CHAPTER 2

RATES OF CHANGE AND THE CHAIN RULE

2.1 Rates of Change and the Second Derivative

PREREQUISITES

1. Recall how to differentiate polynomials, products, and quotients (Sections 1.4 and 1.5).
2. Recall how velocity and slopes are related to the derivative (Section 1.1).

PREREQUISITE QUIZ

1. Differentiate:

   (a) $5x^2 + x - 6$

   (b) $(x + 3)(x^2 - 2)$

   (c) $(x - 5)/(x^2 + 2)$
2. An object's position is given by $y = x^2 - 3x + 2$. What is its velocity at $x_0 = 5$?
3. Explain how slopes are related to the derivative.

GOALS

1. Be able to relate rates of change with the derivative.
2. Be able to compute the second derivative and understand its physical meaning.

STUDY HINTS

1. Rates of change. As previously discussed, an average rate of change is $\Delta y/\Delta x$ over a finite interval, Δx . An instantaneous rate of change is simply the derivative, $f'(x)$. The derivative can represent any rate of change, i.e., a change in one quantity due to a change in another. A linear or proportional change is a special rate of change where $\Delta y = k\Delta x$ for a constant k ; this implies $f'(x)$ is constant for all x .

2. Sign of derivative. Think about the many possible interpretations of the derivative (look at Fig. I.7 on page 4). The sign indicates the direction of the change.

3. Second derivatives. This is simply the derivative of the first derivative function. If you are asked to evaluate $f''(x_0)$, do not substitute x_0 until you compute the second derivative; otherwise, your answer will be zero. Why? If x is time and $y = f(x)$ is position, the interpretation of $f''(x)$ is acceleration.

4. Leibniz notation. The second derivative is denoted d^2y/dx^2 . Note the positions of the "exponents;" this comes from writing $(d/dx)(d/dx)$.

5. Economic applications. Examples 11 and 12 introduce many new terms which are used in economics. In general, the word "marginal" implies a derivative. Ask you instructor how much economic terminology you will be held responsible for.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. If $r = \Delta y/\Delta x$ = slope and $y = y_0$ when $x = x_0$, then $y = y_0 + r(x - x_0)$. In this case, $y = 1 + 5(x - 4) = 5x - 19$.

5. The rate of change of price with respect to time is $\Delta P/\Delta t = (3.2 - 2)$ cents/(1984 - 1982) years = 0.6 cents/year. Now, $P = P_0 + (\Delta P/\Delta t)(t - t_0)$. To determine t when the price is 5 , we solve $5 = 2 + (0.6)(t - 1982)$ to get $t = 1987$. To find the price in 1991, we let $t = 1991$, and so $P = 2 + (0.6)(1991 - 1982) = 2 + 5.4 = 7.4$ cents/kilowatt-hour .

9. The average rate of change is $\Delta f(t)/\Delta t$. $\Delta f(t) = f(3/2) - f(1) = 334-364 = -30$ and $\Delta t = 3/2 - 1 = 1/2$. Thus, the average rate of change is $-30/(1/2) = -60$.

13. The area of the base with radius r is πr^2 and the height is r . Therefore, $V = \pi r^3/3$ and the rate of change of the volume with respect to the radius is $dV/dr = 3\pi r^2/3 = \pi r^2$.

17. The rate of change of H with respect to d is $H'(d) = 56 - 6d$. Hence, at $d = 0.5$, $H'(d) = 53$.

21. The rate of change of the volume with respect to the radius is $V'(r) = 4\pi r^2$, which is the surface area of the sphere.

25. By the product rule, the velocity is $(5t^4)(t + 2) + (t^5 + 1)(1) = 6t^5 + 10t^4 + 1$. Thus, the acceleration is the second derivative, $30t^4 + 40t^3$. At $t = 0.1$, the velocity is 1.00106 and the acceleration is 0.043 .

29. Applying the quotient rule twice, $(d^2/dx^2)[(x^2 + 1)/(x + 2)] = (d/dx)\{[(2x)(x + 2) - (x^2 + 1)(1)]/(x + 2)^2\} = (d/dx)[(x^2 + 4x - 1)/(x^2 + 4x + 4)] = [(2x + 4)(x^2 + 4x + 4) - (x^2 + 4x - 1)(2x + 4)]/(x^2 + 4x + 4)^2 = (10x + 20)/(x + 2)^4 = 10/(x + 2)^3$.

33. $d^2f(x)/dx^2 = d(2x)/dx = 2$.

37. $d^2y/dx^2 = (d/dx)\{[2x(x - 1) - x^2(1)]/(x - 1)^2\} = (d/dx)[(x^2 - 2x)/(x^2 - 2x + 1)] = [(2x - 2)(x^2 - 2x + 1) - (x^2 - 2x)(2x - 2)]/(x^2 - 2x + 1) = 2/(x - 1)^3$.

41. The velocity is $f'(t) = 3$ and $f'(1) = 3$ meters/second . The acceleration is $f''(t) = 0$, so $f''(1)$ is still 0 meters/second^2 .

45. The velocity is $f'(t) = -2 - 0.04t^3$ and $f'(0) = -2$ meters/second . The acceleration is $f''(t) = -0.12t^2$ and $f''(0) = 0$ meters/second^2 .

49. Marginal productivity, the derivative of the output function, is $20 - 2x$. When 5 workers are employed, marginal productivity is $20 - 2(5) = 10$. Thus, productivity would increase by 10 dollars per worker-hour.

53. This exercise is analogous to Example 12. The profit, $P(x)$, is $x(25 - 0.02x) - (4x + 0.02x^2)/(1 + 0.002x^3)$. Therefore, marginal profit is $P'(x) = 25 - 0.04x - [(4 + 0.04x)(1 + 0.002x^3) - (4x + 0.02x^2)(0.006x^2)]/(1 + 0.002x^3)^2 = [25 - 0.04x - (4 + 0.04x - 0.016x^3 + 0.00004x^4)/(1 + 0.002x^3)^2]$ dollars/boot .

57. The rate of change of y with respect to x is the price of fuel in dollars per gallon or cents per liter. Other answers are possible.

61. The average rate of change is $\Delta y/\Delta x = [y(\Delta x) - y(0)]/\Delta x = [4(\Delta x)^2 - 2(\Delta x)]/\Delta x = 4\Delta x - 2$. By the quadratic function rule, the derivative at $x_0 = 0$ is $y'(0) = -2$, where $y'(x) = 8x - 2$. The average rate of change approaches the derivative as Δx gets smaller. For $\Delta x = 0.1$, 0.001 , and 0.000001 , the average rates of change are -1.6 , -1.996 , and -1.999996 , respectively.

65. The area, A , is ℓw , so $dA/dt = \ell(t)w'(t) + \ell'(t)w(t) = (3 + t^2 + t^3)(-1 + 4t) + (2t + 3t^2)(5 - t + 2t^2) = (10t^4 + 4t^3 + 12t^2 + 22t - 3)$ cm^2/sec , which is the rate of change of area with respect to time.

69. Let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and the derivative of this is $f''(x) = 2a$. Hence, $f''(x)$ is equal to zero when $a = 0$ — that is, when $f(x)$ is a linear function $bx + c$.

73. (a) The linear equation is $V = V_0 + (\Delta V/\Delta t)t = 4000 + [(500 - 4000)/10]t = 4000 - (350)t$.

(b) The slope is $\Delta V/\Delta t = -350$ (dollars/year) .

SECTION QUIZ

1. Tell what is wrong with this statement. Suppose $f(x) = -x^2 + 3x + 6$; then $f'(x) = -2x + 3$. Evaluating, we get $f'(-1) = 5$, so $f''(-1) = 0$.

2. Compute $g''(4)$ for $g(x) = 4x - 3$ and for $g(y) = 2 - 3y^2$.

3. Compute the following derivatives:

(a) d^2u/dy^2 for $u = y^5 - y^3/2 + y$.

