Chapter 6

Exponentials and Logarithms

The inverse of $y = e^x$ is $x = \ln y$.

In our work so far, we have studied integer powers (b^n) and rational powers ($b^{m/n}$) as functions of a variable base, i.e., $y = x^m$ or $y = x^{m/n}$. In this chapter, we will study powers as functions of a variable exponent, i.e., $y = b^x$. To do this, we must first *define* b^x when x is not a rational number. This we do in Section 6.1; the rest of the chapter is devoted to the differential and integral calculus of the exponential functions $y = b^x$ and their inverses, the logarithms. The special value $b = e = 2.7182818285\ldots$ leads to especially simple formulas.

6.1 Exponential Functions

Any real number can be used as an exponent if the base is positive.

In Section R.3, we reviewed the properties of the powers b^r, where r was first a positive integer and then a negative number or a fraction. The calculus of the *power function* $g(x) = x^r$ has been studied in Section 2.3. We can also consider b as fixed and r as variable. This gives the function $f(x) = b^x$, whose domain consists of all rational numbers. The following example shows how such *exponential functions* occur naturally and suggests why we would like to have them defined for all real x.

Example 1 The mass of a bacterial colony doubles after every hour. By what factor does the mass grow after: (a) 5 hours; (b) 20 minutes; (c) $2\frac{1}{2}$ hours; (d) x hours, if x is rational?

Solution (a) In 5 hours, the colony doubles five times, so it grows by a factor of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$.
(b) If the colony grows by a factor of k in $\frac{1}{3}$ hour, it grows by a factor $k \cdot k \cdot k = k^3$ in 1 hour. Thus $k^3 = 2$, so $k = 2^{1/3} = \sqrt[3]{2} \approx 1.26$.
(c) In $\frac{1}{2}$ hour, the colony grows by a factor of $2^{1/2}$, so it grows by a factor of $(2^{1/2})^5 = 2^{5/2} \approx 5.66$ in $2\frac{1}{2}$ hours.

(d) Reasoning as in parts (a), (b), and (c) leads to the conclusion that the mass of the colony grows by a factor of 2^x in x hours. ▲

Time is not limited to rational values; we should be able to ask how much the colony in Example 1 grows after $\sqrt{3}$ hours or π hours. Since the colony is increasing in size, we are led to the following mathematical problem: Find a function f defined for all real x such that f is increasing, and $f(x) = 2^x$ for all rational x.

Computing some values of 2^x and plotting, we obtain the graph shown in Fig. 6.1.1. By doing more computations, we can fill in more points between those in Fig. 6.1.1, and the graph looks more and more like a smooth curve. It is therefore plausible that a smooth curve can be drawn through all these points (Fig. 6.1.2).

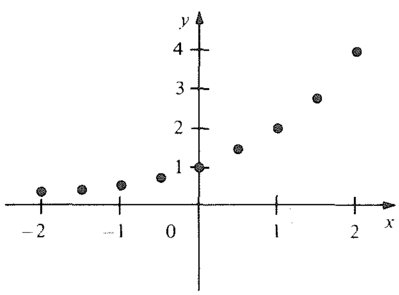

Figure 6.1.1. Some points on the graph $y = 2^x$ for rational x.

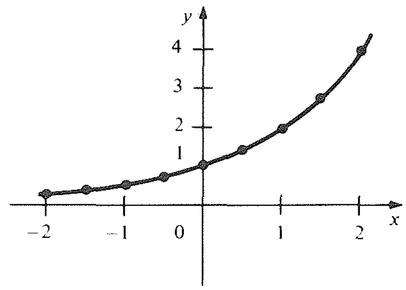

Figure 6.1.2. Interpolating a smooth curve between the points in the previous graph.

The proof that one can really fill in the graph of b^x for irrational x to produce a continuous function defined for all real x is rather technical, so we shall omit it (it is found in more theoretical texts such as the authors' *Calculus Unlimited*). It can also be shown that the laws of exponents, as given in Section R.3 for rational powers, carry over to all real x.

Real Powers

(a) For any $b > 0$, $f(x) = b^x$ is a continuous function.
(b) Let b, c, x, and y be real numbers with $b > 0$ and $c > 0$. Then:
1. $b^{x+y} = b^x b^y$.
2. $b^{xy} = (b^x)^y$.
3. $(bc)^x = b^x c^x$.
4. f is increasing if $b > 1$, constant if $b = 1$, and decreasing if $0 < b < 1$.

Property 4 says that $b^x < b^y$ whenever $b > 1$ and $x < y$; i.e., larger powers of $b > 1$ give larger numbers. If $b < 1$ and $x < y$, then $b^x > b^y$. For example, for $b = 2$, $2^3 < 2^4$, but for $b = \frac{1}{2}$, $(\frac{1}{2})^3 > (\frac{1}{2})^4$. If you examine property 4 for rational powers given in Section R.3, you will see that it corresponds to property 4 here.

Example 2 Simplify (a) $(\sqrt{3^{\pi}})(3^{-\pi/4})$ and (b) $(2^{\sqrt{3}}+2^{-\sqrt{3}})(2^{\sqrt{3}}-2^{-\sqrt{3}})$.

Solution (a) $\sqrt{3^{\pi}}\,3^{-\pi/4}=(3^{\pi})^{1/2}3^{-\pi/4}=3^{\pi/2-\pi/4}=3^{\pi/4}$.
(b) $(2^{\sqrt{3}}+2^{-\sqrt{3}})(2^{\sqrt{3}}-2^{-\sqrt{3}})=(2^{\sqrt{3}})^2-(2^{-\sqrt{3}})^2=2^{2\sqrt{3}}-2^{-2\sqrt{3}}$. ▲

Sometimes the notation $\exp_b x$ is used for b^x; exp stands for "exponential." One reason for this is typographical: an expression like $\exp_b(x^2/2+3x)$ is easier on the eyes and on the typesetter than $b^{(x^2/2+3x)}$. Another reason is mathematical: when we write $\exp_b x$, we indicate that we are thinking of b^x as a *function of x*.

Example 3 Which is larger, $2^{\sqrt{5}}$ or $4^{\sqrt{2}}$? (Do not use a calculator.)

Solution We may write $4^{\sqrt{2}}$ as $(2^2)^{\sqrt{2}}=2^{2\sqrt{2}}=2^{\sqrt{8}}$. Since $\sqrt{8}>\sqrt{5}$, it follows from property 4 in the previous box that $4^{\sqrt{2}}=2^{\sqrt{8}}$ is larger than $2^{\sqrt{5}}$. ▲

Calculator Discussion

When we compute $2^{\sqrt{3}}$ on a calculator, we are implicitly using the continuity of $f(x)=2^x$. The calculator in fact computes a rational power of 2—namely, $2^{1.732050808}$, where 1.732050808 is a decimal approximation to $\sqrt{3}$. Continuity of $f(x)$ means precisely that if the decimal approximation to x is good, then the answer is a good approximation to $f(x)$. The fact that f is increasing gives more information. For example, since

$$\frac{1732}{1000}<\sqrt{3}<\frac{17{,}321}{10{,}000},$$

we can be sure that

$$2^{1732/1000}=3.32188\cdots<2^{\sqrt{3}}<2^{17{,}321/10{,}000}=3.32211\cdots,$$

so $2^{\sqrt{3}}=3.322$ is correct to three decimal places. ▲

Example 4 (a) Sketch the graphs of $\exp_2$, $\exp_{3/2}$, $\exp_1$, $\exp_{2/3}$, and $\exp_{1/2}$. (b) How are the graphs of $\exp_{1/2}$ and $\exp_2$ related?

Solution (a) $1^x=1$ for all x_1, so $\exp_1$ is the constant function with graph $y=1$. The functions $\exp_2$ and $\exp_{3/2}$ are increasing, with $\exp_2 x>\exp_{3/2}x$ for $x>0$ and $\exp_2 x<\exp_{3/2}x$ for $x<0$ (by property 4).

Likewise, $\exp_{2/3}x>\exp_{1/2}x$ for $x>0$ and $\exp_{2/3}$ and $\exp_{1/2}$ are decreasing. Using these facts and a few plotted points, we sketch the graphs in Fig. 6.1.3.

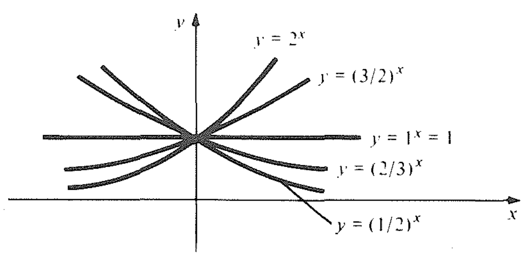

Figure 6.1.3. $y=\exp_b x$ for $b=\frac{1}{2},\frac{2}{3},1,\frac{3}{2}$, and 2.

(b) Using the properties of exponentiation, $\exp_{1/2}x=(\frac{1}{2})^x=2^{-x}=\exp_2(-x)$, so the graph $y=\exp_{1/2}x$ is obtained by reflecting $y=\exp_2 x$ in the y axis; $y=\exp_{2/3}x$ and $y=\exp_{3/2}x$ are similarly related. ▲

Example 5 Match the graphs and functions in Figure 6.1.4.

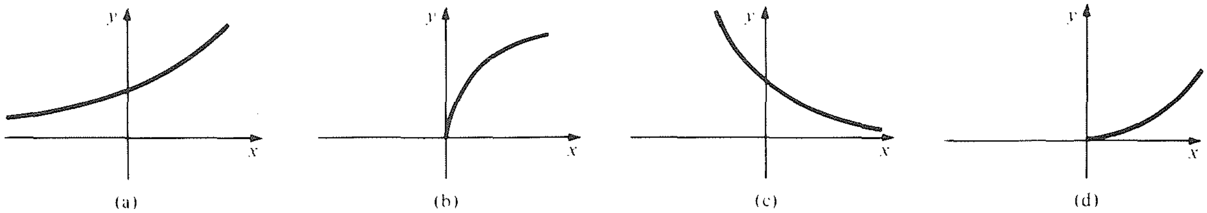

Figure 6.1.4. Match the graphs and functions: (A) $y = x^{\sqrt{3}}$; (B) $y = x^{1/\sqrt{3}}$; (C) $y = (\sqrt{3}\,)^x$; (D) $y = (1/\sqrt{3}\,)^x$.

Solution Only functions (A) and (B) have graphs going through the origin; $x^{\sqrt{3}} < x$ for $x < 1$, so (A) matches (d) and (B) matches (b). The function $y = (\sqrt{3}\,)^x$ is increasing, so (C) matches (a) and (D) matches (c). ▲

Example 6 Match the graphs and functions in Fig. 6.1.5.

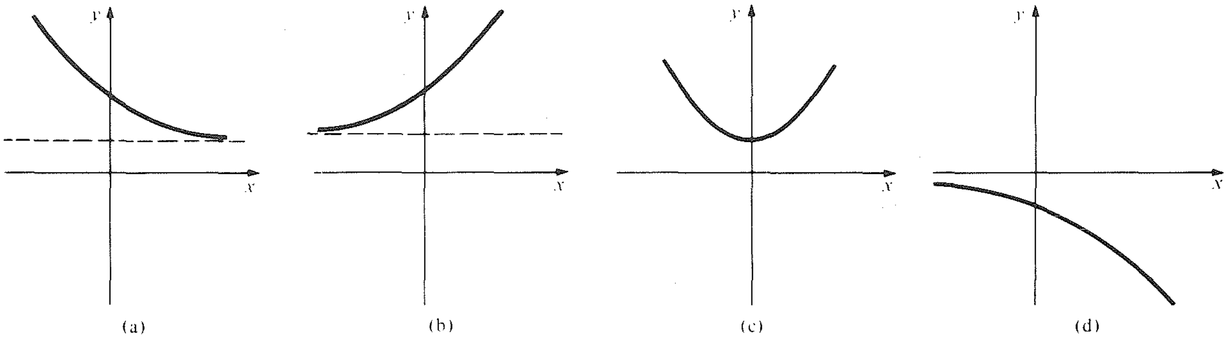

Figure 6.1.5. Match the graphs and functions: (A) $y = -2^x$; (B) $y = x^2 + 1$; (C) $y = 2^{-x} + 1$; (D) $y = 2^x + 1$.

Solution (a) must be the graph of $y = (\frac{1}{2})^x$ shifted up one unit, so it matches (C).
(b) is the graph of 2^x shifted up one unit, so it matches (D).
(c) is a parabola, so it matches (B).
(d) is $y = 2^x$ reflected in the x axis, so it matches (A). ▲

Example 7 A curve whose equation is polar coordinates has the form $r = b^\theta$ for some b is called an *exponential spiral*. Sketch the exponential spiral for $b = 1.1$.

