# Chapter 11

# Limits, L'Hôpital's Rule, and Numerical Methods

*Limits are used in both the theory and applications of calculus.*

Our treatment of limits up to this point has been rather casual. Now, having learned some differential and integral calculus, you should be prepared to appreciate a more detailed study of limits.

The chapter begins with formal definitions for limits and a review of computational techniques for limits of functions, including infinite and one-sided limits. The next topic is l'Hôpital's rule, which employs differentiation to compute limits. Infinite limits are used to study improper integrals. The chapter ends with some numerical methods involving limits of sequences.

## 11.1 Limits of Functions

*There are many kinds of limits, but they all obey similar laws.*

In Section 1.2, we discussed on an intuitive basis what $\lim_{x\to x_0} f(x)$ means and why the limit notion is important in understanding the derivative. Now we are ready to take a more careful look at limits.

Recall that the statement $\lim_{x\to x_0} f(x) = l$ means, roughly speaking, that $f(x)$ comes close to and remains arbitrarily close to $l$ as $x$ comes close to $x_0$. Thus we start with a positive "tolerance" $\varepsilon$ and try to make $|f(x) - l|$ less than $\varepsilon$ by requiring $x$ to be close to $x_0$. The closeness of $x$ to $x_0$ is to be measured by another positive number—mathematical tradition dictates the use of the Greek letter $\delta$ for this number. Here, then, is the famous $\varepsilon$-$\delta$ definition of a limit—it was first stated in this form by Karl Weierstrass around 1850.

**The $\varepsilon$-$\delta$ Definition of $\lim\limits_{x\to x_0} f(x)$**

Let $f$ be a function defined at all points near $x_0$, except perhaps at $x_0$ itself, and let $l$ be a real number. We say that *$l$ is the limit of $f(x)$ as $x$ approaches $x_0$* if, for every positive number $\varepsilon$, there is a positive number $\delta$ such that $|f(x) - l| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$. We write $\lim_{x\to x_0} f(x) = l$.

The purpose of giving the $\varepsilon$-$\delta$ definition is to enable us to be more precise in dealing with limits. Proofs of some of the basic theorems in this chapter and the next require this definition; however, practical computations can often be done without a full mastery of the theory. Your instructor should tell you how much theory you are expected to know.

The $\varepsilon$-$\delta$ definition of limit is illustrated in Figure 11.1.1. We shade the region consisting of those $(x, y)$ for which:

1. $|x - x_0| > \delta$ (region I in Fig. 11.1.1(b));
2. $x = x_0$ (the vertical line II in Fig. 11.1.1(b));
3. $x \neq x_0$, $|x - x_0| < \delta$, and $|y - l| < \varepsilon$ (region III in Fig. 11.1.1(b)).

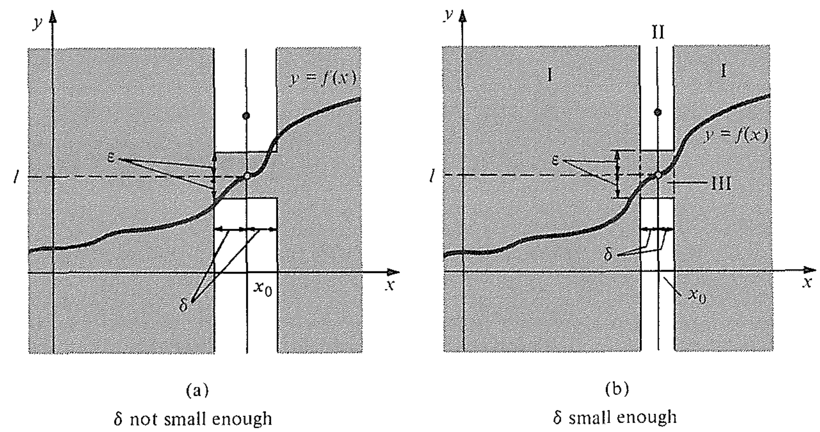



**Figure 11.1.1.** When $\lim_{x\to x_0} f(x) = l$, we can, for any $\varepsilon > 0$, catch the graph of $f$ in the shaded region by making $\delta$ small enough. The value of $f$ at $x_0$ is irrelevant, since the line $x = x_0$ is always "shaded."

If $\lim_{x\to x_0} f(x) = l$, then we can catch the graph of $f$ in the shaded region by making $\delta$ small enough—that is, by making the unshaded strips sufficiently narrow.

Notice the statement $x \neq x_0$ in the definition. This means that the limit depends only upon the values of $f(x)$ for $x$ *near* $x_0$, and not on $f(x_0)$ itself. (In fact, $f(x_0)$ might not even be defined.)

Here are two examples of how the $\varepsilon$-$\delta$ condition is verified.

**Example 1** (a) Prove that $\lim_{x\to 2}(x^2 + 3x) = 10$ using the $\varepsilon$-$\delta$ definition. (b) Prove that $\lim_{x\to a}\sqrt{x} = \sqrt{a}$, where $a > 0$, using the $\varepsilon$-$\delta$ definition.

**Solution** (a) Here $f(x) = x^2 + 3x$, $x_0 = 2$, and $l = 10$. Given $\varepsilon > 0$ we must find $\delta > 0$ such that $|f(x) - l| < \varepsilon$ if $|x - x_0| < \delta$.

A useful general rule is to write down $f(x) - l$ and then to express it in terms of $x - x_0$ as much as possible, by writing $x = (x - x_0) + x_0$. In our case we replace $x$ by $(x - 2) + 2$:

$$\begin{aligned} f(x) - l &= x^2 + 3x - 10 \\ &= (x - 2 + 2)^2 + 3(x - 2 + 2) - 10 \\ &= (x - 2)^2 + 4(x - 2) + 4 + 3(x - 2) + 6 - 10 \\ &= (x - 2)^2 + 7(x - 2). \end{aligned}$$

Now we use the properties $|a + b| \leqslant |a| + |b|$ and $|a^2| = |a|^2$ of the absolute value to note that

$$|f(x) - l| \leqslant |x - 2|^2 + 7|x - 2|.$$

If this is to be less than $\varepsilon$, we should choose $\delta$ so that $\delta^2 + 7\delta \leqslant \varepsilon$. We may require at the outset that $\delta \leqslant 1$. Then $\delta^2 \leqslant \delta$, so $\delta^2 + 7\delta \leqslant 8\delta$. Hence we pick $\delta$ so that $\delta \leqslant 1$ and $\delta \leqslant \varepsilon/8$.

With this choice of $\delta$, we shall now verify that $|f(x) - l| < \varepsilon$ whenever

$|x - x_0| < \delta$. In our case $|x - x_0| < \delta$ means $|x - 2| < \delta$, so for such an $x$,

$$\begin{aligned}|f(x) - l| &\leqslant |x - 2|^2 + 7|x - 2| \\ &< \delta^2 + 7\delta \\ &\leqslant \delta + 7\delta \\ &= 8\delta \\ &\leqslant \varepsilon,\end{aligned}$$

and so $|f(x) - l| < \varepsilon$.

(b) Here $f(x) = \sqrt{x}$, $x_0 = a$, and $l = \sqrt{a}$. Given $\varepsilon > 0$ we must find a $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ when $|x - a| < \delta$. To do this we write $\sqrt{x} - \sqrt{a} = (x - a)/(\sqrt{x} + \sqrt{a})$. Since $f$ is only defined for $x \geqslant 0$, we confine our attention to these $x$'s. Then

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} \leqslant \frac{|x - a|}{\sqrt{a}} \quad \text{(decreasing the denominator increases the fraction).}$$

Thus, given $\varepsilon > 0$ we can choose $\delta = \sqrt{a}\,\varepsilon$; then $|x - a| < \delta$ implies $|\sqrt{x} - \sqrt{a}| < \varepsilon$, as required. ▲

In practice, it is usually more efficient to use the laws of limits than the $\varepsilon$-$\delta$ definition, to evaluate limits. These laws were presented in Section 1.3 and are recalled here for reference.

## Basic Properties of Limits

Assume that $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist:

*Sum rule*:

$$\lim_{x \to x_0} [f(x) + g(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

*Product rule*:

$$\lim_{x \to x_0} [f(x)g(x)] = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x).$$

*Reciprocal rule*:

$$\lim_{x \to x_0} [1/f(x)] = 1/\lim_{x \to x_0} f(x) \qquad \text{if} \qquad \lim_{x \to x_0} f(x) \neq 0.$$

*Constant function rule*:

$$\lim_{x \to x_0} c = c.$$

*Identity function rule*:

$$\lim_{x \to x_0} x = x_0.$$

*Replacement rule*: If the functions $f$ and $g$ agree for all $x$ near $x_0$ (not necessarily including $x = x_0$), then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x).$$

*Rational functional rule*: If $P$ and $Q$ are polynomials and $Q(x_0) \neq 0$, then $P/Q$ is continuous at $x_0$; i.e.,

$$\lim_{x \to x_0} [P(x)/Q(x)] = P(x_0)/Q(x_0).$$

*Composite function rule*: If $h$ is continuous at $\lim_{x \to x_0} f(x)$, then

$$\lim_{x \to x_0} h(f(x)) = h\Big(\lim_{x \to x_0} f(x)\Big).$$

The properties of limits can all be proved using the $\varepsilon$-$\delta$ definition. The theoretically inclined student is urged to do so by studying Exercises 75–77 at the end of this section.