(b) $(d^2/dw^2)(w/3 + 3w^7/14 - 2w^4)$

(c) $g''(t)$ for $g(t) = 5t^5 + 4t^4 + 3t^3 + 2t^2 + t + 81$.

4. You and your spouse are planning to go on a werewolf hunt during the next full moon. In preparation, you do a ballistics test and determine the silver bullet's position as $6x^2 + 3x$ meters after x seconds. Determine the acceleration of the bullet if it hits the werewolf 30 meters away.

5. Careless Christina, during the excitement of her twenty-first birthday, mistakenly provided firecrackers for her birthday cake.

(a) Suppose firecrackers can expend $90x$ units of energy, where x is the number of firecrackers. If the cake can absorb 15 units of energy, write an equation relating the number of firecrackers and the net energy liberated.

(b) Differentiate the function in (a) and give a physical interpretation of the derivative.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $10x + 1$

   (b) $3x^2 + 6x - 2$

   (c) $(-x^2 + 10x + 2)/(x^2 + 2)^2$

2. 7

3. The derivative of a function gives the slope of the tangent line.

ANSWERS TO SECTION QUIZ

1. One should find $f''(x)$ before evaluating; $f''(-1) = -2$.

2. 0 for $g(x)$; -6 for $g(y)$

3. (a) $20y^3 - 3y$

   (b) $9w^5 - 24w^2$

   (c) $100t^3 + 48t^2 + 18t + 4$

4. 27 meters/(second)^2

5. (a) $y = 90x - 15$ where y = energy and x = firecrackers

   (b) 90 ; the derivative is energy/firecracker

2.2 The Chain Rule

PREREQUISITES

1. Recall how to differentiate polynomials, products, quotients, and square roots (Section 1.4 and 1.5).
2. Recall how to find limits (Section 1.2).
3. Recall how to use functional notation (Section R.6).

PREREQUISITE QUIZ

1. Differentiate the following functions with respect to x :
   (a) $\sqrt{x}/(1 + x)$
   (b) $(x^2 + x - 3)(x + 2)$
   (c) $5x^4 - x^3/3$
2. Compute the following limits:
   (a) $\lim_{\Delta x \to 0} \{[(\Delta x)^2 + \Delta x]/\Delta x\}$
   (b) $\lim_{x \to -1} [(x^2 + 4x + 3)/(x + 1)]$
3. Let $f(x) = x^2 + 2x - 4$.
   (a) Find $f(2)$.
   (b) Find $f(y)$.

GOALS

1. Be able to state and apply the chain rule.
2. Be able to use the chain rule for solving word problems.

STUDY HINTS

1. Power of a function rule . Don't bother memorizing this formula since it will soon be covered by the chain rule and the rule $(d/dx)x^n = nx^{n-1}$. Do learn how to apply it, though, as this is important preparation for the chain rule.

2. Composite function notation. Become familiar with $(f\circ g)(x)$. This is the same as $f(g(x))$.

3. Derivation of the chain rule. You will probably not be expected to know the proof of the chain rule. It is much more important to understand how to apply the result.

4. Chain rule. Memorize $(f\circ g)'(x) = f'(g(x)) \cdot g'(x)$. Practice using this formula until you feel comfortable with it. It is probably the most important differentiation formula that you will learn. DO NOT forget the last factor.

5. Leibniz notation. The chain rule demonstrates the usefulness of the Leibniz notation. Notice how the du's appear to cancel in $dy/dx = (dy/du)\cdot(du/dx)$. Remember that dy/dx is a derivative, not a fraction, but here they do behave like fractions.

6. Shifting rule. Geometrically, the shifting rule says that a horizontal displacement of a graph does not alter its slope. Don't memorize the formula. It is just a special case of the chain rule.

7. Word problems. Study Example 11 carefully. It is always a good idea to make a drawing, if possible. Many word problems will involve similar triangles. Notice how each of the rates are determined. Note also that the 8 feet did not enter into the solution of Example 11.

8. Practical application. The chain rule may be related to converting units. For example, suppose we want to convert yards/second into meters/second. Let y be length in yards, let x be length in meters, and let t be time in seconds. Then $dy/dt = (dy/dx)\cdot(dx/dt)$. Here, dy/dx is the number of yards per meter.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Apply the power of a function rule to get $(d/dx)(x + 3)^4 = 4(x + 3)^3 \times (d/dx)(x + 3) = 4(x + 3)^3$.

5. Apply the power of a function rule and the product rule to get $(d/dx)[(x^2 + 8x)^3 x] = [(d/dx)(x^2 + 8x)^3]x + (x^2 + 8x)^3(1) = 3x(x^2 + 8x)^2 \times (d/dx)(x^2 + 8x) + (x^2 + 8x)^3 = 3x(x^2 + 8x)^2(2x + 8) + (x^2 + 8x)^3 = (x^2 + 8x)^2(7x^2 + 32x)$.

9. Apply the power of a function rule and the product rule to get $(d/dy)[(y + 1)^3(y + 2)^2(y + 3)] = [(d/dy)(y + 1)^3]\cdot(y + 2)^2(y + 3) + (y + 1)^3\cdot(d/dy)[(y + 2)^2(y + 3)] = 3(y + 1)^2(y + 2)^2(y + 3)\cdot(d/dy)(y + 1) + (y + 1)^3\cdot\{[(d/dy)(y + 2)^2]\cdot(y + 3) + (y + 2)^2\cdot(d/dy)(y + 3)\} = 3(y + 1)^2(y + 2)^2(y + 3) + 2(y + 1)^3(y + 2)(y + 3) + (y + 1)^3(y + 2)^2 = (y + 1)^2(y + 2)(6y^2 + 26y + 26)$.

13. By definition, $(f\circ g)(x) = f(g(x)) = f(x^3) = (x^3 - 2)^3$. Also, $(g\circ f)(x) = g(f(x)) = g((x - 2)^3) = ((x - 2)^3)^3 = (x - 2)^9$.

17. There is no unique answer. One solution is to let $h(x) = f(g(x))$ with $f(x) = \sqrt{x}$ and $g(x) = 4x^3 + 5x + 3$. Notice that the choice of variable here differs slightly from the answer in the text.

21. We can compute $h(x) = f(g(x)) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$ and then differentiate directly to get $h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$. On the other hand, the chain rule gives $f'(g(x))\cdot g'(x) = 2u\cdot 2x = 2(x^2 - 1)(2x) = 4x(x^2 - 1)$.

25. Let $f(u) = u^3$ and $u = x^2 - 6x + 1$. Then $(d/dx)f(g(x)) = f'(g(x))\cdot g'(x)$, where $u = g(x)$. Thus, the derivative is $3u^2(2x - 6) = 6(x^2 - 6x + 1)^2(x - 3)$.

29. Let $f(x)$ denote the function. Recall that if $u = g(x)$, then $(d/dx)f(g(x)) = f'(g(x))\cdot g'(x)$. Now apply the chain rule twice. First, let $f(u) = u^2$ and $u = (x^2 + 2)^2 + 1$, so $f'(x) = 2u[(d/dx)((x^2 + 2)^2 + 1)]$. Now, let $f(u) = u^2 + 1$ and $u = x^2 + 2$, so $(d/dx)[(x^2 + 2)^2 + 1] = 2u(2x) = 2(x^2 + 2)(2x)$. Thus, $f'(x) = 2[(x^2 + 2)^2 + 1]2(x^2 + 2)(2x) = 8x(x^2 + 2)[(x^2 + 2)^2 + 1]$.

33. If $f(x)$ is the given function and $u = g(x)$, then $(d/dx)f(g(x)) = f'(g(x))\cdot g'(x)$. So let $f(u) = \sqrt{u}$ and $u = 4x^5 + 5x^2$ to get $f'(x) = (1/2\sqrt{u})(20x^4 + 10x) = 5x(2x^3 + 1)/\sqrt{4x^5 + 5x^2}$.

