac{d y}{d x} & =\frac{f(\theta)}{-f(\theta) \tan \theta} \\
& =-\frac{1}{\tan \theta}
\end{aligned}
$$

Since $d y / d x$ is the negative reciprocal of $\tan \theta$, the tangent line is perpendicular to the line from the origin to $(r, \theta)$. (See Fig. 5.6.8.)

Example 7 Find the maxima and minima of $f(\theta)=1+2 \cos \theta$. Sketch the graph of $r=1+2 \cos \theta$ in the $x y$ plane.
Solution Here $d r / d \theta=-2 \sin \theta$, which vanishes if $\theta=0, \pi$. Also, $d^{2} r / d \theta^{2}=-2 \cos \theta$, which is -2 at $\theta=0,+2$ at $\theta=\pi$. Hence $r=3, \theta=0$ is a local maximum and $r=-1, \theta=\pi$ is a local minimum. The tangent lines are vertical there. The curve passes through $r=0$ when $\theta= \pm 2 \pi / 3$. The curve is symmetric in the $x$ axis and can be plotted as in Fig. 5.6.9.


Figure 5.6.9. The maxima and minima of $1+2 \cos \theta$ correspond to the points $P_{1}$ and $P_{2}$.

## Exercises for Section 5.6

In Exercises 1-10, sketch the graph of the given function in polar coordinates. Also convert the given equation to cartesian coordinates.

1. $r=\cos \theta$
2. $r=2 \sin \theta$
3. $r=1-\sin \theta$
4. $r=\sin \left(\frac{\theta}{2}\right)+1$
5. $r=3$
6. $\sin \theta=1$
7. $r^{2}+2 r \cos \theta+1=0$
8. $r^{2} \sin 2 \theta=\frac{1}{2}$
9. $r=\sin 3 \theta$
10. $r=\cos \theta-\sin \theta$
11. Describe the graph of the equation $r=$ constant.
12. What is the equation of a line through the origin?
13. If $f(\pi / 2-\theta)=f(\theta)$, what does this tell you about the graph of $r=f(\theta)$ ?
14. If $f(\theta+\pi)=-f(\theta)$, what does this tell you about the graph of $r=f(\theta)$ ?
Convert the relations in Exercises 15-22 to polar coordinates.
15. $x^{2}+y^{2}=1$
16. $x y=1$
17. $x^{2}+x y+y^{2}=1$
18. $y=x^{5}+x^{3}+2$
19. $y=\frac{1}{1-x^{2}}$
20. $y=\frac{x}{1+x}$
21. $y=x+1$
22. $y=\frac{1}{x}+x^{2}$

In Exercises 23-32, find the slope of the tangent line at the indicated point.

$$
\begin{aligned}
& \text { 23. } r=\tan \theta ; \theta=\frac{2 \pi}{3} \\
& \text { 24. } r=\tan \theta ; \theta=\frac{\pi}{4} \\
& \text { 25. } r=2 \sin 5 \theta ; \theta=\frac{\pi}{2} \\
& \text { 26. } r=1+2 \sin 2 \theta ; \theta=0 . \\
& \text { 27. } r=\cos 4 \theta ; \theta=\frac{\pi}{3} . \\
& \text { 28. } r=2-\sin \theta ; \theta=\frac{\pi}{6} \\
& \text { 29. } r=3 \sin \theta+\cos \left(\theta^{2}\right) ; \theta=0 . \\
& \text { 30. } r=\sin 3 \theta \cos 2 \theta ; \theta=0 . \\
& \text { 31. } r=\theta^{2}+1 ; \theta=5 . \\
& \text { 32. } r=\sec \theta+2 \theta^{3} ; \theta=\frac{\pi}{6}
\end{aligned}
$$

Find the maximum and minimum values of $r$ for the functions in Exercises 33-38. Sketch the graphs in the $x y$ plane.
33. $r=\cos 4 \theta$
34. $r=2 \sin 5 \theta$
35. $r=\tan \theta$
36. $r=2-\sin \theta$
37. $r=1+2 \sin 2 \theta$
38. $r=\theta^{2}+1$
39. Find the maximum and minimum values of $r$ $=\sin 3 \theta \cos 2 \theta$. Sketch its graph in the $x y$ plane.
*40. Sketch the graph of $r=\sin ^{3} 3 \theta$.

