CHAPTER 6

EXPONENTIALS AND LOGARITHMS

6.1 Exponential Functions

PREREQUISITES

1. Recall the laws of exponents for rational powers (Section R.3).

PREREQUISITE QUIZ

1. Simplify the following without using a calculator:

   (a) $6^{25}/6^{23}$

   (b) $(8^3 \cdot 2^3)^{1/2}$

   (c) $(1/8)^{-1/3}$

GOALS

1. Be able to manipulate exponential functions.

2. Be able to recognize the graphs of exponential functions.

STUDY HINTS

1. Exponential growth. In general, if something grows by a factor of b per unit of time, the growth factor after t units of time is b^t.

2. Real powers. The properties are an extension from rational to real powers. The same properties that held for rational powers now hold for real powers.

3. Notation. $\exp_b x$ is the same as b^x .

4. Exponential graphs. You should know the general shapes of the graphs. If $b > 1$, then b^x starts near 0 for $x \to -\infty$, increases until it passes through (0,1) , and continues to increase very steeply toward ∞ as $x \to \infty$. Note that if $0 < b < 1$, then $b^x = (1/b)^{-x}$; therefore, $(1/b)^x$ is a reflection of b^x across the y-axis.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) In 4 hours, the culture triples its mass twice, so it grows by a factor of $3 \cdot 3 = 9$.

   (b) In 6 hours, the mass triples three times or by a factor of $3^3 = 27$.

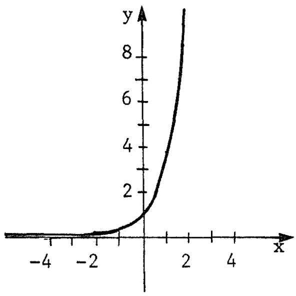
   (c) If it grows by a factor of k in 1 hour, it grows by a factor of $k \cdot k = k^2$ in 2 hours. Thus $k^2 = 3$, so $k = \sqrt{3}$. Therefore, in 7 hours, it grows by a factor of $(\sqrt{3})^7 = 3^{7/2} = 27\sqrt{3}$.

   (d) Using the results of (a), (b), and (c), the culture grows by a factor of $3^{x/2}$ in x hours.

5. Using the law $(b^x)^y = b^{xy}$, we have $(2^{\sqrt{2}})^{\sqrt{2}} = 2^{(\sqrt{2}\cdot\sqrt{2})} = 2^2 = 4$.

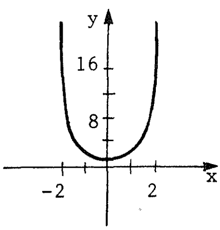
9. Use the laws of exponents to get $5^{\pi/2}\cdot 10^{\pi}/15^{-\pi} = 5^{\pi/2}\cdot(5\cdot 2)^{\pi}\cdot(5\cdot 3)^{\pi} = 5^{5\pi/2}\cdot 2^{\pi}\cdot 3^{\pi} = 5^{5\pi/2}\cdot 6^{\pi}$.

13. Using the laws of exponents, $9^{1/\sqrt{3}} = (3^2)^{1/\sqrt{3}} = 3^{2/\sqrt{3}} = 3^{\sqrt{4/3}}$. Since the base, 3 , is greater than 1 and $\sqrt{2} > \sqrt{4/3}$, $3^{\sqrt{2}}$ is larger.

17. 

Plot a few points and connect them with a smooth curve.

x	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2
3^x	1/9	0.192	1/3	0.577	1	1.732	3	5.196	9

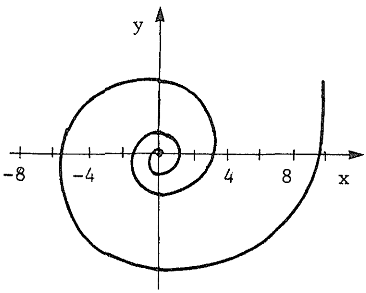
21. 

Note that $f(-x) = f(x)$, so we need only plot the graph for $x \geqslant 0$ and then reflect it.

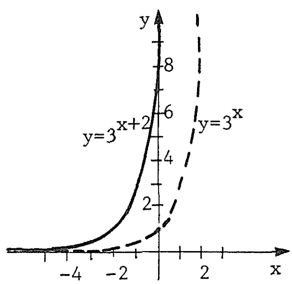
x	0	1/2	1	3/2	7/4	2
$\exp_2 x^2$	1	1.189	2	4.757	8.354	16

25. $\exp_{1/3}x = (1/3)^x = 3^{-x} = \exp_3(-x)$, so the graph of $y = \exp_{1/3}x$ is obtained by reflecting $y = \exp_3 x$ in the y-axis.

29. Functions (A) and (C) are both strictly negative. For x large and positive, $y = -3^{-x}$ approaches 0 , so (A) matches (b) and (C) matches (c) . Similarly, (B) and (D) are both strictly positive and (B) approaches 0 as x approaches $+\infty$, so (B) matches (d) and (D) matches (a) .

33. 

The graph is the same as in Example 7 except that every turn of the spiral is $(1.2)^{2\pi} \approx 3.14$ times as big as the previous one.

37. 

The graph of 3^x is shown in Exercise 17. The graph of $y = 3^{x+2}$ is the same as the graph of $y = 9(3^x)$. This is reasonable since $3^{x+2} = (3^x)(3^2) = 9(3^x)$. In general, shifting the graph of 3^x by k units to the left is the same as graphing $y = 3^{x+k} = (3^k)(3^x)$. Thus, stretching the graph by a factor of 3^k in the y-direction results in the graph of $y = 3^{x+k}$ also.

41. By shifting the graph of 3^x two units to the left, we get the graph of $3^{x+2} = 9(3^x)$. Thus, the area under 3^x between $x = 2$ and $x = 4$ is the same as the area under 3^{x+2} between $x = 0$ and $x = 2$. Therefore, the ratio is $1/9$.

45. Since $b^x > 0$ for all x and all $b > 0$, there is no solution for $2^x = 0$.

SECTION QUIZ

1. Simplify the following expressions:

(a) $[(\exp_2 \pi)(\exp_3 \pi)]^{-1/\pi}$

(b) $5^{\sqrt{3}} \exp_5 \sqrt{9} / 25^{\sqrt{3}}$

(c) $y^x x^2 y^x / \sqrt[3]{y^{2x}} \sqrt{x^3}$

2. For each of the following functions, tell (i) whether it is increasing or decreasing, (ii) where the y-intercept, if any, is located, (iii) whether it has an x-intercept, and (iv) whether it is concave upward or downward.

   (a) $(3/2)^{-x} + 5$

   (b) $(3/2)^{x} - 3$

   (c) $(2/3)^{x} - 3$

   (d) $-(3/2)^{-x} + 5$

   (e) $-(2/3)^{x} - 3$

3. For your tenth wedding anniversary, your six-year-old son presents you with an ant hill enclosed in glass. You observe that the population doubles in about 2 months. Initially, there were 1000 ants.

   (a) Find a function which describes the population after x months.

   (b) Graph the function in part (a).

   26.5 months later, a neighbor gives you a great recipe for chocolate ants, but in your anxiety to try out the recipe, you break the glass enclosure.

   (c) How many ants are there running around your kitchen and up your legs after the accident?

ANSWERS TO PREREQUISITE QUIZ

1. (a) 36

   (b) 64

   (c) 2

ANSWERS TO SECTION QUIZ

1. (a) 1/6

(b) $5^{3-\sqrt{3}}$

(c) $y^{4x/3}\sqrt{x}$

2. (a) (i) decreasing; (ii) (0,6) ; (iii) no; (iv) concave upward

(b) (i) increasing; (ii) (0,-2) ; (iii) yes; (iv) concave upward

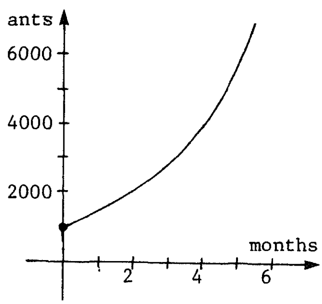
(c) (i) decreasing; (ii) (0,-2) ; (iii) yes; (iv) concave upward

(d) (i) increasing; (ii) (0,4) ; (iii) yes; (iv) concave downward

(e) (i) increasing; (ii) (0,-4) ; (iii) no; (iv) concave downward

3. (a) $(1000)2^{x/2}$

(b)



(c) 9,741,985 ants

6.2 Logarithms

PREREQUISITES

1. Recall the concept of an inverse function (Section 5.3).
2. Recall how to sketch exponential functions (Section 6.1).

PREREQUISITE QUIZ

1. Explain the inverse function test.
2. (a) Sketch the graph of $\exp_{1/2}x$.

   (b) Sketch the graph of the inverse function of $\exp_{1/2}x$.