Let us recall how to use the properties of limits in specific computations.

**Example 2** Using the fact that $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\right)=0$, find $\lim_{\theta\to 0}\cos\left(\frac{1-\cos\theta}{\theta}\right)$.

**Solution** The composite function rule says that $\lim_{x\to x_0}h(f(x))=h(\lim_{x\to x_0}f(x))$ if $h$ is continuous at $\lim_{x\to x_0}f(x)$. We let $f(\theta)=(1-\cos\theta)/\theta$, and $h(\theta)=\cos\theta$ so that $h(f(\theta))=\cos[(1-\cos\theta)/\theta]$. Hence the required limit is

$$\lim_{\theta\to 0}h(f(\theta))=h\left(\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\right)=\cos 0=1,$$

since cos is continuous at $\theta=0$. ▲

**Example 3** Find (a) $\lim_{x\to 2}\left(\frac{x^2-5x+6}{x-2}\right)$ and (b) $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)$.

**Solution** (a) Since the denominator vanishes at $x=2$, we cannot plug in this value. The numerator may be factored, however, and for any $x\neq 2$ our function is

$$\frac{x^2-5x+6}{x-2}=\frac{(x-3)(x-2)}{x-2}=x-3.$$

Thus, by the replacement rule,

$$\lim_{x\to 2}\frac{x^2-5x+6}{x-2}=\lim_{x\to 2}(x-3)=2-3=-1.$$

(b) Again we cannot plug in $x=1$. However, we can rationalize the denominator by multiplying numerator and denominator by $\sqrt{x}+1$. Thus (if $x\neq 1$):

$$\frac{x-1}{\sqrt{x}-1}=\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\frac{(x-1)(\sqrt{x}+1)}{x-1}=\sqrt{x}+1.$$

As $x$ approaches 1, this approaches 2, so $\lim_{x\to 1}[(x-1)/(\sqrt{x}-1)]=2$. ▲

Limits of the form $\lim_{x\to\pm\infty}f(x)$, called *limits at infinity*, are dealt with by a modified version of the ideas above. Let us motivate the ideas by a physical example.

Let $y=f(t)$ be the length, at time $t$, of a spring with a bobbing mass on the end. If no frictional forces act, the motion is sinusoidal, given by an equation of the form $f(t)=y_0+a\cos\omega t$.[1] In reality, a spring does not go on bobbing forever; frictional forces cause *damping*, and the actual motion has the form

$$y=f(t)=y_0+ae^{-bt}\cos\omega t, \tag{1}$$

where $b$ is positive. A graph of this function is sketched in Fig. 11.1.2.

As time passes, we observe that the length becomes and remains arbitrarily near to the equilibrium length $y_0$. (Even though $y=y_0$ already for $t=\pi/2\omega$, this is not the same thing because the length does not yet *remain* near $y_0$.) We express this mathematical property of the function $f$ by writing $\lim_{t\to\infty}f(t)=y_0$. The limiting behavior appears graphically as the fact that the

[1] This is derived in Section 8.1., but if you have not studied that section, you should simply take for granted the formulas given here.

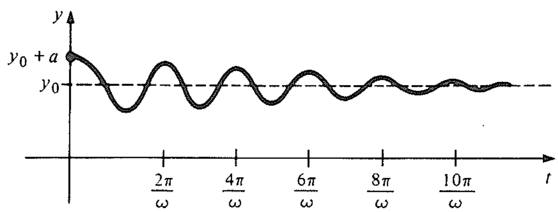



**Figure 11.1.2.** The motion of a damped spring has the form $y = f(t) = y_0 + ae^{-bt}\cos\omega t$.

graph of $f$ remains closer and closer to the line $y = y_0$ as we look farther to the right.

The precise definition is analogous to that for $\lim_{x\to x_0} f(x)$. As is usual in our general definitions, we denote the independent variable by $x$ rather than $t$.

## The $\varepsilon$-$A$ Definition of $\lim_{x\to+\infty} f(x)$

Let $f$ be a function whose domain contains an interval of the form $(a, \infty)$. We say that a real number $l$ *is the limit of* $f(x)$ *as* $x$ *approaches* $\infty$ if, for every positive number $\varepsilon$, there is a number $A > a$ such that $|f(x) - l| < \varepsilon$ whenever $x > A$. We write $\lim_{x\to\infty} f(x) = l$.

A similar definition is used for $\lim_{x\to-\infty} f(x) = l$.

When $\lim_{x\to\infty} f(x) = l$ or $\lim_{x\to-\infty} f(x) = l$, the line $y = l$ is called a *horizontal asymptote* of the graph $y = f(x)$.

We illustrate this definition in Figs. 11.1.3 and 11.1.4 by shading the region consisting of those points $(x, y)$ for which $x \leqslant A$ or for which $x > A$ and $|y - l| < \varepsilon$. If $\lim_{x\to\infty} f(x) = l$, we should be able to "catch" the graph of $f$ in this region by choosing $A$ large enough—that is, by sliding the point $A$ sufficiently far to the right.

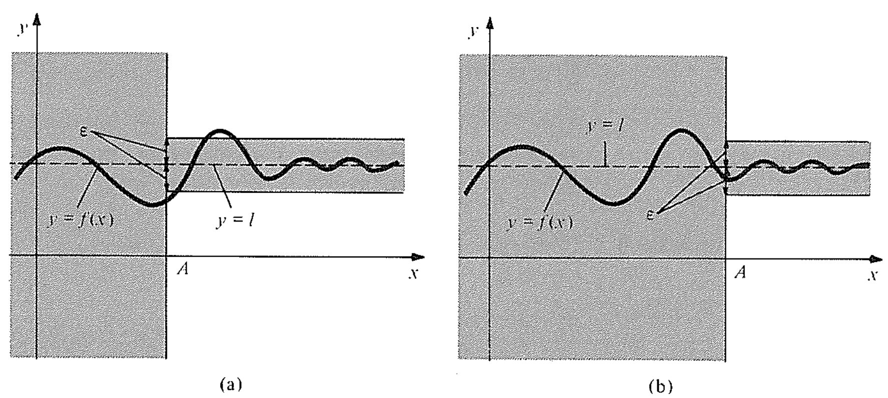



**Figure 11.1.3.** When $\lim_{x\to\infty} f(x) = l$, we can catch the graph in the shaded region by sliding the region sufficiently far to the right. This is true no matter how small $\varepsilon$ may be.

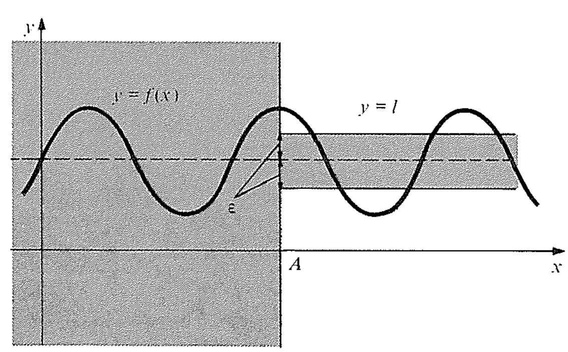



**Figure 11.1.4.** When it is not true that $\lim_{x\to\infty} f(x) = l$, then for some $\varepsilon$, we can never catch the graph of $f$ in the shaded region, no matter how far to the right we slide the region.

There is an analogous definition for $\lim_{x\to-\infty} f(x)$ in which we require a number $A$ (usually large and negative) such that $|f(x) - l| < \varepsilon$ if $x < A$.

**Example 4** Prove that $\lim_{x\to\infty} \dfrac{x^2}{1+x^2} = 1$ by using the $\varepsilon$-$A$ definition.

**Solution** Given $\varepsilon > 0$, we must choose $A$ such that $|x^2/(1+x^2) - 1| < \varepsilon$ for $x > A$. We have

$$\left|\frac{x^2}{1+x^2} - 1\right| = \left|\frac{x^2 - 1 - x^2}{1+x^2}\right| = \frac{1}{|1+x^2|} < \frac{1}{x^2}.$$

To make this less than $\varepsilon$, we observe that $1/x^2 < \varepsilon$ whenever $x > 1/\sqrt{\varepsilon}$, so we may choose $A = 1/\sqrt{\varepsilon}$. (See Fig. 11.1.5.) ▲

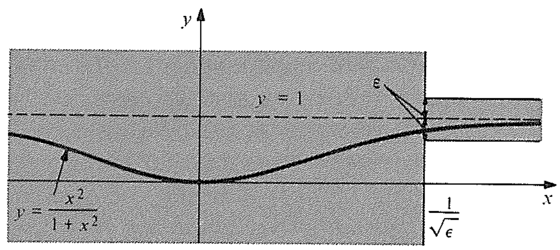



**Figure 11.1.5.** Illustrating the fact that $\lim_{x\to\infty}[x^2/(1+x^2)] = 1$.

At the beginning of Section 6.4. we stated several limit properties for $e^x$ and $\ln x$. Some simple cases can be verified by the $\varepsilon$-$A$ definition; others are best handled by l'Hôpital's rule, which is introduced in the next section.