## Supplement to Chapter 5 Length of days

We outline an application of calculus to a phenomenon which requires no specialized equipment or knowledge for its observation-the setting of the sun.

Using spherical trigonometry or vector methods, one can derive a formula relating the following variables:
$A=$ angle of elevation of the sun above the horizon;
$l=$ latitude of a place on the earth's surface;
$\alpha=$ inclination of the earth's axis ( $23.5^{\circ}$ or 0.41 radian);
$T=$ time of year, measured in days from the first day of summer in the northern hemisphere (June 21);
$t=$ time of day, measured in hours from noon. ${ }^{5}$
The formula reads:

$$
\begin{equation*}
\sin A=\cos l \sqrt{1-\sin ^{2} \alpha \cos ^{2}\left(\frac{2 \pi T}{365}\right)} \cos \left(\frac{2 \pi t}{24}\right)+\sin l \sin \alpha \cos \left(\frac{2 \pi T}{365}\right) . \tag{1}
\end{equation*}
$$

We will derive (1) in the supplement to Chapter 14. For now, we will simply assume the formula and find some of its consequences. ${ }^{6}$

At the time $S$ of sunset, $A=0$. That is,

$$
\begin{equation*}
\cos \left(\frac{2 \pi S}{24}\right)=-\tan l \frac{\sin \alpha \cos (2 \pi T / 365)}{\sqrt{1-\sin ^{2} \alpha \cos ^{2}(2 \pi T / 365)}} . \tag{2}
\end{equation*}
$$

Solving for $S$, and remembering that $S \geqslant 0$ since sunset occurs after noon, we get

$$
\begin{equation*}
S=\frac{12}{\pi} \cos ^{-1}\left[-\tan l \frac{\sin \alpha \cos (2 \pi T / 365)}{\sqrt{1-\sin ^{2} \alpha \cos ^{2}(2 \pi T / 365)}}\right] . \tag{3}
\end{equation*}
$$

For example, let us compute when the sun sets on July 1 at $39^{\circ}$ latitude. We have $l=39^{\circ}, \alpha=23.5^{\circ}$, and $T=11$. Substituting these values in (3), we find $\tan l=0.8098,2 \pi T / 365=0.1894$ (that is, $10.85^{\circ}$ ), $\cos (2 \pi T / 365)$ $=0.9821$, and $\sin \alpha=0.3987$. Therefore, $S=(12 / \pi) \cos ^{-1}(-0.3447)=$ $(12 / \pi)(1.9227)=7.344$. Thus $S=7.344$ (hours after noon); that is, the sun sets at 7:20:38 (if noon is at 12:00).

For a fixed point on the earth, $S$ may be considered a function of $T$. Differentiating (3) and simplifying, we find

$$
\begin{equation*}
\frac{d S}{d T}=-\frac{24}{365} \tan l\left\{\frac{\sin \alpha \sin (2 \pi T / 365)}{\left[1-\sin ^{2} \alpha \cos ^{2}(2 \pi T / 365)\right] \sqrt{1-\sin ^{2} \alpha \cos ^{2}(2 \pi T / 365) \sec ^{2} l}}\right\} . \tag{4}
\end{equation*}
$$

The critical points of $S$ occur when $2 \pi T / 365=0, \pi$, or $2 \pi$; that is, $T=0$, $365 / 2$, or $T=365$-the first day of summer and the first day of winter. For the northern hemisphere, $\tan l$ is positive. By the first derivative test, $T=0$ (or 365 ) is a local maximum and $T=365 / 2$ a minimum. Thus we get the graph shown in Fig. 5.S.1.

[^3]Figure 5.S.1. Sunset time as a function of the date. The sun sets late in summer, early in winter.

Figure 5.S.2. Day length as a function of latitude and day of the year.


