CHAPTER R

REVIEW OF FUNDAMENTALS

R.1 Basic Algebra: Real Numbers and Inequalities

PREREQUISITES

1. There are no prerequisites for this section other than some high school algebra and geometry; however, if the material presented in this section is new to you, it would be a good idea to enroll in a precalculus course. This section is intended to be a review.

PREREQUISITE QUIZ

1. Orientation quizzes A and B in the text will help you evaluate your preparation for this section and this course.

GOALS

1. Be able to factor and expand common mathematical expressions.
2. Be able to complete a square.
3. Be able to use the quadratic formula.
4. Be able to solve equations and inequalities.

STUDY HINTS

1. Common identities. Know how to factor $a^2 - b^2$. It is a good idea to memorize the expansion of $(a + b)^2$ and $(a + b)^3$. Note that $(a - b)^2$ can be obtained by substituting $-b$ for b. $(a - b)^3$ can be similarly expanded. These identities are useful for computing limits in Section 1.2 and Chapter 11.

2. Factoring. This is a technique that is learned best through practice. A good starting point is to find all integer factors of the last term (the constant term). Once you find a factor for the original polynomial, use long division to find a simpler polynomial to factor. This will be important for partial fractions in Chapter 10 and for computing limits.

3. Completing the square. Don't memorize the formula. Practice until you learn the technique. Note that adding $(b/2a)^2$ to $x^2 + bx/a$ forms a perfect square. This technique will be very important for integration techniques introduced in Chapter 10.

4. Quadratic formula. It is recommended that you memorize this formula. It is used in many applications in various disciplines such as engineering, economics, medicine, etc. This formula may also be used to solve equations of the form $Ax^4 + Bx^2 + C = 0$ by solving for $y = x^2$ and taking square roots to get x.

5. Square roots. Note that, unless otherwise stated, square roots are understood to be nonnegative. $\sqrt{0}$ is equal to zero.

6. Inequalities. It is essential to have a good handle on manipulating inequalities. Without this, you will not have a good understanding of some of the basic theory of calculus. Don't forget to reverse the direction of the inequality sign when you multiply by a negative number.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $8/6 - 9/4 = -11/12$ is a rational number. Since the denominator cannot be reduced to one, it is neither a natural number nor an integer.

5. $(a - 3)(b + c) - (ac + 2b) = (ab - 3b + ac - 3c) - (ac + 2b) = ab - 5b - 3c$.

9. We can use Example 2 with b replaced by $-b$ to get $a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. Alternatively, write $(a - b)^3 = (a - b)^2(a - b) = (a^2 -2ab + b^2)(a - b) = a^3 - 3a^2b + 3ab^2 - b^3$.

13. We know that $(x + a)(x + b) = x^2 + (a + b)x + ab$. The factors of 6 are ± 1 , ± 2 , ± 3 , and ± 6 . By choosing $a = 2$ and $b = 3$, we get $a + b = 5$. Thus, $x^2 + 5x + 6 = (x + 2)(x + 3)$.

17. First we factor out 3 to get $3(x^2 - 2x - 8)$. We know that the factors of -8 are ± 1 , ± 2 , ± 4 , and ± 8 . As in Exercise 13, we look for a and b so that $(a + b)x$ is the middle term. In this case, $a = -4$ and $b = 2$. Thus, $3x^2 - 6x - 24 = 3(x - 4)(x + 2)$.

21. $2(3x - 7) - (4x - 10) = 0$ simplifies to $(6x - 14) - (4x - 10) = 2x - 4 = 0$, i.e., $2x = 4$. Dividing by 2 yields $x = 2$.

25. The right-hand side is $(x - 1)(x^2 + x + 1) = x(x^2 + x + 1) - 1(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$, which is the left-hand side.

29. (a) By factoring, we get $x^2 + 5x + 4 = (x + 4)(x + 1) = 0$. If any factor equals zero, the equation is solved. Thus, $x = -4$ or $x = -1$.

(b) By using the method of completing the square, we get $0 = x^2 + 5x + 4 = (x^2 + 5x + 25/4) + (4 - 25/4) = (x + 5/2)^2 - 9/4$. Rearrangement yields $(x + 5/2)^2 = 9/4$, and taking square roots gives $x + 5/2 = \pm 3/2$. Again, $x = -4$ or -1 .

(c) $a = 1$, $b = 5$, and $c = 4$, so the quadratic formula gives $x = (-5\pm\sqrt{25 - 4(1)(4)})/2(1) = (-5 \pm 3)/2 = -4$ or -1 .

33. We use the quadratic formula with $a = -1$, $b = 5$, and $c = 0.3$. This gives $x = (-5 \pm \sqrt{25 + 1.2})/(-2) = (5 \pm \sqrt{26.2})/2$. These are the two solutions for x.

37. $x^2 + 4 = 3x^2 - x$ is equivalent to $0 = 2x^2 - x - 4$. Using the quadratic formula with $a = 2$, $b = -1$, and $c = -4$, we get $x = (1 \pm \sqrt{1 + 32})/4$ $(1 \pm \sqrt{33})/4$.

41. We apply the quadratic formula with $a = 2$, $b = 2\sqrt{7}$, and $c = 7/2$ to get $x = (2\sqrt{7} \pm \sqrt{28 - 28})/4 = \sqrt{7}/2$. Thus, the only solution is $x = \sqrt{7}/2$

45. We add b to both sides to obtain $a + c > 2c$. Then we subtract c to get $a > c$.

49. $b(b + 2) > (b + 1)(b + 2)$ is equivalent to $b^2 + 2b > b^2 + 3b + 2$. Subtracting $b^2 + 2b$ from both sides leaves $0 > b + 2$. Subtracting 2 yields $-2 > b$.

53. (a) Dividing through by a in the general quadratic equation yields $x^2 + (b/a)x + c/a = 0$. Add and subtract $(b/2a)^2$ to get $(x^2 + (b/a)x + b^2/4a^2) + (c/a - b^2/4a^2) = 0 = (x + b/2a)^2 + (4ac/4a^2 - b^2/4a^2)$, i.e., $(x + b/2a)^2 = (b^2 - 4ac)/4a^2$. Taking square roots gives $x + b/2a = \pm\sqrt{b^2 - 4ac}/2a$, and finally $x = (-b \pm \sqrt{b^2 - 4ac})/2a$.

(b) From the quadratic formula, we see that there are no solutions if $b^2 - 4ac < 0$. If $b^2 - 4ac > 0$, there are two distinct roots. However, if $b^2 - 4ac = 0$, there are two roots, which both equal $-b/2a$. This only occurs if $b^2 = 4ac$.