37. (a) It would seem reasonable that $(f\circ g\circ h)(x)$ can be defined as $[f\circ(g\circ h)](x)$. This is $f((g\circ h)(x))$, but since $(g\circ h)(x) = g(h(x))$, it becomes $(f\circ g\circ h)(x) = f(g(h(x)))$.

(b) Let $u = g(h(x))$, then the derivative of $f\circ g\circ h$ is $f'(u)\cdot u'(x)$. Applying the chain rule to u gives $u'(x) = g'(h(x))\cdot h'(x)$. Therefore, $(f\circ g\circ h)'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$.

41. The Leibniz notation is very useful here. We write $dK/dt = (dK/dv) \times (dv/dt) = mv\cdot(dv/dt)$. Since dv/dt is the acceleration, we can substitute the appropriate values, yielding $dK/dt = (10)(30)(5)$ $(\text{gram} - \text{cm}^2/\text{sec}^2)/\text{sec} = 1500 \text{ gram} - \text{cm}^2/\text{sec}^3$.

45. The velocity is the derivative of the position function. $(d/dt)((t^2 + 4)^5) = 5(t^2 + 4)^4(2t)$. At $t = -1$, the velocity is -6250.

49. If $f(x) = (x^4 + 10x^2 + 1)^{98}$, the power of a function rule gives $f'(x) = 98(x^4 + 10x^2 + 1)^{97}(4x^3 + 20x) = 392(x^3 + 5x)(x^4 + 10x^2 + 1)^{97}$. Next, the product rule along with the power of a function rule gives $f''(x) = 392(x^3 + 5x)\cdot(d/dx)(x^4 + 10x^2 + 1)^{97} + 392(x^4 + 10x^2 + 1)^{97} \times (d/dx)(x^3 + 5x) = 392(x^4 + 10x^2 + 1)^{96}[391x^6 + 3915x^4 + 53x^2 + 9700x + 5]$.

53. Applying the chain rule once gives $(d^2/dx^2)(u^n) = (d/dx)(nu^{n-1}(du/dx))$. Now use the constant multiple and product rules to get $n(d/dx)(u^{n-1}(du/dx))$, which becomes $n[(n-1)u^{n-2}(du/dx)(du/dx) + u^{n-1}(d^2u/dx^2)]$. Therefore, $(d^2/dx^2)(u^n) = nu^{n-2}[u(d^2u/dx^2) + (n-1)(du/dx)^2]$.

57. By the chain rule, $(f\circ g)'(x) = f'(g(x))\cdot g'(x)$. Then the product rule is applied: $(f\circ g)''(x) = [f''(g(x))\cdot g'(x)]g'(x) + f'(g(x))\cdot g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x)$.

SECTION QUIZ

1. Given that $f(2) = 2$, $g(2) = 4$, $f(4) = 3$, $g(4) = 5$, $f'(2) = -1$, $g'(2) = -2$, $f'(4) = 4$, $g'(4) = -4$, what is $(d/dx)f(g(x))$ at $x = 2$?

2. $(3x+1)^3 = 27x^3 + 27x^2 + 9x + 1$, so the derivative is $81x^2 + 54x + 9$. On the other hand, the power of a function rule tells us that the derivative is $3(3x+1)^2 = 3(9x^2 + 6x + 1) = 27x^2 + 18x + 3$. Why aren't the derivatives equal?

3. If $f(t) = (t+1)^2$ and $g(t) = t^2 + 1$, what is $(g\circ f)(x)$? $(f\circ g)(x)$?

4. Find dy/dx for the following functions:

   (a) $y = \sqrt{(2x-3)^3 + 1}$

   (b) $y = [(x^3+1)^4 - 5]/[2 - (3x^2 - 2)^3]$

5. Find a formula for the second derivative of $f(g(x))$ with respect to x .

6. A jack-in-the-box suddenly springs up at 10 cm/sec . It is located one meter from a lamp on the floor. The toy casts a scary shadow on the wall 3 meters from the lamp. How fast is the shadow enlarging when the jack-in-the-box has risen 5 cm. ?

7. Wrong-way Willie sometimes gets absent-minded while driving his new Porsche. He has a tendency to drive on the opposite side of the street and to drive in the wrong direction on one-way streets. It is estimated that Willie drives illegally for $0.1x$ mile in each x total miles driven. On the average, he sideswipes w^2 cars after w miles of illegal driving. It is also known that he always drives at 30 miles/hour.
   (a) Use the chain rule to compute the rate of sideswiping per mile of total driving.
   (b) Compute the rate of sideswiping per hour of total illegal driving.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $(1 - x)/2\sqrt{x}(1 + x)^2$
   (b) $3x^2 + 6x - 1$
   (c) $20x^3 - x^2$
2. (a) 1
   (b) 2
3. (a) 4
   (b) $y^2 + 2y - 4$

ANSWERS TO SECTION QUIZ

1. -8
2. In the second method, we forgot to differentiate the function within the parentheses.
3. $(g \circ f)(x) = (x + 1)^4 + 1$; $(f \circ g)(x) = (x^2 + 2)^2$.
4. (a) $3(2x - 3)^2/\sqrt{(2x - 3)^3 + 1}$
   (b) $\{12x^2(x^3 + 1)^3[2 - (3x^2 - 2)^3] + 18x(3x^2 - 2)^2[(x^3 + 1)^4 - 5]\}/$ $[2 - (3x^2 - 2)^3]^2$

5. $f''(g(x))\cdot(g'(x))^2 + f'(g(x))\cdot g''(x)$

6. 30 cm/sec

7. (a) w/10

   (b) 3w

2.3 Fractional Powers and Implicit Differentiation

PREREQUISITES

1. Recall how to differentiate rational functions, especially those which require the use of the chain rule (Section 2.2).

PREREQUISITE QUIZ

1. Differentiate $\sqrt{2x}$.
2. Differentiate $(x^2 + 5)^6$.
3. Suppose $g(t) = t^2 + 3$ and $f(x) = 2x$; what is $(d/dx)g(f(x))$?

GOALS

1. Be able to differentiate functions with fractional exponents.
2. Be able to use the method of implicit differentiation.

STUDY HINTS

1. Rational power rule. This is just an extension of the power rule for integers. The power rule is now valid for all rational numbers for which the derivative is defined.
2. Rational power of a function rule. As with the rational power rule, this is just an extension of a previously learned rule - the power of a function rule for integers.
3. Implicit differentiation. One of the common mistakes in applying this method is forgetting that y is a function of x . Thus, $(d/dx)(y^2) = 2yy'$, not just $2y$; the latter would be $(d/dy)y^2$. Be sure you understand the method of implicit differentiation. You will probably get one of these problems on your exams.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The rational power rule gives $(d/dx)(10x^{1/8}) = (5/4)x^{-7/8} = 5/4x^{7/8}$.

5. By the rational power rule, $(d/dx)(3x^{2/3} - (5x)^{1/2}) = 2x^{-1/3} - (1/2)(5x)^{-1/2}\cdot 5 = 2/x^{1/3} - 5/2\sqrt{5x}$. Actually, the rational power of a function rule was used to differentiate the second term.

9. By the rational power of a function rule, $(d/dx)[(x^5 + 1)^{7/9}] = (7/9)(x^5 + 1)^{-2/9}(5x^4) = 35x^4/9(x^5 + 1)^{2/9}$.

13. Using the quotient rule with the rational power of a function rule, we get $(d/dx)\left[\sqrt{x^2 + 1}/\sqrt{x^2 - 1}\right] = (d/dx)[(x^2 + 1)/(x^2 - 1)]^{1/2} = (1/2)[(x^2 + 1)/(x^2 - 1)]^{-1/2}\cdot[(2x)(x^2 - 1) - (x^2 + 1)(2x)]/(x^2 - 1)^2 = -2x/(x^2 + 1)^{1/2}(x^2 - 1)^{3/2}$.