   (c) How is the sketch in part (a) related to the graph of $(1/2)^x$?

GOALS

1. Be able to define logarithms.
2. Be able to recognize the graphs of and manipulate logarithmic functions.

STUDY HINTS

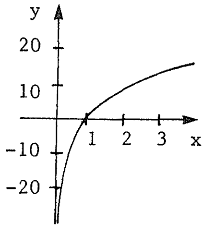
1. Definition. The logarithm is simply the exponent. It is defined so that if $b^x = y$, then $\log_b y = x$.
2. Logarithmic graphs. Since the logarithm is an inverse function, you can easily recognize the graph of a logarithmic function as a "flipped" exponential graph.
3. Laws of logarithms. It is useful to memorize the three laws. Be careful not to be tempted to equate $\log_b(x + y)$ with $\log_b x \cdot \log_b y$ or with $\log_b x + \log_b y$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By definition, $\log_2 4 = x$ implies $2^x = 4$; therefore $x = 2$.

5. By definition, $\log_{10}(0.001) = x$ implies $10^x = 0.001$; therefore, $x = -3$.

9. By definition, $\log_{1/2} 2 = x$ implies $(1/2)^x = 2$; therefore, $x = -1$.

13. 

The graph of $y = \log_2 x$ is shown in Fig. 6.2.2 . Stretch the graph by a factor of 8 in the y-direction.

17. $\log_2(2^8/8^2) = \log_2(2^8/(2^3)^2) = \log_2(2^8/2^6) = \log_2(2^2)$. The logarithm is the exponent, which is 2 in this case.

21. By definition, $\log_2(2^b) = x$ implies $2^x = 2^b$; therefore, $x = b$.

25. Try to write 7.5 in terms of 2 , 3 , and 5 . Then apply the laws of logarithms. $7.5 = 3 \cdot 5/2$, so $\log_7(7.5) = \log_7(3 \cdot 5/2) = \log_7 3 + \log_7 5 - \log_7 2 \approx 0.565 + 0.827 - 0.356 = 1.036$.

29. If $\log_b 10 = 2.5$, then $\exp_b(\log_b 10) = \exp_b(2.5)$ or $10 = b^{2.5} = b^{5/2}$. Raise both sides to the 2/5 power to get $10^{2/5} = (b^{5/2})^{2/5} = b$. Now, take the logarithms of both sides to get $2/5 = \log_{10} b$. Using a table, we find $\log_{10} 2.51 \approx 0.399674$ and $\log_{10} 2.52 \approx 0.401401$; therefore, $b \approx 2.51$.

33. If we let $y = \log_{a^n} x$, then $\exp_{a^n} y = \exp_{a^n}(\log_{a^n} x)$ or $(a^n)^y = a^{ny} = x$. Taking the logarithms in base a , we get $ny = \log_a x$ or $y = (1/n)\log_a x$. Originally, $y = \log_{a^n} x$, so $\log_{a^n} x = (1/n)\log_a x$. Due to the definition of the logarithm, we must restrict a to be positive and unequal to 1 .

37. By using the laws of logarithms, $2\log_b(A\sqrt{1+B}/C^{1/3}B) - \log_b[(B+1)/AC] = 2[\log_b A + \log_b(1+B)^{1/2} - \log_b C^{1/3} - \log_b B] - [\log_b(B+1) - \log_b A - \log_b C] = 2[\log_b A + (1/2)\log_b(1+B) - (1/3)\log_b C - \log_b B] - [\log_b(B+1) - \log_b A - \log_b C] = 3\log_b A - 2\log_b B + (1/3)\log_b C$.

41. $\log_3 x = 2$ implies that $x = 3^2 = 9$.

45. The logarithm is defined only if $x > 0$ and $1 - x > 0$; therefore, we must have $0 < x < 1$. From $\log_x(1-x) = 2$, we get $\exp_x(\log_x(1-x)) = \exp_x 2$ or $1 - x = x^2$. Then $x^2 + x - 1 = 0$ can be solved with the quadratic formula, yielding $x = (-1 \pm \sqrt{1+4})/2$. Since x must be in $(0,1)$, $x = (\sqrt{5} - 1)/2$.

49. Successively substitute 2 , 4 , 8 , 10 , 100 , and 1000 for (I_0/I) in $D = \log_{10}(I_0/I)$ to get $D_2 \approx 0.301$; $D_4 \approx 0.602$; $D_8 \approx 0.903$; $D_{10} = 1$; $D_{100} = 2$; and $D_{1000} = 3$.

53. We want to find b such that $\log_b 3 = 1/3$, i.e., $b^{1/3} = 3$. Raising both sides to the third power, we get $b = 3^3 = 27$.

57. Let $y = \log_2(x-1)$ and solve for x . Exponentiation gives $2^y = x - 1$, so $x = 2^y + 1$. Changing variables gives the inverse function, $g(x) = 2^x + 1$. The domain of $g(x)$ is $(-\infty,\infty)$ since 2^x is defined for all x . Its range is $(1,\infty)$ since $2^x > 0$ for all x .

SECTION QUIZ

1. Determine which statements are true.

   (a) $\ln(x+y) = \ln x + \ln y$

   (b) $\ln\sqrt{x} = (1/2)\ln x$

   (c) $(\ln x)^{1/2} = (1/2)\ln x$

   (d) $\log_b xy = \log_b x + \log_b y$

1. (e) $\ln(x/y) = \ln x - \ln y$

   (f) $\log_b(x + y) = \log_b x \cdot \log_b y$

   (g) $\log_b cx = c \log_b x$, c constant

2. If $\log_b y = x$, then is $y = b^x$ or is $x = b^y$?

3. If $(-4)^3 = -64$, what is $\log_{(-4)}(-64)$?

4. Your idiotic cousin, Irving, notices that you are studying about logarithms. He sees the notation "log" and hears you pronouncing "ln" as "lawn", so he concludes, "must be reading about building houses." A friend tells him that people use the formula $A = P[1 + (r/100n)]^{nt}$ in the housing loan business. (This formula is explained on p. 331 of the text.)

   (a) Irving wants to borrow money to build a home and asks you how much the wood costs. You assume he wants to know $\log_P A$. Write an expression for it.

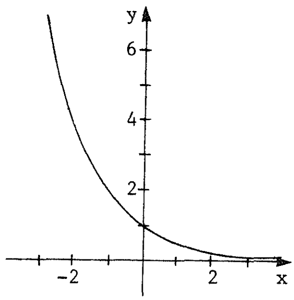
   (b) Next he wants to know how much the grass costs. Compute $\ln A$ for him. Sketch the graph of $\ln A$ versus t if P , r , and n are positive constants.

   (c) What is $\log_B A$ if $B = 1 + r/100n$?

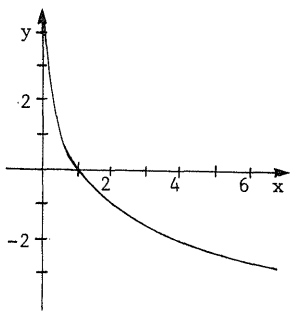
ANSWERS TO PREREQUISITE QUIZ

1. An inverse function exists if the graph of the function is strictly increasing or strictly decreasing.

2. (a)



(b)



(c) They are the same since $\exp_{1/2} x = (1/2)^x$.

ANSWERS TO SECTION QUIZ

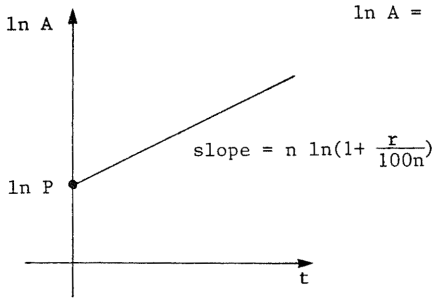
1. b , d , e

2. $y = b^x$

3. 3

4. (a) $\log_p A = 1 + nt \log_p(1 + r/100n)$

4. (b) $\ln A = \ln P + nt \ln(1 + r/100n)$



(c) $\log_B A = \log_B P + nt$

6.3 Differentiation of the Exponential and Logarithm Functions

PREREQUISITES

1. Recall the definition of the derivative as a limit (Section 1.3).
2. Recall how to use the chain rule (Section 2.2).
3. Recall how to compute the derivative of an inverse function (Section 5.3).