SECTION QUIZ

1. Factor: (a) $x^4 + 2x^2 + 1$

   (b) $2x^4 - x^2 - 1$

   (c) $x^6 - 1$

2. Apply the expansion of $(a + b)^3$ to expand $(3x - 2)^3$.

3. Use the quadratic formula to solve $x^5 + 3x^3 - 5x = 0$.

4. Sketch the solution of (a) $x^2 + 3x + 2 < 0$

   (b) $x^2 + 3x + 2 \geq 0$

5. Find the solution set of $x^3 \geq x$.

6. The first King of the Royal Land of Mathematica has decreed that the first young lady to answer the following puzzle shall rule at his side. The puzzle is to compute the product of all solutions to $x^3 + 2x^2 - x - 2 = 0$. Then, divide by the length of the finite interval for which $x^3 + 2x^2 - x - 2 \geq 0$. What answer would make a lady Queen of Mathematica?*

ANSWERS TO SECTION QUIZ

1. (a) $(x^2 + 1)^2$

   (b) $(2x^2 + 1)(x + 1)(x - 1)$

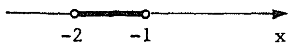
   (c) $(x^3 + 1)(x^2 + x + 1)(x - 1)$

2. $27x^3 - 54x^2 + 36x - 8$

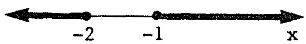
*Dear Reader: I realize that many of you hate math but are forced to complete this course for graduation. Thus, I have attempted to maintain interest with "entertaining" word problems. They are not meant to be insulting to your intelligence. Obviously, most of the situations will never happen; however, calculus has several practical uses and such examples are found throughout Marsden and Weinstein's text. I would appreciate your comments on whether my "unusual" word problems should be kept for the next edition.

3. 0 , $\pm[(-3+\sqrt{29})/2]^{1/2}$

4. (a)



(b)



5. $-1 \leq x \leq 0$ and $x \geq 1$

6. 2

R.2 Intervals and Absolute Values

PREREQUISITES

1. Recall how to solve inequalities. (Section R.1)

PREREQUISITE QUIZ

1. Solve the following inequalities:

   (a) $-x < 1$

   (b) $5x + 2 \geqslant x - 3$

GOALS

1. Be able to express intervals using symbols.
2. Be able to manipulate absolute values in equations and inequalities.

STUDY HINTS

1. Notation. A black dot means that the endpoint is included in the interval. In symbols, a square bracket like this "[" or like this "]" is used. A white circle means that the endpoint is not included in the interval; it is represented by a parenthesis like this "(" or this ")".
2. More notation. In the solution to Example 3, some students will get lazy and write the solution as $-1 > x > 3$. This gives the implication that $-1 > 3$, which is false.
3. Inequalities involving absolute values. Study Example 5 carefully. Such inequalities are often used in Chapter 11. Note that $|x - 8| \leq 3$ is the same as $-3 \leq x - 8 \leq 3$.
4. Triangle inequality. $|x + y| \leq |x| + |y|$ will be useful in proving limit theorems in Chapter 11. The name is derived from the fact that two sides of a triangle are always longer than the third side.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) This is true because $-8 \leqslant -7 \leqslant 1$.

   (b) This is false because $5 < 11/2$.

   (c) This is true because $-4 < 4 \leqslant 6$.

   (d) This is false because $4 < 4 < 6$ is false. 4 is not less than 4 .

   (e) This is true because $3/2 + 7/2 = 5$, which is an integer.

5. $x + 4 \geqslant 7$ implies $x \geqslant 3$. In terms of intervals, $x \in [3,\infty)$.

9. $x^2 + 2x - 3 > 0$ implies $(x + 3)(x - 1) > 0$. In one case, we need $x > -3$ and $x > 1$, i.e., $x > 1$. On the other hand, we can also have $x < -3$ and $x < 1$, i.e., $x < -3$. In terms of intervals, $x \in (-\infty,-3)$ or $x \in (1,\infty)$.

13. The absolute value is the distance from the origin. $|3 - 5| = |-2|$. Since $-2 < 0$, we change the sign to get $|-2| = 2$.

17. The absolute value of $|x| = x$ if $x > 0$. Thus, $|3 \cdot 5| = |15| = 15$.

21. Using the idea of Example 4(d), the only two solutions are $x = \pm 8$.

25. $x^2 - x - 2 > 0$ implies $(x - 2)(x + 1) > 0$. In one case, we have $x > 2$ and $x > -1$, i.e., $x > 2$. On the other hand, we can have $x < 2$ and $x < -1$, i.e., $x < -1$. The solution is all x except for x in $[-1,2]$. Using the idea of Example 6, we want to eliminate $|x - 1/2| \leqslant 3/2$. Therefore, the solution is $|x - 1/2| > 3/2$.

29. $|x| < 5$ implies $-5 < x < 5$, so x belongs to the interval $(-5,5)$.

33. The midpoint of $(-3,3)$ is 0 and the length of half of the interval is 3 . Therefore, by the method of Example 6, we get $|x - 0| = |x| < 3$.

37. The midpoint of $[-8,12]$ is 2 and the length of half of the interval is 10 . Therefore, by the method of Example 6, we get $|x - 2| \leqslant 10$. Equality is allowed as a possibility since the endpoints are included.

41. If $x \geqslant 0$, then $x^3 \geqslant 0$. In this case, the cube root of x^3 is still positive and equals x. If $x < 0$, then $x^3 < 0$. In this case, the cube root of x^3 is negative and equals x. Thus, independent of the sign of x, we have $x = \sqrt[3]{x^3}$.

SECTION QUIZ

1. Express the possible solutions of $x^3 - x \geqslant 0$ in terms of intervals.
2. Which of the following is true?

   (a) $\sqrt{-1} = -1$. (b) $\sqrt{0} = 0$.

   (c) $\sqrt{x^2} = x$ if $x < 0$. (d) $\sqrt{x^2} = |x|$ for all $x \neq 0$.
3. Solve $|x - 5| \geqslant 5$.
4. The school bully has selected you to help him do his homework problem, which is to solve $x^2 - 6x + 5 \geqslant 0$. You determine that the solution is $x \leqslant 1$ or $x \geqslant 5$. The bully understands, but then you turn the tables on him and tell him, "Look, $1 \geqslant x$ and $x \geqslant 5$, so $1 \geqslant x \geqslant 5$." He turns in this answer.

   (a) Explain why $1 \geqslant x \geqslant 5$ is incorrect.

   (b) Write the correct answer in terms of intervals.