17. By the quotient rule with the rational power rule, we get $(d/dx)[\sqrt{x}/(3 + x + x^3)] = [(1/2)x^{-1/2}(3 + x + x^3) - \sqrt{x}(1 + 3x^2)]/(3 + x + x^3)^2 = [(3 + x + x^3 - 2x - 6x^3)/2\sqrt{x}]/(3 + x + x^3)^2 = (3 - x - 5x^3)/2\sqrt{x}(3 + x + x^3)^2$.

21. Combining the quotient rule with the rational power rule, we get $(d/dx)[\sqrt[3]{x}/(x^2 + 2)] = [(1/3)x^{-2/3}(x^2 + 2) - \sqrt[3]{x}(2x)]/(x^2 + 2)^2 = [(x^2 + 2 - 6x^2)/3x^{2/3}]/(x^2 + 2)^2 = (2 - 5x^2)/3x^{2/3}(x^2 + 2)^2$.

25. By the rational power of a function rule, $(d/dx)(x^2 + 5)^{7/8} = (7/8)(x^2 + 5)^{-1/8}(2x) = 7x/4(x^2 + 5)^{1/8}$.

29. The sum rule and the rational power rule give $f'(x) = (3/11)x^{-8/11} - (1/5)x^{-4/5}$.

33. Applying the rational power of a function rule gives $\ell'(x) = (1/2)[(x^{1/2} + 1)/(x^{1/2} - 1)]^{-1/2}[(1/2)x^{-1/2}(x^{1/2} - 1) - (x^{1/2} + 1)(1/2)x^{-1/2}]/(x^{1/2} - 1)^2 = -1/2\sqrt{x}(\sqrt{x} - 1)^{3/2}(\sqrt{x} + 1)^{1/2}$.

37. (a) Differentiating with respect to x, we get $4x^3 + 2y(dy/dx) + dy/dx = 0$, i.e., $(dy/dx)(2y + 1) = -4x^3$. Hence $dy/dx = -4x^3/(2y + 1)$.

(b) $(dy/dx)|_{x=1,y=1} = (-4)/(2 + 1) = -4/3$.

(c) $y^2 + y + (x^4 - 3) = 0$ implies $y = [-1 \pm \sqrt{1 - 4(1)(x^4 - 3)}]/2 = [-1 \pm \sqrt{13 - 4x^4}]/2$, so $dy/dx = \pm[1(-16x^3)/4\sqrt{13 - 4x^4}] = \pm[4x^3/\sqrt{13 - 4x^4}]$. Note that $2y + 1 = -1 \pm \sqrt{13 - 4x^4} + 1 = \pm\sqrt{13 - 4x^4}$. Hence $dy/dx = -4x^3/(2y + 1)$. The answer checks.

41. Using implicit differentation, $4x^3 + 4y^3(dy/dx) = 0$. When $x = y = 1$, we get $4 + 4(dy/dx) = 0$ or $dy/dx = -1$. The equation of the tangent line is $y = 1 + (-1)(x - 1) = -x + 2$.

45. $dy/dx = -2x/2\sqrt{1 - x^2} = -x/\sqrt{1 - x^2}$, so $(dy/dx)|_{x = \sqrt{3}/2} = -(\sqrt{3}/2)/(1/2) = -\sqrt{3}$. Thus, the equation of the tangent line is $y = 1/2 - \sqrt{3}(x - \sqrt{3}/2)$ or $y = -\sqrt{3}x + 2$.

49. Use the linear approximation $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. Let $f(x) = x^{1/4}$, so $f'(x) = (1/4)x^{-3/4}$, and so $f(15.97) = f(16 - 0.03) \approx f(16) + f'(16)(-0.03) = 2 + (1/4)(1/8)(-0.03) = 2 - 0.03/32 = 1.9990625$.

53. Applying the rational power rule with the chain rule yields $dM/dx = 24(4)(2 + x^{1/3})^3(1/3)x^{-2/3} = [32(2 + x^{1/3})^3/x^{2/3}]$ kg/unit distance.

57. The rate of change of period with respect to tension is $dP/dT = (1/2)(32/T)^{-1/2}(-32/T^2) = -2\sqrt{2}/T^{3/2}$; at $T = 9$, $dP/dT = -2\sqrt{2}/27$ seconds/pound.

SECTION QUIZ

1. $x^2y + 2y = (x + y)^3 - 5$ describes a differentiable curve. Find the tangent line to the curve at $(1,1)$.

2. Find dy/dx if $y = (x^{1/2} + x^{1/3})/(1 + \sqrt{x})^{2/3}$.

3. Find d^2f/dw^2 if $f = (3w^{1/3} - \sqrt{w^3})/w^{2/5}$.

4. For the curve in Question 1, compute d^2y/dx^2 at $(1,1)$.

5. Four eyes Frankie could hardly see even with his glasses on. Thus, when he got into his motor boat, he steered it along the path described by $(x^2y + 4)^{3/2} = 22x/\sqrt{y + 2} + 2x^3 + 11x^{1/2}y^2/2$. He kept this up for an hour until he lost control at $(4,2)$ and went sailing off along the tangent line. What is this tangent line?

ANSWERS TO PREREQUISITE QUIZ

1. $1/\sqrt{2x}$

2. $12x(x^2 + 5)^5$

3. $8x$

ANSWERS TO SECTION QUIZ

1. $9y + 10x = 19$

2. $[(3/2 + \sqrt{x}/\sqrt[3]{x^2})(1 + \sqrt{x}) - (\sqrt{x} + \sqrt[3]{x})]/3\sqrt{x}(1 + \sqrt{x})^{5/3}$.

3. $48/225w^{31/15} - 11/100w^{9/10}$

4. $-70/81$

5. $211y + 207x = 1250$

2.4 Related Rates and Parametric Curves

PREREQUISITES

1. Recall how to differentiate by using the chain rule (Section 2.2).

PREREQUISITE QUIZ

1. Differentiate $\sqrt{x^2 - 3}$.
2. Differentiate $(x - 3)^5(x^2 + 1)^4$.

GOALS

1. Be able to solve related rates word problems.
2. Be able to calculate the slope of a parametric curve.
3. Be able to sketch simple parametric curves.

STUDY HINTS

1. Related rates. Don't forget to use the chain rule when you differentiate x and y with respect to t . You may want to substitute $x = f(t)$ and $y = g(t)$ to help remind yourself to use the chain rule.
2. Parametric curves. Two curves may appear to be related by the same equation and yet they are not the same curve. For example, $x = t^6$ and $y = t^3$ are related by $y^2 = x$, which is an entire parabola; y may take on negative values. Compare this with Example 2.
3. Finding an xy-relationship. Sometimes it is useful to find an equation involving only x and y when you are asked to sketch a parametric curve. This can often be done by solving for t in one equation and substituting into the other.

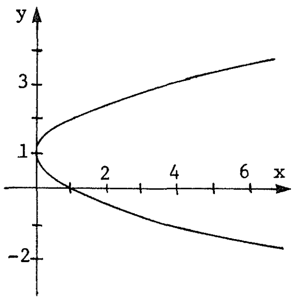
4. Slopes of parametric curves. Remember that $dy/dx = (dy/dt)/(dx/dt)$. As with the chain rule, the dt's appear to cancel, but remember they are not really fractions.

5. Word problems. Many word problems involve related rates. Draw a picture, if possible. Look for a relationship between the variables. Sometimes you will have to derive a relationship as in Example 6. Differentiate both sides of your relationship with respect to time. Finally, substitute in the given values. A few minutes spent in studying Examples 6, 7, and 8 should prove worthwhile.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Differentiate x and y with respect to t as in Example 1 to get $2x(dx/dt) - 2y(dy/dt) = 0$, so $2y(dy/dt) = 2x(dx/dt)$ or $dy/dt = (x/y)(dx/dt)$.