PREREQUISITE QUIZ

1. Suppose the derivative of $y = f(x)$ at $x = 4$ is $dy/dx = 3$, what is the derivative of $f^{-1}(x)$ at the point $f(4)$?
2. State the definition of the derivative as a limit.
3. Write a formula for the derivative of $f(g(x))$ in terms of $f(x)$, $g(x)$, $f'(g(x))$, $f(g(x))$, $g'(x)$, and $f'(x)$.

GOALS

1. Be able to differentiate exponential and logarithmic functions.
2. Be able to use the technique of logarithmic differentiation to differentiate certain functions.

STUDY HINTS

1. The number e. Similar to π, e has a specific value; it is not a variable. Its value is approximately 2.718... .
2. Notation and definitions. Logarithms to the base e are called natural logarithms; it is usually denoted by $\ln x$ rather than $\log_e x$. $\log x$, without a base written, is understood to be the common logarithm, which uses 10 as its base.* $\exp x$, without an associated base, is understood to be e^x, whereas $\exp_b x$ means b^x.

* Except in more advanced courses, where log x means ln x, since $\log_{10} x$ is rarely used.

3. Derivatives of exponential functions. $d(e^x)/dx = e^x$ is something that should be memorized. We do not commonly differentiate b^x, so its derivative is easy to forget. Therefore, if we remember that $b = \exp(\ln b)$, then $b^x = \exp(x \ln b)$ and the chain rule may be used to get $d(b^x)/dx = b^x \ln b$. WARNING: n^x and x^n, for constant n, are differentiated differently.

4. Derivatives of logarithmic functions. Memorize $(d/dx)(\ln x) = 1/x$. As with exponentials, logarithms in bases other than e are not often used. One can derive $(d/dx)\log_b x = 1/(\ln b)x$ by writing $b^y = x$ as $e^{(\ln b)y} = x$, so $\ln x = (\ln b)y = (\ln b)\log_b x$, or $\log_b x = \ln x/\ln b$. Differentiating yields $(d/dx)\log_b x = 1/x \ln b$.

5. Logarithmic differentiation. This technique is used mainly to differentiate x to some variable power or complex expressions involving numerical powers. The technique is to take the logarithm of both sides, differentiate, and then solve for dy/dx. Note that $(d/dx)(\ln y) = (1/y)(dy/dx)$. Don't forget to express your answer in terms of the original variable. Study Examples 7 and 8.

6. Integration. As usual, once you have learned the differentiation formulas, you can easily recover the antidifferentiation formulas. Note that we do not have a special antidifferentiation formula to recover $\log_b x$. This is because if we integrate $(d/dx)\log_b x = 1/(\ln b)x$, we get $[1/(\ln b)]\int(dx/x)$, which is one of the given formulas. However, it is still correct to write $\int[dx/(\ln b)x] = \log_b x + C$. Also, note that $\int(dx/x) = \ln|x| + C$, not just $\ln x + C$, unless x is restricted to be positive.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. As in Example 1, use $\exp_b'(x) = \exp_b'(0)\exp_b(x)$. Here, $f'(3) = f'(0)f(3) = f'(0) = 8f'(0)$, so f is increasing 8 times as fast at $x = 3$ than at $x = 0$.

5. Use the law that $\ln(e^x) = x$ to get $\ln(e^{x+1}) + \ln(e^2) = (x + 1) + (2) = x + 3$.

9. Use the law that $\ln(e^x) = x$ to get $e^{4x}[\ln(e^{3x-1}) - \ln(e^{1-x})] = e^{4x}[(3x - 1) - (1 - x)] = e^{4x}(4x - 2)$.

13. Using the chain rule and the fact that $(d/dx)e^x = e^x$, we get $(d/dx)(e^{1-x^2} + x^3) = \exp(1 - x^2)\cdot(d/dx)(1 - x^2) + 3x^2 = -2x\exp(1 - x^2) + 3x^2$.

17. Use the fact that $(d/dx)b^x = (\ln b)b^x$ to get $(d/dx)(3^x - 2^{x-1}) = (\ln 3)3^x - (\ln 2)2^{x-1}(d/dx)(x - 1) = (\ln 3)3^x - (\ln 2)2^{x-1}$.

21. Use the fact that $(d/dx)\ln x = 1/x$ and the quotient rule to get $(d/dx)(\ln x/x) = [(1/x)x - \ln x(1)]/x^2 = (1 - \ln x)/x^2$.

25. Use the chain rule and the fact that $(d/dx)\ln x = 1/x$ to get $(d/dx)\ln(2x + 1) = [1/(2x + 1)](d/dx)(2x + 1) = 2/(2x + 1)$.

29. Use the chain rule, quotient rule, and the fact that $(d/dx)\ln x = 1/x$ to get $[(1/\tan 3x)(d/dx)(\tan 3x)(1 + \ln x^2) - (\ln(\tan 3x))(1/x^2)\times(d/dx)(x^2)]/(1 + \ln x^2)^2 = [3\sec^2 3x(1 + \ln x^2)/\tan 3x - (2/x)\ln(\tan 3x)]/(1 + \ln x^2)^2 = [3x\sec^2 3x(1 + \ln x^2) - 2(\tan 3x)\ln(\tan 3x)]/[x\tan 3x\times(1 + \ln x^2)^2]$.

33. Taking logarithms yields $\ln y = x\ln(\sin x)$. Using the chain rule, differentiation gives $(dy/dx)/y = \ln(\sin x) + (x/\sin x)(d/dx)(\sin x) = \ln(\sin x) + x\cot x$. Hence, $dy/dx = y(\ln(\sin x) + x\cot x) = (\sin x)^x[\ln(\sin x) + x\cot x]$.

37. Taking logarithms yields $\ln y = (2/3)\ln(x - 2) + (8/7)\ln(4x + 3)$. Using the chain rule, differentiation gives $(dy/dx)/y = (2/3)/(x - 2) + (8/7)4/(4x + 3) = 2/3(x - 2) + 32/7(4x + 3) = 2/(3x - 6) + 32/(28x + 21)$. Hence, $dy/dx = y[2/(3x - 6) + 32/(28x + 21)] = (x - 2)^{2/3}(4x + 3)^{8/7} \times [2/(3x - 6) + 32/(28x + 21)]$.

41. By the chain rule and product rule, $(d/dx)\exp(x \sin x) = (\sin x + x \cos x)\exp(x \sin x)$.

45. By the chain rule and the fact that $(d/dx)b^x = (\ln b)b^x$, we get $(d/dx)14^{x^2-8 \sin x} = (2x - 8 \cos x)(\ln 14)14^{x^2-8 \sin x}$.

49. Let $u = x^{\sin x}$, so $(d/dx)\cos(x^{\sin x}) = (d/du)\cos u \cdot (du/dx)$. Now use logarithmic differentiation to get du/dx. Begin with $\ln u = \sin x(\ln x)$, so $(du/dx)/u = \cos x\,(\ln x) + \sin x/x$. Thus, $du/dx = (x^{\sin x})[\cos x\,(\ln x) + \sin x/x]$ and $(d/dx)\cos(x^{\sin x}) = -[\sin(x^{\sin x})](x^{\sin x})[\cos x\,(\ln x) + \sin x/x]$.

53. Use the fact that $\exp(\ln x) = x$ to get $(1/x)^{\tan x^2} = x^{-\tan x^2} = e^{-\ln x(\tan x^2)}$. Thus, the derivative is $[-(1/x)\tan x^2 + (-\ln x) \times (2x \sec^2 x^2)]\exp(-\ln x \cdot \tan x^2) = -(1/x)^{\tan x^2}(\tan x^2/x + 2x(\ln x)\sec^2 x^2)$.