   (c) Write your answer in the form $|x - a| \geqslant b$ for constants a and b.
5. An architect is building an arched doorway whose height at x feet from the left base is $x - 0.1x^2$.

   (a) How wide is the doorway at position $x = 1.5$?

   (b) Express the width at $x = 1.5$ in terms of absolute values.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $x > -1$

   (b) $x \geqslant -5/4$

ANSWERS TO SECTION QUIZ

1. $[-1,0]$ and $[1,\infty)$

2. b and d

3. $x \leqslant 0$ and $x \geqslant 10$

4. (a) The implication is that $5 \leqslant 1$.

   (b) $(-\infty,1]$ and $[5,\infty)$

   (c) $|x - 3| \geqslant 2$

5. (a) 7

   (b) $|x - 5| \leqslant 7/2$

R.3 Laws of Exponents

PREREQUISITIES

1. There are no prerequisites for this section beyond some high school algebra; however, if the material presented in this section is new to you, it would be a good idea to enroll in a precalculus course. This section is intended to be a review.

GOALS

1. Be able to simplify expressions involving exponents.

STUDY HINTS

1. Integer exponents. The laws of exponents should be memorized. The "common sense" method preceding Example 1 serves as a useful check if you are unsure of your answer; however, it can slow you down during an exam.
2. Definitions. Know that $b^{-n} = 1/b^n$, $b^0 = 1$, and $b^{1/n} = \sqrt[n]{b}$.
3. Rational exponents. These laws are the same as those for integer powers.
4. Rationalization. Example 6 demonstrates a technique which is useful in the study of limits. The idea is to use the fact that $(a + b)(a - b) = a^2 - b^2$ to eliminate radicals from the denominator.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the law $(bc)^n = b^n c^n$, we have $3^2(1/3)^2 = (3\cdot 1/3)^2 = (1)^2 = 1$.
5. $[(4\cdot 3)^{-6}\cdot 8]/9^3 = 8/4^6\cdot 3^6\cdot (3^2)^3 = 2^3/(2^2)^6\cdot 3^6\cdot 3^6 = 2^3/2^{12}\cdot 3^{12} = 1/2^9\cdot 3^{12}$.
9. $9^{1/2} = \sqrt{9} = 3$.
13. $2^{5/3}/4^{7/3} = 2^{5/3}/(2^2)^{7/3} = 2^{5/3}/2^{14/3} = 2^{5/3 - 14/3} = 2^{-3} = 1/2^3 = 1/8$.

17. Apply to distributive law to get $(x^{3/2} + x^{5/2})x^{-3/2} = (x^{3/2})(x^{-3/2}) + (x^{5/2})(x^{-3/2}) = x^{3/2 - 3/2} + x^{5/2 - 3/2} = x^0 + x^1 = 1 + x$.

21. By using the laws of rational exponents, we get $\sqrt[a]{\sqrt[b]{x}} = (\sqrt[b]{x})^{1/a} = (x^{1/b})^{1/a} = x^{1/ab} = \sqrt[ab]{x}$.

25. Since the price doubles in 10 years, it will double again after 20 years and the factor is $2\cdot 2 = 4$. In thirty years, the factor is $2\cdot 2\cdot 2 = 8$, and in 50 years, the factor of increase is $2\cdot 2\cdot 2\cdot 2\cdot 2 = 32$.

29. The first term factors into $\sqrt{x}\cdot\sqrt{x}$ and $(\sqrt{x} + a)(\sqrt{x} + b) = x^2 + (a + b)\sqrt{x} + ab$. The last term has factors ± 1 , ± 2 , ± 4 , and ± 8 . We want factors a and b such that $ab = -8$ and $a + b = -2$. Thus $a = -4$ and $b = 2$. Therefore, the solution is $(\sqrt{x} - 4)(\sqrt{x} + 2) = (x^{1/2} - 4)(x^{1/2} + 2)$.

33. We will use the corresponding rule for integer powers several times. Let $p = m/n$ and $q = k/l$, so $b^{pq} = b^{(m/n)(k/l)} = b^{mk/nl} = \sqrt[nl]{b^{mq}}$. Thus, rasing the equation to the nl power gives $(b^{pq})^{nl} = b^{mk}$. Now, $(b^p)^q = (b^{m/n})^{k/l} = \sqrt[l]{(\sqrt[n]{b^m})^k}$, so $((b^p)^q)^{nl} = (\sqrt[l]{(\sqrt[n]{b^m})^k})^{ln} = ((\sqrt[n]{b^m})^k)^n = (\sqrt[n]{b^m})^{nk} = ((\sqrt[n]{b^m})^n)^k = (b^m)^k = b^{mk}$. Therefore, $(b^{pq})^{nl} = ((b^p)^q)^{nl}$, and taking the nl^{th} root of both sides gives Rule 2.

For Rule 3, let $p = m/n$. Then $(bc)^p = (bc)^{m/n} = \sqrt[n]{(bc)^m} = \sqrt[n]{b^m c^m}$, so $((bc)^p)^n = b^m c^m$. Now, $b^p c^p = b^{m/n} c^{m/n} = (\sqrt[n]{b^m})(\sqrt[n]{c^m})$, so $(b^p c^p)^n = ((\sqrt[n]{b^m})(\sqrt[n]{c^m}))^n = (\sqrt[n]{b^m})^n(\sqrt[n]{c^m})^n = b^m c^m$. Hence $(bc)^p$ and $b^p c^p$ have the same n^{th} power, so they are equal.

SECTION QUIZ

1. Eliminate the radicals from the denominator:

   (a) $(3\sqrt{5} + 2)/(\sqrt{5} - 1)$

   (b) $1/(\sqrt{x} + \sqrt{a})$, a is a constant.

   (c) $1/(\sqrt[3]{x} - \sqrt[3]{4})$

2. Simplify:

   (a) $36^{3/2}6^{-3/2}/6^{-1/2}\sqrt{9}$

   (b) $9^{-2}3^{3/2}/(\sqrt{81})^2$

3. As part of the class assignment, a young couple was asked to simplify $[(x + 2)^2(y + x)^2]^3/(x^2 + xy + 2x + 2y)^5$. The young man suggests expanding the entire expression and then using long division. However, unknown to him, his girlfriend is secretly disguised as Super-Brain and shows him a much easier method. Explain the easier method and find the answer.