Example Use (4) and (5) to compute how many minutes earlier the sun will set on July 2 at latitude $39^{\circ}$ than on July 1.

Solution Substituting $T=11$ into (4), we obtain $\Delta S \approx-0.0055$; that is, the sun should set 0.0055 hour, or 20 seconds, earlier on July 2 than on July 1. Computing the time of sunset on July 2 by formula (3), with $T=12$, we obtain $S=7.338$, or

[^4]7:20:17, which is 21 seconds earlier than the time computed by (4) and (5) for sunset on July 2. The error in the first-order approximation to $\Delta S$ is thus about 1 part in 20 , or $5 \%$.

Differentiating formula (4), we find that the extreme values of $d S / d T$ occur when $2 \pi T / 365=\pi / 2$ or $3 \pi / 2$. (These are the inflection points in Fig. 5.6.1.) When $2 \pi T / 365=\pi / 2$ (the first day of fall), $d S / d T$ is equal to $-(24 / 365) \tan l \sin \alpha$; at this time, the days are getting shorter most rapidly. ${ }^{8}$ When $2 \pi T / 365=3 \pi / 2$ (the first day of spring), the days are lengthening most rapidly, with $d S / d T=(24 / 365) \tan / \sin \alpha$.

It is interesting to note how this maximal rate, $(24 / 365) \tan l \sin \alpha$, depends on latitude. Near the equator, $\tan l$ is very small, so the rate is near zero, corresponding to the fact that seasons don't make much difference near the equator. Near the poles, $\tan l$ is very large, so the rate is enormous. This large rate corresponds to the sudden switch from nearly 6 months of sunlight to nearly 6 months of darkness. At the poles, the rate is "infinite." (Of course, in reality the change isn't quite sudden because of the sun's diameter, the fact that the earth isn't a perfect sphere, refraction by the atmosphere, and so forth.)

The reader who wishes to explore these topics further should read the supplements to Section 9.5 and Chapter 14 and try the following exercises.

## Exercises ior the Supplement to Chapter 5

1. According to the Los Angeles Times for July 12, 1975, the sun set at 8:06 P.M. The latitude of Los Angeles is $33.57^{\circ}$ North. Guess what time the July 13 paper gave for sunset? What about July 14?
2. Determine your latitude (approximately) by measuring the times of sunrise and sunset.
3. Calculate $d^{2} S / d T^{2}$ to confirm that the inflection points of $S$ occur at the first days of spring and fall.
4. Derive formula (4) by differentiating (3). (It may help you to use the chain rule with $u=$ $\sin \alpha \cos (2 \pi T / 365)$ as the intermediate variable.)
5. At latitude $l$ on the earth, on what day of the year is the day 13 hours long? Sketch the relation between $T$ and $l$ in this case.
6. Near springtime in the temperate zone (near $45^{\circ}$ ), show that the sunsets are getting later at a rate of about 1.6 minutes a day (or 11 minutes a week).
7. Planet VCH revolves about its sun once every 590 VCH "days." Each VCH "day" $=19$ earth hours $=1140$ earth minutes. Planet VCH's axis is inclined at $31^{\circ}$. What time is sunset, at a latitude of $12^{\circ}, 16 \mathrm{VCH}$ days after the first day of summer? (Assume that each VCH "day" is divided into 1440 "minutes.")
8. For which values of $T$ does the sun never set if $l>90^{\circ}-\alpha$ (that is, near the North Pole)? Discuss.
9. (a) When the sun rises at the equator on June 21, how fast is its angle of elevation changing?
(b) Using the linear approximation, estimate how long it takes for the sun to rise $5^{\circ}$ above the horizon.
10. Let $L$ be the latitude at which the sun has the highest noon elevation on day $T$. (a) Find a formula for $L$ in terms of $T$. (b) Graph $L$ as a function of $T$.
[^5]
## Review Exercises ior Chapter 5

Perform the conversions in Exercises 1-6.