57. Use the fact that $\exp(\ln x) = x$, so $3x^{\sqrt{x}} = 3 \exp(\sqrt{x} \ln x)$. Thus, the derivative is $3 \exp(\sqrt{x} \ln x)[(1/2\sqrt{x})\ln x + \sqrt{x}/x] = 3x^{\sqrt{x}}(\ln x/2\sqrt{x} + 1/\sqrt{x})$.

61. Using the fact that $\exp(\ln x) = x$, we have $(\cos x)^x = \exp(x \ln(\cos x))$, so $d(\cos x)^x/dx = \exp(x \ln(\cos x)) \cdot (\ln(\cos x) - x \tan x) = (\cos x)^x(\ln(\cos x) - x \tan x)$. Now, let $y = (\sin x)^{[(\cos x)^x]}$, so by logarithmic differentiation, $\ln y = (\cos x)^x \ln(\sin x)$, and therefore, $(dy/dx)/y = (\cos x)^x(\ln(\cos x) - x \tan x)\ln(\sin x) + (\cos x)^x \cot x$. Hence, the derivative is $dy/dx = (\sin x)^{[(\cos x)^x]}(\cos x)^x \times [(\ln(\cos x) - x \tan x)\ln(\sin x) + \cot x]$.

65. From Example 9(a), we have $\int e^{ax}dx = (1/a)e^{ax} + C$. Thus, $\int(\cos x + e^{4x})dx = \sin x + e^{4x}/4 + C$.

69. Use division to get $\int[(x^2 + 1)/2x]\,dx = (1/2)\int(x + 1/x)dx = (1/2)(x^2/2 + \ln|x|) + C = x^2/4 + (1/2)\ln|x| + C$.

73. Using the formula $\int b^x dx = b^x/\ln b + C$, we get $\int 3^x dx = 3^x/\ln 3 + C$.

77. $\int_0^1 (x^2 + 3e^x)dx = (x^3/3 + 3e^x)\big|_0^1 = 1/3 + 3e - 3 = 3e - 8/3$.

81. Use the formula $\int b^x dx = b^x/\ln b + C$ to get $\int_0^1 2^x dx = (2^x/\ln 2)\big|_0^1 = (2^1 - 2^0)/\ln 2 = 1/\ln 2$.

85. (a) $(d/dx)x\ln x = \ln x + 1$.

(b) From part (a), $\int(\ln x + 1)dx = \int\ln x\,dx + \int dx = \int \ln x\,dx + x = x\ln x + C$. Rearrangement yields $\int\ln x\,dx = x\ln x - x + C$. Note that the absolute value was not necessary since we were integrating $\ln x$, which we assume exists, so $x > 0$ anyway.

89. According to the fundamental theorem of calculus, the derivative of the integral must equal the integrand.

(a) $(d/dx)[\ln(x + \sqrt{1 + x^2}) + C] = [1/(x + \sqrt{1 + x^2})][1 + (1/2)(1 + x^2)^{-1/2}(2x)] = [1/(x + \sqrt{1 + x^2})][1 + x/\sqrt{1 + x^2}] = [1/(x + \sqrt{1 + x^2})] \times [(\sqrt{1 + x^2} + x)/\sqrt{1 + x^2}] = 1/\sqrt{1 + x^2}$, which is the integrand.

(b) $(d/dx)[-\ln|(1 + \sqrt{1 - x^2})/x| + C] = -[x/(1 + \sqrt{1 - x^2})][(1/2)(1 - x^2)^{-1/2}(-2x)x - (1 + \sqrt{1 - x^2})]/x^2 = -[x/(1 + \sqrt{1 - x^2})][-x^2 - \sqrt{1 - x^2} - (1 - x^2)]/x^2\sqrt{1 - x^2} = -[x/(1 + \sqrt{1 - x^2})][(-1)(1 + \sqrt{1 - x^2})]/x^2\sqrt{1 - x^2} = 1/x\sqrt{1 - x^2}$, which is the integrand.

93. We choose $\Delta x = 0.0001$ and denote $f'(\Delta x) = (b^{\Delta x} - 1)/\Delta x$. We know that $f'(\Delta x) = 0.69$ for $b = 2$ and $f'(\Delta x) = 1.10$ for $b = 3$ and we wish to find b so that $f'(\Delta x) = 1.00$. Using the method of bisection (see Section 3.1), we get the following:

b	2.5	2.75	2.625	2.69	2.70	2.71	2.72	2.73
$f'(\Delta x)$)	0.916	1.012	0.965	0.990	0.993	0.997	1.001	1.004

Thus, $e \approx 2.72$.

97. (a) From Exercise 85, $\int_{\varepsilon}^{2} \ln x\, dx = (x \ln x - x)\Big|_{\varepsilon}^{2}$. For $\varepsilon = 1$, 0.1 , and 0.01 , the integral is approximately 0.38629, -0.28345 , and -0.55765 , respectively.

(b) Define $\int_0^2 \ln x\, dx$ by $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{2} \ln x\, dx = \lim_{\varepsilon\to 0} (x \ln x - x)\Big|_{\varepsilon}^{2} = 2 \ln 2 - 2 - \lim_{\varepsilon\to 0}(\varepsilon \ln \varepsilon - \varepsilon) = 2 \ln 2 - 2 - \lim_{\varepsilon\to 0} \varepsilon \ln \varepsilon$. Numerical evaluation of $\lim_{x\to 0} x \ln x$ yields the following table:

x	1	0.1	0.01	0.001	0.0001	0.00001	1×10^{-10}
x ln x	0	-0.23	-0.046	-0.0069	-0.00092	-0.00011	-2.3×10^{-9}

Thus, we conclude that $\lim_{x\to 0} x \ln x = 0$ and $\int_0^2 \ln x\, dx = 2 \ln 2 - 2$.

(c) The integral doesn't exist in the ordinary sense because ln x approaches a negatively infinite value as x approaches 0. Thus, ln x has no lower sums on [0,2] .

101. $\ln y = n_1 \ln [f_1(x)] + n_2 \ln [f_2(x)] + \dots + n_k \ln [f_k(x)]$. Differentiation yields $(dy/dx)/y = n_1 f_1'(x)/f_1(x) + n_2 f_2'(x)/f_2(x) + \dots + n_k f_k'(x)/f_k(x)$, so $dy/dx = y \sum_{i=1}^{k} [n_i f_i'(x)/f_i(x)]$.

SECTION QUIZ

1. Differentiate the following with respect to x :

   (a) x^e

   (b) $\ln e$

   (c) $1/\ln x$

   (d) $3e^2$

   (e) $1/x$

   (f) $1/2^x$

   (g) $(1/2)^x$

   (h) $\log_{10}(3/2)$

   (i) $\log_7 2x$

2. Perform the following integrations:

   (a) $\int e^3 dx$

   (b) $\int x^e dx$

   (c) $\int (3/4x) dx$

   (d) $\int (2/5^x) dx$

3. Find the minimum of $x^{\sqrt[3]{x}}$, $x > 0$.

4. In the midst of a galactic war, an enemy spacecraft was lasered. Your computer determines that the disabled ship spirals along the curve given by $y = (5x + 3)^3(7x - 8)^4(6x^2 + 12x - 1)^5/(5x^3 - 7x^2 - x + 4)^6$. When it reaches $x = 0$, another laser thrust pushes it out of the galaxy along the tangent line. What is the tangent line?

ANSWERS TO PREREQUISITE QUIZ

1. $1/3$

2. $f'(x_0) = \lim_{\Delta x \to 0}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\}$

3. $f'(g(x)) \cdot g'(x)$

ANSWERS TO SECTION QUIZ

1. (a) ex^{e-1}

   (b) 0

   (c) $-1/x(\ln x)^2$

   (d) 0

   (e) $-1/x^2$

   (f) $-\ln 2/2^x$

   (g) $-\ln 2/2^x$

   (h) 0

   (i) $1/x \ln 7$

2. (a) $e^3x + C$

   (b) $x^{e+1}/(e+1) + C$

   (c) $(3/4)\ln |x| + C$

   (d) $-(2 \ln 5)(1/5)^x + C$

3. $e^{-3/e}$

4. $y = 1539x - 27$

6.4 Graphing and Word Problems

PREREQUISITES

1. Recall how derivatives are used as aids to graphing (Section 3.4).

PREREQUISITE QUIZ

1. What conclusions about the graph of $y = f(x)$ can you draw if you know that:

   (a) $f''(x) < 0$?

   (b) $f'(x) > 0$?