ANSWERS TO SECTION QUIZ

1. (a) $(17 + 5\sqrt{5})/4$

   (b) $(\sqrt{x} - \sqrt{a})/(x - a)$

   (c) $(\sqrt[3]{x^2} + \sqrt[3]{4x} + \sqrt[3]{16})/(x - 4)$

2. (a) 12

   (b) $3^{-13/2}$

3. Use the laws of exponents. The numerator is $[(x + 2)^2(y + x)^2]^3 = [((x + 2)(y + x))^2]^3 = ((x^2 + xy + 2x + 2y)^2)^3 = (x^2 + xy + 2x + 2y)^6$. Alternatively, you can factor the denominator: $(x^2 + xy + 2x + 2y)^5 = [x(x + y) + 2(x + y)]^5 = [(x + y)(x + 2)]^5$. The answer is $(x + y)(x + 2)$.

R.4 Straight Lines

PREREQUISITES

1. There are no prerequisites for this section other than some high school analytic geometry; however, if the material presented in this section is new to you, it would be a good idea to enroll in a precalculus course. This section is intended to be a review.

GOALS

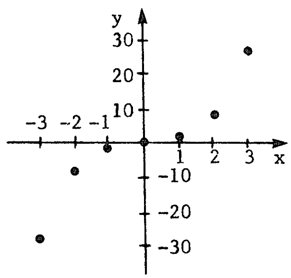
1. Be able to find the distance between two points.
2. Be able to write and manipulate equations of the line in its various forms.
3. Be able to write the equation for perpendicular lines.

STUDY HINTS

1. Distance formula. Don't be concerned about which point is (x_1,y_1) and which is (x_2,y_2). The squaring process eliminates the need to make such a distinction. Remembering that the formula is derived from the Pythagorean theorem makes it easier to recall.
2. Slope formula. As with the distance formula, don't be concerned about which point is (x_1,y_1) and which is (x_2,y_2). The sign will correct itself when the division is performed.
3. Point-slope form. Replacing (x_2,y_2) by (x,y) gives the slope as $m = (y - y_1)/(x - x_1)$. Rearrangement yields $y = y_1 + m(x - x_1)$.
4. Slope-intercept form. Choosing $(x_1,y_1) = (0,b)$ in the point-slope form of the line yields $y = mx + b$.
5. Point-point form. Substituting $m = (y_2 - y_1)/(x_2 - x_1)$ yields $y = y_1 + [(y_2 - y_1)/(x_2 - x_1)](x - x_1)$.

6. <u>Perpendicular lines</u>. Many instructors try to write harder exam problems by asking for equations of lines perpendicular to a given line, so you will benefit by remembering that slopes of perpendicular lines are negative reciprocals of each other, i.e., if a line has slope m , then the perpendicular line has slope $-1/m$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

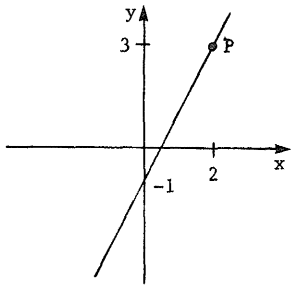
For each point (x,y) , go x units along the x-axis and the go y units along the y-axis.

5. The distance from P_1 to P_2 is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. In this case, the distance is $\sqrt{(1 - 1)^2 + (1 - (-1))^2} = \sqrt{4} = 2$.

9. The distance from P_1 to P_2 is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. In this case, the distance is $\sqrt{(43721 - 3)^2 + (56841 - 56841)^2} = \sqrt{(43718)^2} = 43718$.

13. The distance from P_1 to P_2 is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. In this case, the distance is $\sqrt{(x - 3x)^2 + (y - (y + 10))^2} = \sqrt{(-2x)^2 + (-10)^2} = \sqrt{4x^2 + 100} = 2\sqrt{x^2 + 25}$.

17. The slope of a line through (x_1,y_1) and (x_2,y_2) is $m = (y_2 - y_1)/(x_2 - x_1)$. In this case, $m = (6 - 3)/(2 - 1) = 3$.

21. 

The equation of the line with slope m passing through (x_1, y_1) is $y = y_1 + m(x - x_1)$. In this case, $y = 3 + 2(x - 2) = 2x - 1$.

25. The equation of the line through (x_1, y_1) and (x_2, y_2) is $y = y_1 + [(y_2 - y_1)/(x_2 - x_1)](x - x_1)$. In this case, $y = 7 + [(4 - 7)/(-1 - 5)](x - 5) = 7 + (1/2)(x - 5) = x/2 + 9/2$ or $2y = x + 9$.

29. We want to write the line in the form $y = mx + b$. Then m is the slope and b is the y-intercept. $x + 2y + 4 = 0$ is the same as $2y = -x - 4$ or $y = -x/2 - 2$. Thus, the slope is $-1/2$ and the y-intercept is -2 .

33. Write the line in the form $y = mx + b$. Then m is the slope and b is the y-intercept. $13 - 4x = 7(x + y) = 7x + 7y$ is equivalent to $13 - 11x = 7y$ or $y = -(11/7)x + 13/7$. Thus, the slope is $-11/7$ and the y-intercept is $13/7$.

37. (a) $4x + 5y - 9 = 0$ implies $5y = -4x + 9$, i.e., $y = -(4/5)x + 9/5$. This is in the form $y = mx + b$, so the slope is $-4/5$.

(b) A perpendicular line has slope $5/4$. When this line passes through $(1,1)$, its equation is $y = 1 + (5/4)(x - 1) = (5x - 1)/4$ or $4y = 5x - 1$.

41. Using the point-point form of the line, we get the equation $y = 2 + [(4 - 2)/(2 - 4)](x - 4) = 2 + (-1)(x - 4) = -x + 6$.

SECTION QUIZ

1. Sketch the line $3x + 2y = 1$.

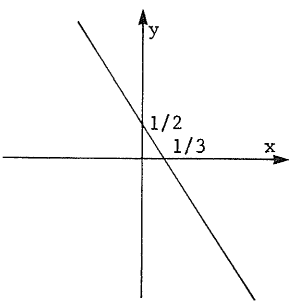
2. What is the slope of the line $5x - 8y = 4$?

3. Find the equation of line passing through $(3,1/2)$ and perpendicular to the line going through $(98,3)$ and $(98,-10)$.

4. One of your psychotic math friends has seen a little green space ship and a little red one land in her sink. According to her estimation, the green ship landed at $(-3,1)$ and the red one landed at $(2,4)$. She sees the aliens attacking each other with toothbrushes. The sink is divided by the perpendicular passing through the midpoint between the spacecrafts. What is the equation of the line? What is the distance between the two ships?

ANSWERS TO SECTION QUIZ

1.



2. 5/8

3. $y = 1/2$

4. Line: $3y + 5x + 5$; distance: $\sqrt{34}$.

R.5 Circles and Parabolas

PREREQUISITES

1. There are no prerequisites for this section other than some high school analytic geometry; however, if the material presented in this section is new to you, it would be a good idea to enroll in a precalculus course. This section is intended to be a review.