Solution We observe that $r \to \infty$ as $\theta \to \infty$ and that $r \to 0$ as $\theta \to -\infty$. To graph the spiral, we note that r increases with θ; we then plot several points (using a calculator) and connect them with a smooth curve. (See Fig. 6.1.6.) Every turn of the spiral is $(1.1)^{2\pi} \approx 1.82$ times as big as the previous one. ▲

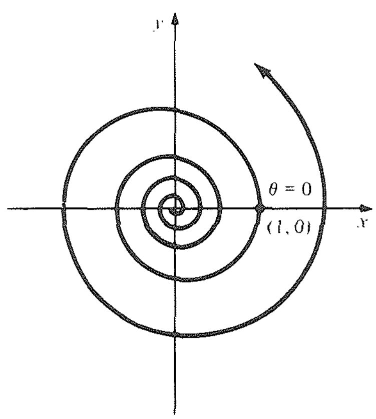

Figure 6.1.6. The exponential spiral $r = (1.1)^\theta$.

Exercises for Section 6.1

1. The mass of a certain bacterial culture triples every 2 hours. By what factor does the mass grow after (a) 4 hours; (b) 6 hours, (c) 7 hours, (d) x hours?
2. The amount of a radioactive substance in an ore sample halves every 5 years. How much is left after (a) 10 years, (b) 30 years, (c) 45 years, (d) x years?
3. The amount of money in a bank account increases by 8% after being deposited for 1 year. How much is there in the account after (a) 2 years (b) 10 years, (c) x years?
4. A lender supplies an amount P to a borrower at an annual interest rate of r. After t years with interest compounded n times a year, the borrower will owe the lender the amount $A = P[1 + (r/n)]^{nt}$ (compound interest). Suppose $P = 100$, $r = 0.06$, $t = 2$ years. Find the amount owed for interest compounded: (a) monthly, (b) weekly, (c) daily, (d) twice daily. Draw a conclusion.

Simplify the expressions in Exercises 5–12.

5. $(2^{\sqrt{2}})^{\sqrt{2}}$
6. $(2^{1/\sqrt{2}})^2$
7. $\dfrac{3^{\sqrt{3}} + 3^{2\sqrt{3}}}{3^{\sqrt{3}}}$
8. $\dfrac{\pi^{-\sqrt{2}} - \pi^{\sqrt{2}}}{\pi^{\sqrt{2}} + 1}$
9. $\dfrac{5^{\pi/2} \cdot 10^{\pi}}{15^{-\pi}}$
10. $\dfrac{8^{\sqrt{3}} 2^{-\sqrt{12}}}{4^{2\sqrt{3}}}$
11. $(3^{\pi} - 2^{(3^{1/2})})(3^{-\pi} + 2^{-(3^{1/2})})$
12. $\dfrac{(\sqrt{3})^{\pi} - (\sqrt{2})^{\sqrt{5}}}{\sqrt[4]{3^{\pi}} + 2^{\sqrt{5}/4}}$

In Exercises 13–16, decide which number is larger without using a calculator.

13. $3^{\sqrt{2}}$ or $9^{1/\sqrt{3}}$
14. $8^{\sqrt{\pi}}$ or $2^{3\pi}$
15. $2^{\sqrt{3}}$ or $3^{\sqrt{2}}$
16. $10^{\sqrt{8}}$ or $8^{\sqrt{10}}$

Sketch the graphs of the functions in Exercises 17–24.

17. $f(x) = \exp_3(x)$
18. $f(x) = \exp_{1/3}(x)$
19. $f(x) = \exp_{4/3}(x)$
20. $f(x) = \exp_{3/4}(x)$
21. $y = 2^{(x^2)}$
22. $y = (2^x)^2$
23. $y = 2^{\sqrt{x}}$
24. $y = 2^{1/x}$
25. How are the graphs of $\exp_3 x$ and $\exp_{1/3}(x)$ related?
26. How are the graphs of $\exp_{4/3}(x)$ and $\exp_{3/4}(x)$ related?
27. How are the graphs of $\exp_3(x)$ and $\exp_3(-x)$ related?
28. How are the graphs of $\exp_5(x)$ and $\exp_5(-x)$ related?
29. Match the graphs and functions in Fig. 6.1.7.

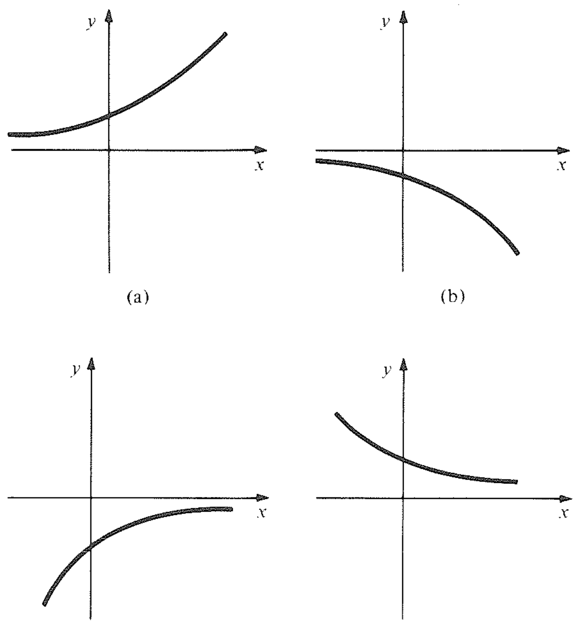

Figure 6.1.7. Match the graphs and functions:
(A) $y = -3^x$; (B) $y = 3^{-x}$;
(C) $y = -3^{-x}$; (D) $y = 3^x$.

30. Match the graphs and functions in Fig. 6.1.8.

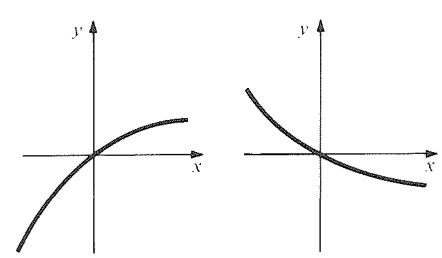

Figure 6.1.8. Match the graphs and functions:
(A) $y = -2^{-x} + 1$; (B) $y = 2^x - 1$;
(C) $y = -2^x + 1$; (D) $y = 2^{-x} - 1$.

31. Match the graphs and functions in Fig. 6.1.9.

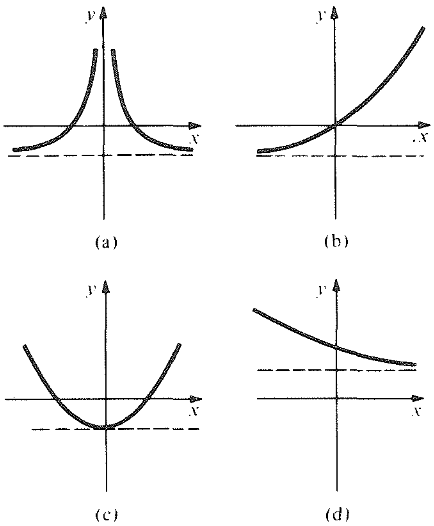

Figure 6.1.9. Match the graphs and functions: (A) $y = x^2 - 1$; (B) $y = 2^x - 1$; (C) $y = 2^{-x} + 1$; (D) $y = x^{-2} - 1$.

32. Match the graphs and functions in Fig. 6.1.10.

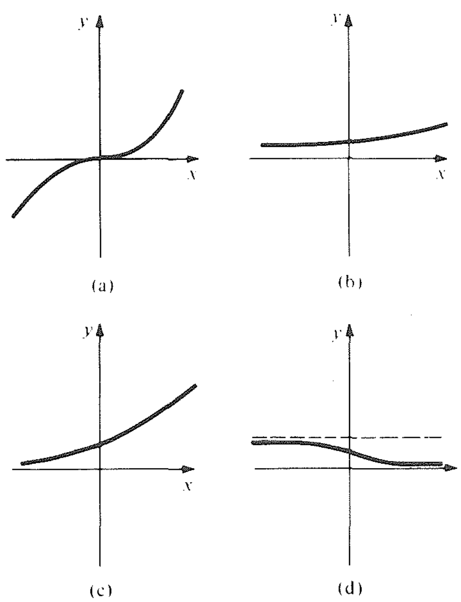

Figure 6.1.10. Match the graphs and functions: (A) $y = x^3$ (B) $y = \sqrt{3^x}$ (C) $y = (2^x + 1)^{-1}$ (D) $y = 10^{x/100}$

33. Graph the exponential spiral $r = (1.2)^\theta$.
34. Graph the exponential spiral $r = (1/1.1)^\theta$.
35. Graph the exponential spiral $r = (1.1)^{2\theta}$.
36. Graph the exponential spiral $r^2 = (1.1)^\theta$.
37. Graph $y = 3^{x+2}$ by "shifting" the graph of $y = 3^x$ by 2 units to the left. Graph $y = 9(3^x)$ by "stretching" the graph of $y = 3^x$ by a factor of 9 in the direction of the y axis. Compare the two results. In general, how does shifting the graph of $y = 3^x$ by k units to the left compare with stretching the graph by a factor of 3^k in the direction of the y axis?
38. Carefully graph the following functions on one set of axes: (a) $f(x) = 2^x$, (b) $g(x) = x^2 + 1$, (c) $h(x) = x + 1$. Can you see why $f'(1)$ should be between 1 and 2?
39. From the graph of $f(x) = 2^x$, make a reasonable sketch of what the function $f'(x)$ might look like.
40. Answer the question in Exercise 39 for $f(x) = 2^{-x}$.
41. Compute the ratio of the area under the graph of $y = 3^x$ between $x = 0$ and $x = 2$ to that between $x = 2$ and $x = 4$ (see Exercise 37).
42. Compare the areas under the graph of $y = 3^x$ between $x = 1$ and $x = 2$ and between $x = 2$ and $x = 3$ (see Exercise 37).

Solve for x in Exercises 43–46.

43. $10^x = 0.001$
44. $5^x = 1$
45. $2^x = 0$
46. $x - 2\sqrt{x} - 3 = 0$ (*Hint*: factor)

6.2 Logarithms

The function $\log_b$ *is the inverse of* $\exp_b$.

If $b > 1$, the function $\exp_b x = b^x$ is positive, increasing, and continuous. As $x \to \infty$, $\exp_b x$ becomes arbitrarily large, while as $x \to -\infty$, $\exp_b x$ decreases to zero. (See Review Exercise 85 for an outline of a proof of these facts.) Thus the range of $\exp_b$ is $(0, \infty)$. It follows from the inverse function test in Section 5.3 that $\exp_b$ has a unique inverse function with domain $(0, \infty)$ and range $(-\infty, \infty)$. This function is called $\log_b$. By the definition of an inverse function, $\log_b y$ *is that number* x *such that* $b^x = y$. The number b is called the *base* of the logarithm.

Example 1 Find $\log_3 9$, $\log_{10} 10^a$, and $\log_9 3$.

Solution Let $x = \log_3 9$. Then $3^x = 9$. Since $3^2 = 9$, x must be 2. Similarly, $\log_{10} 10^a$ is a, and $\log_9 3 = \frac{1}{2}$ since $9^{1/2} = 3$. ▲

The graph of $\log_b x$ for $b > 1$ is sketched in Fig. 6.2.1 and is obtained by flipping over the graph of $\exp_b x$ along the diagonal. As usual with inverse functions, the label y in $\log_b y$ is only temporary and merely stresses the fact that $\log_b y$ is the inverse of $y = \exp_b x$. From now on we will usually use the variable name x and write $\log_b x$. In Fig. 6.2.1, the negative y axis is a vertical asymptote for $y = \log_b x$.

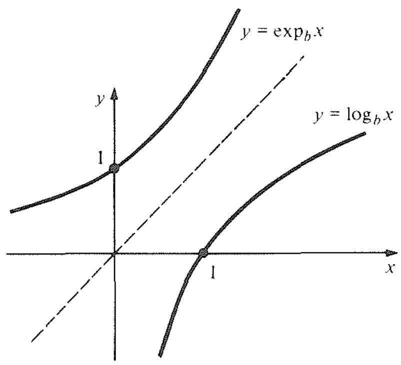

Figure 6.2.1. The graphs of $y = \exp_b x$ and $y = \log_b x$ for $b > 1$.

Example 2 Sketch the graphs of $\log_2 x$ and $\log_{1/2} x$.

Solution This is done by flipping the graphs of 2^x and $(\frac{1}{2})^x$, as shown in Fig. 6.2.2. The graphs of $\log_2 x$ and $\log_{1/2} x$ are reflections of one another in the x axis. ▲

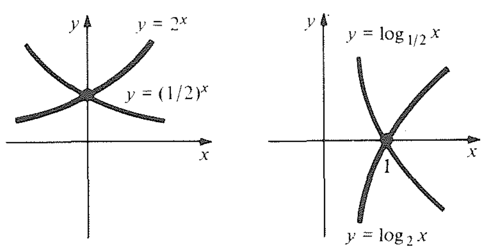

Figure 6.2.2. Exponential and logarithm functions with base $= 2 > 1$ and base $= \frac{1}{2} < 1$.