1. $66^{\circ}$ to radians.
2. $\pi / 10$ radians to degrees.
3.. $(4, \pi / 2)$ from polar to cartesian coordinates.
3. $(3,6)$ from cartesian to polar coordinates.
4. The equation $y=x^{2}$ from cartesian to polar coordinates.
5. The equation $r^{2}=\cos 2 \theta$ from polar to cartesian coordinates.
6. Prove that

$$
\tan (\theta+\phi)=\frac{\tan \theta+\tan \phi}{1-\tan \theta \tan \phi}
$$

8. Prove that

$$
\tan \theta=\frac{\sin (\theta+\phi)+\sin (\theta-\phi)}{\cos (\theta+\phi)+\cos (\theta-\phi)}
$$

Refer to Fig. 5.R. 1 for Exercises 9-12.
9. Find $a$.
10. Find $b$.

11. Find $c$.


12. Find $d$.


Figure 5.R.1. Find $a, b, c$, and $d$.
13. Side $B C$ of the equilateral triangle $A B C$ is trisected by points $P$ and $Q$. What is the angle between $A P$ and $A Q$ ?
14. A 100 -meter building and a 200 -meter building stand 500 meters apart. Where between the buildings should an observer stand so that the taller building subtends twice the angle of the shorter one?
Differentiate the functions in Exercises 15-34.
15. $y=-3 \sin 2 x$.
16. $y=8 \tan 10 x$.
17. $y=x+x \sin 3 x$.
18. $y=x^{2} \cos x^{2}$.
19. $f(\theta)=\theta^{2}+\frac{\theta}{\sin \theta}$.
20. $g(x)=\sec x+\left[\frac{1}{x \cos (x+1)}\right]$.
21. $h(y)=y^{3}+2 y \tan \left(y^{3}\right)+1$.
22. $x(\theta)=\left[\sin \left(\frac{\theta}{2}\right)\right]^{4 / 7}+\theta^{9}+11 \sqrt{\theta}+4$.
23. $y(x)=\cos \left(x^{8}-7 x^{4}-10\right)$.
24. $f(y)=\left(2 y^{3}-3 \csc \sqrt{y}\right)^{1 / 3}$.
25. $f(x)=\sec ^{-1}\left[(x+\sin x)^{2}\right]$.
26. $f(x)=\cot ^{-1}(20-\sqrt[4]{x})$.
27. $r(\theta)=6 \cos ^{3}\left(\theta^{2}+1\right)+1$.
28. $r(\theta)=\frac{7 \sin a \theta}{\sin b \theta+\cos c \theta} ; a, b, c$ constants.
29. $\sin ^{-1}(\sqrt{x})$.
30. $\tan ^{-1}\left(\sqrt{1+x^{2}}\right)$.
31. $\tan (\sin \sqrt{x})$.
32. $\tan (\cos x+\csc \sqrt{x})$.
33. $\sin ^{-1}(\sqrt{x}+\cos 3 x)$.
34. $\frac{\sin \sqrt{x}}{\cos ^{-1}(\sqrt{x}+1)}$.
35. Let $y=x^{2}+\sin (2 x+1)$ and $x=t^{3}+1$. Find $d y / d x$ and $d y / d t$.
36. Let $g=1 / r^{2}+\left(r^{2}+4\right)^{1 / 3}$ and $r=\sin 2 \theta$. Find $d g / d r$ and $d g / d \theta$.
37. Let $h=x \sin ^{-1}(x+1)$ and $x=y-y^{3}$. Find $d h / d x$ and $d h / d y$.
38. Let $f=x^{3 / 5}+\sqrt{2 x^{4}+x^{2}-6}$ and $x=y+\sin y$. Find $d f / d x$ and $d f / d y$.
39. Let $f=\tan ^{-1}\left(2 x^{3}\right)$ and $x=a+b t$, where $a$ and $b$ are constants. Find $d f / d x$ and $d f / d t$.
40. Let $y=\sin ^{-1}\left(u^{2}\right)$ and $u=\frac{\cos x+1}{x^{2}+1}$. Find $d y / d u$ and $d y / d x$.
Find the antiderivatives in Exercises 41-50.
41. $\int \sin 3 x d x$.
42. $\int\left(x^{3 / 2}+\cos 2 x\right) d x$.
43. $\int(4 \cos 4 x-4 \sin 4 x) d x$.
44. $\int(\cos x+\sin 2 x+\cos 3 x) d x$.
45. $\int\left(3 x^{2} \sin x^{3}+2 x\right) d x$.
46. $\int\left[\left(\frac{4}{x^{2}}\right) \sin \left(1+\frac{1}{x}\right)\right] d x$.
47. $\int \sin (u+1) d u$.
48. $\int\left(x^{3}+3 \sec ^{2} x\right) d x$.
49. $\int\left[\frac{1}{\left(4+y^{2}\right)}\right] d y$.
50. $\int\left[\frac{1}{\left(s^{2}+1\right)}-2 \cos 5 s\right] d s$