GOALS

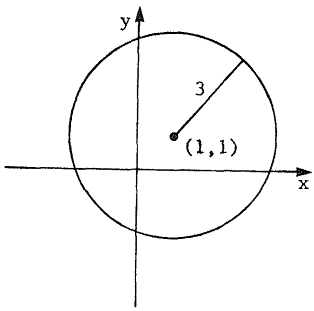
1. Be able to recognize equations for circles and parabolas, and be able to describe their graphs.
2. Be able to solve simultaneous equations to find intersection points.

STUDY HINTS

1. Circles. The general equation is $(x - a)^2 + (y - b)^2 = r^2$, where (a,b) is the center of the circle and r is the radius. Any equation of the form $Ax^2 + Ay^2 + Bx + Cy + D = 0$ may be written in the general form of a circle by completing the square. If $r^2 > 0$, then the equation describes a circle. The coefficient of x^2 and y^2 must be the same.
2. Parabolas. The general equation is $y - q = a(x - p)^2$, where (p,q) is the vertex. The parabola opens upward if $a > 0$, downward if $a < 0$. The graphs of all quadratic equations of the form $y = ax^2 + bx + c$ are parabolas.
3. Simultaneous equations. The solutions represent points of intersection. One method of solution is to multiply equations by a constant factor and then subtract equations to eliminate a variable as in Example 6 . The other method uses substitution to eliminate a variable as in Example 7.

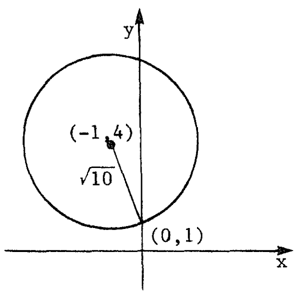
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



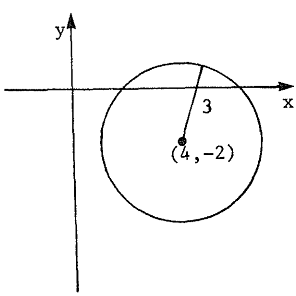
A circle with center at (a,b) and radius r has the equation $(x - a)^2 + (y - b)^2 = r^2$. With center $(1,1)$ and radius 3, the equation is $(x - 1)^2 + (y - 1)^2 = 9$ or $x^2 - 2x + y^2 - 2y = 7$.

5.



A circle with center at (a,b) and radius r has the equation $(x - a)^2 + (y - b)^2 = r^2$. With center at $(-1,4)$, the equation is $(x +1)^2 + (y - 4)^2 = r^2$. Substituting in $(0,1)$ yields $1^2 + (-3)^2 = r^2 = 10$, so the desired equation is $(x + 1)^2 + (y - 4)^2 = 10$ or $x^2 + 2x + y^2 - 8y + 7 = 0$.

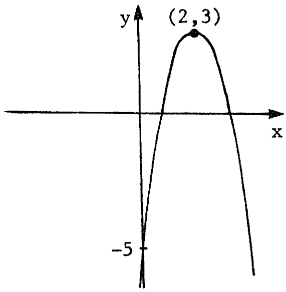
9.



We complete the square to find the center and radius. $-x^2 - y^2 + 8x - 4y - 11 = 0$ is equivalent to $x^2 + y^2 - 8x + 4y + 11 = 0$, i.e., $(x^2 - 8x + 16) + (y^2 + 4y + 4) = -11 + 16 + 4 = 9 = (x - 4)^2 + (y + 2)^2$. Thus, the center is $(4,-2)$ and the radius is 3.

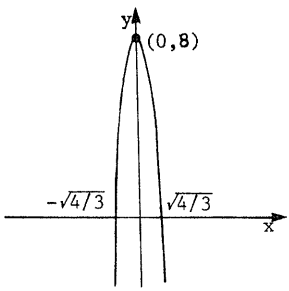
13. The parabola with vertex at (p,q) has the general equation $y - q = a(x - p)^2$. In this case, $y - 5 = a(x - 5)^2$. Substituting $(0,0)$ yields $-5 = a(-5)^2$, so $a = -1/5$ and the equation becomes $y = -(x - 5)^2/5 + 5 = -x^2/5 + 2x$.

17.



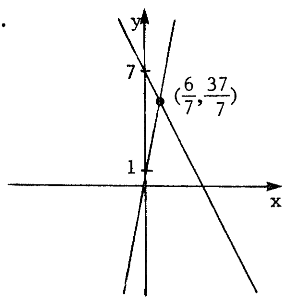
We complete the square to get the form $y - q = a(x - p)^2$, where (p,q) is the vertex. $y = -2x^2 + 8x - 5$ implies $(y + 5) - 8 = -2(x^2 - 4x + 4) = -2(x - 2)^2 = y - 3$. Thus, the vertex is $(2,3)$ and since $a = -2 < 0$, the parabola opens downward.

21.



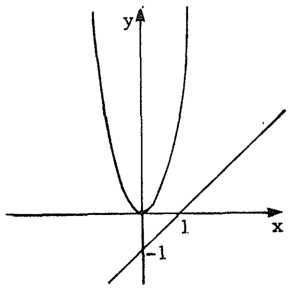
$y = -6x^2 + 8$ is the same as $y - 8 = -6(x - 0)^2$. This has the form $y - q = a(x - p)^2$, so the vertex is (p,q) or $(0,8)$. Since $a = -6 < 0$, the parabola opens downward.

25.



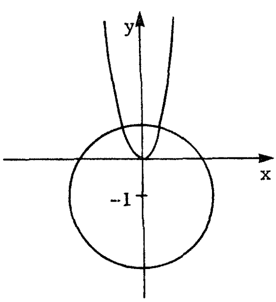
To find the point of intersection, solve the equations simultaneously. $-2x + 7 = y = 5x + 1$ implies $6 = 7x$, i.e., $x = 6/7$. Substituting back into one of the equations, we get $y = 5(6/7) + 1 = 37/7$, so the point of intersection is $(6/7,37/7)$.

To graph the equations, we see that $y = -2x + 7$ is a line with slope -2 and y-intercept 7. $y = 5x + 1$ has slope 5 and y-intercept 1.

29. 

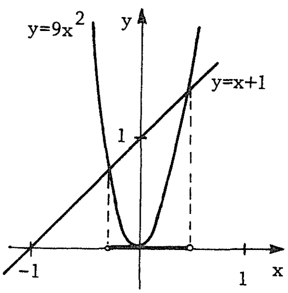
To find the point of intersection, solve the equations simultaneously. Thus, $y - x + 1 = 0$ becomes $3x^2 - x + 1 = 0$. The quadratic formula yields $x = (1 \pm \sqrt{1 - 12})/6$. Therefore, the parabola and the line do not intersect.