Notice that for $b > 1$, $\log_b x$ is increasing. If $b < 1$, $\exp_b x$ is decreasing and so is $\log_b x$. However, while $\exp_b x$ is always positive, $\log_b x$ can be either positive or negative. Since $\exp_b 0 = 1$, we can conclude that $\log_b 1 = 0$; since $\exp_b 1 = b$, $\log_b b = 1$. These properties are summarized in the following box.

Properties of $\log_b x$

Definition: $\log_b x$ is that number y such that $b^y = x$; i.e., $b^{\log_b x} = x$.

1. $\log_b x$ is defined for $x > 0$ and $b > 0$ (but $\log_b x$ can be positive or negative).
2. $\log_b 1 = 0$.
3. If $b < 1$, $\log_b x$ is a decreasing function of x; if $b > 1$, $\log_b x$ is increasing.

Example 3 Match the graphs and functions in Fig. 6.2.3.

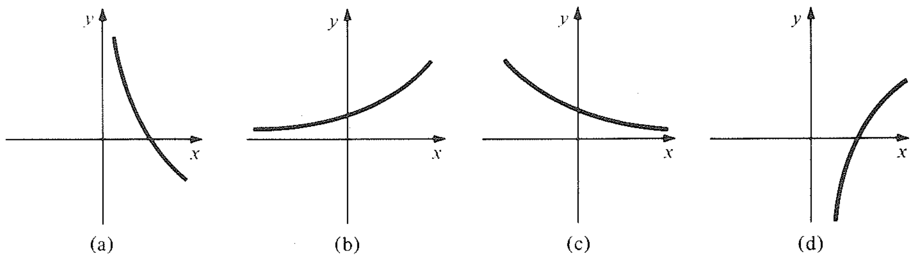

Figure 6.2.3. Match the graphs and functions:
(A) $y = 3^x$;
(B) $y = \log_3 x$;
(C) $y = \log_{1/3} x$;
(D) $y = (\frac{1}{3})^x$.

Solution The functions (A) and (B) are increasing, but only (A) is defined for all x, so (A) matches (b) and (B) matches (d). Of (C) and (D), only (D) is defined for all x, so (C) matches (a) and (D) matches (c). ▲

From the laws of exponents given in Section 6.1, we can read off the corresponding laws for $\log_b x$.

Laws of Logarithms

1. $\log_b(xy) = \log_b x + \log_b y$ and $\log_b(x/y) = \log_b x - \log_b y$.
2. $\log_b(x^y) = y \log_b x$.
3. $\log_b x = (\log_b c)(\log_c x)$.

To prove Law 1, for instance, we remember that $\log_b(xy)$ is that number u such that $\exp_b u = xy$. So we must check that

$$\exp_b(\log_b x + \log_b y) = xy.$$

By the rule $\exp_b(v + w) = \exp_b v \cdot \exp_b w$, the left hand side is given by $\exp_b(\log_b x)\exp_b(\log_b y) = xy$, as is the right side. The other laws are proved in the same way (see Exercises 31 and 32.)

Example 4 Simplify $\log_{10}[(100^{3.2})\sqrt{10}\,]$ without using a calculator.

Solution

$$\begin{aligned} \log_{10}(100^{3.2}\sqrt{10}\,) &= \log_{10}100^{3.2} + \log_{10}\sqrt{10} && \text{(Law 1)} \\ &= 3.2\log_{10}100 + \tfrac{1}{2}\log_{10}10 && \text{(Law 2)} \\ &= 3.2\log_{10}10^2 + \tfrac{1}{2} \\ &= 6.4 + \tfrac{1}{2} = 6.9. \ ▲ \end{aligned}$$

Example 5 What is the relationship between $\log_b c$ and $\log_c b$?

Solution Substituting b for x in Law 3, we get

$$\log_b b = (\log_b c)(\log_c b).$$

But $\log_b b = 1$, so $\log_b c = 1/\log_c b$. ▲

Example 6 Solve for x: (a) $\log_x 5 = 0$, (b) $\log_2(x^2) = 4$, (c) $2\log_3 x + \log_3 4 = 2$.

Solution (a) $\log_x 5 = 0$ means $x^0 = 5$. Since any number to the zero power is 1, there is no solution for x.
(b) $\log_2(x^2) = 4$ means $2^4 = x^2$. This is the same as $16 = x^2$. Hence, $x = \pm 4$.
(c) Solving for $\log_3 x$, we get

$$\log_3 x = 1 - \tfrac{1}{2}\log_3 4 = 1 - \log_3 2.$$

Thus,

$$x = 3^{\log_3 x} = 3^{1-\log_3 2} = 3 \cdot 3^{-\log_3 2} = \tfrac{3}{2}. \; ▲$$

We conclude this section with a word problem involving exponentials and logarithms.

Example 7 The number N of people who contract influenza t days after a group of 1000 people are put in contact with a single person with influenza can be modeled by $N = 1000/(1 + 999 \cdot 10^{-0.17t})$.

(a) How many people contract influenza after 20 days?
(b) Will everyone eventually contract the disease?
(c) In how many days will 600 people contract the disease?

Solution (a) According to the given model, we substitute $t = 20$ into the fomrula for N to give

$$N = \frac{1000}{1 + 999 \cdot 10^{-0.17 \cdot 20}} = \frac{1000}{1 + 999 \cdot 10^{-3.4}} = \frac{1000}{1.398} \approx 715.$$

Thus 715 people will contract the disease after 20 days. (The calculation was done on a calculator.)
(b) "Eventually" is interpreted to mean "for t very large." For t large, $-0.17t$ will be a large negative number and so $10^{-0.17t}$ will be nearly zero (equivalently $10^{-0.17t} = 1/10^{0.17t}$ and $10^{0.17t}$ will be very large if t is very large). Thus the denominator in N will be nearly 1 and so N itself is nearly 1000. For instance, it is eventually larger than 999.9999. Thus, according to the model, all of the 1000 will eventually contract the disease.
(c) We must find the t for which $N = 600$:

$$600 = \frac{1000}{1 + 999 \cdot 10^{-0.17t}}, \qquad \text{so} \quad (600)(1 + 999 \cdot 10^{-0.17t}) = 1000.$$

Thus $1 + 999 \cdot 10^{-0.17t} = \frac{10}{6} = \frac{5}{3}$. Solving for $10^{-0.17t}$, $10^{-0.17t} = (2/3)/999$. Therefore, $-0.17t = \log_{10}((2/3)/999) \approx -3.176$ (from our calculator) and so $t = 3.176/0.17 \approx 18.68$ days. ▲

Exercises for Section 6.2

Compute the logarithms in Exercises 1–10.

1. $\log_2 4$
2. $\log_3 81$
3. $\log_{10} 0.01$
4. $\log_{10}(10^{-8})$
5. $\log_{10}(0.001)$
6. $\log_{10}(1000)$
7. $\log_3 3$
8. $\log_5 125$
9. $\log_{1/2} 2$
10. $\log_{1/3} 9$

Sketch the graphs of the functions in Exercises 11–14.

11. $y = \log_{10} x$
12. $y = \log_{1/10} x$
13. $y = 8\log_2 x$
14. $y = \log_{1/2}(x+1)$

15. Match the graphs and functions in Fig. 6.2.4.

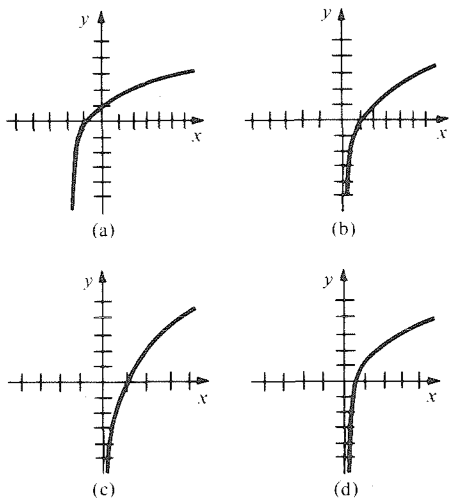

Figure 6.2.4. Match the graphs and functions:
(A) $y = \log_2 x$; (B) $y = 2\log_2 x$;
(C) $y = \log_2(x+2)$; (D) $y = \log_2(2x)$.

16. Match the graphs and functions in Fig. 6.2.5.

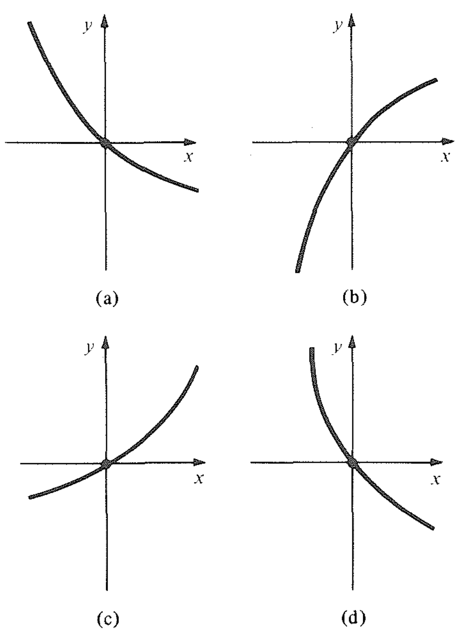

Figure 6.2.5. Match the graphs with:
(A) $y = \log_2(x+1)$;
(B) $y = \log_{1/2}(x+1)$;
(C) $y = (\frac{1}{2})^x - 1$;
(D) $y = 2^x - 1$.

Simplify the expressions in Exercises 17–24 without using a calculator.

17. $\log_2(2^8/8^2)$
18. $\log_2(3^5 \cdot 4^{-6} \cdot 9^{-5.2})$
19. $2^{\log_2 4}$
20. $2^{\log_2 b}$
21. $\log_2(2^b)$
22. $\log_b[b^2 \cdot (2b)^3 \cdot (8b)^{-2/3}]$
23. $\log_b(b^{2x}/2b)$
24. $(\log_b c^2)(\log_c b^2)$

Given that $\log_7 2 \approx 0.356$, $\log_7 3 \approx 0.565$, and $\log_7 5 \approx 0.827$, calculate the quantities in Exericises 25–28 without using a calculator.

25. $\log_7(7.5)$
26. $\log_7 6$
27. $\log_7(3.333\cdots)$
28. $\log_7(1.5)$

29. Suppose that $\log_b 10 = 2.5$. Use a $\log_{10}$ table or a calculator to find an approximate value for b.
30. Which is larger, $\log_{11} 2$ or $\log_{10} 2$?. How about $\log_{1/2} 2$ or $\log_{1/4} 2$? Do not use a calculator.

Use the definition of $\log_b x$ to prove the identities in Exercises 31 and 32.

31. $\log_b(x^y) = y\log_b x$.
32. $\log_b x = (\log_b c)(\log_c x)$.
33. Verify the formula $\log_{a^n} x = (1/n)\log_a x$. What restrictions must you make on a?
34. Prove that $\log_{b^n}(x^m) = (m/n)\log_b x$.

Write the expressions in Exercises 35–38 as sums of (rational) multiples of $\log_b A$, $\log_b B$, and $\log_b C$.

35. $\log_b(A^2B/C)$.
36. $\log_b(\sqrt{AB^3}\,/C^4B^2)$.
37. $2\log_b(A\sqrt{1+B}\,/C^{1/3}B) - \log_b[(B+1)/AC]$.
38. $\log_{b^2}(A^{-1}B^3) - \log_{b^{-1}}(C^{-1}B^2)$. [*Hint*: Use Exercise 34.]

Solve for x in Exercises 39–46.

39. $\log_x 9 = 2$
40. $\log_x 27 = 3$
41. $\log_3 x = 2$
42. $\log_3 x = 3$
43. $\log_2 x = \log_2 5 + 3\log_2 3$
44. $\log_{25}(x+1) = \log_5 x$
45. $\log_x(1-x) = 2$
46. $\log_x(2x-1) = 2$

47. A biologist measures culture growth and gets the following data: After 1 day of growth the count is 1750 cells. After 2 days it is 3065 cells. After 4 days it is 9380 cells. Finish filling out the following table by using a table or calculator:

$x =$ number of days of growth	1	2	4
$n =$ number of cells	1750	3065	9380
$y = \log_{10} n$			

(a) Verify that the data fit a curve of the form $n = Mb^x$ by examining the linear equation $y = (\log_{10} b)x + \log_{10} M$ (with respect to the y and x values in the table). Using the slope and y intercept to evaluate M and b. If the biologist counts the culture on the fifth day, predict how many cells will be found.