   (c) $\lim_{x\to\infty} f(x) = a$?

GOALS

1. Be able to graph functions involving exponentials and logarithms.
2. Be able to define e as a limit.
3. Be able to compute actual interest rates from compounded interest rates.

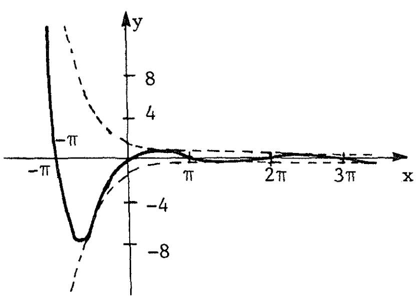
STUDY HINTS

1. Limiting behavior. You should know the three limits listed in the box preceding Example 1. Since you will be learning simpler proofs in Chapter 11, you probably will not be held responsible for the methods of the proofs. Ask your instructor.
2. Graphing. The same techniques that were introduced in Chapter 3 are used to sketch exponentials and logarithms. Use the limiting behaviors discussed above to complete the graphs.
3. Relative rate changes. This quantity is simply the derivative of $\ln f(x)$, i.e., f'/f.

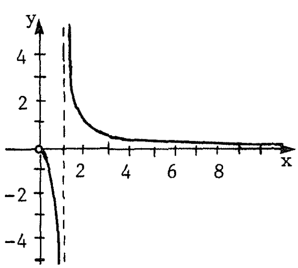
4. <u>e as a limit</u>. If you remember that $e = \lim_{h\to 0}(1 + h)^{1/h}$, then letting $h = 1/n$ and $h = -1/n$ yields the other two formulas in the box on p. 330.

5. <u>Compound interest</u>. You may find it easier to derive the formula rather than memorizing it. At $r\%$ interest compounded n times annually, the percent interest during a single period is r/n. Thus, the growth factor is $1 + r/100n$. For n periods (1 year), the growth factor is $(1 + r/100n)^n$, and over t years, it is $(1 + r/100n)^{nt}$. The growth factor $\exp(rt/100)$ is obtained by using the definition of e as a limit. Continuously compounded interest is thus a special case of exponential growth.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x)$, so the critical points occur where $\cos x = \sin x$ or $\tan x = 1$ or $x = \pi/4 + n\pi$, where n is an integer. The zeros of $f(x)$ occur where $\sin x = 0$ or $x = n\pi$, where n is an integer. Since $-1 \leqslant \sin x \leqslant 1$, $e^{-x}\sin x$ lies between the graphs of e^{-x} and $-e^{-x}$.

5. 

Beginning with $f(x) = y = \log_x 2$, we exponentiate to get $x^y = 2$. Then take the natural logarithm, giving $y \ln x = \ln 2$ or $y = \log_x 2 = \ln 2/\ln x$. The domain is $x > 0$, $x \neq 1$. $f'(x) = -\ln 2/x(\ln x)^2$, so $f(x)$ is always decreasing. $f''(x) = -\ln 2\,(\ln x + 2)/x^2(\ln x)^3$, so the inflection point occurs at $x = 1/e^2$.

9. Using the results of Example 3, $(1/p)(dp/dt) = 0.3 - 0.002t$. Substituting $t = 2$ corresponding to January 1, 1982, we get $(1/p)(dp/dt) = 0.3 - (0.002)(2) = 0.296$. Therefore, on January 1, 1982, the company's profits are increasing at a rate of 29.6% per year.

13. Use the formula $e^a = \lim_{n\to\infty}(1 + a/n)^n$. Using $\exp(\ln x) = x$, we have $3^{\sqrt{2}} = [\exp(\ln 3)]^{\sqrt{2}} = \exp(\sqrt{2}\ln 3)$. Therefore, $3^{\sqrt{2}} = \lim_{n\to\infty}(1 + \sqrt{2}\ln 3/n)^n$ since $a = \sqrt{2}\ln 3$.

17. The annual percentage increase of funds invested at $r\%$ per year compounded continuously is $100(e^{r/100} - 1)\%$. Therefore, if $r = 8$, then the actual yield is $100(e^{0.08} - 1) \approx 8.33\%$.

21. The tangent line is $y = y_0 + (dy/dx)|_{x_0}(x - x_0)$. $dy/dx = 2xe^{2x} + e^{2x}$, so $(dy/dx)|_1 = 3e^2$. Also, at $x = 1$, $y = e^2$, so the tangent line is $y = e^2 + (3e^2)(x - 1) = 3e^2x - 2e^2$.

25. The tangent line at (x_0,y_0) is $y = y_0 + (dy/dx)|_{x_0}(x - x_0)$. $dy/dx = 2x/(x^2 + 1)$, so $(dy/dx)|_1 = 1$. Also, at $x = 1$, $y = \ln 2$, so the tangent line is $y = \ln 2 + 1(x - 1) = x + (\ln 2 - 1)$.

29. Using $\exp(\ln x) = x$, we get $y = x^x = \exp(x \ln x)$, so $y' = (\ln x + 1)\exp(x \ln x)$. This vanishes when $x = 1/e$. At $x = 1/e$, $y = (1/e)^{1/e} = e^{-1/e} \approx 0.692$. Since this is the only critical point and the limits of x^x at 0 and ∞ are larger, 0.692 is the minimum value.

33. Apply the hint by first finding $dp/dx = -2116(0.0000318)\exp(-0.0000318x)$, which is approximately $-0.0672\exp(-0.0636) \approx -0.0631$ when $x = 2000$. At the time in question, $dx/dt = 10$, so $dp/dt = (dp/dx)(dx/dt) \approx -(0.0631)(10) = -0.631$.

37. (a) $g'(x) = f(1 + 1/x) + xf'(1 + 1/x)(-1/x^2) = f(1 + 1/x) - f'(1 + 1/x)/x$. The mean value theorem states that there exists some x_0 in $(1,1 + 1/x)$ at which $f'(x_0) = [f(1 + 1/x) - f(1)]/(1/x)$. Since $f(1) = 0$, $f'(x_0)/x = f(1 + 1/x)$. Substitute this into $g'(x) = [f'(x_0) - f'(1 + 1/x)]/x$. Since $f'(x)$ is a decreasing function and $x_0 < 1 + 1/x$, $f'(x_0) - f'(1 + 1/x) > 0$. If $x \geqslant 1$, $g'(x) > 0$, so $g(x)$ is increasing on $[1,\infty)$.

(b) $g(x) = x \ln(1 + 1/x)$ is increasing on $[1,\infty)$.

(c) Let $x = n/a$. Then $g(x) = (n/a) \ln(1 + a/n) = (1/a)\ln(1 + a/n)^n$ is increasing on $[1,\infty)$. To show that $(1 + a/n)^n$ is increasing, multiply by a and exponentiate, giving $e^{ag(x)} = (1 + a/n)^n$. Since $a > 0$ and $g(x)$ is increasing, $e^{ag(x)}$ and therefore $(1 + a/n)^n$ are also increasing.

SECTION QUIZ

1. Sketch the graph of $y = x \ln x^2$.
2. Sketch the graph of $y = \sqrt[3]{x}\, e^x$.
3. Express e^a as a limit.
4. Having eyed a $2 million mansion, you decide that you must have it in ten years. Long-term interest rates will pay 12% annually with continuous compounding. How much must be put into the savings account to get $2 million in ten years?

ANSWERS TO PREREQUISITE QUIZ

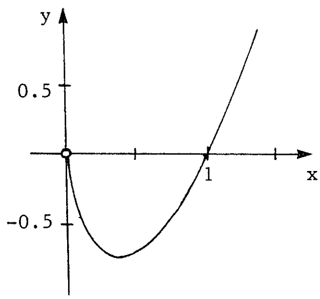
1. (a) $f(x)$ is concave downward.

   (b) $f(x)$ is increasing.

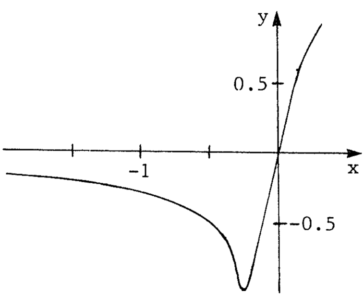
   (c) $y = a$ is a horizontal asymptote.

ANSWERS TO SECTION QUIZ

1.



2.