**Example 5** Use the $\varepsilon$-$A$ definition to show that for $k < 0$, $\lim_{x\to\infty} e^{kx} = 0$.

**Solution** First of all, we note that $f(x) = e^{kx}$ is a decreasing positive function. Given $\varepsilon > 0$, we wish to find $A$ such that $x > A$ implies $e^{kx} < \varepsilon$. Taking logarithms of the last inequality gives $kx < \ln\varepsilon$, or $x > (\ln\varepsilon)/k$. So we may let $A = (\ln\varepsilon)/k$. (If $\varepsilon$ is small, $\ln\varepsilon$ is a large negative number.) ▲

The examples above illustrate the $\varepsilon$-$A$ method, but limit computations are usually done using laws analogous to those for limits as $x \to x_0$, which are stated in the box on the facing page.

**Example 6** Find (a) $\lim_{x\to\infty}\left(\dfrac{1}{x} + \dfrac{3}{x^2} + 5\right)$ and (b) $\lim_{x\to\infty}\dfrac{8x+2}{3x-1}$.

**Solution** (a) We have

$$\lim_{x\to\infty}\left(\frac{1}{x} + \frac{3}{x^2} + 5\right) = \lim_{x\to\infty}\frac{1}{x} + 3\left(\lim_{x\to\infty}\frac{1}{x}\right)^2 + \lim_{x\to\infty} 5 = 0 + 3\cdot 0^2 + 5 = 5.$$

(b) We cannot simply apply the quotient rule, since the limits of the numerator and denominator do not exist. Instead we use a trick: if $x \neq 0$, we can multiply the numerator and denominator by $1/x$ to obtain

$$\frac{8x+2}{3x-1} = \frac{8 + (2/x)}{3 - (1/x)} \quad \text{for} \quad x \neq 0.$$

By the replacement rule (with $A = 0$), we have

$$\lim_{x\to\infty}\frac{8x+2}{3x-1} = \lim_{x\to\infty}\frac{8+(2/x)}{3-(1/x)} = \frac{8+0}{3-0} = \frac{8}{3}.$$

(The values of $(8x+2)/(3x-1)$ for $x = 10^2, 10^4, 10^6, 10^8$ are $2.682\ldots$, $2.66682\ldots$, $2.6666682\ldots$, $2.666666682\ldots$.) ▲

## Limits of Functions as $x$ Approaches $\infty$

*Constant function rule*:

$$\lim_{x\to\infty} c = c.$$

*$1/x$ rule*:

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Assuming that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, we have these additional rules:

*Sum rule*:

$$\lim_{x\to\infty}\left[f(x)+g(x)\right] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

*Product rule*:

$$\lim_{x\to\infty}\left[f(x)g(x)\right] = \lim_{x\to\infty} f(x) \lim_{x\to\infty} g(x).$$

*Quotient rule*: If $\lim_{x\to\infty} g(x) \neq 0$, then

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}.$$

*Replacement rule*: If for some real number $A$, the functions $f(x)$ and $g(x)$ agree for all $x > A$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x).$$

*Composite function rule*: If $h$ is continuous at $\lim_{x\to\infty} f(x)$, then

$$\lim_{x\to\infty} h(f(x)) = h\left(\lim_{x\to\infty} f(x)\right).$$

All these rules remain true if we replace $\infty$ by $-\infty$ (and "$> A$" by "$< A$" in the replacement rule).

The method used in Example 6 also shows that

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0} = \frac{a_n}{b_n}$$

as long as $b_n \neq 0$.

**Example 7** Find $\lim_{x\to\infty}(\sqrt{x^2+1} - x)$. Interpret the result geometrically in terms of right triangles.

**Solution** Multiplying the numerator and denominator by $\sqrt{x^2+1} + x$ gives

$$\sqrt{x^2+1} - x = \left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}$$

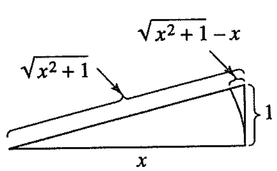



**Figure 11.1.6.** As the length $x$ goes to $\infty$, the difference $\sqrt{x^2+1} - x$ between the lengths of the hypotenuse and the long leg goes to zero.

As $x \to \infty$, the denominator becomes arbitrarily large, so we find that $\lim_{x\to\infty}(\sqrt{x^2+1} - x) = 0$. For a geometric interpretation, see Fig. 11.1.6. ▲

**Example 8** Find the horizontal asymptotes of $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. Sketch.

**Solution** We find

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+1/x^2}} = 1$$

and

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{-\sqrt{x^2}}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{1+1/x^2}} = -1$$

(in the second limit we may take $x<0$, so $x = -\sqrt{x^2}$). Hence the horizontal asymptotes are the lines $y = \pm 1$. See Fig. 11.1.7. ▲

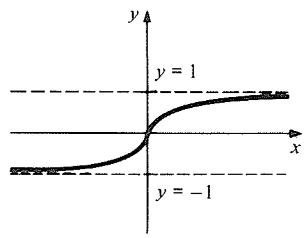
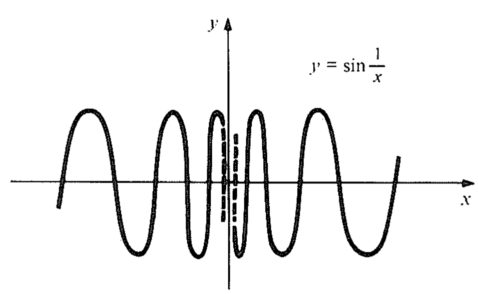



**Figure 11.1.7.** The curve $y = x/\sqrt{x^2+1}$ has the lines $y=-1$ and $y=1$ as horizontal asymptotes.

Consider the limits $\lim_{x\to 0}\sin(1/x)$ and $\lim_{x\to 0}(1/x^2)$. Neither limit exists, but the functions $\sin(1/x)$ and $1/x^2$ behave quite differently as $x\to 0$. (See Fig. 11.1.8.) In the first case, for $x$ in the interval $(-\delta, \delta)$, the quantity $1/x$ ranges



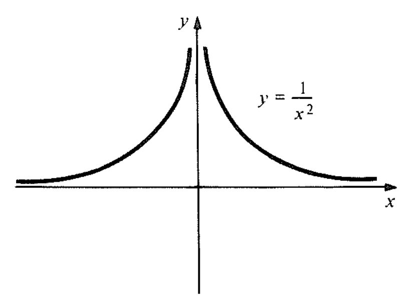




**Figure 11.1.8.** $\lim_{x\to 0} f(x)$ does not exist for either of these functions.

over all numbers with absolute value greater than $1/\delta$, and $\sin(1/x)$ oscillates back and forth infinitely often. The function $\sin(1/x)$ takes each value between $-1$ and 1 infinitely often but remains close to no particular number. In the case of $1/x^2$, the value of the function is again near no particular number, but there is a definite "trend" to be seen; as $x$ comes nearer to zero, $1/x^2$ becomes a larger positive number; we may say that $\lim_{x\to 0}(1/x^2) = \infty$.

Here is a precise definition.

### The $B$-$\delta$ Definition of $\lim_{x\to x_0} f(x) = \infty$

Let $f$ be a function defined in an interval about $x_0$, except possibly at $x_0$ itself. We say that $f(x)$ *approaches* $\infty$ as $x$ *approaches* $x_0$ if, given any real number $B$, there is a positive number $\delta$ such that for all $x$ satisfying $|x - x_0| < \delta$ and $x \neq x_0$, we have $f(x) > B$. We write $\lim_{x\to x_0} f(x) = \infty$.

The definition of $\lim_{x\to x_0} f(x) = -\infty$ is similar: replace $f(x) > B$ in the $B$-$\delta$ definition by $f(x) < B$.

**Remarks**

1. In the preceding definition, we usually think of $\delta$ as being small, while $B$ is large positive if the limit is $\infty$ and large negative if the limit is $-\infty$.
2. If $\lim_{x\to x_0} f(x)$ is equal to $\pm\infty$, we still may say that "$\lim_{x\to x_0} f(x)$ does not exist," since it does not approach any particular number.
3. One can define the statements $\lim_{x\to\infty} f(x) = \pm\infty$ in an analogous way.

The following test provides a useful technique for detecting "infinite limits."

### Reciprocal Test for $\lim_{x\to x_0} f(x) = \infty$

Let $f$ be defined in an open interval about $x_0$, except possibly at $x_0$ itself. Then $\lim_{x\to x_0} f(x) = \infty$ if:

1. For all $x \neq x_0$ in some interval about $x_0$, $f(x)$ is positive; and
2. $\lim_{x\to x_0}[1/f(x)] = 0$.

Similarly, if $f(x)$ is negative and $\lim_{x\to x_0}[1/f(x)] = 0$, then $\lim_{x\to x_0} f(x) = -\infty$.

The complete proof of the reciprocal test is left to the reader in Exercise 79. However, the basic idea is very simple: $f(x)$ is very large if and only if $1/f(x)$ is very small.

A similar result is true for limits of the form $\lim_{x\to\infty} f(x)$; namely, if $f(x)$ is positive for large $x$ and $\lim_{x\to\infty}[1/f(x)] = 0$, then $\lim_{x\to\infty} f(x) = \infty$.