(b) Suppose that you had originally known that the data would satisfy a relation of the form $n = Mb^x$. Solve for M and b without using logarithms.

48. Color analyzers are constructed from photomultipliers and various electronic parts to give a scale reading of light intensities falling on a light probe. These scales read relative densities directly, and the scale reading S can be given by $S = k\log_{10}(I/I')$ where I is a reference intensity, I' is the new intensity, and k is a positive constant.

(a) Show that the scale reads zero when $I = I'$.

(b) Assume the needle is vertical on the scale when $I = I'$. Find the sign of S when $I' = 2I$ and $I' = I/2$.

(c) In most photographic applications, the range of usable values of I' is given by $I/8 \leqslant I' \leqslant 8I$. What is the scale range?

49. The *opacity* of a photographic negative is the ratio I_0/I, where I_0 is the reference light intensity and I the intensity transmitted through the negative. The *density* of a negative is the quantity $D = \log_{10}(I_0/I)$. Find the density for opacities of 2, 4, 8, 10, 100, 1000.

50. The loudness, in decibels (dB), of a sound of intensity I is $L = 10\log_{10}(I/I_0)$, where I_0 is the threshold intensity for human hearing.

(a) Conversations have intensity $(1{,}000{,}000)I_0$. Find the dB level.

(b) An increase of 10 dB doubles the loudness of a particular sound. What is the effect of this increase on the intensity I?

(c) A jet airliner on takeoff has sound intensity $10^{12}I_0$. Levels above 90 dB are considered dangerous to the ears. Is this level dangerous?

51. The Richter scale for earthquake magnitude uses the formula $R = \log_{10}(I/I_0)$, where I_0 is a minimum reference intensity and I is the earthquake intensity.

(a) Compare the Richter scale magnitudes of the 1906 earthquake in San Francisco, $I = 10^{8.25}I_0$, and the 1971 earthquake in Los Angeles, $I = 10^{6.7}I_0$.

(b) Show that the difference between the Richter scale magnitude of the earthquakes depends only on the *ratio* of the intensities.

52. The pH value of a substance is determined by the concentration of $[\mathrm{H}^+]$ of the hydrogen ions in the substance in moles per liter, via the formula $\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$. The pH of distilled water is 7; acids have pH < 7; bases have pH > 7.

(a) Tomatoes have $[\mathrm{H}^+] = (6.3)\cdot 10^{-5}$. Are tomatoes acidic?

(b) Milk has $[\mathrm{H}^+] = 4\cdot 10^{-7}$. Is milk acidic?

(c) Find the hydrogen ion concentration of a skin cleanser of rated pH value 5.5.

53. The graph of $y = \log_b x$ contains the point $(3, \frac{1}{3})$. What is b?

54. Graph and compare the following functions: $f(x) = 2\log_2 x$; $g(x) = \log_2(x^2)$; $h(x) = 2\log_2|x|$. Which (if any) are the same?

★55. Give the domain of the following functions. Which (if any) are the same?

(a) $f(x) = \log_{10}\left[\dfrac{(1-x^2)^4}{\sqrt{(x+5)/(x^2+1)}}\right]$

(b) $g(x) = 4\log_{10}(1-x) + 4\log_{10}(1+x) + \frac{1}{2}\log_{10}(x^2+1) - \frac{1}{2}\log_{10}(x+5)$

(c) $h(x) = 4\log_{10}|1-x| + 4\log_{10}|1+x| + \frac{1}{2}\log_{10}(x^2+1) - \frac{1}{2}\log_{10}(x+5)$

★56. Give the domain and range of the following functions:

(a) $f(x) = \log_{10}(x^2 - 2x - 3)$,

(b) $g(x) = \log_2[(2x+1)/2]$,

(c) $h(x) = \log_{10}(1 - x^2)$.

★57. Let $f(x) = \log_2(x-1)$. Find a formula for the inverse function g of f. What is its domain and range?

★58. Is the logarithm to base 2 of an irrational number ever rational? If so, find an example.

6.3 Differentiation of the Exponential and Logarithm Functions

When a special number e is used as the base, the differentiation rules for the exponential and logarithm functions become particularly simple.

Since we have now defined b^x for all real x, we can attempt to differentiate with respect to x. The result is that $\exp_b$ reproduces itself up to a constant multiple when differentiated. Choosing b properly, we can make the constant equal to 1. The derivative of the corresponding logarithm function turns out to be simply $1/x$.

Consider the function $f(x) = \exp_b(x) = b^x$ defined in Section 6.1. If we assume that f is differentiable at zero, we can calculate $f'(x)$ for all x as follows:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{b^{x+\Delta x} - b^x}{\Delta x} = \frac{b^x b^{\Delta x} - b^x}{\Delta x} = b^x\left(\frac{b^{\Delta x} - 1}{\Delta x}\right);$$

thus,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} b^x\left(\frac{b^{\Delta x} - 1}{\Delta x}\right) \\ &= b^x \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = b^x f'(0). \end{aligned}$$

One can show that $f'(0)$ really does exist,[1] so it follows by the preceding argument that f is differentiable everywhere.

Derivative of b^x

If $b > 0$, then $\exp_b(x) = b^x$ is differentiable and

$$\exp_b'(x) = \exp_b'(0)\exp_b(x).$$

That is,

$$\frac{d}{dx} b^x = kb^x,$$

where $k = \exp_b'(0)$ is a number depending on b.

Notice that when we differentiate an exponential function, we reproduce it, multiplied by a constant k. If $b \neq 1$, then $k \neq 0$, for otherwise $\exp_b'(x)$ would be zero for all x, and $\exp_b$ would be constant.

Example 1 Let $f(x) = 3^x$. How much faster is f increasing at $x = 5$ than at $x = 0$?

Solution By the preceding display,

$$f'(5) = f'(0)f(5) = f'(0) \cdot 3^5.$$

Thus at $x = 5$, f is increasing $3^5 = 243$ times as fast as at $x = 0$. ▲

To differentiate effectively, we still need to find $\exp_b'(0)$ and see how it depends upon b. It would be nice to be able to adjust b so that $\exp_b'(0) = 1$, for then we would have simply $\exp_b'(x) = \exp_b(x)$. To find such a b, we

[1] For the proof of this fact, see Chapter 10 of *Calculus Unlimited* by the authors.

numerically compute the derivative of b^x for $b = 1, 2, 3, 4, 5$ at $x = 0$. These derivatives are obtained by computing $(b^{\Delta x} - b^0)/\Delta x = (b^{\Delta x} - 1)/\Delta x$ for various small values of Δx. The results are as follows:

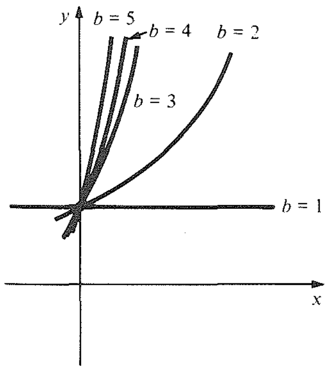

Figure 6.3.1. $y = b^x$ for $b = 1, 2, 3, 4, 5$.

b \ $\frac{b^{\Delta x} - 1}{\Delta x}$ (Δx →)	1	0.1	0.01	0.001	0.0001	Derivative at 0 Equals Approximately
1	0	0	0	0	0	0
2	1	0.72	0.70	0.69	0.69	0.69
3	2	1.16	1.10	1.10	1.10	1.10
4	3	1.49	1.40	1.39	1.39	1.39
5	4	1.75	1.62	1.61	1.61	1.61

The graphs of b^x for these values of b are shown in Fig. 6.3.1. The slopes of the graphs at $x = 0$ are given by the corresponding derivatives computed in the above table. We see that $b = 2$ gives a slope less than 1, while $b = 3$ gives a slope larger than 1. Since it is plausible that the slopes increase steadily with b, it is also plausible that there is a unique number somewhere between 2 and 3 that will give a slope exactly equal to 1. The number is called e, and further numerical experimentation shows that the value of e is approximately 2.718. (Exercise 92 shows another way to find e, and a formula for e in terms of limits is given on page 330.)

The Number e

The number e is chosen so that $\exp_e'(0) = 1$, that is, so that

$$\frac{d}{dx} e^x = e^x.$$

Logarithms to the base e are called *natural logarithms*. We denote $\log_e x$ by $\ln x$. (The notation $\log x$ is generally used in calculus books for the common logarithm $\log_{10} x$.) Since $e^1 = e$, we have the formula $\ln e = 1$.

Natural Logarithms and e

$\ln x$ means $\log_e x$ (natural logarithm).
$\log x$ means $\log_{10} x$ (common logarithm).
$\exp x$ means e^x.
$\ln(\exp x) = x$; $\exp(\ln x) = x$.
$\ln e = 1$, $\ln 1 = 0$.

Most scientific calculators have buttons for evaluating e^x and $\ln x$, but one can sometimes get answers faster and more accurately by hand, as the next example illustrates.

Example 2 Simplify $\ln[e^{205}/(e^{100})^2]$.

Solution By the laws of logarithms,

$$\ln\left[\frac{e^{205}}{(e^{100})^2}\right] = \ln(e^{205}) - 2\ln(e^{100}) = 205 - 2 \cdot 100 = 5. \blacktriangle$$

We can now complete the differentiation formula for the general exponential function $\exp_b x$. Since $b = e^{\ln b}$ we have $b^x = e^{x \ln b}$. Using the chain rule we find, since $\ln b$ is a constant,

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x \ln b} = e^{x \ln b} \frac{d}{dx}(x \ln b) = e^{x \ln b} \ln b = b^x \ln b.$$

Thus, the unknown factor $\exp_b'(0)$ turns out to be just the natural logarithm of the base b.

Differentiation of the Exponential

$$\frac{d}{dx} e^x = e^x,$$

$$\frac{d}{dx} b^x = (\ln b) b^x.$$

Example 3 Differentiate: (a) $f(x) = e^{3x}$; (b) $g(x) = 3^x$.

Solution (a) Let $u = 3x$ so $e^{3x} = e^u$ and use the chain rule:

$$\frac{d}{dx} e^u = \frac{d}{du}(e^u) \frac{du}{dx} = e^u \cdot 3 = 3e^{3x}.$$

(b) $$\frac{d}{dx} 3^x = 3^x \ln 3,$$

taking $b = 3$ in the preceding box. This expression cannot be simplified further; one can find the value $\ln 3 \approx 1.0986$ in a table or with a calculator. (Compare the third line of the table on p. 319.) ▲

Example 4 Differentiate the following functions: (a) xe^{3x}, (b) $\exp(x^2 + 2x)$, (c) x^2, (d) $e^{\sqrt{x}}$, (e) $e^{\sin x}$, (f) $2^{\sin x}$.

Solution (a) $$\frac{d}{dx}(xe^{3x}) = \frac{dx}{dx} e^{3x} + x \frac{d}{dx} e^{3x} = e^{3x} + x \cdot 3e^{3x} = (1 + 3x)e^{3x};$$

(b) $$\frac{d}{dx} \exp(x^2 + 2x) = \exp(x^2 + 2x) \frac{d}{dx}(x^2 + 2x) = \left[\exp(x^2 + 2x)\right](2x + 2);$$

(c) $$\frac{d}{dx} x^2 = 2x;$$

(d) $$\frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{1}{2} x^{-1/2} e^{\sqrt{x}};$$

(e) $$\frac{d}{dx} e^{\sin x} = e^{\sin x} \frac{d}{dx} \sin x = e^{\sin x} \cos x;$$

(f) $$\frac{d}{dx} 2^{\sin x} = \frac{d}{du}(2^u) \frac{du}{dx} \qquad (\text{with } u = \sin x)$$
$$= \ln 2 \cdot 2^u \cdot \cos x$$
$$= \ln 2 \cdot 2^{\sin x} \cdot \cos x. \text{ ▲}$$

We can differentiate the logarithm function by using the inverse function rule of Section 5.3. If $y = \ln x$, then $x = e^y$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

Hence

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

For other bases, we use the same process; setting $y = \log_b x$ and $x = b^y$:

$$\frac{d}{dx}\log_b x = \frac{1}{(d/dy)b^y} = \frac{1}{\ln b \cdot b^y} = \frac{1}{\ln b \cdot x}.$$

That is,

$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}.$$

The last formula may also be proved by using Law 3 of logarithms given in Section 6.2:

$$\ln x = \log_e x = \log_b x \cdot \ln b,$$

so

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\left(\frac{1}{\ln b}\ln x\right)$$
$$= \frac{1}{\ln b}\frac{d}{dx}\ln x = \frac{1}{(\ln b)x}.$$

Our discussion so far can be summarized as follows:

Derivative of the Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x}, \qquad x > 0;$$
$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}, \qquad x > 0.$$

Example 5 Differentiate: (a) $\ln(3x)$, (b) $xe^x\ln x$, (c) $8\log_3 8x$.