Find the definite integrals in Exercises 51-54.
51. $\int_{0}^{\pi}\left(\sin \frac{\theta}{2}+\sin 2 \theta\right) d \theta$.
52. $\int_{-\pi}^{3 \pi} \cos ^{2} u d u$ [Hint: Use a trigonometric identity.]
53. $\int_{-1}^{1}\left[\frac{1}{\sqrt{4-x^{2}}}\right] d x$.
54. $\int_{2}^{3}\left[\frac{4}{\left(5+6 x^{2}\right)}\right] d x$.
55. (a) Verify that

$$
\int \sin ^{-1} x d x=x \sin ^{-1} x+\sqrt{1-x^{2}}+C
$$

(b) Differentiate $f(x)=\cos ^{-1} x+\sin ^{-1} x$ to conclude that $f$ is constant. What is the constant?
(c) Find $\int \cos ^{-1} x d x$.
(d) Find $\int \sin ^{-1} 3 x d x$.
56. (a) If $F$ and $G$ are antiderivatives for $f$ and $g$, show that $F(x) G(x)+C$ is the antiderivative of $f(x) G(x)+F(x) g(x)$.
(b) Find the antiderivative of $x \sin x-\cos x$.
(c) Find the antiderivative of $x \sin (x+3)-\cos (x+3)$.
57. Let $f(x)=x^{3}-3 x+7$.
(a) Find an interval containing zero on which $f$ is invertible.
(b) Denote the inverse by $g$. What is the domain of $g$ ?
(c) Calculate $g^{\prime}(7)$.
58. (a) Show that the function $f(x)=\sin x+x$ has an inverse $g$ defined on the whole real line.
(b) Find $g^{\prime}(0)$.
(c) Find $g^{\prime}(2 \pi)$.
(d) Find $g^{\prime}\left(1+\frac{\pi}{2}\right)$.
59. Let $f$ be a function such that $f^{\prime}(x)=1 / x$. (We will find such a function in the next chapter.) Show that the inverse function to $f$ is equal to its own derivative.
60. Find a formula for the inverse function to $y$ $=\sin ^{2} x$ on $[0, \pi / 2]$. Where is this function differentiable?
61. A balloon is released from the ground 10 meters from the base of a 30 -meter lamp post. The balloon rises steadily at 2 meters per second. How fast is the shadow of the balloon moving away from the base of the lamp after 4 seconds?
62. Three runners are going around a track which is an equilateral triangle with sides 50 meters long. If the runners are equally spread and all running counterclockwise at 20 kilometers per hour, at what rate is the distance between a pair of them changing when they are:
(a) leaving the vertices?
(b) arriving at the vertices?
(c) at the midpoints of the sides?
63. A pocket watch is swung counterclockwise on the end of its chain in a vertical circle; it undergoes circular motion, but not uniform, and the tension $T$ in the chain is given by $T=$ $m\left[\left(v^{2} / R\right)+g \cos \theta\right]$, where $\theta$ is the angle from the downward direction. Suppose the length is $R=0.5$ meter, the watch mass is $m=0.1$ kilograms, and the tangential velocity is $v=$ $f(\theta) ; g=9.8$ meters per second per second.
(a) At what points on the circular path do you expect $d v / d \theta=0$ ?
(b) Compute $d T / d \theta$ when $d v / d \theta=0$.
(c) If the speed $v$ is low enough at the highest point on the path, then the chain will become slack. Find the critical speed $v_{c}$ below which the chain becomes slack.
64. A wheel of unit radius rolls along the $x$ axis uniformly, rotating one half-turn per second. A point $P$ on the circumference at time $t$ (in seconds) has coordinates ( $x, y$ ) given by $x=\pi t-$ $\sin \pi t, y=1-\cos \pi t$.
(a) Find the velocities $d x / d t, d y / d t$ and the accelerations $d^{2} x / d t^{2}, d^{2} y / d t^{2}$.
(b) Find the speed $\sqrt{(d x / d t)^{2}+(d y / d t)^{2}}$ when $y$ is a maximum.
65. Refer to Fig. 5.R.2. A girl at point $G$ on the riverbank wishes to reach point $B$ on the opposite side of the river as quickly as possible. She starts off in a rowboat which she can row at 4 kilometers per hour, and she can run at 16 kilometers per hour. What path should she take? (Ignore any current in the river.)