5. Differentiation of x and y with respect to t yields $dx/dt + 2y(dy/dt) = dy/dt$, so $(1 - 2y)(dy/dt) = dx/dt$ or $dy/dt = (dx/dt)/(1 - 2y)$.

9. 

$y = 1 - t$ implies $t = 1 - y$, so the curve is $x = (1 - y)^2$, which is a parabola symmetric about $y = 1$.

13. The slope of the tangent line is $(dy/dt)/(dx/dt) = 3t^2/2t = 3t/2$. Thus, at $t = 5$, the slope is $15/2$, $x = 25$, and $y = 125$. Hence, the equation of the line is $y = 125 + (15/2)(x - 25)$ or $y = (15x - 125)/2$.

17. Differentiating with respect to t, we get $(dx/dt)y + (dy/dt)x = 0$. Substituting $x = 8$ and $y = 1/2$ yields $(1/2)(dx/dt) + 8(dy/dt) = 0$, and so $dy/dt = (1/2)(dx/dt)/(-8) = -(dx/dt)/16$.

21. Let $r(t)$ be the radius at time t and let $h(t)$ be the height at time t. We have volume = $V = 1000 = \pi(r(t))^2h(t)$. Differentiating with respect to t, we get $0 = \pi[2r(t)r'(t)h(t) + (r(t))^2h'(t)]$. At the instance when $r(t) = 4$, $r'(t) = 1/2$ and $h(t) = 1000/\pi(4)^2 = 62.5/\pi$. Substituting in all of these known values, we get $0 = \pi[2(4)(1/2)(62.5/\pi) + (4)^2h'(t)]$, i.e., $h'(t) = -250/16\pi = -(125/8\pi)$ cm/sec.

25. (a) By the distance formula, we have $\sqrt{x^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$. Squaring and rearranging yields $3x^2 + 3(y - 4/3)^2 = 4/3$, which is a circle centered at $(0,4/3)$ with radius $2/3$.

(b) We want to know dy/dt at $(0,2/3)$. By implicit differentiation, we have $6x(dx/dt) + 6(y - 4/3)(dy/dt) = 0$. At $(0,2/3)$, the equation is $4(dy/dt) = 0$, so $dy/dt = 0$.

(c) Rearranging the equation in part (b), we get $(dy/dt)/(dx/dt) = 1 = -x/(y - 4/3)$; therefore, we need $x = 4/3 - y$. Substitute into the equation in part (a): $3(4/3 - y)^2 + 3(y - 4/3)^2 = 6(y - 4/3)^2 = 4/3$, so $y - 4/3 = \pm 2/3\sqrt{2}$. Therefore, $y = (\pm\sqrt{2} + 4)/3$, and the points are $(\sqrt{2}/3,(-\sqrt{2} + 4)/3)$ and $(-\sqrt{2}/3,(\sqrt{2} + 4)/3)$.

29. Let ℓ and w denote the length of the rectangle's sides. We want to know what $d\ell/dw$ or $dw/d\ell$ is when $\ell = w = 5$. Differentiating $\ell w = 25$ with respect to ℓ yields $w + \ell(dw/d\ell) = 0$ or $dw/d\ell = -w/\ell = -5/5 = -1$. On the other hand, differentiating with respect to w, $(d\ell/dw)w + \ell = 0$ yields $-\ell/w = d\ell/dw = -5/5 = -1$.

33. Let y be the rainfall rate, R be the radius of the tank, and H be the height of the tank. The other variables have the same meaning as in Example 8. dV/dt becomes $\pi R^2 y$, $r/h = R/H$, and $r = Rh/H$. Therefore, $V = \pi r^2 h/3 = \pi(Rh/H)^2 h/3 = \pi R^2 h^3/3H^2$. Differentiation yields $dV/dt = \pi R^2 h^2 (dh/dt)/H^2 = \pi R^2 y$; therefore, $(dh/dt)/y = H^2/h^2$. But from $r/h = R/H$, we get $R^2/r^2 = H^2/h^2$ or $H^2/h^2 = \pi R^2/\pi r^2 = (dh/dt)/y$, which is the desired result.

SECTION QUIZ

1. A curve is described by $x = t^4$, $y = t^2$ and another curve is described by $x = t^6$, $y = t^3$. Sketch the two curves.

2. (a) If $x = t^3 - 2t^2$ and $y = t^2 - 4$, what is dy/dx whenever the curve crosses the x-axis?

   (b) At what points is the tangent line horizontal?

   (c) At what points is the tangent line vertical?

3. (a) Airbelly Alice just got a job perfectly suited for her rotund tummy. Her new job is blowing up balloons for the circus. If the spherical balloon inflates at a rate of 5 cc/min., how fast is the diameter increasing when the radius is $\sqrt{5}$ cm. ?

   (b) Airbelly Alice's tummy has the shape of a circular cylinder. If all of the air used for blowing up the balloons comes from her belly which is 10 cm. high, how fast is Alice's waistline decreasing when her tummy is 15 cm. in radius?

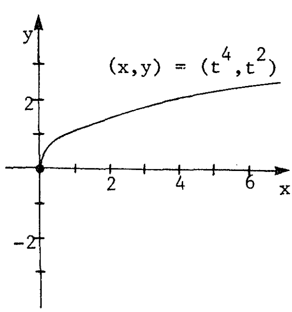
4. Queer Mr. Q, who enjoys giraffeback riding, needed a new fence to prevent his giraffe from running away. He ordered the fence installer to put up the fence according to the following specifications: For $-1 \leqslant t \leqslant 3$, $x = t - 1$ and $y = 5 - t^2 - 2t$. Then for $3 \leqslant t \leqslant 7$, $x = 5 - t$ and $y = (5 - t)^4 - 14$.
   (a) Suppose at $t = 4$, the giraffe runs away along the tangent line. What path does it follow?
   (b) The frightened fence installer ran off at a perpendicular at $t = 4$. He is sprinting at 9 kilometers/hour , while the giraffe is running at 12 kilometers/hour . How fast is their distance increasing after 15 minutes?
   (c) Make a sketch of the completed giraffe pen.

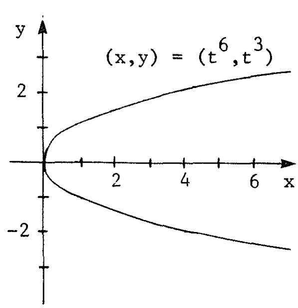
ANSWERS TO PREREQUISITE QUIZ

1. $x/\sqrt{x^2 - 3}$
2. $(x - 3)^4(x^2 + 1)^3(13x^2 - 24x + 5)$

ANSWERS TO SECTION QUIZ

1.





2. (a) 1 when $t = 2$ and $-1/5$ when $t = -2$.

(b) At no points

(c) $(-32/27, -20/9)$ when $t = 4/3$.

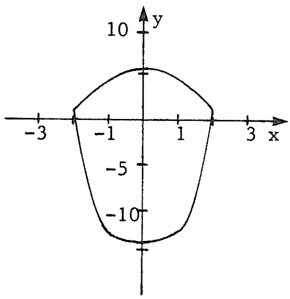
3. (a) $1/2\pi$ cm/min

(b) The radius decreases at $1/60\pi$ cm/min., so her waistline is decreasing at $1/30$ cm/min .

4. (a) $y - 4x + 17 = 0$

(b) 15 km/hr

(c)



2.5 Antiderivatives

PREREQUISITES

1. Recall how to differentiate a polynomial (Section 1.4).
2. Recall how to differentiate a composite function (Section 2.2).
3. Recall how position and velocity are related by the derivative (Section 1.1).

PREREQUISITE QUIZ

1. Differentiate $x^{48} - 5x^5 + x^3 - 3x + 25$.
2. What is $(d/dx)f(g(x))$?
3. Differentiate $(3x + 2)^4$.
4. Suppose $y = 3x^3 - 2x^2 + 4x - 4$ describes a particle's position y at time x .
   (a) What is dy/dx ?
   (b) What is the physical interpretation of dy/dx ?