Solution (a) Setting $u = 3x$ and using the chain rule:

$$\frac{d}{dx}\ln 3x = \frac{d}{du}(\ln u)\cdot\frac{du}{dx} = \frac{1}{3x}\cdot 3 = \frac{1}{x}.$$

Alternatively, $\ln 3x = \ln 3 + \ln x$, so the derivative with respect to x is $1/x$.
(b) By the product rule:

$$\frac{d}{dx}(xe^x\ln x) = x\frac{d}{dx}(e^x\ln x) + e^x\ln x = xe^x\ln x + e^x + e^x\ln x.$$

(c) From the formula $(d/dx)\log_b x = 1/[(\ln b)x]$ with $b = 3$,

$$\frac{d}{dx}8\log_3 8x = 8\frac{d}{dx}\log_3 8x = 8\left(\frac{d}{du}\log_3 u\right)\frac{du}{dx} \qquad [u = 8x]$$
$$= 8\cdot\frac{1}{(\ln 3)\cdot u}\cdot 8 = \frac{64}{(\ln 3)8x} = \frac{8}{(\ln 3)x}. \blacktriangle$$

Example 6 Differentiate: (a) $\ln(10x^2+1)$; (b) $\sin(\ln x^3)\exp(x^4)$.

Solution (a) By the chain rule with $u = 10x^2 + 1$, we get

$$\frac{d}{dx}\ln(10x^2+1) = \frac{d}{du}\ln u\,\frac{du}{dx} = \frac{1}{u}\cdot 20x = \frac{20x}{10x^2+1}.$$

(b) By the product rule and chain rule,

$$\begin{aligned}\frac{d}{dx}&\left(\sin(\ln x^3)\exp(x^4)\right)\\ &= \left(\frac{d}{dx}\sin(\ln x^3)\right)\exp(x^4) + \sin(\ln x^3)\frac{d}{dx}\exp(x^4)\\ &= \cos(\ln x^3)\cdot\frac{3x^2}{x^3}\cdot\exp(x^4) + \sin(\ln x^3)\cdot 4x^3\exp(x^4)\\ &= \exp(x^4)\left(\frac{3}{x}\cos(\ln x^3) + 4x^3\sin(\ln x^3)\right). \blacktriangle\end{aligned}$$

Previously, we knew the formula $(d/dx)x^n = nx^{n-1}$ for rational n. Now we are in a position to prove it for all n, rational or irrational, and $x > 0$. Indeed, write $x^n = e^{(\ln x)n}$ and differentiate using the chain rule and the laws of exponents:

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{(\ln x)\cdot n} = \frac{n}{x}\cdot e^{(\ln x)n} = \frac{n}{x}x^n = nx^{n-1}.$$

For example, $(d/dx)x^\pi = \pi x^{\pi-1}$.

In order to differentiate complex expressions involving powers, it is sometimes convenient to begin by taking logarithms.

Example 7 Differentiate the functions (a) $y = x^x$ and (b) $y = x^x\cdot\sqrt{x}$.

Solution (a) We take natural logarithms,

$$\ln y = \ln(x^x) = x\ln x.$$

Next, we differentiate using the chain rule, remembering that y is a function of x:

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x = 1 + \ln x.$$

Hence

$$\frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x).$$

Alternatively, we could have written $x^x = e^{x\ln x}$. Thus, by the chain rule,

$$\frac{d}{dx}x^x = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1).$$

(b) $y = x^x\cdot\sqrt{x} = x^{x+1/2}$, so $\ln y = (x+\frac{1}{2})\ln x$. Thus

$$\frac{1}{y}\frac{dy}{dx} = \left(x+\frac{1}{2}\right)\frac{1}{x} + \ln x,$$

and so

$$\frac{dy}{dx} = x^{x+1/2}\left[\left(x+\frac{1}{2}\right)\frac{1}{x} + \ln x\right]. \blacktriangle$$

This method of differentiating functions by first taking logarithms and then differentiating is called *logarithmic differentiation.*

Example 8 Use logarithmic differentiation to calculate dy/dx, where

$$y = (2x+3)^{3/2}/\sqrt{x^2+1}\,.$$

Solution $\ln y = \ln[(2x+3)^{3/2}/(x^2+1)^{1/2}] = \frac{3}{2}\ln(2x+3) - \frac{1}{2}\ln(x^2+1)$, so

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{3}{2}\cdot\frac{2}{2x+3} - \frac{1}{2}\cdot\frac{2x}{x^2+1}\\ &= \frac{3}{2x+3} - \frac{x}{x^2+1} = \frac{(x^2-3x+3)}{(2x+3)(x^2+1)}\,,\end{aligned}$$

and hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+3)^{3/2}}{(x^2+1)^{1/2}}\cdot\frac{(x^2-3x+3)}{(2x+3)(x^2+1)}\\ &= \frac{(x^2-3x+3)(2x+3)^{1/2}}{(x^2+1)^{3/2}}\,. \blacktriangle\end{aligned}$$

Since the derivative of $\ln x$ is $1/x$, $\ln x$ is an antiderivative of $1/x$; that is

$$\int \frac{1}{x}\,dx = \ln x + C \qquad \text{for} \quad x > 0.$$

This integration rule fills an important gap in our earlier formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

from Section 2.5, which was valid only for $n \neq -1$.

Antidifferentiation Formulas for exp and log

1. $\int e^x\,dx = e^x + C$;
2. $\int b^x\,dx = \dfrac{b^x}{\ln b} + C \qquad (b > 0)$;
3. $\int \dfrac{1}{x}\,dx = \ln|x| + C \qquad (x \neq 0)$.

To prove integration formula 3 in the preceding box, consider separately the cases $x > 0$ and $x < 0$. For $x > 0$, it is the inverse of our basic formula for differentiating the logarithm. For $x < 0$, $(d/dx)(\ln|x|) = (d/dx)[\ln(-x)] = [1/(-x)]\cdot[-1] = 1/x$, so $\ln|x|$ is an antiderivative for $1/x$, for $x \neq 0$.

Example 9 Find the indefinite integrals: (a) $\int e^{ax}\,dx$; (b) $\int\left(\dfrac{1}{3x+2}\right)dx$.

Solution (a) $(d/dx)e^{ax} = ae^{ax}$, by the chain rule, so $\int e^{ax}\,dx = (1/a)e^{ax} + C$.
(b) Differentiate $\ln|3x+2|$ by the chain rule, setting $u = 3x+2$. We get $(d/dx)\ln|u| = (d/du)\ln|u|\cdot du/dx = (1/u)\cdot 3 = 3/(3x+2)$; hence

$$\int \frac{1}{(3x+2)}\,dx = \frac{1}{3}\ln|3x+2| + C. \blacktriangle$$

Example 10 Integrate

$$\text{(a)} \int_0^1 \frac{1}{x+1}\,dx \qquad \text{(b)} \int_1^2 \frac{x^3+3x+2}{x}\,dx.$$

Solution (a) Since $(d/dx)\ln x = 1/x$, for $x > 0$, the chain rule gives

$$\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}.$$

(Here $x + 1 > 0$, so we can omit the absolute value signs.) Thus

$$\int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

$$\text{(b)} \int_1^2 \frac{x^3+3x+2}{x}\,dx = \int_1^2\left(x^2+3+\frac{2}{x}\right)dx = \left(\frac{x^3}{3}+3x+2\ln x\right)\Bigg|_1^2$$

$$= \frac{7}{3}+3+2\ln 2 = \frac{16}{3}+2\ln 2. \blacktriangle$$

Example 11 Verify the formula $\int e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + C$.

Solution We must check that the right-hand side is an antiderivative of the integrand. We compute, using the product rule:

$$\frac{d}{dx}\frac{1}{2}e^x(\sin x + \cos x)$$

$$= \frac{1}{2}\left(\frac{de^x}{dx}\right)(\sin x + \cos x) + \frac{1}{2}e^x\left(\frac{d}{dx}(\sin x + \cos \times)\right)$$

$$= \frac{1}{2}e^x(\sin x + \cos x) + \frac{1}{2}e^x(\cos x - \sin x) = e^x \cos x$$

Thus the formula is verified. ▲

Exercises for Section 6.3

1. How much faster is $f(x) = 2^x$ increasing at $x = 3$ than at $x = 0$?
2. How much faster is $f(x) = 4^x$ increasing at $x = 2$ than at $x = 0$?
3. How much faster is $f(x) = (\frac{1}{2})^x$ increasing at $x = \frac{1}{2}$ than at $x = 0$?
4. How much faster is $f(x) = (\frac{1}{4})^x$ increasing at $x = -3$ than at $x = 0$?

Simplify the expressions in Exercises 5–10.

5. $\ln(e^{x+1}) + \ln(e^2)$
6. $\ln(e^{\sin x}) - \ln(e^{\cos x})$
7. $e^{\ln x + \ln x^2}$
8. $e^{\ln \sin x}e^{-\ln \cos x}$
9. $e^{4x}[\ln(e^{3x-1}) - \ln(e^{1-x})]$
10. $e^{x \ln 3 + \ln 2^x}$

Differentiate the functions in Exercises 11–32.

11. e^{x^2+1}
12. $(e^{3x^3+x})(1-e^x)$
13. $e^{1-x^2} + x^3$
14. $e^{2x} - \cos(x + e^{2x})$
15. $2^x + x$
16. $3^x + x^{-x}$
17. $3^x - 2^{x-1}$
18. $\tan(3^{2x})$
19. $\ln 10x$
20. $\ln x^2$
21. $\dfrac{\ln x}{x}$
22. $(\ln x)^3$
23. $\ln(\sin x)$
24. $\ln(\tan x)$
25. $\ln(2x+1)$
26. $\ln(x^2 - 3x)$
27. $(\sin x)\ln x$
28. $(x^2 - 2x)\ln(2x+1)$
29. $\dfrac{\ln(\tan 3x)}{1 + \ln x^2}$
30. $x^{\sqrt{2}} + (\ln \cos x)^{\sqrt{3}}$
31. $\log_5 x$
32. $\log_7(2x)$

Use logarithmic differentiation to differentiate the functions in Exercises 33–40.

33. $y = (\sin x)^x$.
34. $y = x^{\sin x}$.
35. $y = (\sin x)^{\cos x}$.
36. $y = (x^3+1)^{x^2-2}$.
37. $y = (x-2)^{2/3}(4x+3)^{8/7}$.
38. $y = (x+2)^{5/8}(8x+9)^{10/13}$.
39. $y = x^{(x^x)}$.
40. $y = x^{3x}$.

Differentiate the functions in Exercises 41–62.

41. $e^{x \sin x}$
42. x^e
43. $\ln(x^{-5} + x)$

44. $6\ln(x^3 - xe^x) + e^x \ln x$
45. $14^{x^2 - 8\sin x}$
46. $\log_2[\sin(x^2)]$
47. $\ln(x + \ln x)$
48. $e^x \sin(\ln x + 1)$
49. $\cos(x^{\sin x})$
50. $\sin(x^{\cos x})$
51. $x^{(x^2)}$
52. x^{e^x}
53. $(1/x)^{\tan x^2}$
54. $\ln(x^{\sec x^2})$
55. $\sin(x^4 + 1) \cdot \log_8(14x - \sin x)$
56. $\log_{5/3}(\cos 2x)$
57. $3x^{\sqrt{x}}$
58. $3x^{x/2}$
59. $\sin(x^x)$
60. $\ln(x^{x+1})$
61. $(\sin x)^{(\cos x)^x}$
62. 2^{2^x}

Find the indefinite integrals in Exercises 63–76.

63. $\int e^{2x}\,dx$
64. $\int (x^2 + e^x)\,dx$
65. $\int (\cos x + e^{4x})\,dx$
66. $\int 4e^{-2x}\,dx$
67. $\int \left(s^2 + \frac{2}{s}\right) ds$
68. $\int \left(s^2 + s + 1 + \frac{1}{s} + \frac{1}{s^2}\right) ds$
69. $\int \left(\frac{x^2+1}{2x}\right) dx$
70. $\int \left(e^{4x} - \frac{2}{x}\right) dx$
71. $\int \left(\frac{x}{x-1}\right) dx$ [*Hint*: Divide.]
72. $\int \left(\frac{x}{x+3}\right) dx$
73. $\int 3^x\,dx$
74. $\int x^3\,dx$
75. $\int \left(\frac{x^2+2x+2}{x-8}\right) dx$ [*Hint*: Divide.]
76. $\int \left(\frac{y-1}{y^2-1}\right) dy$

Find the definite integrals in Exercises 77–84.