Figure 5.R.2. Find the best
path from $G$ to $B$.
66. The angle of deviation $\delta$ of a light ray entering a prism of Snell's index $n$ and apex angle $A$ is given by

$$
\delta=\sin ^{-1}(n \sin \rho)+\sin ^{-1}(n \sin (A-\rho))-A
$$

where $\rho$ depends on the angle of incidence $\phi$ of the light ray. (See Fig. 5.R.3.)


Figure 5.R.3. Light passing through a prism.
(a) Find $d \delta / d \rho$.
(b) Show that $d \delta / d \rho=0$ occurs for $\rho=A / 2$. This is a minimum value for $\delta$.
(c) By Snell's law, $n=\sin \phi / \sin \rho$. Verify that $n=\sin [(A+\delta) / 2] / \sin (A / 2)$ when $\rho=$ $A / 2$.
67. A man is driving on the freeway at 50 miles per hour. He sees the sign in Fig. 5.R.4. How far from the sign is $\theta$ maximum? How fast is $\theta$ changing at this time?


Figure 5.R.4. At what
distance is $\theta$ biggest?
68. Two lighthouses on a straight coastline are 10 kilometers apart. A ship sees the two lights, and the lines of sight make an angle of $120^{\circ}$ with one another. How far from the shore could the ship possibly be?
69. Where is $f(x)=1+2 \sin x \cos x$ concave upward? Concave downward? Find its inflection points and sketch its graph.
70. Sketch the graph of $y=x+\sin x$ on $[-2 \pi, 2 \pi]$.
71. Prove that $f(x)=x-1-\cos x$ is increasing on $[0, \infty)$. What inequality can you deduce?
72. Suppose that the graph of $r=f(\theta)$ is symmetric in the line $x=y$. What does this imply about $f$ ?
Sketch the graphs in the $x y$ plane of the functions in
Exercises 73-76 given in polar coordinates.

> 73. $r=\cos 6 \theta$
> 74. $r=1+3 \cos \theta$
> 75. $r=\sin \theta+\cos \theta$

䍖76. $r=\theta^{2} \cos ^{2} \theta$. (Use your calculator to locate the zeros of $d r / d \theta$. )

In Exercises 77 and 78, find a formula for the tangent line to the graph at the indicated point.

$$
\text { 77. } r=\cos 4 \theta ; \theta=\pi / 4
$$

78. $r=\frac{1}{1-\sin ^{2} \theta} ; \theta=\pi / 2$.
79. (a) Using a calculator, try to determine whether $\tan ^{-1}\left[\tan \left(\pi-10^{-20}\right)\right]$ is in the interval $(0,1)$. (b) Do part (a) without using a calculator. (c) Do some other calculator experiments with trigonometric functions. How else can you "fool" your calculator (or vice versa!)?
80. On a calculator, put any angle in radians on the display and successively press the buttons " $\sin$ " and "cos," alternatively, until you see the numbers 0.76816 and 0.69481 appear on the display.
(a) Try to explain this phenomenon from the graphs of $\sin x$ and $\cos x$, using composition of functions.
(b) Can you guess the solutions $x, y$ of the equations $\sin (\cos x)=x, \cos (\sin y)=y$ ?