GOALS

1. Be able to find antiderivatives for polynomials and simple composite functions.
2. Be able to interpret the meaning of an antiderivative.

STUDY HINTS

1. Antiderivatives. Remember that an antiderivative is not unique unless an extra condition is given. Always remember to include the arbitrary constant. It is a common mistake to forget the arbitrary constant.
2. Power rule. $n = -1$ is excluded because the antiderivative would require division by 0 .

3. Polynomial rule. This rule incorporates the sum rule, the constant multiple rule, and the power rule. You should learn the basic parts well and be able to derive the polynomial rule by yourself.

4. Antidifferentiating composite functions. A systematic method will be introduced in Chapter 7. For now, think of the quantity inside the parenthesis as a single variable when you guess an antiderivative. Then differentiate as in Example 7.

5. Physical interpretation. To help you understand Example 11, recall that differentiation yields a rate. Antidifferentiation will yield the original function. Therefore, antidifferentiating the water flow rate should give the total amount of water.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Apply the polynomial rule for antidifferentiation to get $F(x) = x^2/2 + 2x + C$.

5. Apply the power rule for antidifferentiation to get $F(t) = t^{-3+1}/(-3 + 1) + C = -1/2t^2 + C$.

9. By the result of Example 10, the position function is $F(t) = \int v dt$. By the polynomial rule for antidifferentation, $F(t) = 4t^2 + 2t + C$. Therefore, $F(0) = 0$ implies $C = 0$, and so $F(1) = 4(1)^2 + 2(1) + 0 = 6$.

13. Using the polynomial rule for antidifferentiation, $F(x) = (3/2)x^2 + C$.

17. Use the formula $\int (ax + b)^n dx = (ax + b)^{n+1}/a(n + 1) + C$. Here, $a = 1$ and $b = 1$, so the general antiderivative is $F(x) = 2(x + 1)^{3/2}/3 + C$.

21. The acceleration is 9.8 near the earth's surface, so $v = 9.8t + C$, which is v_0 at $t = 0$. Thus, $v = 9.8t + v_0$, and the position function becomes $x = 4.9t^2 + v_0t + D$. $x = x_0$ at $t = 0$, so $x = 4.9t^2 + v_0t \quad x_0$. Since $v_0 = 1$, we have $v = 9.8t + 1$, and since $x_0 = 2$, $x = 4.9t^2 + t + 2$.

25. It is not true. For a counterexample, take $f(x) = x$ and $g(x) = 1$. Then $\int f(x)g(x)dx = \int xdx = x^2/2$. (For simplicity, let all constants be $C = 0$.) Now $\int f(x)\,dx = x^2/2$ and $\int g(x)dx = x$, so $[\int f(x)dx]g(x) + f(x)[\int g(x)dx] = (x^2/2)(1) + x(x) = (3/2)x^2$. This is not equal to $\int f(x)g(x)dx = x^2/2$.

29. By the polynomial rule for antidifferentiation, $\int(x^2 + 3x + 2)dx = x^3/3 + 3x^2/2 + 2x + C$.

33. Using $\int(ax + b)^n dx = (ax + b)^{n+1}/a(n + 1) + C$, $F(t) = (8t + 1)^{-1}/8(-1) + C = -1/8(8t + 1) + C$.

37. Using the polynomial rule for antidifferentiation, $\int(1/x^4 + x^4)dx = \int(x^{-4} + x^4)dx = x^{-3}/(-3) + x^5/5 + C = -1/3x^3 + x^5/5 + C$.

41. By using the polynomial rule for antidifferentiation, $\int(x^3 + 3x)dx = x^4/4 + 3x^2/2 + C$.

45. Use the formula $\int(ax + b)^n dx = (ax + b)^{n+1}/a(n + 1) + C$ to get $\int(8x + 3)^{1/2}dx = (8x + 3)^{3/2}/8(3/2) + C = (8x + 3)^{3/2}/12 + C$.

49. Simplification gives $\int[(\sqrt{x - 1} + 3)/(x - 1)^{1/2}]\,dx = \int[1 + 3(x - 1)^{-1/2}]\,dx$. Using the sum rule for antidifferentiation and $\int(ax + b)^n dx = (ax + b)^{n+1}/a(n + 1) + C$, the antiderivative is $x + 3(x - 1)^{1/2}/(1/2) + C = x + 6\sqrt{x - 1} + C$.

53. From Example 4, $x = 4.9t^2 + v_0t + x_0$ where $v_0 = 10$ meters/sec and $x_0 = 0$, i.e., $x = 4.9t^2 + 10t$. We want to find t such that $x = 150 = 4.9t^2 + 10t$. Using the quadratic formula in solving $4.9t^2 + 10t - 150 = 0$, we find $t = \left[-10 + \sqrt{10^2 + 4(150)(4.9)}\right]/2(4.9) = 4.6$ sec.

57. From Example 4, we have the formula $x = 4.9t^2 + v_0t + x_0$, where v_0 is the downward velocity which is -19.6 for this problem. We want the time when $x = x_0$, so we solve $0 = 4.9t^2 - 19.6t = t(4.9t - 19.6)$. This has solutions 0 and 4, but 0 does not make sense, so $t = 4$ seconds.

61. (a) By the power of a function rule, $(d/dx)(x^4 + 1)^{20} = 20(x^4 + 1)^{19}(4x^3) = 80(x^4 + 1)^{19}x^3$.

(b) By the sum rule for antidifferentiation and part (a), the integral is $(x^4 + 1)^{20}/80 + 9x^{5/3}/5 + C$.

65. By the polynomial rule for antidifferentiation, $F(x) = x^4/4 + x^3 + 2x + C$. $F(0) = 1$ implies that $C = 1$, so $F(x) = x^4/4 + x^3 + 2x + 1$.

SECTION QUIZ

1. Calculate the following antiderivatives:

(a) $\int(x + 3)(x + 1)dx$

(b) $\int[(x^3 - 3x^2)/x^{3/2}]\,dx$

(c) $\int(-2t - 5)dt$

(d) $\int -389\ dy$

2. (a) Differentiate $(x^4 + 4x)^3$.

(b) Find the antiderivative $F(z)$ of $f(z) = 2(z^4 + 4z)^2(z^3 + 1)$ such that $F(0) = 5$.

3. Evaluate $\int(3t + 7)^5dt$.

4. A rich stranger has just dropped his gold plated credit card into a tank of lobsters. Fearful of being pinched, he hires you to reach in and retrieve his credit card. He offers you \$1,000, but his daddy always told him, "Time is money." Thus, he will decrease your pay at a rate of 50t dollars per minute, i.e., after x minutes, you will lose $\int_0^x 50t\,dt$ dollars. How much time can you use to retrieve the card and still earn \$800 ?

ANSWERS TO PREREQUISITE QUIZ

1. $48x^{47} - 25x^4 + 3x^2 - 3$

2. $f'(g(x))\cdot g'(x)$

3. $12(3x + 2)^3$

4. (a) $9x^2 - 4x + 4$

   (b) Velocity

ANSWERS TO SECTION QUIZ

1. (a) $x^3/3 + 2x^2 + 3x + C$

   (b) $2x^{5/2}/5 - 2x^{3/2} + C$

   (c) $-t^2 - 5t + C$

   (d) $-389y + C$

2. (a) $12(x^4 + 4x)^2(x^3 + 1)$

   (b) $F(z) = (z^4 + 4z)^3/6 + 5$

3. $(3t + 7)^6/18 + C$

4. $\sqrt{8}$ minutes

2.R Review Exercises for Chapter 2

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Apply the power of a function rule to get $3(6x + 1)^2 \cdot 6 = 18(6x + 1)^2$.