**Example 9** Find the following limits: (a) $\lim_{x\to 1} \dfrac{1}{(x-1)^2}$; (b) $\lim_{x\to\infty} \dfrac{1-x^2}{x^{3/2}}$.

**Solution** (a) We note that $1/(x-1)^2$ is positive for all $x \neq 1$. We look at the reciprocal: $\lim_{x\to 1}(x-1)^2 = 0$; thus, by the reciprocal test, $\lim_{x\to 1}[1/(x-1)^2] = \infty$.
(b) For $x > 1$, $(1-x^2)/x^{3/2}$ is negative. Now we have

$$\lim_{x\to\infty} \frac{x^{3/2}}{1-x^2} = \lim_{x\to\infty} \frac{1}{x^{-3/2}-x^{1/2}} = \lim_{x\to\infty} \frac{1}{x^{1/2}} \frac{1}{1/x^2-1}$$

$$= \lim_{x\to\infty} \frac{1}{x^{1/2}} \lim_{x\to\infty} \frac{1}{1/x^2-1} = 0\cdot(-1) = 0,$$

so $\lim_{x\to\infty}[(1-x^2)/x^{3/2}] = -\infty$, by the reciprocal test. ▲

If we look at the function $f(x) = 1/(x-1)$ near $x_0 = 1$ we find that $\lim_{x\to 1}[1/f(x)] = 0$, but $f(x)$ has different signs on opposite sides of 1, so $\lim_{x\to 1}[1/(x-1)]$ is neither $\infty$ nor $-\infty$. This example suggests the introduction of the notion of a "one-sided limit." Here is the definition.

### One-Sided Limits

Let $f$ be defined for all $x$ in an interval of the form $(x_0, b)$. We say that $f(x)$ *approaches* $l$ *as* $x$ *approaches* $x_0$ *from the right* if, for any positive number $\varepsilon$, there is a positive number $\delta$ such that for all $x$ such that $x_0 < x < x_0 + \delta$, we have $|f(x) - l| < \varepsilon$. We write $\lim_{x\to x_0+} f(x) = l$.

A similar definition holds for the limit of $f(x)$ as $x$ approaches $x_0$ from the left; this limit is written as $\lim_{x\to x_0-} f(x) = l$.

In the definition of a one-sided limit, only the values of $f(x)$ for $x$ on one side of $x_0$ are taken into account. Precise definitions of statements like $\lim_{x\to x_0+} f(x) = \infty$ are left to you. We remark that the reciprocal test extends to one-sided limits.

**Example 10** Find (a) $\lim_{x\to 1+} \dfrac{1}{(1-x)}$, (b) $\lim_{x\to 1-} \dfrac{1}{(1-x)}$,

(c) $\lim_{x\to 0+} \dfrac{(x^2+2)|x|}{x}$, and (d) $\lim_{x\to 0-} \dfrac{(x^2+2)|x|}{x}$.

**Solution** (a) For $x > 1$, we find that $1/(1-x)$ is negative, and we have $\lim_{x\to 1}(1-x) = 0$, so $\lim_{x\to 1+}[1/(1-x)] = -\infty$. Similarly, $\lim_{x\to 1-}[1/(1-x)] = +\infty$, so we get $+\infty$ for (b).
(c) For $x$ positive, $|x|/x = 1$, so $(x^2+2)|x|/x = x^2+2$ for $x > 0$. Thus the limit is $0^2 + 2 = 2$.
(d) For $x < 0$, $|x|/x = -1$, so

$$\lim_{x\to 0-} \frac{(x^2+2)|x|}{x} = -\lim_{x\to 0-}\left[x^2+2\right] = -2. \blacktriangle$$

If a one-sided limit of $f(x)$ at $x_0$ is equal to $\infty$ or $-\infty$, then the graph of $f$ lies closer and closer to the line $x = x_0$; we call this line a *vertical asymptote* of the graph.

**Example 11** Find the vertical asymptotes and sketch the graph of

$$f(x) = \frac{1}{(x-1)(x-2)^2}.$$

**Solution** Vertical asymptotes occur where $\lim_{x\to x_0\pm}[1/f(x)] = 0$; in this case, they occur at $x_0 = 1$ and $x_0 = 2$. We observe that $f(x)$ is negative on $(-\infty, 1)$, positive on $(1, 2)$, and positive on $(2, \infty)$. Thus we have $\lim_{x\to 1-} f(x) = -\infty$, $\lim_{x\to 1+} f(x) = \infty$, $\lim_{x\to 2-} f(x) = \infty$, and $\lim_{x\to 2+} f(x) = \infty$. The graph of $f$ is sketched in Fig. 11.1.9. $\blacktriangle$

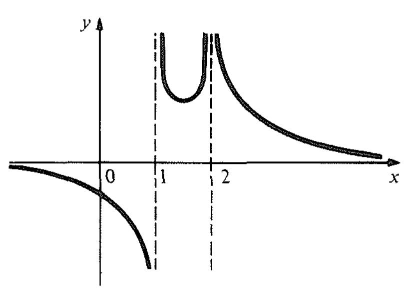



**Figure 11.1.9.** The graph $y = 1/(x-1)(x-2)^2$ has the lines $x = 1$ and $x = 2$ as vertical asymptotes.

We conclude this section with an additional law of limits. In the next sections we shall consider various additional techniques and principles for evaluating limits.

## Comparison Test

1. If $\lim_{x\to x_0} f(x) = 0$ and $|g(x)| \leqslant |f(x)|$ for all $x$ near $x_0$ with $x \neq x_0$, then $\lim_{x\to x_0} g(x) = 0$.
2. If $\lim_{x\to\infty} f(x) = 0$ and $|g(x)| \leqslant |f(x)|$ for all large $x$, then $\lim_{x\to\infty} g(x) = 0$.

Some like to call this the "sandwich principle" since $g(x)$ is sandwiched between $-|f(x)|$ and $|f(x)|$ which are squeezing down on zero as $x \to x_0$ (or $x \to \infty$ in case 2).

**Example 12** (a) Establish comparison test 1 using the $\varepsilon$-$\delta$ definition of limit.

(b) Show that $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] = 0$.

**Solution** (a) Given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x)| < \varepsilon$ if $|x - x_0| < \delta$, by the assumption that $\lim_{x\to x_0} f(x) = 0$. Given that $\varepsilon > 0$, this same $\delta$ also gives $|g(x)| < \varepsilon$ if $|x - x_0| < \delta$ since $|g(x)| \leqslant |f(x)|$. Hence $g$ has limit zero as $x \to x_0$ as well.

(b) Let $g(x) = x\sin(1/x)$ and $f(x) = x$. Then $|g(x)| \leqslant |x|$ for all $x \neq 0$, since $|\sin(1/x)| \leqslant 1$, so the comparison test applies. Since $x$ approaches 0 as $x \to 0$, so does $g(x)$. ▲

## Exercises for Section 11.1

Verify the limit statements in Exercises 1–4 using the $\varepsilon$-$\delta$ definition.

1. $\lim_{x\to a} x^2 = a^2$
2. $\lim_{x\to 3}(x^2 - 2x + 4) = 7$
3. $\lim_{x\to 3}(x^3 + 2x^2 + 2) = 47$
4. $\lim_{x\to 3}(x^3 + 2x) = 33$
5. Using the fact that $\lim_{\theta\to 0}[(\tan\theta)/\theta] = 1$, find $\lim_{\theta\to 0}\exp[(3\tan\theta)/\theta]$.
6. Using the fact that $\lim_{\theta\to 0}[(\sin\theta)/\theta] = 1$, find $\lim_{\theta\to 0}\cos[(\pi\sin\theta)/(4\theta)]$.

Find the limits in Exercises 7–12.

7. $\lim\limits_{x\to 3}(x^2 - 2x + 2)$
8. $\lim\limits_{x\to -2}\dfrac{(x^2-4)}{x^2+4}$
9. $\lim\limits_{x\to 2}\dfrac{(x^2-4)}{(x^2-5x+6)}$
10. $\lim\limits_{x\to 27}\dfrac{\sqrt[3]{x}-3}{x-27}$
11. $\lim\limits_{x\to 0}\dfrac{(3+x)^2-9}{x}$
12. $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-3x+2}$

Verify the limit statements in Exercises 13–16 using the $\varepsilon$-$A$ definition.

13. $\lim\limits_{x\to\infty}\dfrac{1+x^3}{x^3} = 1$
14. $\lim\limits_{x\to\infty}\dfrac{3x}{x^2+2} = 0$
15. $\lim\limits_{x\to\infty}(1 + e^{-3x}) = 1$
16. $\lim\limits_{x\to\infty}\dfrac{1}{\ln x} = 0$

Find the limits in Exercises 17–24.