3. $\lim_{h\to 0}(1 + h)^{a/h} = \lim_{n\to\infty}(1 + 1/n)^{an} = \lim_{n\to\infty}(1 - 1/n)^{-an}$

4. \$602,388.43

6.R Review Exercises for Chapter 6

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using $a^{x+y} = a^x a^y$ and $(a + b)(a - b) = a^2 - b^2$, we get $(x^{\pi} + x^{-\pi})(x^{\pi} - x^{-\pi}) = x^{2\pi} - x^{-2\pi}$.

5. Using the fact that $\ln(e^x) = x$, we get $\ln(e^3) + (1/2)\ln(e^{-5}) = 3 + (1/2)(-5) = 1/2$.

9. Use the chain rule and $(d/dx)e^x = e^x$ to get $(d/dx)\exp(x^3) = 3x^2\exp(x^3)$.

13. Use the chain rule and $(d/dx)e^x = e^x$ to get $(d/dx)\exp(\cos 2x) = (-2 \sin 2x)\exp(\cos 2x)$.

17. Use the chain rule and $(d/dx)e^x = e^x$ to get $(d/dx)\exp(6x) = 6\exp(6x)$.

21. Use the quotient rule to get $(d/dx)[\sin(e^x)/(e^x + x^2)] = [((e^x) \cos (e^x))(e^x + x^2) - (\sin (e^x))(e^x + 2x)]/(e^x + x^2)^2$.

25. Use the chain rule and $(d/dx)e^x = e^x$ to get $(d/dx)\exp(\cos x + x) = (-\sin x + 1)\exp(\cos x + x)$.

29. Use the chain rule, product rule, and $(d/dx) \ln x = 1/x$ to get $(d/dx)x \ln(x + 3) = \ln(x + 3) + x/(x + 3)$.

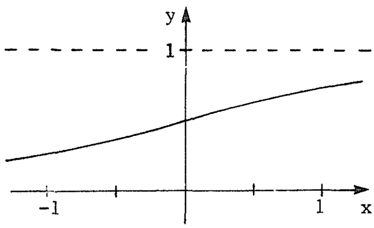
33. Exponentiate to get $5x = 3^y = \exp (y \ln 3)$. Take natural logarithms to get $\ln 5 + \ln x = y \ln 3$. Differentiate to get $1/x = \ln 3(dy/dx)$, i.e., $dy/dx = 1/x \ln 3$.

37. Use the reciprocal rule to get $-[2(\ln t)/t]/[(\ln t)^2 + 3]^2 = -2 \ln t/t[(\ln t)^2 + 3]^2$.

41. Use the formula $\int(dx/x) = \ln |x| + C$ to get $\int(\cos x + 1/3x)dx = \sin x + (1/3)\ln |x| + C$.

45. Use division to get $\int_1^2[(x + x^2 \sin \pi x + 1)/x^2]\,dx = \int_1^2(x^{-1} + \sin \pi x + x^{-2})dx = (\ln |x| - \cos \pi x/\pi - 1/x)|_1^2 = \ln 2 - 2/\pi + 1/2$. $\sin \pi x$ was integrated by guessing it was $a \cos \pi x$ and differentiation was used to find a .

49. Let $y = (\ln x)^x$, so $\ln y = x \ln(\ln x)$. Now apply the chain rule to get $(dy/dx)/y = \ln(\ln x) + (x/\ln x)(1/x) = \ln (\ln x) + 1/\ln x$. Therefore, $dy/dx = (\ln x)^x[\ln(\ln x) + 1/\ln x]$.

53. 

We have $dy/dx = [e^x(1 + e^x) - e^x(e^x)]/(1 + e^x)^2 = e^x/(1 + e^x)^2$, which is positive for all x , so y is always increasing. Also, $d^2y/dx^2 = [e^x(1 + e^x)^2 - e^x(2) \times (1 + e^x)e^x]/(1 + e^x)^4 = e^x(1 - e^x)/(1 + e^x)^3$, so $x = 0$ is an inflection point. $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$, so there are two horizontal asymptotes.

57. Rearrange to get $e^{xy} = 1 + xy$. Then, differentiate implicitly to get $(y + x(dy/dx))e^{xy} = y + x(dy/dx)$. Rearrange again to get $xe^{xy}(dy/dx) - x(dy/dx) = y - ye^{xy} = (dy/dx)(xe^{xy} - x)$. Therefore, $dy/dx = (y - ye^{xy})/(xe^{xy} - x) = y(1 - e^{xy})/x(e^{xy} - 1) = -y/x$.

61. The tangent line is $y = y_0 + m(x - x_0)$. Here, $m = (dy/dx)|_0 = [(1)\exp(3x^2 + 4x) + (x + 1)(6x + 4)\exp(3x^2 + 4x)]|_0 = 5$. Hence, the tangent line is $y = 5x + 1$.

65. The derivative is $[e^x(x + 1) - e^x(1)]/(x + 1)^2 = xe^x/(x + 1)^2$; therefore, $\int [xe^x/(x + 1)^2]\,dx = e^x/(x + 1) + C$.

69. Apply the formula $e^a = \lim_{n\to\infty}(1 + a/n)^n$. Here, $a = 10$, so the limit is e^{10} .

73. With continuous compounding, the interest rate is $100(e^{r/100} - 1)\%$. Thus, we get $e^{r/100} - 1 = 0.08$ or $e^{r/100} = 1.08$, i.e., $r = 100 \ln(1.08) \approx 7.70\%$.

77. (a) After one interval, $t = 1$ and $A(t) = A(1) = A_0$. At the end of the second interval, $A(2) = A_0 + A_0(1 + i/n)^n$. At the end of the third interval, $A(3) = A_0 + A(2)(1 + i/n)^n = A_0 + A_0(1 + i/n)^n + A_0(1 + i/n)^{2n}$. Similarly, $A(t) = A_0 \sum_{k=0}^{t-1} (1 + i/n)^{nk}$. Using the hint, $\sum_{k=0}^{t-1} (1 + i/n)^{nk} = ((1 + i/n)^{nt} - 1)/((1 + i/n)^n - 1)$, so $A(t) = A_0((1 + i/n)^{nt} - 1)/((1 + i/n)^n - 1)$.

(b) Here, $A_0 = 400$, $n = 1$, $i = 0.07/4$ and $t = 24$. Hence $A = 400((1 + 7/400)^{24} - 1)/((1 + 7/400) - 1) \approx (400)^2(0.516/7) \approx \$11,804.41$.

81. (a) Differentiate $P(t)$: $dP/dt = -a(b + ((a/P_0) - b)e^{-at})^{-2}(a/P_0 - b) \times (-ae^{-at})$. Add $0 = a^2b - a^2b$ to get $dP/dt = [(a^2(a/P_0 - b)e^{-at} + a^2b) - a^2b]\cdot(b + (a/P_0 - b)e^{-at})^{-2}$. Use the distributive law, yielding $dP/dt = a^2((a/P_0 - b)e^{-at} + b)(b + (a/P_0 - b)e^{-at})^{-2} - b(a/(b + (a/P_0 - b)e^{-at}))^2$. Cancel the factors of a^2 and substitute P, giving $dP/dt = a^2/(b + (a/P_0 - b)e^{-at}) - bP^2 = aP - bP^2 = P(a - bP)$. When $t = 0$, $P(0) = a/(b + (a/P_0 - b)) = a/(a/P_0) = P_0$.

(b) $\lim_{t\to\infty} P(t) = a/(b + (a/P_0 - b)\cdot 0) = a/b$.

85. (a) Use mathematical induction. If $n = 1$, the relation is true, since $b \geqslant 1 + b - 1 = b$. Suppose the relation holds for some n: $b \geqslant 1 + n(b - 1)$. Multiply by b to get $b\cdot b^n = b^{n+1} \geqslant b + nb(b - 1) = 1 + (b - 1) + nb(b - 1) = 1 + (nb + 1)(b - 1)$. Since $b > 1$, $nb + 1 > n + 1$, so $b^{n+1} \geqslant 1 + (n + 1)(b - 1)$. Thus if the relation holds for some n, it also holds for $n + 1$. Since

85. (a) (continued)

it also holds for $n = 1$, it holds for all n. Take reciprocals to show that $b^{-n} \leqslant 1/(1 + n(b - 1))$.