5. Using the power rule, $(d/dx)(6/x) = (d/dx)(6x^{-1}) = -6/x^2$.

9. Combine the quotient and chain rules to get $[13(x^2 + 1)^{12}(2x)(x^2 - 1)^{14} - (x^2 + 1)^{13}(14)(x^2 - 1)^{13}(2x)]/(x^2 - 1)^{28} = (-2x^3 - 54x)(x^2 + 1)^{12}/(x^2 - 1)^{15}$.

13. By using the quotient rule, the derivative is $[A'(x)D(x) - A(x)D'(x)]/[D(x)]^2 = [(3x^2 - 2x - 2)(x^2 + 8x + 16) - (x^3 - x^2 - 2x)(2x + 8)]/(x^2 + 8x + 16)^2 = [(3x^2 - 2x - 2)(x + 4) - (x^3 - x^2 - 2x)(2)]/(x + 4)^3 = (x^3 + 12x^2 - 6x - 8)/(x + 4)^3$.

17. Recall that the tangent line is $y = f(x_0) + f'(x_0)(x - x_0)$. By the power of a function rule, $f'(x) = (1/3)[A(x)]^{-2/3}A'(x) = (x^3 - x^2 - 2x)^{-2/3} \times (3x^2 - 2x - 2)/3$, so $f'(1) = -(1/3)(1/4)^{1/3} = -\sqrt[3]{2}/6$. Also, $f(1) = -1$, so the tangent line is $y = -1 - \sqrt[3]{2}(x - 1)/6$.

21. By the rational power rule, $f'(x) = (5/3)x^{2/3}$.

25. Applying the quotient and rational power rules gives $f'(x) = [(3/2)x^{1/2}(1 - x^{3/2}) - (-3/2)x^{1/2}(1 + x^{3/2})]/(1 - x^{3/2})^2 = 3\sqrt{x}/(1 - x^{3/2})^2$.

29. Use the quotient rule to get $f'(x) = [(1)(x^2 + 2bx + c) - (x - a)(2x + 2b)]/(x^2 + 2bx + c)^2 = (-x^2 + 2ax + c + 2ab)/(x^2 + 2bx + c)^2$; $f''(x) = [(-2x + 2a)(x^2 + 2bx + c)^2 - (-x^2 + 2ax + c + 2ab)(2) \times (x^2 + 2bx + c)(2x + 2b)]/(x^2 + 2bx + c)^4 = [(-2x + 2a)(x^2 + 2bx + c) - (-x^2 + 2ax + c + 2ab)\cdot(4x + 4b)]/(x^2 + 2bx + c)^3 = [2x^3 - 6ax^2 - (6c + 12ab)x + (2ac - 8ab^2 - 4bc)]/(x^2 + 2bx + c)^3$.

33. Combine the sum rule, the power rule, and the quotient rule to get $h'(r) = 13r^{12} - 4\sqrt{2}r^3 - [(1)(r^2 + 3) - (r)(2r)]/(r^2 + 3)^2 = 13r^{12} - 4\sqrt{2}r^3 - (-r^2 + 3)/(r^2 + 3)^2$; $h''(r) = 156r^{11} - 12\sqrt{2}r^2 - [(-2r)(r^2 + 3)^2 - (-r^2 + 3)(2)(r^2 - 3)(2r)]/(r^2 + 3)^4 = 156r^{11} - 12\sqrt{2}r^2 - [(-2r)(r^2 + 3) - (-r^2 + 3)(4r)]/(r^2 + 3)^3 = 156r^{11} - 12\sqrt{2}r^2 - (2r^3 - 18r)/(r^2 + 3)^3$.

37. Apply the power of a function rule with the product rule to get $h'(x) = 4(x - 2)^3(x^2 + 2) + (x - 2)^4(2x) = 2(x - 2)^3(3x^2 - 2x + 4)$; $h''(x) = 6(x - 2)^2(3x^2 - 2x + 4) + 2(x - 2)^3(6x - 2) = 2(x - 2)^2 \times (15x^2 - 20x + 16)$.

41. Mathematically, the first statement says: $dV/dt = kS$, where V is the volume, k is the proportionality constant and S is the surface area, $4\pi r^2$. By the chain rule, $dV/dt = (dV/dr)(dr/dt) = kS$. Since $V = 4\pi r^3/3$, we have $4\pi r^2(dr/dt) = k(4\pi r^2)$, which simplifies to $dr/dt = k$.

45. Denote the length of the legs by a and b , so the perimeter is $P = a + b + \sqrt{a^2 + b^2}$. Differentiate with respect to time: $dP/dt = da/dt + db/dt + (1/2)(a^2 + b^2)^{-1/2}(2da/dt + 2db/dt)$. At the moment in question, $a = b$, and $10^{-6} = (1/2)ab$, so $a = \sqrt{2} \cdot 10^{-3} = b$. Also, since the area is constant, one leg is decreasing its length while the other increases. Thus, $dP/dt = -10^{-4} + 10^{-4} + (1/2)(4 \cdot 10^{-6})^{-1/2} \times [(2)(-10^{-4}) + (2)(10^{-4})] = 0$.

49. Using the figure, we have $A = (\text{side})^2 - (1/2)(\text{base})(\text{height}) = (5x)^2 - (1/2)(2x)(3x) = 22x^2$. Therefore, $dA/dx = 44x$ and $d^2A/dx^2 = 44$.

53. Using the Pythagorean theorem, the hypotenuse of the triangle has length $13\sqrt{x}$. Then, the perimeter is $4(5x) + 3x + 2x + \sqrt{13}x =$ $25x + \sqrt{13}x$. Solve for x and substitute into A from Exercise 49. $x = P/(25 + \sqrt{13})$ implies $A = 22P^2/(25 + \sqrt{13})^2$. Therefore, $dA/dP = 44P/(25 + \sqrt{13})^2$ and $dP/dx = 15 + \sqrt{13}$.

57. (a) Marginal cost is defined as $dC/dx = [5 - (0.02)x]$ dollars/case.

(b) $(dC/dx)|_{84} = 5 - (0.02)(84) = \$3.32$.

(c) According to part (a), marginal cost is a linear function with slope -0.02 , a decreasing function of x .

(d) It is unreasonable for total cost to be less than or equal to zero. The quadratic formula, applied to $C(x)$, results in $x \approx 503.97$; therefore, when $x \geqslant 504$, the formula cannot be applicable.

61. The quotient rule gives $f'(x) = [(3x^2)(x^3 + 11) - (x^3 - 7)(3x^2)]/$ $(x^3 + 11)^2 = 54x^2/(x^3 + 11)^2$, which is $54(2)^2/((2)^3 + 11)^2 =$ $216/361$ at $x_0 = 2$. Thus, the tangent line is $y = y_0 +$ $f'(x_0)(x - x_0) = 1/19 + (216/361)(x - 2)$.

65. The tangent line is given by $y = y(2) + (dy/dx)|_{t=2}(x - x(2))$, where $dy/dx = (dy/dt)/(dx/dt)$. $y(2) = 1 + \sqrt[3]{2} + 2 = 3 + \sqrt[3]{2}$; $x(2) = \sqrt{2} + 4 + 1/2 = 9/2 + \sqrt{2}$; $(dy/dt)/(dx/dt) = [(1/3)t^{-2/3} + 1]/$ $[(1/2)t^{-1/2} + 2t - t^{-2}]$, and at $t = 2$, $dy/dx = [1/3(4)^{1/3} + 1]/$ $[1/2\sqrt{2} + 15/4]$. Therefore, the tangent line is $y = (3 + \sqrt[3]{2}) +$ $[(1 + 3\sqrt[3]{4})\ 4\sqrt{2}/3\sqrt[3]{4}(15\sqrt{2} + 2)](x - 9/2 - \sqrt{2})$.

69. (a) The linear approximation is given by $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. $f'(x) = [(40x^{39})(x^{29} + 1) - (x^{40} - 1)(29x^{28})]/(x^{29} + 1)^2$ and $f'(1) = [(40)(2) - (0)(29)]/(2)^2 = 20$. Also, $f(1) = 0/2 = 0$. Therefore, the linear approximation to $(x^{40} - 1)/(x^{29} + 1)$ at $x_0 = 1$ is $20\Delta x$.