17. $\lim\limits_{x\to\infty}\left(\dfrac{3}{x} + \dfrac{5}{x^2} - 2\right)$
18. $\lim\limits_{x\to\infty}\left(\dfrac{8}{x^2} - \dfrac{1}{x^3} + 5\right)$
19. $\lim\limits_{x\to\infty}\dfrac{10x^2-2}{15x^2-3}$
20. $\lim\limits_{x\to\infty}\dfrac{-4x+3}{x+2}$
21. $\lim\limits_{x\to\infty}\dfrac{3x^2+2x+4}{5x^2+x+7}$
22. $\lim\limits_{x\to\infty}\dfrac{x^2+xe^{-x}}{6x^2+2}$
23. $\lim\limits_{x\to\infty}\dfrac{x+2+1/x}{2x+3+2/x}$
24. $\lim\limits_{x\to\infty}\dfrac{x-3-1/x^2}{2x+5+1/x^2}$
25. Find $\lim_{x\to\infty}[\sqrt{x^2+a^2} - x]$ and interpret your answer geometrically.
26. Find $\lim_{x\to\infty}[\sqrt{c^2x^2+1} - cx]$ and interpret your answer geometrically.
27. Find the horizontal asymptotes of the graph of $\sqrt{x^2+1} - (x+1)$. Sketch.
28. Find the horizontal asymptotes of the graph $y = (x+1)/\sqrt{x^2+2}$. Sketch.

Find the limits in Exercises 29–32 using the reciprocal test.

29. $\lim\limits_{x\to 2}\dfrac{1}{(x-2)^2}$
30. $\lim\limits_{x\to 2}\dfrac{x^2}{(x-2)^4}$
31. $\lim\limits_{x\to\infty}\dfrac{x^2+2}{\sqrt{x}}$
32. $\lim\limits_{x\to\infty}\dfrac{x^4+8}{x^{5/2}}$

Find the one-sided limits in Exercises 33–40.

33. $\lim\limits_{x\to 2+}\dfrac{x^2-4}{(x-2)^2}$
34. $\lim\limits_{x\to 2-}\dfrac{x^2-4}{(x-2)^2}$
35. $\lim\limits_{x\to 0-}\dfrac{(x-1)(x-2)}{x(x+1)(x+2)}$
36. $\lim\limits_{x\to 1+}\dfrac{x(x+3)}{(x-1)(x-2)}$
37. $\lim\limits_{x\to 0+}\dfrac{(x^3-1)|x|}{x}$
38. $\lim\limits_{x\to 0-}\dfrac{(x^4+2)|x|}{x}$
39. $\lim\limits_{x\to \frac{1}{2}-}\dfrac{2x-1}{\sqrt{(2x-1)^2}}$
40. $\lim\limits_{x\to \frac{1}{2}+}\dfrac{\sqrt{(2x-1)^2}}{x-1/2}$

Find the vertical and horizontal asymptotes of the functions in Exercises 41–44 and sketch their graphs.

41. $f(x) = \dfrac{1}{x^2-5x+6}$
42. $f(x) = \dfrac{1}{2x+3}$
43. $f(x) = \dfrac{1}{x^2-1}$
44. $f(x) = \dfrac{x^2}{x^2-1}$
45. (a) Establish the comparison test 2 using the $\varepsilon$-$A$ definition of limit. (b) Use (a) to find
$$\lim_{x\to\infty}\left[\frac{1}{x}\sin\left(\frac{1}{x}\right)\right].$$
46. (a) Use the $B$-$\delta$ definition of limit to show that if $\lim_{x\to x_0} f(x) = \infty$ and $g(x) \geqslant f(x)$ for $x$ close to $x_0$, $x \neq x_0$, then $\lim_{x\to x_0} g(x) = \infty$. (b) Use (a) to show that $\lim_{x\to 1}[(1+\cos^2 x)/(1-x)^2] = \infty$.

Find the limits in Exercises 47–60.

47. $\lim_{x\to 1} \dfrac{3+4x}{4+5x}$
48. $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$
49. $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$
50. $\lim_{x\to 2} \dfrac{x-2}{x^2+3x+2}$
51. $\lim_{x\to -3} \dfrac{x^2+2x-3}{x^2+x-6}$
52. $\lim_{x\to 1} \dfrac{x^n-1}{x-1}$
53. $\lim_{x\to -1} \dfrac{x^{2n+1}+1}{x+1}$
54. $\lim_{x\to 2} \dfrac{x-2}{\sqrt{x}-\sqrt{2}}$
55. $\lim_{x\to 2} \dfrac{x^2+3x+6}{9x-1}$
56. $\lim_{x\to\infty} \sin\left(\dfrac{1}{x}\right)$
57. $\lim_{x\to 1+} \dfrac{e^x-1}{x-1}$
58. $\lim_{x\to\infty} \sin\left(\dfrac{\pi x^2+4}{6x^2+9}\right)$
59. $\lim_{x\to 1-} \dfrac{\ln 2x}{x-1}$
60. $\lim_{x\to -\infty} \ln(x^2)$

Find the horizontal and vertical asymptotes of the functions in Exercises 61–64.

61. $y = \dfrac{x}{x^2-1}$
62. $y = \dfrac{(x+1)(x-1)}{(x-2)x(x+2)}$
63. $y = \dfrac{e^x+2x}{e^x-2x}$
64. $y = \dfrac{\ln x - 1}{\ln x + 1}$

65. Let $f(x)$ and $g(x)$ be polynomials such that $\lim_{x\to\infty}[f(x)/g(x)] = l$. Prove that the limit $\lim_{x\to-\infty} f(x)/g(x)$ is equal to $l$ as well. What happens if $l = \infty$ or $-\infty$?
66. How close to 3 does $x$ have to be to ensure that $|x^3 - 2x - 21| < \frac{1}{1000}$?
67. Let $f(x) = |x|$.
    (a) Find $f'(x)$ and sketch its graph.
    (b) Find $\lim_{x\to 0-} f'(x)$ and $\lim_{x\to 0+} f'(x)$.
    (c) Does $\lim_{x\to 0} f'(x)$ exist?
68. (a) Give a precise definition of this statement: $\lim_{x\to\infty} f(x) = -\infty$. (b) Draw figures like Figs. 11.1.1, 11.1.3, and 11.1.4 to illustrate your definition.
69. Draw figures like Figs. 11.1.1, 11.1.3, and 11.1.4 to illustrate the definition of these statements: (a) $\lim_{x\to x_0+} f(x) = l$; (b) $\lim_{x\to x_0+} f(x) = \infty$. [*Hint*: The shaded region should include all points with $x \leqslant x_0$.]
70. (a) Graph $y = f(x)$, where
$$f(x) = \begin{cases} |x|/x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$
    Does $\lim_{x\to 0} f(x)$ exist?
    (b) Graph $y = g(x)$, where
$$g(x) = \begin{cases} x+1, & x < 0, \\ 2x-1, & x \geqslant 0. \end{cases}$$
    Does $\lim_{x\to 0} g(x)$ exist?
    (c) Let $f(x)$ be as in part (a) and $g(x)$ as in part (b). Graph $y = f(x) + g(x)$. Does $\lim_{x\to 0}[f(x)+g(x)]$ exist? Conclude that the limit of a sum can exist even though the limits of the summands do not.
71. The number $N(t)$ of individuals in a population at time $t$ is given by
$$N(t) = N_0 \frac{e^{3t}}{(3/2) + e^{3t}}.$$
    Find the value of $\lim_{t\to\infty} N(t)$ and discuss its biological meaning.
72. The current in a certain RLC circuit is given by $I(t) = \{[(1/3)\sin t + \cos t]e^{-t/2} + 4\}$ amperes. The value of $\lim_{t\to\infty} I(t)$ is called the *steady-state current*; it respresents the current present after a long period of time. Find it.
73. The temperature $T(x,t)$ at time $t$ at position $x$ of a rod located along $0 \leqslant x \leqslant l$ on the $x$ axis is given by the rule $T(x,t) = B_1 e^{-\mu_1 t}\sin\lambda_1 x + B_2 e^{-\mu_2 t}\sin\lambda_2 x + B_3 e^{-\mu_3 t}\sin\lambda_3 x$, where $\mu_1$, $\mu_2$, $\mu_3$, $\lambda_1$, $\lambda_2$, $\lambda_3$ are all positive. Show that $\lim_{t\to\infty} T(x,t) = 0$ for each fixed location $x$ along the rod. The model applies to a rod without heat sources, with the heat allowed to radiate from the right end of the rod; zero limit means all heat eventually radiates out the right end.
74. A psychologist doing some manipulations with testing theory wishes to replace the reliability factor
$$R = \frac{nr}{1+(n-1)r} \qquad (\textit{Spearman–Brown formula})$$
    by unity, because someone told her that she could do this for large extension factors $n$. She formally replaces $n$ by $1/x$, simplifies, and then sets $x = 0$, to obtain 1. What has she done, in the language of limits?

★75. Study this $\varepsilon$-$\delta$ proof of the sum rule: Let $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. Given $\varepsilon > 0$, choose $\delta_1 > 0$ such that $|x - x_0| < \delta_1$ and $x \neq x_0$ implies $|f(x) - L| < \varepsilon/2$; choose $\delta_2 > 0$ such that $|x - x_0| < \delta_2$, $x \neq x_0$, implies that $|g(x) - M| < \varepsilon/2$. Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. Then $|x - x_0| < \delta$, and $x \neq x_0$ implies $|(f(x) + g(x)) - (L + M)| \leqslant |f(x) - L| + |g(x) - M|$ (by the triangle inequality $|x + y| \leqslant |x| + |y|$). This is less than $\varepsilon/2 + \varepsilon/2 = \varepsilon$, and therefore $\lim_{x\to x_0}[f(x) + g(x)] = L + M$.