77. $\int_0^1 (x^2 + 3e^x)\,dx$
78. $\int_1^2 e^{-x}\,dx$
79. $\int_2^3 (x^3 + e^{2x})\,dx$
80. $\int_{50}^{100} (4/x)\,dx$
81. $\int_0^1 2^x\,dx$
82. $\int_{-1}^1 3^x\,dx$
83. $\int_0^1 \frac{dx}{x+2}$
84. $\int_0^1 \frac{x}{x^2+2}\,dx$ [*Hint*: differentiate $\ln(x^2+2)$.]
85. (a) Differentiate $x\ln x$. (b) Find $\int \ln x\,dx$.
86. (a) By differentiating $\ln(\cos x)$, find $\int \tan x\,dx$.
    (b) Find $\int \cot x\,dx$.
87. (a) Verify the integration formula
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C.$$
    (b) Find a similar formula for
$$\int e^{ax}\cos bx\,dx.$$
88. Verify the following:
    (a) $\int (x^n e^x + nx^{n-1}e^x)\,dx = x^n e^x + C.$
    (b) $\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$
89. Verify the following integration formulas:
    (a) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln\left(x + \sqrt{1+x^2}\right) + C;$
    (b) $\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C.$
90. Use Exercises 88 and 89 to evaluate
    (a) $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$
    and
    (b) $\int_0^1 x^2 e^x\,dx.$
91. Express the derivatives of the following in terms of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$:
    (a) $f(x) \cdot e^x + g(x)$;
    (b) $e^{f(x) + x^2}$;
    (c) $f(x) \cdot e^{g(x)}$;
    (d) $f(e^x + g(x))$;
    (e) $f(x)^{g(x)}$.

★92. This exercise shows how to adjust b to make $\exp_b'(x) = \exp_b(x)$. We start with the base 10 of common logarithms and find another base b for which $\exp_b'(0) = 1$ as follows. By the definition of the logarithm, $b = 10^{\log_{10} b}$.
    (a) Deduce that $\exp_b(x) = \exp_{10}(x\log_{10} b)$.
    (b) Differentiate (a) to show that $\exp_b'(0) = \exp_{10}'(0) \cdot \log_{10} b$. To have $\exp_b'(0) = 1$, we should pick b in such a way that $\log_{10} b = 1/\exp_{10}'(0)$.
    (c) Deduce that $e = \exp_{10}[1/\exp_{10}'(0)]$ satisfies the condition $\exp_b'(0) = 1$ and so $\exp_e'(x) = \exp_e(x)$.
    (d) Show that for any b, $\exp_b[1/\exp_b'(0)] = e$.

★93. By calculating $(b^{\Delta x} - 1)/\Delta x$ for small Δx and various values of b, as at the beginning of this section, estimate e to within 0.01.

★94. Suppose that you *defined* $\ln x$ to be $\int_1^x dt/t$.
(a) Use the fundamental theorem to show that $(d/dx)\ln x = 1/x$.
(b) Define e^x to be the inverse function of $\ln x$ and show $(d/dx)e^x = e^x$.

★95. (a) Use the definition of $\ln x$ in Exercise 94 to show that $\ln xy = \ln x + \ln y$ by showing that for a given fixed x_0,

$$\frac{d}{dx}(\ln(xx_0) - \ln x - \ln x_0) = 0.$$

(b) Deduce from (a) that $e^{x+y} = e^x e^y$.
(c) Prove $e^{x+y} = e^x e^y$ by assuming only that $(d/dx)e^x = e^x$ and $e^0 = 1$.

★96. (a) Compute $\int_0^x e^{-t}\,dt$ for $x = 1$, 10, and 100.
(b) How would you define $\int_0^\infty e^{-t}\,dt$? What number would this integral be?
(c) Interpret the integral in (b) as an area.

★97. (a) Compute $\int_\epsilon^2 \ln x\,dx$ (see Exercise 85) for $\epsilon = 1$, 0.1, and 0.01.
(b) How would you define $\int_0^2 \ln x\,dx$? Compute it by evaluating $\lim_{x\to 0}(x\ln x)$ numerically.
(c) Why doesn't this integral exist in the ordinary sense?

★98. What do you see if you rotate an exponential spiral about the origin at a uniform rate? Compare with the spiral $r = \theta$.

★99. Differentiate $y = f(x)g(x)$ by writing the logarithm of y as a sum of logarithms. Show that you recover the product rule.

★100. Differentiate $y = f(x)/g(x)$ logarithmically to recover the quotient rule.

★101. Find a formula for the derivative of $f_1(x)^{n_1}f_2(x)^{n_2}\cdots f_k(x)^{n_k}$ using logarithmic differentiation.

6.4 Graphing and Word Problems

Money grows exponentially when interest is compounded continuously.

Now we turn to applications of the exponential and logarithm functions in graphing and word problems. Additional applications involving growth and decay are given in Chapter 8.

We begin by studying some useful facts about limits of exponential and logarithm functions.

We shall first show that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0. \tag{1}$$

Intuitively, this means that for large x, x is much larger than $\ln x$. Indeed this is plausible from their graphs (Fig. 6.4.1).

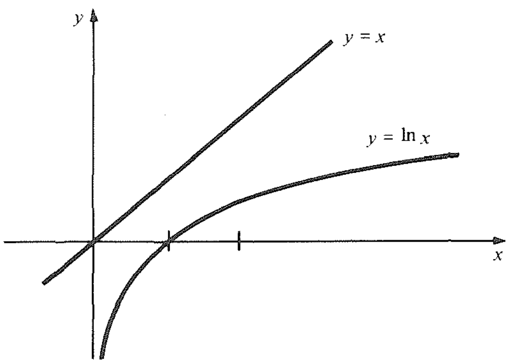

Figure 6.4.1. x is much larger than $\ln x$ for large x.

To prove (1), note that for any fixed integer n,

$$\frac{d}{dx}(x - n\ln x) = 1 - \frac{n}{x} \to 1 \quad \text{as} \quad x\to\infty.$$

Thus

$$x - n\ln x \to \infty \qquad \text{as} \quad x \to \infty$$

(since its slope is nearly 1 for large x). In particular,

$$x - n\ln x > 0 \qquad \text{for large } x,$$

$$\frac{\ln x}{x} < \frac{1}{n} \qquad \text{for large } x.$$

This shows that $\ln x / x$ becomes arbitrarily small for x large, so (1) holds.

Now let $y = x^a$ for $a > 0$. Then $y \to \infty$ as $x \to \infty$, and so

$$\frac{\ln x}{x^a} = \frac{\ln(y^{1/a})}{y} = \frac{1}{a}\frac{\ln y}{y} \to 0$$

as $y \to \infty$ by (1), so we get

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = 0 \quad \text{or} \quad \lim_{x\to\infty} \frac{x^a}{\ln x} = \infty. \tag{2}$$

Thus, not only does x become much larger than $\ln x$, but so does any positive power of x. For example, with $a = \frac{1}{2}$,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$$

Calculator Discussion

The validity of (1) and (2) for various a can also be readily checked by performing numerical calculations on a calculator. For example, on our calculator we got the following data:

x	1	2	3	5	50	500	50,000	10^6	10^{10}	10^{20}	10^{30}
$\ln x$	0	.69	1.09	1.609	3.912	6.214	10.8	13.8	23.02	46.05	69.08
$x^{0.1}$	1	1.07	1.11	1.17	1.47	1.86	2.95	3.98	10	100	1000

It takes $x^{0.1}$ a while to overtake $\ln x$, but eventually it does. ▲

If we let $y = 1/x$, then $y \to \infty$ as $x \to 0$, so

$$x^a \ln x = y^{-a}\ln\left(\frac{1}{y}\right) = -\frac{\ln y}{y^a} \to 0$$

by (2). Thus for $a > 0$, we have the limit

$$\lim_{x\to 0} x^a \ln x = 0. \tag{3}$$

This means that $\ln x$ approaches $-\infty$ more slowly than x^a approaches zero as $x \to 0$.

Finally, write

$$\frac{e^x}{x^n} = e^{(x - n\ln x)}.$$

As we have seen in deriving (1), $x - n\ln x \to \infty$ as $x \to \infty$. Thus,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty. \tag{4}$$

This says that the exponential function grows more rapidly than any power of x.

Alternative proofs of (1)–(4) are given in Review Exercises 86–91 at the end of this chapter; simple proofs also follow from l'Hôpital's rule given in Chapter 11.

Limiting Behavior of exp and log

1. $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for any n.

e^x grows more rapidly as $x\to\infty$ than any power of x (no matter how large).

2. $\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$ for any $a > 0$.

$\ln x$ grows more slowly as $x\to\infty$ than any positive power of x (no matter how small).

3. $\lim_{x\to 0} x^a \ln|x| = \lim_{x\to 0} \frac{\ln|x|}{x^{-a}} = 0$ for any $a > 0$.

$\ln x$ is dominated as $x\to 0$ by any negative power of x.

Example 1 Sketch the graph $y = x^2e^{-x}$.

Solution We begin by noting that y is positive except at $x = 0$. Thus $x = 0$ is a minimum. There are no obvious symmetries. The positive x axis is an asymptote, since $x^2e^{-x} = x^2/e^x = 1/(e^x/x^2)$, and $e^x/x^2 \to \infty$ as $x\to\infty$ by item 1 in the previous box. For $x\to-\infty$, both x^2 and e^{-x} become large, so $\lim_{x\to-\infty} x^2e^{-x} = \infty$.

The critical points are obtained by setting $dy/dx = 0$; here $dy/dx = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x}$. Thus $dy/dx = 0$ when $x = 0$ and $x = 2$; y is decreasing on $(-\infty, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. The second derivative is

$$\frac{d^2y}{dx^2} = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x},$$

which is positive at $x = 0$ and negative at $x = 2$. Thus 0 is a minimum and 2 is a maximum. There are inflection points where $d^2y/dx^2 = 0$; i.e., at $x = 2 \pm \sqrt{2}$. This information, together with the plot of a few points, enables us to sketch the graph in Fig. 6.4.2. ▲

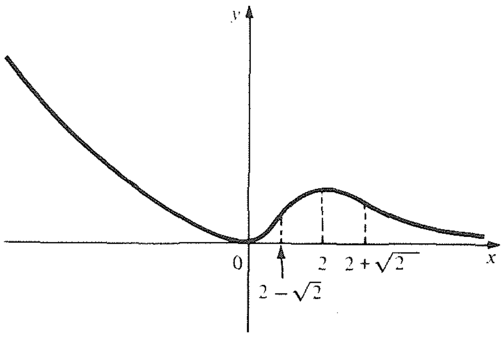

Figure 6.4.2. $y = x^2e^{-x}$.

Example 2 Sketch the graph of $y = x\ln x$.

Solution The function is defined for $x > 0$. As $x\to 0$, $x\ln x\to 0$ by item 3 in the preceding box, so the graph approaches the origin. As $x\to\infty$, $x\ln x\to\infty$. The function changes sign from negative to positive at $x = 1$. $dy/dx = \ln x + 1$, which is zero when $x = 1/e$ and changes sign from negative to positive there, so $x = 1/e$ is a local minimum point. We also note that dy/dx approaches $-\infty$ as $x\to 0$, so the graph becomes "vertical" as it approaches the origin.

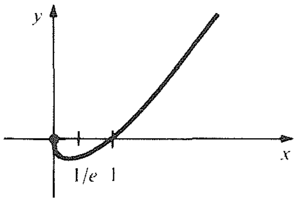

Figure 6.4.3. $y = x \ln x$.

Finally, we see that $d^2y/dx^2 = 1/x$ is positive for all $x > 0$, so the graph is everywhere concave upward. The graph is sketched in Fig. 6.4.3. ▲

Next we turn to an application of logarithmic differentiation. The expression $(d/dx)\ln f(x) = f'(x)/f(x)$ is called the *logarithmic derivative* of f. The quantity $f'(x)/f(x)$ is also called the *relative rate of change* of f, since it measures the rate of change of f per unit of f itself. This idea is explored in the following application.

Example 3 A certain company's profits are given by $P = 5000\exp(0.3t - 0.001t^2)$ dollars where t is the time in years from January 1, 1980. By what percent per year were the profits increasing on July 1, 1981?

Solution We compute the relative rate of change of P by using logarithmic differentiation.

$$\frac{1}{P}\frac{dP}{dt} = \frac{d}{dt}(\ln P) = \frac{d}{dt}(\ln 5000 + 0.3t - 0.001t^2) = 0.3 - 0.002t.$$

Substituting $t = 1.5$ corresponding to July 1, 1981, we get

$$\frac{1}{P}\frac{dP}{dt} = 0.3 - (0.002)(1.5) = 0.2970.$$

Therefore on July 1, 1981, the company's profits are increasing at a rate of 29.7% per year. ▲

We shall be examining compound interest shortly, but before doing so, we shall need some further information about the number e.