* (c) Using the mean value and intermediate value theorems, show that the equations in (b) have exactly one solution.

8 81. If $f$ is differentiable with a differentiable inverse, and $g(x)=f^{-1}(\sqrt{x})$, what is $g^{\prime}(x) ?$
*82. Consider water waves impinging on a breakwater which has two gaps as in Fig. 5.R.5. With the notation in the figure, analyze the maximum and minimum points for wave amplitude along the shore. The two wave forms emanating from $P$ and $Q$ can be described at any point $R$ as $\alpha \cos (k \rho-\omega t)$, where $\rho$ is the distance from the source $P$ or $Q ; k, \omega, \alpha$ are constant. The net wave is described by their sum; the amplitudes do not add; ignore complications such as reflections of waves off the beach. ${ }^{9}$


Figure 5.R.5. Find the wave pattern on the shore.

* 83. Find a formula for the second derivative of an inverse function.
$* 84$. Prove that the function

$$
f(x)= \begin{cases}x^{2} \sin \frac{1}{x} & x \neq 0 \\ 0 & x=0\end{cases}
$$

is differentiable for all $x$, but that $f^{\prime}(x)$ is not continuous at zero.
*85. Show that the function

$$
f(x)= \begin{cases}x^{4} \sin \frac{1}{x} & x \neq 0 \\ 0 & x=0\end{cases}
$$

is twice differentiable but that the second derivative is not continuous.

[^6]
[^0]:    ${ }^{2}$ Polar coordinates were first used successfully by Newton (1671) and Jacques Bernoulli (1691). The definitive treatment of polar coordinates in their modern form was given by Leonhard Euler in his 1748 textbook Introductio in analysis infinitorium. See C. B. Boyer. "The foremost textbook of modern times," American Mathematical Monthly 58 (1951): 223-226.

[^1]:    ${ }^{4}$ For more details on how calculators do those computations, see "Calculator Function Approximation" by C. W. Schelin, Am. Math. Monthly Vol. 90 (1983), 317-325.

[^2]:    ${ }^{4}$ Although the notation $\sin ^{2} y$ is commonly used to mean $(\sin y)^{2}, \sin ^{-1} y$ does not mean $(\sin y)^{-1}=1 / \sin y$. Sometimes the notation arcsin $y$ is used for the inverse sine function to avoid confusion.

[^3]:    ${ }^{5}$ By noon we mean the moment at which the sun is highest in the sky. To find out when noon occurs in your area, look in a newspaper for the times of sunrise and sunset, and take the midpoint of these times. It will probably not be $12: 00$, but it should change only very slowly from day to day (except when daylight savings time comes or goes).
    ${ }^{6}$ If $\pi / 2-\alpha<|l|<\pi / 2$ (inside the polar circles), there will be some values of $t$ for which the right-hand side of formula (1) does not lie in the interval $[-1,1]$. On the days corresponding to these values of $t$, the sun will never set ("midnight sun").

    If $l= \pm \pi / 2$, then $\tan l=\infty$, and the right-hand side does not make sense at all. This reflects the fact that, at the poles, it is either light all day or dark all day, depending upon the season.

[^4]:    ${ }^{7}$ Note that, in this figure, the date is measured from December 21st rather than June 21st. Copyright 1985 Springer-Verlag. All rights reserved.

[^5]:    ${ }^{8}$ One of the authors was stimulated to do these calculations by the observation that he was most aware of the shortening of the days at the beginning of the school year. This calculation provides one explanation for the observation; perhaps the reader can think of others.

[^6]:    ${ }^{9}$ We recommend the book Waves and Beaches by W. Bascom (Anchor Books, 1965; revised, 1980) as a fascinating study of the mathematics, physics, engineering, and aesthetics of water waves.