Now prove that $\lim_{x\to x_0}[af(x) + bg(x)] = a\lim_{x\to x_0} f(x) + b\lim_{x\to x_0} g(x)$.

★76. Study this $\varepsilon$-$\delta$ proof of the product rule: If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then $\lim_{x\to x_0} f(x)g(x) = LM$.

*Proof*: Let $\varepsilon > 0$ be given. We must find a number $\delta > 0$ such that $|f(x)g(x) - LM| < \varepsilon$ whenever $|x - x_0| < \delta$, $x \neq x_0$. Adding and subtracting $f(x)M$, we have

$$\begin{aligned} &|f(x)g(x) - LM| \\ &\quad = |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\quad \leqslant |f(x)|\cdot|g(x) - M| + |f(x) - L||M|. \end{aligned}$$

The closeness of $g(x)$ to $M$ and $f(x)$ to $L$ must depend upon the size of $f(x)$ and $|M|$. Choose $\delta_1$ such that $|f(x) - L| <$ whenever $|x - x_0| < \delta_1$, $x \neq x_0$. Also, choose $\delta_2$ such that $|x - x_0| < \delta_2$, $x \neq x_0$, implies that $|f(x) - L|$

$\varepsilon/(2|M|)$ if $M \neq 0$

$< 1$, which in turn implies that $|f(x)| < |L| + 1$ (since $|f(x)| = |f(x) - L + L| \leqslant |f(x) - L| + |L| < 1 + |L|$). Finally, choose $\delta_3 > 0$ such that $|g(x) - M| < \varepsilon/[2(|L|+1)]$ whenever $|x - x_0| < \delta_3$, $x \neq x_0$. Let $\delta$ be the smallest of $\delta_1$, $\delta_2$, and $\delta_3$. If $|x - x_0| < \delta$, $x \neq x_0$, then $|x - x_0| < \delta_1$, $|x - x_0| < \delta_2$, and $|x - x_0| < \delta_3$, so by the choice of $\delta_1, \delta_2, \delta_3$, we have

$$\begin{aligned}
&|f(x)g(x) - LM| \\
&\quad \leqslant |f(x)|\,|g(x) - M| + |f(x) - L|\,|M| \\
&\quad < (|L|+1)\frac{\varepsilon}{2(|L|+1)} + \frac{\varepsilon}{2|M|}\cdot|M| \\
&\quad = \varepsilon,
\end{aligned}$$

and so $|f(x)g(x) - LM| < \varepsilon$.

Now prove the quotient rule for limits.

★77. Study the following proof of the one-sided composite function rule: If $\lim_{x\to x_0+} f(x) = L$ and $g$ is continuous at $L$, then $g(f(x))$ is defined for all $x$ in some interval of the form $(x_0, b)$, and $\lim_{x\to x_0+} g(f(x)) = g(L)$.

*Proof*: Let $\varepsilon > 0$. We must find a positive number $\delta$ such that whenever $x_0 < x < x_0 + \delta$, $g(f(x))$ is defined and $|g(f(x)) - g(L)| < \varepsilon$. Since $g$ is continuous at $L$, there is a positive number $\rho$ such that whenever $|y - L| < \rho$, $g(y)$ is defined and $|g(y) - g(L)| < \varepsilon$. Now since $\lim_{x\to x_0+} f(x) = L$, we can find a positive number $\delta$ such that whenever $x_0 < x < x_0 + \delta$, $|f(x) - L| < \rho$. For such $x$, we apply the previously obtained property of $\rho$, with $y = f(x)$, to conclude that $g(f(x))$ is defined and that $|g(f(x)) - g(L)| < \varepsilon$.

Now prove the composite function rule.

★78. Use the $\varepsilon$-$A$ definition to prove the sum rule for limits at infinity.

★79. Use the $B$-$\delta$ definition to prove the reciprocal test for infinite limits.

★80. Suppose that a function $f$ is defined on an open interval $I$ containing $x_0$, and that there are numbers $m$ and $K$ such that we have the inequality $|f(x) - f(x_0) - m(x - x_0)| \leqslant K|x - x_0|^2$ for all $x$ in $I$. Prove that $f$ is differentiable at $x_0$ with derivative $f'(x_0) = m$.

★81. Show that $\lim_{x\to\infty} f(x) = l$ if and only if $\lim_{y\to 0+} f(1/y) = l$. (This reduces the computation of limits at infinity to one-sided limits at zero.)

# 11.2 L'Hôpital's Rule

*Differentiation can be used to evaluate limits.*

L'Hôpital's rule[2] is a very efficient way of using differential calculus to evaluate limits. It is not necessary to have mastered the theoretical portions of the previous sections to use l'Hôpital's rule, but you should review some of the computational aspects of limits from either Section 11.1 or Section 1.3.

L'Hôpital's rule deals with limits of the form $\lim_{x\to x_0}[f(x)/g(x)]$, where $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ are both equal to zero or infinity, so that the quotient rule cannot be applied. Such limits are called *indeterminate forms*. (One can also replace $x_0$ by $\infty$, $x_0+$, or $x_0-$.)

Our first objective is to calculate $\lim_{x\to x_0}[f(x)/g(x)]$ if $f(x_0) = 0$ and $g(x_0) = 0$. Substituting $x = x_0$ gives us $\frac{0}{0}$, so we say that we are dealing with an *indeterminate form of type* $\frac{0}{0}$. Such forms occurred when we considered the derivative as a limit of difference quotients; in Section 1.3 we used the limit rules to evaluate some simple derivatives. Now we can work the other way around, using our ability to calculate derivatives in order to evaluate quite complicated limits: l'Hôpital's rule provides the means for doing this.

The following box gives the simplest version of l'Hôpital's rule.

[2] In 1696, Guillaume F. A. l'Hôpital published in Paris the first calculus textbook: *Analyse des Infiniment Petits (Analysis of the infinitely small)*. Included was a proof of what is now referred to as l'Hôpital's rule; the idea, however, probably came from J. Bernoulli. This rule was the subject of some work by A. Cauchy, who clarified its proof in his *Cours d'Analyse (Course in analysis)* in 1823. The foundations were in debate until almost 1900. See, for instance, the very readable article, "The Law of the Mean and the Limits $\frac{0}{0}$, $\frac{\infty}{\infty}$," by W. F. Osgood, *Annals of Mathematics*, Volume 12 (1898–1899), pp. 65–78.

### L'Hôpital's Rule: Preliminary Version

Let $f$ and $g$ be differentiable in an open interval containing $x_0$; assume that $f(x_0) = g(x_0) = 0$. If $g'(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

To prove this, we use the fact that $f(x_0) = 0$ and $g(x_0) = 0$ to write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{[f(x) - f(x_0)]/(x - x_0)}{[g(x) - g(x_0)]/(x - x_0)}.$$

As $x$ tends to $x_0$, the numerator tends to $f'(x_0)$, and the denominator tends to $g'(x_0) \neq 0$, so the result follows from the quotient rule for limits.

Let us verify this rule on a simple example.

**Example 1** Find $\lim_{x \to 1} \left[ \frac{x^3 - 1}{x - 1} \right]$.

**Solution** Here we take $x_0 = 1$, $f(x) = x^3 - 1$, and $g(x) = x - 1$. Since $g'(1) = 1$, the preliminary version of l'Hôpital's rule applies to give

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \frac{f'(1)}{g'(1)} = \frac{3}{1} = 3.$$

We know two other ways (from Chapter 1) to calculate this limit. First, we can factor the numerator:

$$\frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1 \qquad (x \neq 1).$$

Letting $x \to 1$, we again recover the limit 3. Second, we can recognize the function $(x^3 - 1)/(x - 1)$ as the different quotient $[h(x) - h(1)]/(x - 1)$ for $h(x) = x^3$. As $x \to 1$, this different quotient approaches the derivative of $h$ at $x = 1$, namely 3. ▲

The next example begins to show the power of l'Hôpital's rule in a more difficult limit.

**Example 2** Find $\lim_{x \to 0} \frac{\cos x - 1}{\sin x}$.

**Solution** We apply l'Hôpital's rule with $f(x) = \cos x - 1$ and $g(x) = \sin x$. We have $f(0) = 0$, $g(0) = 0$, and $g'(0) = 1 \neq 0$, so

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin x} = \frac{f'(0)}{g'(0)} = \frac{-\sin(0)}{\cos(0)} = 0. \ ▲$$

This method does not solve all $\frac{0}{0}$ problems. For example, suppose we wish to find

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

If we differentiate the numerator and denominator, we get $(\cos x - 1)/3x^2$, which becomes $\frac{0}{0}$ when we set $x = 0$. This suggests that we use l'Hôpital's rule again, but to do so, we need to know that $\lim_{x\to x_0}[f(x)/g(x)]$ is equal to $\lim_{x\to x_0}[f'(x)/g'(x)]$, even when $f'(x_0)/g'(x_0)$ is again indeterminate. The following strengthened version of l'Hôpital's rule is the result we need. Its proof is given later in the section.