In our previous discussion, the number e was obtained in an implicit way. Using limits, we can derive the more explicit expression

$$e = \lim_{h \to 0}(1 + h)^{1/h}. \tag{5}$$

To prove this, we write

$$\begin{aligned}(1 + h)^{1/h} &= \exp\left[\ln(1 + h)^{1/h}\right] \\ &= \exp\left[\frac{1}{h}\ln(1 + h)\right] \\ &= \exp\left[\frac{1}{h}(\ln(1 + h) - \ln 1)\right] \qquad (\text{since } \ln 1 = 0);\end{aligned}$$

but

$$\lim_{h \to 0}\left[\frac{1}{h}(\ln(1 + h) - \ln 1)\right] = \lim_{\Delta x \to 0}\left(\frac{\ln(1 + \Delta x) - \ln 1}{\Delta x}\right) = \frac{d}{dx}\ln x\Big|_{x=1} = 1.$$

Substituting this in our expression for $(1 + h)^{1/h}$ and using continuity of the exponential function gives

$$\lim_{h \to 0}(1 + h)^{1/h} = \exp\left[\lim_{h \to 0}\frac{1}{h}(\ln(1 + h) - \ln 1)\right] = \exp(1) = e.$$

which proves (5).

One way to get approximations for e is by letting $h = \pm(1/n)$, where n is a large integer. We get

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty}\left(1 - \frac{1}{n}\right)^{-n}. \tag{6}$$

Notice that the numbers $(1 + 1/n)^n$ and $(1 - 1/n)^{-n}$ are all rational, so e is the limit of a sequence of rational numbers. It is known that e itself is irrational.[2]

[2] A proof is given in Review Exercise 129, Chapter 12.

e as a Limit

$$e = \lim_{h\to 0}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} \tag{6}$$

Calculator Discussion

Let us calculate $(1-1/n)^{-n}$ and $(1+1/n)^n$ for various values of n. By (6), the numbers should approach $e = 2.71828\ldots$ in both cases, as n becomes large.

One can achieve a fair degree of accuracy before round off errors make the operations meaningless. For example, on our calculator we obtained the following table. (Powers of 2 were chosen for n to permit computing the nth powers by repeated squaring.)

n	$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n$	$\left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$	
2	4.000000	2.250000	
4	3.160494	2.441406	
8	2.910285	2.565785	
16	2.808404	2.637928	
32	2.762009	2.676990	
64	2.739827	2.697345	
128	2.728977	2.707739	
256	2.723610	2.712992	
512	2.720941	2.715632	
1024	2.719610	2.716957	
2048	2.718947	2.717617	
4096	2.718611	2.717954	
8192	2.718443	2.718109	
16,384	2.718370	2.718192	
32,768	2.718367	2.718278	
65,536	2.718299	2.718299	
131,072	2.718131	2.718131	
262,144	2.718492	2.718492	Calculator
524,288	2.717782	2.717782	roundoff
1,048,576	2.719209	2.719209	errors
16,777,216	2.736372	2.736372	significant
536,870,912	2.926309	2.926309	beyond
2,147,483,648	1.00000	1.00000	↓ here

▲

Example 4 Express $\ln b$ as a limit by using the formula $\ln b = \exp_b'(0)$.

Solution By the definition of the derivative as a limit,

$$\ln b = \exp_b'(0) = \lim_{\Delta x\to 0}\frac{\exp_b(\Delta x)-\exp_b(0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{b^{\Delta x}-1}{\Delta x},$$

or $\ln b = \lim_{h\to 0}\left(\frac{b^h-1}{h}\right)$. ▲

The limit formula (6) for e has the following generalization: for any real number a,

$$e^a = \lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = \lim_{n\to\infty}\left(1 - \frac{a}{n}\right)^{-n}. \tag{7}$$

To prove (7), write $h = a/n$. Then $h \to 0$ as $n \to \infty$, so

$$\left(1 + \frac{a}{n}\right)^n = (1+h)^{a/h} = \left((1+h)^{1/h}\right)^a$$

which tends to e^a by (5). The second half of (7) follows by taking $h = -a/n$.

Formula (7) has an interpretation in terms of *compound interest*. If a bank offers $r\%$ interest on deposits, compounded n times per year, then any invested amount will grow by a factor of $1 + r/100n$ during each compounding period and hence by a factor of $(1 + r/100n)^n$ over a year. For instance, a deposit of \$1000 at 6% interest will become, at the end of the year,

$$1000(1 + 6/400)^4 = \$1061.36 \qquad \text{with quarterly compounding,}$$

$$1000(1 + 6/36500)^{365} = \$1061.8314 \qquad \text{with daily compounding,}$$

and

$$1000\left[1 + 6/(24\cdot 36500)\right]^{24\cdot 365} = \$1061.8362 \quad \text{with hourly compounding.}$$

Two lessons seem to come out of this calculation: the final balance is an increasing function of the number of compounding periods, but there may be an upper limit to how much interest could be earned at a given rate, even if the compounding period were to be decreased to the tiniest fraction of a second.

In fact, applying formula (7) with $a = r/100$ gives us

$$\lim_{n\to\infty}\left(1 + \frac{r}{100n}\right)^n = e^{r/100},$$

and so \$1000 invested at 6% interest can never grow in a year to more than $1000e^{0.06} = \$1061.8366$, no matter how frequent the compounding. (Strictly speaking, this assertion depends on the fact that $(1 + r/100n)^n$ is really an increasing function of n. This is intuitively clear from the compound interest interpretation; a proof is outlined in Exercise 37.)

In general, if P_0 dollars are invested at $r\%$ interest, compounded n times a year, then the account balance after a year will be $P_0(1 + r/100n)^n$, and the limit of this as $n \to \infty$ is $P_0e^{r/100}$. This limiting case is often referred to as *continuously compounded interest*. The actual fraction by which the funds increase during each one year period with continuous compound interest is $(P_0e^{r/100} - P_0)/P_0 = e^{r/100} - 1$.

Compound Interest

1. If an initial principal P_0 is invested at r percent interest compounded n times per year, then the balance after one year is $P_0(1 + r/100n)^n$.
2. If $n \to \infty$, so that the limit of continuous compounding is reached, then the balance after one year is $P_0e^{r/100}$.
3. The annual percentage increase on funds invested at $r\%$ per year compounded continuously is

$$100(e^{r/100} - 1)\%. \tag{8}$$

Example 5 What is the yearly percent increase on a savings account with 5% interest compounded continuously?

Solution By formula (8), the fraction by which funds increase is $e^{r/100} - 1$. Substituting $r = 5$ gives $e^{r/100} - 1 = e^{0.05} - 1 = 0.0513 = 5.13\%$. ▲

Continuous compounding of interest is an example of *exponential growth*, a topic that will be treated in detail in Section 8.1.

Reasoning as above, we find that if P_0 dollars is invested at r percent compounded annually and is left for t years, the amount accumulated is $P_0(1 + r/100)^t$. If it is compounded n times a year, the amount becomes $P_0(1 + r/n \cdot 100)^{nt}$, and if it is compounded continuously, it is $P_0 e^{rt/100}$.

Example 6 Manhattan Island was purchased by the Dutch in 1626 for the equivalent of \$24. Assuming an interest rate of 6%, how much would the \$24 have grown to be by 1984 if (a) compounded annually? (b) Compounded continuously?

Solution (a) $24 \cdot (1.06)^{1984-1626} = 24 \cdot (1.06)^{358} \approx \$27.5$ billion.
(b) $24 \cdot e^{(0.06)(1984-1626)} = 24e^{(0.06)(358)} \approx \$51.2$ billion. ▲

Exercises for Section 6.4

Sketch the graphs of the functions in Exercises 1 – 8.

1. $y = e^{-x}\sin x$.
2. $y = (1 + x^2)e^{-x}$.
3. $y = xe^{-x}$.
4. $y = e^x/(1 + x^2)$.
5. $y = \log_x 2$. [*Hint*: $y = \ln 2/\ln x$.]
6. $y = (\ln x)/x$.
7. $y = x^\pi (x > 0)$.
8. $y = e^{-1/x^2}$.
9. By what percentage are the profits of the company in Example 3 increasing on January 1, 1982?
10. A certain company's profits are given by $P = 50{,}000\exp(0.1t - 0.002t^2 + 0.00001t^3)$, where t is the time in years from July 1, 1975. By what percentage are the profits growing on January 1, 1980?
11. For $0 \leqslant t < 1000$, the height of a redwood tree in feet, t years after being planted, is given by $h = 300(1 - \exp[-t/(1000 - t)])$. By what percent per year is the height increasing when the tree is 500 years old?
12. A company truck and trailer has salvage value $y = 120{,}000e^{-0.1x}$ dollars after x years of use. (a) Find the rate of depreciation in dollars per year after five years. (b) By what percentage is the value decreasing after three years?
13. Express $3^{\sqrt{2}}$ as a limit.
14. Express e^{a+1} as a limit.
15. Express $3\ln b$ as a limit.
16. Express $\ln(\frac{1}{2})$ as a limit.
17. A bank offers 8% per year compounded continuously and advertises an actual yield of 8.33%. Verify that this is correct.
18. (a) What rate of interest compounded annually is equivalent to 7% compound continuously?
(b) How much money would you need to invest at 7% to see the difference between continuous compounding and compounding by the minute over a year?
19. The amount A for principal P compounded continuously for t years at an annual interest rate of r is $A = Pe^{rt}$. Find the amount after three years for \$100 principal compounded continuously at 6%.
20. (a) How long does it take to double your money at 6% compounded anually?
(b) How long does it take to double your money at 6% compounded continuously?

Find the equation of the tangent line to the graph of the given functions at the indicated points in Exercises 21–26.

21. $y = xe^{2x}$ at $x = 1$.
22. $y = x^2e^{x/2}$ at $x = 2$.
23. $y = \cos(\pi e^x/4)$ at $x = 0$.
24. $y = \sin(\ln x)$ at $x = 1$.
25. $y = \ln(x^2 + 1)$ at $x = 1$.
26. $y = x^{\ln x}$ at $x = 1$.
27. (a) Show that the first-order approximation to b^x, for x near zero, is $1 + x\ln b$.
(b) Compare $2^{0.01}$ with $1 + 0.01\ln 2$; compare $2^{0.0001}$ with $1 + 0.0001\ln 2$. (Use a calculator or tables.)
(c) By writing $e = (e^{1/n})^n$ and using the first-order approximation for $e^{1/n}$, obtain an approximation for e.
28. Show that $\ln b = \lim_{n\to\infty} n(\sqrt[n]{b} - 1)$.
29. Find the minimum value of $y = x^x$ for x in $(0, \infty)$.

30. Let f be a function satisfying $f'(t_0) = 0$ and $f(t_0) \neq 0$. Show that the relative rate of change of $P = \exp[f(t)]$ is zero at $t = t_0$.
31. One form of the Weber–Fechner law of mathematical psychology is $dS/dR = c/R$, where S = perceived sensation, R = stimulus strength. The law says, for example, that adding a fixed amount to the stimulus is less perceptible as the total stimulus is greater.
   (a) Show that $S = c \cdot \ln(R/R_0)$ satisfies the Weber–Fechner law and that $S(R_0) = 0$. What is the meaning of R_0?
   (b) The loudness L in decibels is given by $L = 10\log_{10}(I/I_0)$, where I_0 is the least audible intensity. Find the value of the constant c in the Weber–Fechner law of loudness.
32. The rate of damping of waves in a plasma is proportional to $r^3e^{-r^2/2}$, where r is the ratio of wave velocity to "thermal" velocity of electrons in the plasma. Find the value of r for which the damping rate is maximized.
33. The atmospheric pressure p at x feet above sea level is approximately given by $p = 2116e^{-0.0000318x}$. Compute the decrease in outside pressure expected in one second by a balloon at 2000 feet which is rising at 10 feet per second. [*Hint*: Use $dp/dt = (dp/dx)(dx/dt)$.]
34. The pressure P in the aorta during the diastole phase—period of relaxation—can be modeled by the equation

$$\frac{dP}{dt} + \frac{C}{W}P = 0, \qquad P(0) = P_0.$$

   The numbers C and W are positive constants.
   (a) Verify that $P = P_0e^{-Ct/W}$ is a solution.
   (b) Find $\ln(P_0/P)$ after 1 second.
35. The pressure P in the aorta during systole can be given by

$$P = \left(P_0 + \frac{CAW^2B}{C^2 + W^2B^2}\right)e^{-Ct/W} + \frac{CAW}{C^2 + W^2B^2}[C\sin Bt + (-WB)\cos Bt].$$

   Show that $P(0) = P_0$ and $dP/dt + (C/W)P = CA\sin Bt$.
36. A rich uncle makes an endowment to his brother's firstborn son of $10,000, due on the child's twenty-first birthday. How much money should be put into a 9% continuous interest account to secure the endowment? [*Hint*: Use the formula $P = P_0e^{kt}$, solving for P_0.]
★37. Let $a > 0$. Show that $[1 + (a/n)]^n$ is an increasing function of n by following this outline:
   (a) Suppose that $f(1) = 0$ and $f'(x)$ is positive and decreasing on $[1, \infty)$. Then show that $g(x) = xf(1 + (1/x))$ is increasing on $[1, \infty)$. [*Hint*: Compute $g'(x)$ and use the mean value theorem to show that it is positive.]
   (b) Apply the result of (a) to $f(x) = \ln(x)$.
   (c) Apply the result of (b) to

$$\frac{1}{a}\ln\left[\left(1 + \frac{a}{n}\right)^n\right].$$

★38. Let $r = b^\theta$ be an exponential spiral.
   (a) Show that the angle ϕ between the tangent line at any point of the spiral and the line from that point to the origin is the same for all points of the spiral. (Use the formula for the tangent line in polar coordinates given in Section 5.6.) Express ϕ in terms of b.
   (b) The tangent lines to a certain spiral make an angle of 45° with the lines to the origin. By what factor does the spiral grow after one turn about the origin?
★39. Determine the number of (real) solutions of the equation $x^3 - 4x + \frac{1}{2} = 2^x$ by a graphical method.
★40. (a) For any positive integer n, show by using calculus that $(1/n)e^x - x \geqslant 0$ for large x.
   (b) Use (a) to show that $\lim_{x\to\infty}(e^x/x) = \infty$.