To graph the equations, we see that $y = 3x^2$ is an upward opening parabola with vertex at $(0,0)$. $y - x + 1 = 0$ is equivalent to $y = x - 1$, which is a line with slope 1 and y-intercept -1.

33. 

We solve the equations simultaneously. $y = 4x^2$ becomes $y/4 = x^2$, so $x^2 + 2y + y^2 - 3 = 0 = y/4 + 2y + y^2 - 3 = y^2 + 9y/4 - 3$. By the quadratic formula, $y = (-9/4 \pm \sqrt{81/6 + 12})/2 = (-9 \pm \sqrt{273})/8$. From $y = 4x^2$, we have $x = \pm\sqrt{y/4}$, and y must be non-negative. Thus, the intersection points are $([(-9 + \sqrt{273})/32]^{1/2}, (-9 + \sqrt{273})/8)$ and $(-[(-9 + \sqrt{273})/32]^{1/2}, (-9 + \sqrt{273})/8)$.

$y = 4x^2$ is a parabola opening upward with vertex at the origin. $x^2 + 2y + y^2 - 3 = 0$ is equivalent to $x^2 + (y^2 + 2y + 1) = 4$ or $x^2 + (y + 1)^2 = 2^2$. This is a circle centered at $(0,-1)$ with radius 2.

37. 

$9x^2 < x + 1$ implies $9x^2 - x - 1 < 0$. The solution of $9x^2 - x - 1 = 0$ is $(1 \pm \sqrt{1 + 36})/18$. $1/18$ satisfies $9x^2 < x + 1$, and since $1/18$ lies between the solutions of $9x^2 - x - 1 = 0$, the solution of the inequality is $x \in ((1 - \sqrt{37})/18, (1 + \sqrt{37})/18)$.

Geometrically, the solution interval is where the line $x + 1$ lies above the parabola $9x^2$.

SECTION QUIZ

1. The curves $y - 1 = x^4$ and $y = 2x^2$ intersect at two points. Find them.

2. They called her Melody the Marcher because she always got her employees to march off to work. After working a day for the Marcher, you no longer needed to be told, "Get to work!" Her glaring eyes told you, "You better march back to work!" She wanted a circular fence of radius 7 to protect a prized tree at (2,3). One of the hired hands came up with a devious plan. He decided to make a parabolic fence passing through (9,3),(-5,3) and (2,-4). After the fence was half finished, he asked her to stand at the focus (see Fig. R.5.4). When the Marcher arrived, the other employees threw garbage at the wall to reflect back at the focus.

   (a) What is the equation of the parabola?

   (b) What is the equation of the circle?

   (c) Sketch both curves on the same set of axes.

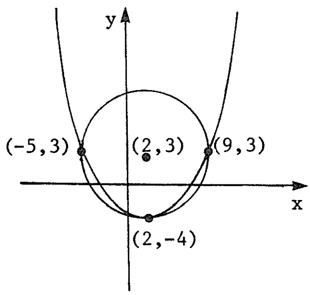
ANSWERS TO SECTION QUIZ

1. $(-1,2),(1,2)$

2. (a) $7y = x^2 - 4x - 24$

   (b) $x^2 - 4x + y^2 - 6y = 36$

   (c)



R.6 Functions and Graphs

PREREQUISITES

1. There are no special prerequisites for this section; however, if the material presented in this section is new to you, it would be a good idea to enroll in a precalculus course. This section is intended to be a review.

GOALS

1. Be able to evaluate a function at a given point.
2. Be able to sketch a graph by plotting.
3. Be able to recognize the graph of a function.

STUDY HINTS

1. Teminology. Sometimes, x and y are referred to as the independent and dependent variables, respectively. A good way to remember this is that normally x is chosen independently and then, the value of y depends on x.
2. Calculator errors. A calculator is accurate only up to a certain number of digits. Round-off errors may accumulate, so be careful! Calculators do not always give the correct answer.
3. Domain. The domain of a function f is simply all of the x values for which $f(x)$ is defined. Note the emphasis on is; it is not can be.
4. More on terminology. Although many of us may get sloppy with the useage of f, $f(x)$, and the graph of f, you should know their distinctions. f is the function itself. $f(x)$ refers to the function value when the variable is x; the variable may just as well be any other letter. Finally, the graph of f is a drawing which depicts f.

5. Graphing. For now, just be aware that plotting points at smaller intervals gives a more accurate drawing.

6. Calculator plotting. A programmable calculator can be very useful for plotting Fig. R.6.8. Depending on your calculator, a variation of the following program may be used. $\boxed{\text{STO}}$ $\boxed{\text{RCL}}$ $\boxed{x^2}$ $\boxed{x^2}$ $\boxed{\times}$ $\boxed{0}$ $\boxed{.}$ $\boxed{3}$ $\boxed{-}$ $\boxed{\text{RCL}}$ $\boxed{x^2}$ $\boxed{\times}$ $\boxed{0}$ $\boxed{.}$ $\boxed{2}$ $\boxed{-}$ $\boxed{0}$ $\boxed{.}$ $\boxed{1}$ $\boxed{=}$. x^2 is used twice to represent x^4 because most calculators will not raise negative numbers to a power other than two.

7. Graphs of functions. Another way of recognizing graphs of functions is to retrace the curve starting from the left. If you do not have to go straight up or down, or go backwards to the left, then the graph depicts a function.

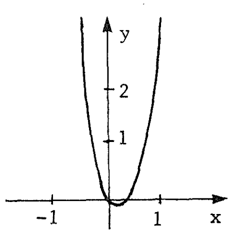
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $f(-1) = 5(-1)^2 - 2(-1) = 7$; $f(1) = 5(1)^2 - 2(1) = 3$.

5. $f(-1) = -(-1)^3 + (-1)^2 - (-1) + 1 = 4$; $f(1) = -(1)^3 + (1)^2 - (1) + 1 = 0$.

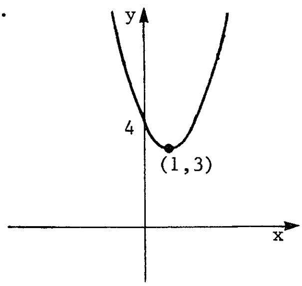
9. The domain of a function are the x-values for which there exists y-values. In this case, we need the expression under the radical sign to be non-negative. We want $1 - x^2 \geqslant 0$, so the domain is $-1 \leqslant x \leqslant 1$. $f(10)$ is not real.