(b) $\lim_{x\to\infty} b^x \geqslant \lim_{x\to\infty} [1 + x(b - 1)] = \lim_{x\to\infty} 1 + \lim_{x\to\infty} x(b - 1) = \lim_{x\to\infty} 1 + (b - 1)\lim_{x\to\infty} x = \lim_{x\to\infty} (1 + x) = \lim_{x\to\infty} x = \infty$; therefore, $\lim_{x\to\infty} b^x = \infty$. We know that $0 < b^{-n} \leqslant 1/[1 + n(b - 1)]$, so $\lim_{x\to-\infty} b^x = \lim_{x\to\infty} b^{-x} \leqslant \lim_{x\to\infty} \{1/[1 + x(b - 1)]\} = 1/\lim_{x\to\infty} [1 + x(b - 1)] = 1/[\lim_{x\to\infty} 1 + (b - 1)\lim_{x\to\infty} x] = 1/\lim_{x\to\infty} (1 + x) = 1/\lim_{x\to\infty} x = \lim_{x\to\infty} (1/x) = 0$; therefore, $\lim_{x\to-\infty} b^x = 0$.

89. (a) By taking logarithms, we get $x \geqslant n \ln x$ for $x \geqslant (n + 1)!$ Rearrangement yields $x/\ln x \geqslant n$.

(b) Taking the reciprocal of the inequality in part (a) gives $\ln x/x \leqslant 1/n$ when $x \geqslant (n + 1)!$ When $n \to \infty$, $x \to \infty$; and both $\ln x$ and x are positive, so $0 \leqslant \lim_{x\to\infty} (\ln x/x) \leqslant \lim_{n\to\infty} (1/n) = 0$. Consequently, $\lim_{x\to\infty} (\ln x/x) = 0$.

93. Take the natural logarithm of both sides. We want to show that $(x - 2) \ln 3 > \ln 2 + 2 \ln x$ or $(x - 2) > \ln 2/\ln 3 + (2/\ln 3) \ln x$ or $x - (2/\ln 3) \ln x > \ln 2/\ln 3 + 2$. Let $f(x) = x - (2/\ln 3) \ln x$, so what we want to prove is $f(x) > 2 + \ln 2/\ln 3 \simeq 2.63$ if $x \geqslant 7$. Note that $f'(x) = 1 - (2/\ln 3)(1/x)$, so $f'(x) > 0$ for $x \geqslant 2/\ln 3 \simeq 1.82$. Thus, since f is increasing, $f(x) > f(7)$ if $x \geqslant 7$. But $f(7) = 7 - (2/\ln 3)\ln 7 \simeq 3.46 > 2.63$, so we get our result. We also find $f(6) = 6 - (2/\ln 3)\ln 6 \simeq 2.74 > 2.63$, so in fact the statement holds if $x \geqslant 6$. (Numerically experimenting shows that it is actually valid if $x \geqslant 5.8452\ldots$.)

TEST FOR CHAPTER 6

1. True or false.

   (a) $\exp a + \exp b = \exp(a + b)$.

   (b) The domain of $\ln |x|$ is only $x > 0$.

   (c) $\exp |x|$ has symmetry in the y-axis.

   (d) $\int_{-e}^{e} (dx/x) = \ln |x| \Big|_{-e}^{e}$.

   (e) $\log_a 1 = 0$ for all real $a > 0$.

2. Differentiate the following functions:

   (a) e/x

   (b) $(\exp x)(\log_{10} x)$

   (c) $\sqrt{5^x}$

   (d) $\ln (4x)$

3. Evaluate the following:

   (a) $\int 8e^{8t} dt$

   (b) $\int [4\, dx/(x - 5)]$

   (c) $\int_1^{\exp 2} (e/x) dx$

4. Sketch the graph of $y = \ln(x^2 + 1)$.

5. Write an equation of the form $y = \pm A \exp(\pm Bx)$ with the following properties, where $A > 0$ and $B > 0$ are constants.

   (a) The graph is increasing and concave downward.

   (b) The graph is decreasing and concave downward.

   (c) The graph is increasing and concave upward.

   (d) The graph is decreasing and concave upward.

6. Without using a calculator, approximate the following by using the fact that $\ln 2 \approx 0.693$, $\ln 3 \approx 1.099$, and $\ln 5 \approx 1.609$.

(a) $[(d/dx)\int_4^x t \ln t\, dt]|_3$

(b) $(d/du)(\ln u/u)|_{10}$

(c) $\int_1^5 x(\ln 10)dx$

7. Compute the following:

(a) $(d/dt)(\exp t + \ln t)$

(b) $(d/dx)[(4x^3 + 9)^2(8x^5)(2/x + 3)^3/(6)^5(x^3 - x)^6]$

(c) $(d/dx)[\exp(\ln(5x))]$

(d) $(d/dy)[e^y(y + 5)^3(y^2 - y)(y/e^{2y})^3]$

8. Multiple choice. More than one may be correct.

(a) The graph of $y = -\ln(x/2)$ has a _____ asymptote; it is _____ .

(i) vertical, $y = 0$

(ii) horizontal, $y = 0$

(iii) vertical, $x = 0$

(iv) horizontal, $x = 0$

(b) Which is the reflection of $e^{3x} + 3$ across the y-axis?

(i) $(1/e)^{3x} + 3$

(ii) $(1/e)^{3x} - 3$

(iii) $-e^{3x} + 3$

(iv) $e^{-3x} + 3$

9. If P dollars is invested at $r\%$ interest compounded daily, the amount after t years is given by $A = P(1 + r/365)^{365t}$.

(a) Make a graph to show how long it takes to double your money at $r\%$ interest.

(b) Suppose 365 is replaced by n in the formula. Simplify the formula if $n \to \infty$.

10. One day, the Gabber went to see his doctor because of a sore tongue. It was discovered that the cells had been cancerous due to excess gossiping. Also, it was known that the cell population doubled for every five pieces of juicy news coming out of the Gabber's mouth.

    (a) Find the growth factor for the cancerous cells in terms of the number of pieces of news n.

    (b) There are now 10^9 cells. At what rate are the cells growing?

    (c) 10^{12} cancer cells is lethal. How much more gossiping can the Gabber do?

ANSWERS TO CHAPTER TEST

1. (a) False; $\exp a + \exp b = \exp ab$

   (b) False; the domain includes $x < 0$.

   (c) True

   (d) False; $\ln |x|$ is not continuous at $x = 0$.

   (e) True

2. (a) $-e/x^2$

   (b) $(\exp x)(\log_{10}x + 1/x \ln 10)$

   (c) $(\ln 5)\sqrt{5^x}/2$

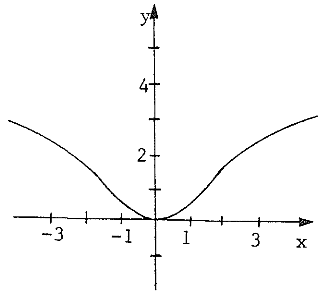
   (d) $1/x$

3. (a) $e^{8t} + C$

   (b) $4 \ln |x - 5| + C$

   (c) $2e$

4.



5. (a) $y = -A \exp(-Bx)$

(b) $y = -A \exp(Bx)$

(c) $y = A \exp(Bx)$

(d) $y = A \exp(-Bx)$

6. (a) 3.297

(b) -0.01302

(c) 27.624

7. (a) $\exp t + 1/t$

(b) $[(4x^3 + 9)^2(8x^5)(2/x + 3)^3/(6)^5(x^3 - x)^6][24x^2/(4x^3 + 9) + 5/x - (6/x^2)/(2/x + 3) - 6(3x^2 - 1)/(x^3 - x)]$

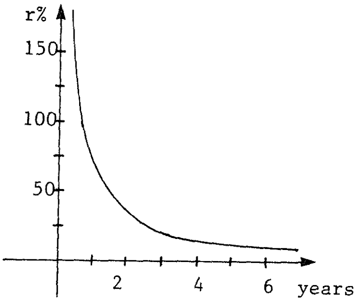
(c) 5

(d) $[e^y(y + 5)^3(y^2 - y)(y/e^{2y})^3][1 + 3/(y + 5) + (2y - 1)/(y^2 - y) + 3/y - 6]$

8. (a) iii

(b) i, iv

9. (a)



(b) $A = Pe^{rt}$

10. (a) $2^{n/5}$

(b) $(\ln 2/5)10^9$

(c) $5 \ln 1000/\ln 2$ pieces of news.