Review Exercises for Chapter 6

Simpify the expressions in Exercises 1–8.

1. $(x^\pi + x^{-\pi})(x^\pi - x^{-\pi})$
2. $[(x^{-3/2})^2]^{1/4}$
3. $\log_2(8^3)$
4. $\log_3(9^2)$
5. $\ln(e^3) + \frac{1}{2}\ln(e^{-5})$
6. $\dfrac{3e^{-\ln 4}}{\ln e^4}$
7. $\ln\exp(-36)$
8. $\exp(\ln(\exp 3 + \exp 4) + \ln(8))$

Differentiate the functions in Exercises 9–38.

9. e^{x^3}
10. $(e^x)^3$
11. $e^x\cos x$
12. $\cos(e^x)$
13. $e^{\cos 2x}$
14. $e^{\cos x}$
15. x^2e^{10x}
16. $xe^{(x+2)^3}$
17. e^{6x}
18. $xe^x - e^x$
19. $\dfrac{e^{\cos x}}{\cos(\sin x)}$
20. $\dfrac{x^2 + 2x}{1 + e^{\cos x}}$
21. $\dfrac{\sin(e^x)}{e^x + x^2}$
22. $\tan(\sin(e^x))$
23. $\cos\sqrt{1 + e^x}$
24. $e^x\cos(x^{3/2})$

25. $e^{\cos x + x}$
26. $\exp((\sin x) - x^2)$
27. $\dfrac{e^{-x^2}}{1 + x^2}$
28. $\cos(e^{x^2+2})$
29. $x \ln(x + 3)$
30. $x \ln x$
31. $\ln(\cos x)$
32. $\ln(\sqrt{x}\,)$
33. $\log_3(5x)$
34. $\log_2(3x)$
35. $\cos^{-1}(x + e^{-x})$
36. $\sin^{-1}(e^x - 1)$
37. $\dfrac{1}{(\ln t)^2 + 3}$
38. $\sin\left[(\ln t)^3 + \dfrac{\pi}{6}\right]$

Compute the integrals in Exercises 39–48.

39. $\int e^{3x}\,dx$
40. $\int (e^{6x} + e^{-6x})\,dx$
41. $\int \left(\cos x + \dfrac{1}{3x}\right) dx$
42. $\int \dfrac{1}{x + 2}\,dx$
43. $\int \left(\dfrac{x + 1}{x}\right) dx$
44. $\int \dfrac{x^2 + x + 2}{x}\,dx$
45. $\int_1^2 \dfrac{x + x^2 \sin \pi x + 1}{x^2}\,dx$
46. $\int_1^2 \left(\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}\right) dx$
47. $\int_1^2 (x - \cos x - e^x)\,dx$
48. $\int_0^5 e^{-5x}\,dx$

Use logarithmic differentiation to differentiate the functions in Exercises 49–52.

49. $(\ln x)^x$
50. $(\ln x)^{\exp x}$
51. $\dfrac{(x + 3)^{7/2}(x + 8)^{5/3}}{(x^2 + 1)^{6/11}}$
52. $(3x + 2)^{1/2}(8x^2 - 6)^{3/4}(\sin x - 3)^{6/17}$

Sketch the graphs of the functions in Exercises 53–56.

53. $y = \dfrac{e^x}{1 + e^x}$.
54. $y = \sin(\ln x)$.
55. $y = \dfrac{e^{-x}}{1 + x}$.
56. $y = \dfrac{1}{(\ln t)^2 + 1}$.

Find dy/dx in Exercises 57–60.

57. $e^{xy} = x + y$
58. $x^y + y = 3$.
59. $e^{-x} + e^{-y} = 2$.
60. $e^x + e^y = 1$.
61. Find the equation of the tangent line to the graph of $y = (x + 1)e^{(3x^2+4x)}$ at $(0, 1)$.
62. Find the tangent line at $(0, \ln 3)$ to the graph of the curve defined implicitly by the equation $e^y - 3 + \ln(x + 1)\cos y = 0$.

Differentiate the functions in Exercises 63–66 and write the corresponding integral formulas.

63. $\dfrac{1}{36}\sin(6x) - \dfrac{x}{6}\cos(6x)$
64. $2x^2 \ln x - x^2$
65. $\dfrac{e^x}{x + 1}$
66. $\frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x$

Find the limits in Exercises 67–70.

67. $\lim_{n\to\infty}\left(1 + \dfrac{8}{n}\right)^n$
68. $\lim_{n\to\infty}\left(1 - \dfrac{3}{2n}\right)^{-2n}$
69. $\lim_{n\to\infty}\left(1 + \dfrac{10}{n}\right)^n$
70. $\lim_{n\to\infty}\left(1 - \dfrac{6}{n}\right)^{2n}$
71. The radius of a bacterial colony is growing with a percentage rate of 20% per hour. (a) If the colony maintains a disk shape, what is the growth rate of its area? (b) What if the colony is square instead of round, and the lengths of the sides are increasing at a percentage rate of 20% per hour?
72. If a quantity $f(t)$ is increasing by 10% a year and quantity $g(t)$ is decreasing by 5%, how is the product $f(t)g(t)$ changing?
73. What annual interest rate gives an effective 8% rate after continuous compounding?
74. How much difference does continuous versus daily compounding make on a one-million dollar investment at 10% per year?
75. The velocity of a particle moving on the line is given by $v(t) = 37 + 10e^{-0.07t}$ meters per second. (a) If the particle is at $x = 0$ at $t = 0$, how far has it travelled after 10 seconds? (b) How important is the term $e^{-0.07t}$ in the first 10 seconds of motion? In the second 10 seconds?
76. If \$1000 is to double in 10 years, at what rate of interest must it be invested if interest is compounded (a) continuously, and (b) quarterly?
77. If a deposit of A_0 dollars is made t times and is compounded n times during each deposit interval at an interest rate of i, then

$$A = A_0\left\{\frac{(1 + i/n)^{nt} - 1}{(1 + i/n)^n - 1}\right\}$$

is the amount after t intervals of deposit. (Deposits occur at the *end* of each deposit interval.)

(a) Justify the formula. [*Hint*:

$$x^{t-1} + \cdots + x + 1 = (x^t - 1)/(x - 1).]$$

(b) A person deposits \$400 every three months, to be compounded quarterly at 7% per annum. How much is in the bank after 6 years?

78. The *transmission density* of a test area in a color slide is $D = \log_{10}(I/I_0)$, where I_0 is a reference intensity and I is the intensity of light transmitted through the slide. Rewrite this equation in terms of the natural logarithm.
79. The salvage value of a tugboat is given by $y = 260{,}000\,e^{-0.15x}$ dollars after x years of use. What is the expected depreciation during the fifth year?
80. Find the marginal revenue of a commodity with demand curve $p = (1 + e^{-0.05x})10^3$ dollars per unit for x units produced. [Revenue equals (number of units)(price per unit) $= xp$; the marginal revenue is the derivative of the revenue with respect to x.]
81. A population model which takes birth and death rates into account is the *logistic model* for the population $P: dP/dt = P(a - bP)$. The constants a, b, where $a > 0$ and $b \neq 0$, are the *vital constants.*
    (a) Let $P(0) = P_0$. Check by differentiation that $P(t) = a/[b + ([a/P_0] - b)e^{-at}]$ is a solution of the logistic equation.
    (b) Show that the population size approaches a/b as t tends to ∞.
82. We have seen that the exponential function $\exp(x)$ satisfies $\exp(x) > 0$, $\exp(0) = 1$, and $\exp'(x) = \exp(x)$. Let $f(x)$ be a function such that $0 \leqslant f'(x) \leqslant f(x)$ and $f(0) = 0$. Prove that $f(x) = 0$ for all x. [*Hint*: Consider $g(x) = f(x)/\exp(x)$.]
83. Show that for any $x \neq 0$ there is a number c between zero and x such that $e^x = 1 + e^c x$. Deduce that $e^x > 1 + x$.
84. Fig. 6.R.1 shows population data from each U.S. census from 1790 to 1970.

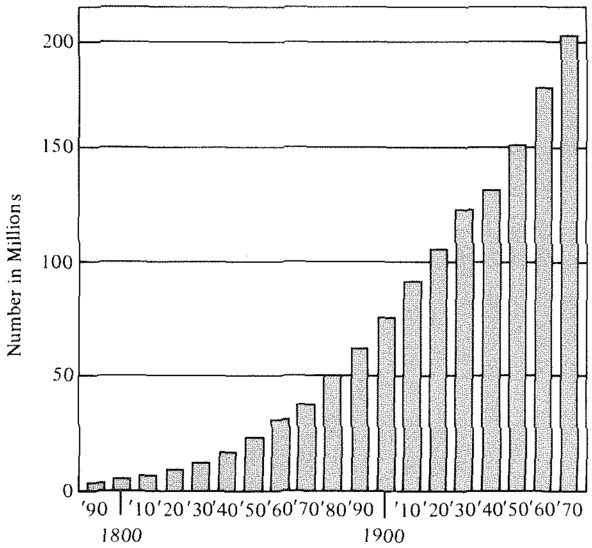

Figure 6.R.1. Population of the United States. Total number of persons in each census: 1790–1970.

(a) Fit the data to an exponential curve $y = Ae^{\gamma t}$, where t is the time in years from 1900; i.e., find A and γ numerically.
(b) Use (a) to "predict" the 1980 census. How close was your prediction? (The actual 1980 census figure was 226,500,000.)
(c) Use (a) to predict when the U.S. population will be 400 million.

★85. (a) Show that if $b > 1$ and n is a positive integer, then
$$b^n \geqslant 1 + n(b-1)$$
and
$$b^{-n} \leqslant \frac{1}{1 + n(b-1)}.$$
[*Hint*: Write $b^n = [1 + (b-1)]^n$ and expand.]
(b) Deduce from these inequalities that $\lim_{x\to\infty} b^x = \infty$ and $\lim_{x\to-\infty} b^x = 0$.

Exercises 86–91 form a unit.

★86. Show that $e^x \geqslant x^n/n!$ for all integers $n \geqslant 0$ and all real $x \geqslant 1$, as follows: (recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ and $0! = 1$).
    (a) Let n be fixed, let x be variable, and let $f_n(x) = e^x - x^n/n!$ Show that $f_n'(x) = f_{n-1}(x)$.
    (b) Show that $f_0(x) > 0$ for $x > 1$, and conclude that $f_1(x)$ is increasing on $[1, \infty)$.
    (c) Conclude that $f_1(x) > 0$ for $x \geqslant 1$.
    (d) Repeat the argument to show that $f_2(x) > 0$ for $x \geqslant 1$.
    (e) Finish the proof.

★87. Show that $e^x \geqslant x^n$ for $x \geqslant (n+1)!$

★88. Show that $\lim_{x\to\infty}(e^x/x^n) = \infty$.

★89. (a) By taking logarithms in the inequality derived in Review Exercise 87, show that $x/\ln x \geqslant n$ when $x \geqslant (n+1)!$
(b) Conclude that $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$.

★90. Use the result of Review Exercise 89 to show that $\lim_{x\to\infty}(\ln x/x^a) = 0$ for any $a > 0$. (*Hint*: Let $y = x^a$.)

★91. Use the result of Review Exercise 90 to show that $\lim_{x\to 0}(\ln x/x^{-a}) = 0$ for any $a > 0$.

★92. Look at Exercise 43 in Section 3.1. Explain why it takes, on the average, 3.32 bisections for every decimal place of accuracy.

★93. Prove that $3^{x-2} > 2x^2$ if $x \geqslant 7$. Can you improve this result?