## L'Hôpital's Rule

Let $f$ and $g$ be differentiable on an open interval containing $x_0$, except perhaps at $x_0$ itself. Assume:

(i) $g(x) \neq 0$,
(ii) $g'(x) \neq 0$ for $x$ in an interval about $x_0$, $x \neq x_0$,
(iii) $f$ and $g$ are continuous at $x_0$ with $f(x_0) = g(x_0) = 0$, and
(iv) $\displaystyle\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l.$

Then $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = l.$

**Example 3** Calculate $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

**Solution** This is in $\frac{0}{0}$ form, so by l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x}$$

if the latter limit can be shown to exist. However, we can use l'Hôpital's rule again to write

$$\lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2}.$$

Now we may use the continuity of $\cos x$ to substitute $x = 0$ and find the last limit to be $-\frac{1}{2}$; thus

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

To keep track of what is going on, some students like to make a table:

| | form | type | limit |
|---|---|---|---|
| $\frac{f}{g}$ | $\frac{\cos x - 1}{x^2}$ | $\frac{0}{0}$ indeterminate | ? |
| $\frac{f'}{g'}$ | $\frac{-\sin x}{2x}$ | $\frac{0}{0}$ indeterminate | ? |
| $\frac{f''}{g''}$ | $\frac{-\cos x}{2}$ | determinate | $\boxed{-\frac{1}{2}}$ |

▲

Each time the numerator and denominator are differentiated, we must check the type of limit; if it is $\frac{0}{0}$, we proceed and are sure to stop when the limit becomes determinate, that is, when it can be evaluated by substitution of the limiting value.

**Warning** If l'Hôpital's rule is used when the limit is determinate, incorrect answers can result. For example, $\lim_{x\to 0}[(x^2+1)/x]=\infty$ but l'Hôpital's rule would lead to $\lim_{x\to 0}(2x/1)$ which is zero (and is incorrect).

**Example 4** Find $\lim_{x\to 0}\dfrac{\sin x - x}{\tan x - x}$.

**Solution** This is in $\frac{0}{0}$ form, so we use l'Hôpital's rule:

| | form | type | limit |
|---|---|---|---|
| $\dfrac{f}{g}$ | $\dfrac{\sin x - x}{\tan x - x}$ | $\dfrac{0}{0}$ | ? |
| $\dfrac{f'}{g'}$ | $\dfrac{\cos x - 1}{\sec^2 x - 1}$ | $\dfrac{0}{0}$ | ? |
| $\dfrac{f''}{g''}$ | $\dfrac{-\sin x}{2\sec x(\sec x\tan x)}$ | $\dfrac{0}{0}$ | ? |
| $\dfrac{f'''}{g'''}$ | $\dfrac{-\cos x}{4\sec^2 x\tan^2 x + 2\sec^4 x}$ | determinate | $\boxed{-\frac{1}{2}}$ |

Thus $\lim_{x\to 0}\dfrac{\sin x - x}{\tan x - x} = -\dfrac{1}{2}$. ▲

L'Hôpital's rule also holds for one-sided limits, limits as $x\to\infty$, or if we have indeterminates of the form $\frac{\infty}{\infty}$. To prove the rule for the form $\frac{0}{0}$ in case $x\to\infty$, we use a trick: set $t=1/x$, so that $x=1/t$ and $t\to 0+$ as $x\to+\infty$. Then

$$\begin{aligned}
\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} &= \lim_{t\to 0+}\frac{f'(1/t)}{g'(1/t)}\\
&= \lim_{t\to 0+}\frac{-t^2 f'(1/t)}{-t^2 g'(1/t)}\\
&= \lim_{t\to 0+}\frac{(d/dt)f(1/t)}{(d/dt)g(1/t)} && \text{(by the chain rule)}\\
&= \lim_{t\to 0+}\frac{f(1/t)}{g(1/t)} && \text{(by l'Hôpital's rule)}\\
&= \lim_{x\to+\infty}\frac{f(x)}{g(x)}.
\end{aligned}$$

It is tempting to use a similar trick for the $\frac{\infty}{\infty}$ form as $x\to x_0$, but it does not work. If we write

$$\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)},$$

which is in the $\frac{0}{0}$ form, we get

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{-g'(x)/[g(x)]^2}{-f'(x)/[f(x)]^2}$$

which is no easier to handle. For the correct proof, see Exercise 42.

The use of l'Hôpital's rule is summarized in the following display.

## L'Hôpital's Rule

To find $\lim_{x\to x_0}[f(x)/g(x)]$ where $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ are both zero or both infinite, differentiate the numerator and denominator and take the limit of the new fraction; repeat the process as many times as necessary, checking each time that l'Hôpital's rule applies.

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ (or each is $\pm\infty$), then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

($x_0$ may be replaced by $\pm\infty$ or $x_0\pm$).

The result of the next example was stated at the beginning of Section 6.4. The solution by l'Hôpital's rule is much easier than the one given in Review Exercise 90 of Chapter 6.

**Example 5** Find $\lim_{x\to\infty}\frac{\ln x}{x^p}$, where $p > 0$.

**Solution** This is in the form $\frac{\infty}{\infty}$. Differentiating the numerator and denominator, we find

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = \lim_{x\to\infty}\frac{1/x}{px^{p-1}} = \lim_{x\to\infty}\frac{1}{px^p} = 0,$$

since $p > 0$. ▲

Certain expressions which do not appear to be in the form $f(x)/g(x)$ can be put in that form with some manipulation. For example, the indeterminate form $\infty \cdot 0$ appears when we wish to evaluate $\lim_{x\to x_0} f(x)g(x)$ where $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = 0$. This can be converted to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \quad \text{or} \quad f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

**Example 6** Find $\lim_{x\to 0+} x\ln x$.

**Solution** We write $x\ln x$ as $(\ln x)/(1/x)$, which is now in $\frac{\infty}{\infty}$ form. Thus

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-1/x^2} = \lim_{x\to 0+}(-x) = 0. ▲$$

Indeterminate forms of the type $0^0$ and $1^\infty$ can be handled by using logarithms:

**Example 7** Find (a) $\lim_{x\to 0+} x^x$ and (b) $\lim_{x\to 1} x^{1/(1-x)}$.

**Solution** (a) This is of the form $0^0$, which is indeterminate because zero to any power is zero, while any number to the zeroth power is 1. To obtain a form to which l'Hôpital's rule is applicable, we write $x^x$ as $\exp(x\ln x)$. By Example 6, we have $\lim_{x\to 0+} x\ln x = 0$. Since $g(x) = \exp(x)$ is continuous, the composite function rule applies, giving $\lim_{x\to 0+}\exp(x\ln x) = \exp(\lim_{x\to 0+} x\ln x) = e^0 = 1$, so $\lim_{x\to 0+} x^x = 1$. (Numerically, $0.1^{0.1} = 0.79$, $0.001^{0.001} = 0.993$, and $0.00001^{0.00001} = 0.99988$.)

(b) This has the indeterminate form $1^\infty$. We have $x^{1/(x-1)} = e^{(\ln x)/(x-1)}$; applying l'Hôpital's rule gives

$$\lim_{x\to1} \frac{\ln x}{x-1} = \lim_{x\to1} \frac{1/x}{1} = 1,$$

so

$$\lim_{x\to1} x^{1/(x-1)} = \lim_{x\to1} e^{(\ln x)/(x-1)} = e^{\lim_{x\to1}[(\ln x)/(x-1)]} = e^1 = e.$$

If we set $x = 1 + (1/n)$, then $x \to 1$ when $n \to \infty$; we have $1/(x-1) = n$, so the limit we just calculated is $\lim_{n\to\infty}(1 + 1/n)^n$. Thus l'Hôpital's rule gives another proof of the limit formula $\lim_{n\to\infty}(1+1/n)^n = e$. ▲

The next example is a limit of the form $\infty - \infty$.

**Example 8** Find $\lim_{x\to0}\left(\frac{1}{x\sin x} - \frac{1}{x^2}\right)$.

**Solution** We can convert this limit to $\frac{0}{0}$ form by bringing the expression to a common denominator:

| | form | type | limit |
|---|---|---|---|
| | $\frac{1}{x\sin x} - \frac{1}{x^2}$ | $\infty - \infty$ | ? |
| $\frac{f}{g}$ | $\frac{x - \sin x}{x^2 \sin x}$ | $\frac{0}{0}$ | ? |
| $\frac{f'}{g'}$ | $\frac{1-\cos x}{2x\sin x + x^2\cos x}$ | $\frac{0}{0}$ | ? |
| $\frac{f''}{g''}$ | $\frac{\sin x}{2\sin x + 4x\cos x - x^2 \sin x}$ | $\frac{0}{0}$ | ? |
| $\frac{f'''}{g'''}$ | $\frac{\cos x}{6\cos x - 6x\sin x - x^2\cos x}$ | determinate | $\boxed{\frac{1}{6}}$ |

Thus $\lim_{x\to0}\left(\frac{1}{x\sin x} - \frac{1}{x^2}\right) = \frac{1}{6}$. ▲

Finally, we shall prove l'Hôpital's rule. The proof relies on a generalization of the mean value theorem.