13.



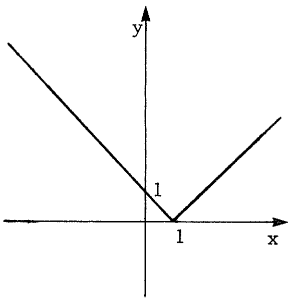
We complete the square as follows: $y = 5x^2 - 2x = 5(x^2 - 2x/5 + 1/25) - 1/5 = 5(x - 1/5)^2 - 1/5$. This is a parabola opening upwards with vertex at $(1/5,-1/5)$.

17.



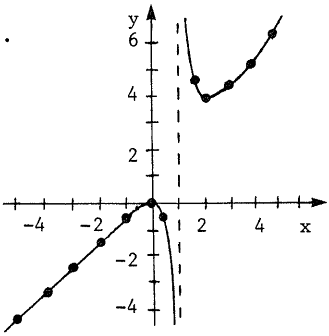
$y = (x - 1)^2 + 3$ is a parabola opening upwards with vertex at $(1,3)$.

21.



$y = |x - 1|$ is $y = x - 1$ if $x - 1 \geqslant 0$, i.e., if $x \geqslant 1$. Also, $y = |x - 1|$ is $y = -(x - 1) = -x + 1$ if $x - 1 < 0$, i.e., if $x < 1$. Thus, the graph consists of two line segments.

25.



The entire graph is shown at the left. The 10 points should lie on the graph. Some arbitrary points include: (-10,-9.1), (-8,7.1), (-6,-5.1), (-4,-3.2), (-2,1.3), (-1, -0.5), (0,0), (0.5,-0.5), (1.5,4.5), (3,4.5), (5,6.25), and (6,7.2) .

29. A graph of a function has only one y-value for any given x-value, i.e., each vertical line intersects the graph at only one point. Thus, only (a) and (c) represent functions. (d) is not a function due to the vertical line segment in the graph.

33. $y = \sqrt{x^2 - 1}$ has only one value for any given x , so it is a function. We want $x^2 - 1 \geqslant 0$, i.e., $x \leqslant -1$ and $x \geqslant 1$ make up the domain.

SECTION QUIZ

1. Suppose $g(x) = x^3 - x + 1$. What is $g(-1)$? $g(0)$? $g(y)$?

2. What is the domain of f if $f(x) = \begin{cases} 1/(1-x) & x < 0 \\ x^2 & x \geq 0 \end{cases}$?

3. What is the domain of f if $f(x) = \begin{cases} 1/(1-x) & x > 0 \\ x^2 & x \leq 0 \end{cases}$?

4. In Questions 2 and 3, which are functions?

5. In the midst of a nightmare, your roommate believes a giant black mouse is chasing him. Sometimes, he tries to fool the mouse by running backwards. Thus, when he awakens in a cold sweat, his position from the point of origin is given by $f(x) = 36x - x^2$.
   (a) If $f(x)$ must be nonnegative, what is the domain of f ?
   (b) Sketch $f(x)$, given the domain in part (a).

ANSWERS TO SECTION QUIZ

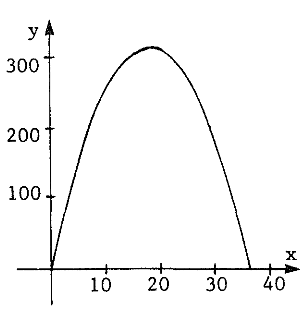
1. 1, 1 , and $y^3 - y + 1$.

2. $(-\infty,\infty)$.

3. $x \neq 1$.

4. Both

5. (a) $x \in [0,36]$
   (b)



R.R Review Exercises for Chapter R

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Subtract 2 from both sides to get $3x = -2$. Then divide by 3 to get $x = -2/3$.

5. Expansion gives $(x + 1)^2 - (x - 1)^2 = 2 = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$. Divide both sides by 4 to get $x = 1/2$.

9. $8x + 2 > 0$ is equivalent is $8x > -2$. Divide both sides by 8 to get $x > -1/4$.

13. Expand to get $x^2 - (x - 1)^2 = x^2 - (x^2 - 2x + 1) = 2x - 1 > 2$. Add 1 to both sides to get $2x > 3$ and divide by 2 to get $x > 3/2$.

17. $x^2 < 1$ is equivalent to $x^2 - 1 < 0$ or $(x + 1)(x - 1) < 0$. One possible solution is $x < -1$ and $x > 1$, but this is not possible. Another possibility is $x > -1$ and $x < 1$. Thus, the solution is $-1 < x < 1$, i.e., $x \in (-1,1)$.

21. $|x - 1|^2 \geqslant 2$ means $|x - 1| \geqslant \sqrt{2}$, i.e., $x - 1 \geqslant \sqrt{2}$ or $x - 1 \leqslant -\sqrt{2}$. For $x - 1 \geqslant \sqrt{2}$, we get $x \geqslant 1 + \sqrt{2}$. For $x - 1 \leqslant -\sqrt{2}$, we get $x \leqslant -\sqrt{2} + 1$. Therefore, the solution is $x \geqslant 1 + \sqrt{2}$ or $x \leqslant 1 - \sqrt{2}$, i.e., $x \in (-\infty, 1 - \sqrt{2}]$ or $x \in [1 + \sqrt{2}, \infty)$.

25. $x^3 \in (-8,27)$ implies $x \in (-2,3)$, so $x < 10$ and $x \in (-2,3)$ means $-2 < x < 3$; i.e., $x \in (-2,3)$.

29. $2(7 - x) \geqslant 1$ is equivalent to $14 - 2x \geqslant 1$ or $-2x \geqslant -13$. Dividing by -2 reverses the inequality and yields $x \leqslant 13/2$. Also, $3x - 22 > 0$ is equivalent to $3x > 22$ or $x > 22/3$. Thus, the solution is $x \in (-\infty,13/2]$ or $x \in (22/3,\infty)$.

33. $|-8| = 8$, so $|-8| + 5 = 13$.

37. $\sqrt{2} \cdot 2^{-1/2} = \sqrt{2}/2^{1/2} = \sqrt{2}/\sqrt{2} = 1$.

41. $x^{1/4} \sqrt{x}\, y/x^{1/2}\, y^{3/4} = x^{1/4+1/2-1/2}\, y^{1-3/4} = x^{1/4}\, y^{1/4} = \sqrt[4]{xy}$.