(b) x_0 and the function are the same as in part (a) . $\Delta x = 0.021$, so the approximate value is $20(0.021) = 0.42$.

73. Applying the power of a function rule once gives the derivative as $n[f(x)^m]^{n-1}(d/dx)[f(x)^m]$. Applying the rule again gives $n[f(x)^m]^{n-1}m[f(x)^{m-1}]f'(x) = nm[f(x)^{mn-m+m-1}]f'(x) = nm[f(x)^{mn-1}]f'(x)$. Applying the rule to the right-hand side gives $mn[f(x)^{mn-1}]f'(x)$, which is the same.

77. The polynomial rule for antidifferentiation gives $\int(4x^3 + 3x^2 + 2x + 1)dx = 4x^4/4 + 3x^3/3 + 2x^2/2 + x + C = x^4 + x^3 + x^2 + x + C$.

81. This simplifies to $\int(-x^{-2} - 2x^{-3} - 3x^{-4} - 4x^{-5})dx$. The sum and power rules for antidifferentiation may now be applied to get $-x^{-1}/(-1) - 2x^{-2}/(-2) - 3x^{-3}/(-3) - 4x^{-4}/(-4) + C = 1/x + 1/x^2 + 1/x^3 + 1/x^4 + C$.

85. The sum and power rules for antidifferentiation gives $\int(x^{3/2} + x^{-1/2})dx = x^{5/2}/(5/2) + x^{1/2}/(1/2) + C = 2x^{5/2}/5 + 2\sqrt{x} + C$.

89. Use the formula $\int(ax + b)^n dx = (ax + b)^{n+1}/a(n+1) + C$ to get $\int\sqrt{x - 1}dx = (x - 1)^{3/2}/(3/2) + C = 2(x - 1)^{3/2}/3 + C$.

93. Apply the sum rule for antiderivatives along with the formula $\int(ax + b)^n dx = (ax + b)^{n+1}/a(n + 1) + C$. This gives $\int[(x - 1)^{1/2} - (x - 2)^{5/2}]dx = (x - 1)^{3/2}/(3/2) - (x - 2)^{7/2}/(7/2) + C = 2(x - 1)^{3/2}/3 - 2(x - 2)^{7/2}/7 + C$.

97. The use of the chain rule gives $f'(x) = (1/2)x^{-1/2} - (1/2)[(x-1)/(x+1)]^{-1/2}[(x+1)-(x-1)]/(x+1)^2 = 1/2\sqrt{x} - (1/2)[(x+1)/(x-1)]^{1/2}(2)/(x+1)^2 = 1/2\sqrt{x} - 1/(x+1)^{3/2}\sqrt{x-1}$. By the definition of antiderivatives, $\int[1/2\sqrt{x} - 1/(x+1)^{3/2}\sqrt{x-1}]dx = \sqrt{x} - \sqrt{(x-1)/(x+1)} + C$.

101. The use of the chain rule gives $f'(x) = (1/2)[(x^2+1)/(x^2-1)]^{-1/2} \times [(2x)(x^2-1) - (x^2+1)(2x)]/(x^2-1)^2 = (1/2)[(x^2-1)/(x^2+1)]^{1/2}(-4x)/(x^2-1)^2 = -2x/(x^2-1)^{3/2}\sqrt{x2+1}$. By the definition of antiderivatives, $\int[-2x/(x^2-1)^{3/2}\sqrt{x^2-1}]dx = [(x^2+1)/(x^2-1)]^{1/2} + C$.

105. By the chain rule, we have $dD/dt = (dD/dv)\cdot(dv/dt) = 7(12) = 84$ pounds/second.

109. The proof is by induction on k. For $k = 1$, if $r'(x) = 0$, then r is constant by Review Exercise 108. Hence, it is a polynomial.

Suppose the statement is true for $k-1$. If $0 = (d^k/dx^k)(r(x)) = (d^{k-1}/dx^{k-1})(r'(x))$, then $r'(x)$ is a polynomial by the induction hypothesis. Let $g(x)$ be a polynomial such that $g'(x) = r'(x)$ (by the antiderivative rule for polynomials). Then $(g-r)'(x) = 0$, so $g - r$ is constant, and hence, r is a polynomial.

TEST FOR CHAPTER 2

1. True or false:

(a) If $f'(x)$ exists, then $f''(x)$ also exists.

(b) The parametric equations $x = t^3$ and $y = t^3 + 8$ describe a straight line.

(c) If $y = 3x^2 + 2x$, then $d^2y/dx^2 = (6x+2)^2$.

(d) For differentiable functions f and g, the second derivative of $f + g$ is $f'' + g''$.

(e) The curve described by $x = t^3 - 3$ and $y = -2t^3 + 2$ has a constant slope.

2. Find dy/dx in each case:

(a) $xy^2 = 2y/x + 2$

(b) $3xy - \sqrt{xy} = y + x$

(c) $x(y + 3) + y/x - y^3 = y(x + 3)$

3. In each case, find dy/dx in terms of t :

(a) $x = 9t^3 + 8$, $y = 7t^2 - 8$

(b) $x = t^{3/2}$, $y = 8$

(c) $y = (t + 4)^2$, $x = \sqrt{t^2 - 3}$

4. Find a general formula for the second derivative of f/g . Assume f , f' , g , and g' are differentiable and $g \neq 0$.

5. Let $F(x)$ be a cubic function. If $F(-1) = 3$, $F'(0) = 3$, $F''(1) = 3$, and $F'''(3) = 3$, what is $F(x)$?

6. Differentiate $((1 + y^2)^3 + 1)^{-1/2}$ with respect to y .

7. Compute the second derivatives of the following functions:

(a) $f(x) = x^6 - 5x^3 + 3$

(b) $f(x) = -x + 6$

(c) $f(x) = (x + 4)(x^2 + 2)$

(d) $f(x) = 1/(3 - x)$

8. Suppose a square's side is increasing by 5 cm/sec. How fast is the area increasing when the length of a side is 10 cm. ?

9. $y = (x^{3/2} + 1)^{3/4}$ is a particle's position. What is the acceleration at time x ?

10. You're at the top of a 26 m ladder painting an office building. The other end of the ladder is being held by your partner who is 10 m from the building. At precisely 3 PM, your partner runs off for his coffee break leaving you to fall with the ladder at 50m/min. How fast is the other end of the ladder moving when you are halfway to breaking your bones?

ANSWERS TO CHAPTER TEST

1. (a) False; suppose $y = x^2$ if $x \leqslant 0$ and $y = 0$ if $x \geqslant 0$.

   (b) True

   (c) False; $d^2y/dx^2 = 6$.

   (d) True

   (e) True

2. (a) $(x^2y^2 + 2y)/(2x - 2x^3y)$

   (b) $(6y\sqrt{xy} - 2\sqrt{xy} - y)/(-6x\sqrt{xy} + 2\sqrt{xy} + x)$

   (c) $(3x^2 - y)/(3y^2x^2 - x + 3x^2)$

3. (a) $14/27t$

   (b) 0

   (c) $2(t + 4)\sqrt{t^2 - 3}/t$

4. $[g(f''g - g''f) - 2g'(f'g - g'f)]/g^3$

5. $F(x) = x^3/2 + 3x + 13/2$

6. $-3y(1 + y^2)^2/((1 + y^2)^3 + 1)^{3/2}$

7. (a) $30x^4 - 30x$

   (b) 0

   (c) $6x + 8$

   (d) $2/(3 - x)^3$

8. $100 cm^2/sec$

9. $-9[3/4(x^{3/2} + 1) - 1/x^{3/2}]x/16(x^{3/2} + 1)^{1/4}$

10. $600/\sqrt{532}$ m/min