COMPREHENSIVE TEST FOR CHAPTERS 1-6 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) The area between the x-axis and the curve $x^2 - x$ on $[0,1]$ is $\int_0^1(x^2 - x)dx$.

   (b) $x^4 - 3x - 2 = 0$ has a solution for x between 1 and 2 .

   (c) $(d/dx)(1-x)^3 = 3(1 - x)^2$.

   (d) If f and g are both increasing on $[a,b]$, then $f + g$ is also increasing on $[a,b]$.

   (e) In general, $\int_a^b f(x)g(x)dx = \int_a^b f(x)dx \cdot \int_a^b g(x)dx$.

   (f) All continuous functions on $[a,b]$ are integrable.

   (g) If $n > 0$ is constant, then $(d/dx)n^x$ is xn^{x-1} .

   (h) $y = \sin x$ satisfies $y'' + y = 0$.

   (i) The slope of $r = \sin 2\theta$, graphed in the xy-plane, is $(2 \cos 2\theta)|_{\pi/5}$ at $\theta = \pi/5$.

2. Fill in the blank.

   (a) The fundamental theorem of calculus states that $\int_a^b f(x)dx =$ ________________ , where $F'(x) = f(x)$.

   (b) The point with cartesian coordinates $(-3,3)$ has polar coordinates ____________________ .

   (c) Simplify: $\exp(4 \ln(\exp x^2)) \cdot \ln(\exp(\exp(4x^2))) =$ ________________ .

   (d) dr/dw is the derivative of ____________________ with respect to ____________________ .

   (e) The point with polar coordinates $(-2,2)$ lies in the ________________ quadrant.

3. Differentiate the following functions of x.

(a) $(\sin 2x \cos x)^{3/2}$

(b) $e^{3x}/\ln(x+2)$

(c) The inverse of $x^5 + x^3 + 1$ at $x = 1$.

4. Multiple choice.

(a) The derivative of $\cos^{-1}x$ is:

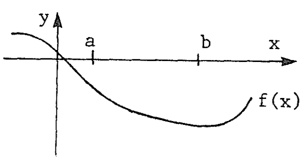
(i) $-\sin x$

(ii) $-1\sqrt{1 - x^2}$

(iii) $-1/\sqrt{x^2 - 1}$

(iv) $1/\sqrt{1 - x^2}$

(b)



According to the figure, $\int_a^b f(x)dx$ is:

(i) positive

(ii) negative

(iii) does not exist

(iv) unknown; insufficient information

(c) The antiderivative F of $f(x) = x^5 - x^3 + x - 2$ such that $F'(x) = f(x)$ and $F(1) = 0$ is:

(i) $x^6/6 - x^4/4 + x^2/2 - 2x + C$

(ii) $5x^4 - 3x^2 + 1 + C$

(iii) $x^6/6 - x^4/4 + x^2/2 - 2x - 19/12$

(iv) $x^6/6 - x^4/4 + x^2/2 - 2x + 19/12$

(d) If $f'(a) = 0$ and $f''(a) > 0$, then:

(i) $f(a)$ is a local minimum.

(ii) $f(a)$ is a local maximum.

(iii) $f(x)$ is increasing at $x = a$.

(iv) no conclusion can be made.

4. (e) Suppose $g'(t) > 0$ on $[a,b]$. Then $\int_b^a g(t)dt$ is:

   (i) positive

   (ii) negative

   (iii) zero

   (iv) unknown; need more information

5. (a) Differentiate x^x .

   (b) Evaluate $\int_1^2 [(3x^2 + 2x)/x]\,dx$.

   (c) Differentiate $3^x/\sin^{-1}x$.

   (d) Evaluate $\int [3dy/(1 + y^2)]$.

6. Consider the function $f(x) = (x + 3)/(x - 1)$.

   (a) Discuss its asymptotes .

   (b) Discuss its critical points.

   (c) Where is $f(x)$ increasing? Decreasing?

   (d) Where is $f(x)$ concave upward? Downward?

   (e) Sketch the graph of $f(x)$.

7. Short answer questions.

   (a) Compute $\sum_{i=6}^{101} [(i - 1)^2 - i^2]$.

   (b) Find $(d/dt)\int_t^3 xe^x \cos(x + 2)dx$.

   (c) Approximate $(1.11)^5$ by using the linear approximation.

   (d) Find dy/dx if $(y + 3)x = x^2y^3 - 5$.

   (e) Sketch the graph of $r = 3 \cos \theta$ in the xy-plane.

8. A tightrope walker needs to walk from the top of a 10 m. building to the ground and then back up to the top of a 20 m. building. The bases of the buildings are separated by 50 m. Where should the rope be placed on the ground between the buildings to minimize the distance walked?

9. Integration word problems.

(a) Find the area of the region bounded by $y = |x| + 2$ and $y = x^2$.

(b) A millionaire is spending money at the rate of $(e^x/3 + \cos x + 3)$ thousand dollars per hour. How much money does he spend between hours 2 and 3 ?

10. Mount Olympus is located 400 m. above sea level. Sitting on his throne, Zeus spots Mercury running off at sea level at the rate of 50m./min. When Zeus spots Mercury, they are separated by 500 m. How fast must Zeus rotate his head upward (in radians/minute) to keep his eyes on Mercury?

ANSWERS TO COMPREHENSIVE TEST

1. (a) False; $x^2 - x \leqslant 0$ on $[0,1]$, so the area is $-\int_0^1(x^2 - x)dx$.

(b) True; use the intermediate value theorem.

(c) False; the chain rule requires another factor of -1.

(d) False; let $f(x) = g(x) = x$ on $[-1,0]$.

(e) False; let $f(x) = x$ and $g(x) = x$ on $[0,1]$.

(f) True

(g) False; $(d/dx)n^x = (\ln n)n^x$.

(h) True

(i) False; the slope is $[\tan(\pi/5)(2\cos(2\pi/5)) + \sin(2\pi/5)]/[2\cos(2\pi/5) - \sin(2\pi/5)\tan(\pi/5)]$.

2. (a) $F(b) - F(a)$

(b) $(3\sqrt{2}, 3\pi/4)$

(c) $\exp(8x^2)$

(d) r;w

(e) fourth

3. (a) $(3/2)(\sin 2x \cos x)^{1/2}(2 \cos 2x \cos x - \sin 2x \sin x)$

 (b) $e^{3x}[3(x + 2)\ln(x + 2) - 1]/(x + 2)[\ln(x + 2)]^2$.

 (c) 1/8

4. (a) ii

 (b) ii

 (c) iv

 (d) i

 (e) iv

5. (a) $x^x(\ln x + 1)$

 (b) 13/2

 (c) $[(\ln 3)3^x \sin^{-1}x - 3^x/\sqrt{1 - x^2}]/(\sin^{-1}x)^2$

 (d) $3 \tan^{-1}y + C$

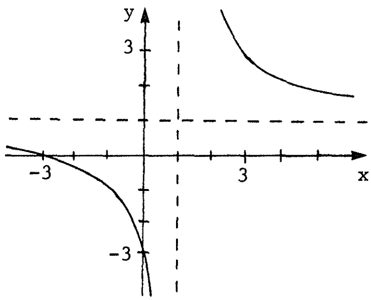
6. (a) Horizontal asymptote: $y = 1$; vertical asymptote: $x = 1$.

 (b) There are no critical points.

 (c) Decreasing on $(-\infty,1)$ and $(1,\infty)$

 (d) Concave upward on $(1,\infty)$; concave downward on $(-\infty,1)$

 (e)



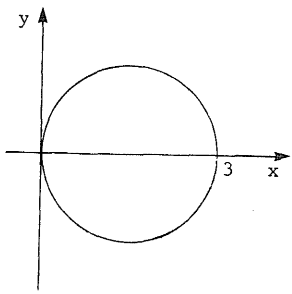
7. (a) -10176

 (b) $-te^t\cos(t + 2)$

 (c) 1.55

7. (d) $(y + 3 - 2xy^3)/(3x^2y^2 - x)$

(e)



8. 50/3 meters from the 10 m. building.

9. (a) 20/3

(b) $[(e^3 - e^2)/3 + \sin(3) - \sin(2) + 3]$ thousand dollars $\approx$ 6464 dollars

10. 0.08 radians/minute.