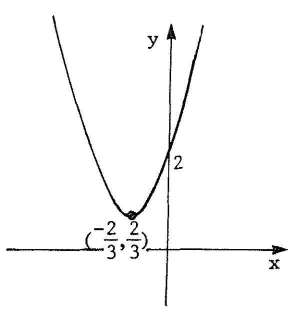
45. The distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In this case, it is $\sqrt{(2 - (-1)^2 + (0 - 1)^2} = \sqrt{9 + 1} = \sqrt{10}$.

49. The point-point form of a line is $y = y_1 + [(y_2 - y_1)/(x_2 - x_1)\ (x - x_1)]$. In this case, the line is $y = -1 + [(3 - (-1))/(7 - 1/2)](x - 1/2) = -1 + [4/(13/2)](x - 1/2) = -1 + (8/13)(x - 1/2) = 8x/13 - 17/13$, i.e., $13y - 8x + 17 = 0$.

53. The point-slope form of a line is $y = y_1 + m(x - x_1)$. In this case, the line is $y = 13 + (-3)(x - 3/4) = -3x + 61/4$ or $4y + 3x = 61$.

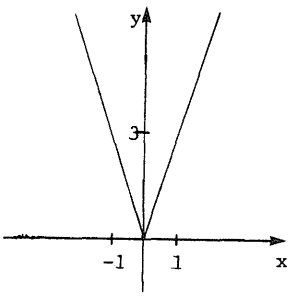
57. The slope of $5y + 8x = 3$ is $-8/5$. Thus, the slope of a perpendicular line is $5/8$, and the line passing through $(1,1)$ is $y = 1 + (5/8)(x - 1)$, using point-slope form of the line. The line is $y = 5x/8 + 3/8$ or $8y - 5x = 3$.

61. The equation of the circle with center (a,b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$. In this case, the circle is $(x - 12)^2 + (y - 5)^2 = 8^2$ or $(x^2 - 24x + 144) + (y^2 - 10y + 25) = 64$ or $x^2 - 24x + y^2 - 10y + 105 = 0$.

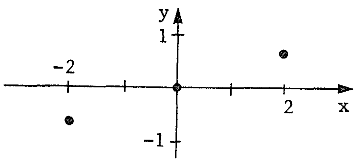
65. 

Complete the square to get $y = 3(x^2 + 4x/3 + 4/9) + 2 - 4/3 = 3(x + 2/3)^2 + 2/3$. This is a parabola opening upward with vertex at $(-2/3,2/3)$.

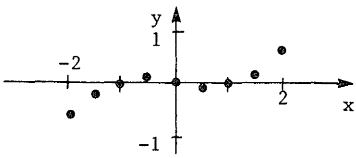
69. Solve the equations simultaneously. Substitute $y = x$ into $x^2 + y^2 = 4$ to get $2x^2 = 4$, i.e., $x^2 = 2$ or $x = \pm\sqrt{2}$. Thus, the points of intersection are $(-\sqrt{2},-\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$.

73. 

This is the graph of $y = 3x$ if $x \geqslant 0$; it is $y = -3x$ if $x < 0$.

77. (a) 

$f(-2) = -0.6$; $f(0) = 0$; $f(2) = 0.6$.

(b) 

In addition to the points in part (a), we have $f(-1.5) = -0.1875$; $f(-1) = 0$; $f(-0.5) = 0.0375$; $f(0.5) = -0.0375$; $f(1) = 0$; $f(1.5) = 0.1875$.

81. If a vertical line passes through two points of the graph, it is not a function. Thus, (a) and (c) are functions.

TEST FOR CHAPTER R

1. True or false:

(a) If $a > b > 0$, then $1/a > 1/b$.

(b) The interval $(2,4)$ contains 3 integers.

(c) If $a > b > 0$, then $ac > bc$ for constant c .

(d) The domain of $\sqrt{x}$ is $x \geqslant 0$.

(e) The line $x = 2$ has slope zero.

2. Express the solution set of $x^2 + 3x + 2 \geqslant 0$ in terms of absolute values.

3. Write equations for the following lines:

   (a) The line going through $(1,3)$ and $(2,4)$.

   (b) The line with slope 5 and passing through $(-3,2)$.

   (c) The line with slope -1 and y-intercept $1/2$.

4. Sketch the graph of $y = ||2x| - 2|$.

5. Do the following equations describe a circle or a parabola?

   (a) $x^2 + y^2 + 5x - 4y - 6 = 0$.

   (b) $y = x^2 - 2x + 3$.

   (c) $x + y^2 = 0$.

6. (a) Complete the square for the equation $y^2 + 4y + 3 = 0$.

   (b) Solve the equation in part (a) .

7. Sketch the graph of $y = |x^2 - 1|$.

8. (a) Find the points of intersection of the graphs of $y = x^2$ and $y = 5x - 6$.

   (b) Find the distance between the intersection points.

9. Factor the following expressions.

   (a) $x^2 + 2x - 15$

   (b) $x^3 - xy^2$

10. Dumb Donald had heard how wonderful chocolate mousse tasted, so he decided to go on a hunting trip at Moose Valley. When he finally spotted a moose, he chased the moose in circles around a tree located at $(-4,-2)$. Unfortunately for Dumb Donald, the moose ran much faster, caught up with Dumb Donald, and trampled him. Their path was radius 7 from the tree. What is the equation of the circle?

ANSWERS TO CHAPTER TEST

1. (a) False; $1/a < 1/b$.

   (b) False; 2 and 4 are excluded from the interval.

   (c) False; c must be positive for $ac > bc$.

   (d) True.

   (e) False; vertical lines do not have slopes.

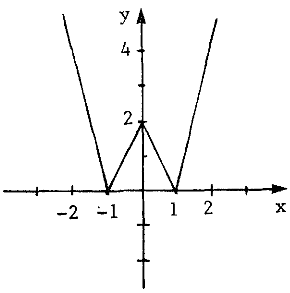
2. $|x - 3/2| \geqslant 1/2$.

3. (a) $y = x + 2$.

   (b) $y = 5x + 17$.

   (c) $y = -x + 1/2$.

4.



5. (a) Circle

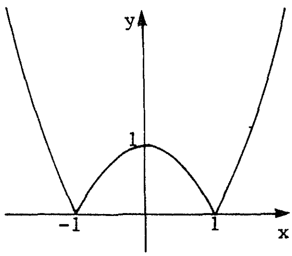
   (b) Parabola

   (c) Parabola

6. (a) $(y + 2)^2 - 1 = 0$.

   (b) $y = -3$ or $y = -1$.

7.



8. (a) $(2,4)$ and $(3,9)$

   (b) $\sqrt{26}$

9. (a) $(x + 5)(x - 3)$

   (b) $x(x + y)(x - y)$

10. $x^2 + y^2 + 8x + 4y = 29$.