**Cauchy's mean value theorem** *Suppose that $f$ and $g$ are continuous on $[a,b]$ and differentiable on $(a,b)$ and that $g(a) \neq g(b)$. Then there is a number $c$ in $(a,b)$ such that* and $g'$ does not vanish on $(a,b)$.

$$g'(c)\frac{f(b)-f(a)}{g(b)-g(a)} = f'(c).$$

**Proof** First note that if $g(x) = x$, we recover the mean value theorem in its usual form. The proof of the mean value theorem in Section 3.6 used the function

$$l(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a}.$$

For the Cauchy mean value theorem, we replace $x - a$ by $g(x) - g(a)$ and look at

$$h(x) = f(a) + [\,g(x) - g(a)\,]\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Notice that $f(a) = h(a)$ and $f(b) = h(b)$. By the horserace theorem (see Section 3.6), there is a point $c$ such that $f'(c) = h'(c)$; that is,

$$f'(c) = g'(c)\left[\frac{f(b) - f(a)}{g(b) - g(a)}\right],$$

which is what we wanted to prove. ■

We now prove the final version of l'Hôpital's rule. Since $f(x_0) = g(x_0) = 0$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)}, \tag{1}$$

where $c_x$ (which depends on $x$) lies between $x$ and $x_0$. Note that $c_x \to x_0$ as $x \to x_0$. Since, by hypothesis, $\lim_{x\to x_0}[f'(x)/g'(x)] = l$, it follows that we also have $\lim_{x\to x_0}[f'(c_x)/g'(c_x)] = l$, and so by equation (1), $\lim_{x\to x_0}[f(x)/g(x)] = l$. ■

## Exercises for Section 11.2

Use the preliminary version of l'Hôpital's rule to evaluate the limits in Exercises 1–4.

1. $\lim_{x\to 3} \frac{x^4 - 81}{x - 3}$
2. $\lim_{x\to 2} \frac{3x^2 - 12}{x - 2}$
3. $\lim_{x\to 0} \frac{x^2 + 2x}{\sin x}$
4. $\lim_{x\to 1} \frac{x^3 + 3x - 4}{\sin(x - 1)}$

Use the final version of l'Hôpital's rule to evaluate the limits in Exercises 5–8.

5. $\lim_{x\to 0} \frac{\cos 3x - 1}{5x^2}$
6. $\lim_{x\to 0} \frac{\cos 10x - 1}{8x^2}$
7. $\lim_{x\to 0} \frac{\sin 2x - 2x}{x^3}$
8. $\lim_{x\to 0} \frac{\sin 3x - 3x}{x^3}$

Evaluate the $\frac{\infty}{\infty}$ forms in Exercises 9–12.

9. $\lim_{x\to\infty} \frac{e^x}{x^{375}}$
10. $\lim_{x\to\infty} \frac{x^4 + \ln x}{3x^4 + 2x^2 + 1}$
11. $\lim_{x\to 0^+} \frac{\ln x}{x^{-2}}$
12. $\lim_{x\to 0+} \frac{e^{1/x}}{1/x}$

Evaluate the $0 \cdot \infty$ forms in Exercises 13–16.

13. $\lim_{x\to 0} [x^4 \ln x]$
14. $\lim_{x\to 1} [\tan \frac{\pi x}{2} \ln x]$
15. $\lim_{x\to 0} [x^\pi e^{-\pi x}]$
16. $\lim_{x\to\pi} [(x^2 - 2\pi x + \pi^2)\csc^2 x]$

Evaluate the limits in Exercises 17–36.

17. $\lim_{x\to 0} [(\tan x)^x]$
18. $\lim_{x\to\infty} \left[\left(1 + \frac{1}{x}\right)^x\right]$
19. $\lim_{x\to 0} (\csc x - \cot x)$
20. $\lim_{x\to\infty} [\ln x - \ln(x - 1)]$
21. $\lim_{x\to 0} \frac{\sqrt{1 + x^2} - 1}{\sin 2x}$
22. $\lim_{x\to 1} \frac{\sqrt{x^2 - 1}}{\cos(\pi x/2)}$
23. $\lim_{x\to 1} \frac{1 - x^2}{1 + x^2}$
24. $\lim_{x\to 0} \frac{x + \sin 2x}{2x + \sin 3x}$
25. $\lim_{x\to\infty} \frac{x}{x^2 + 1}$
26. $\lim_{x\to\infty} \frac{(2x + 1)^3}{x^3 + 2}$
27. $\lim_{x\to 5+} \frac{\sqrt{x^2 - 25}}{x - 5}$
28. $\lim_{x\to 5+} \frac{\sqrt{x^2 - 25}}{x + 5}$
29. $\lim_{x\to -1} \frac{x^2 + 2x + 1}{x^2 - 1}$
30. $\lim_{x\to 1-} x^{1/(1-x^2)}$
31. $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4}$
32. $\lim_{x\to 1} \frac{\ln x}{e^x - 1}$
33. $\lim_{x\to\pi} \frac{1 + \cos x}{x - \pi}$
34. $\lim_{x\to\pi/2} (x - \frac{\pi}{2})\tan x$
35. $\lim_{x\to 0} \frac{\sin x - x + (1/6)x^3}{x^5}$
36. $\lim_{x\to\infty} \frac{x^3 + \ln x + 5}{5x^3 + e^{-x} + \sin x}$
37. Find $\lim_{x\to 0+} x^p \ln x$, where $p$ is positive.
38. Use l'Hôpital's rule to show that as $x \to \infty$, $x^n/e^x \to 0$ for any integer $n$; that is, $e^x$ goes to infinity faster than any power of $x$. (This was proved by another method in Section 6.4.)

★39. Give a geometric interpretation of the Cauchy mean value theorem. [*Hint*: Consider the curve given in parametric form by $y = f(t)$, $x = g(t)$.]

★40. Suppose that $f$ is continuous at $x = x_0$, that $f'(x)$ exists for $x$ in an interval about $x_0$, $x \neq x_0$, and that $\lim_{x\to x_0} f'(x) = m$. Prove that $f'(x_0)$ exists and equals $m$. [*Hint*: Use the mean value theorem.]

★41. Graph the function $f(x) = x^x$, $x > 0$.

★42. Prove l'Hôpital's rule for $x_0 = \infty$ as follows:
(i) Let $f$ and $g$ be differentiable on $(a, \infty)$ with $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x > a$. Use Cauchy's mean value theorem to prove that for every $\varepsilon > 0$, there is an $M > a$ such that for $y > x > M$,
$$\left| \frac{f(y) - f(x)}{g(y) - g(x)} - l \right| < \varepsilon$$
(ii) Write
$$\frac{f(x)}{g(x)} = \lim_{y \to \infty} \frac{f(x) - f(y)}{g(x) - g(y)}$$
and choose $y$ sufficiently large, to conclude that
$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon.$$
(iii) Complete the proof using (ii).

★43. (a) Find $\lim_{a \to 0} \frac{1}{a} \ln\left(\frac{e^a - 1}{a}\right)$
(b) Find $\lim_{a \to \infty} \frac{1}{a} \ln\left(\frac{e^a - 1}{a}\right)$
(c) Are your results consistent with the computations of Exercise 30, Section 9.4?

## 11.3 Improper Integrals

*The area of an unbounded region is defined by a limiting process.*

The definite integral $\int_a^b f(x)\,dx$ of a function $f$ which is non-negative on the interval $[a, b]$ equals the area of the region under the graph of $f$ between $a$ and $b$. If we let $b$ go to infinity, the region becomes unbounded, as in Fig. 11.3.1. One's first inclination upon seeing such unbounded regions may be to assert that their areas are infinite; however, examples suggest otherwise.

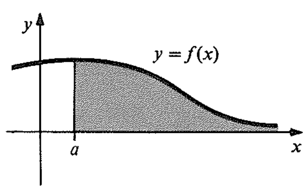



**Figure 11.3.1.** The region under the graph of $f$ on $[a, \infty)$ is unbounded.

**Example 1** Find $\displaystyle\int_1^b \frac{1}{x^4}\,dx$. What happens as $b$ goes to infinity?

**Solution** We have
$$\int_1^b \frac{dx}{x^4} = \int_1^b x^{-4}\,dx = \left.\frac{x^{-3}}{-3}\right|_1^b = \frac{1/b^3 - 1}{-3} = \frac{1 - 1/b^3}{3}.$$
As $b$ becomes larger and larger, this integral always remains less than $\frac{1}{3}$; furthermore, we have
$$\lim_{b \to \infty} \int_1^b \frac{dx}{x^4} = \lim_{b \to \infty} \frac{1 - 1/b^3}{3} = \frac{1}{3}. \ \blacktriangle$$

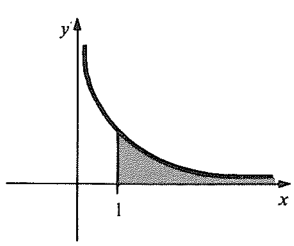



**Figure 11.3.2.** The region under the graph of $1/x^4$ on $[1, \infty)$ has finite area. It is $\int_1^\infty (dx/x^4) = \frac{1}{3}$.

Example 1 suggests that $\frac{1}{3}$ is the area of the unbounded region consisting of those points $(x, y)$ such that $1 \leqslant x$ and $0 \leqslant y \leqslant 1/x^4$. (See Fig. 11.3.2.) In accordance with our notation for finite intervals, we denote this area by $\int_1^\infty (dx/x^4)$. Guided by this example, *we define integrals over unbounded intervals as limits of integrals over finite intervals.* The general definition follows.

## Integrals over Unbounded Intervals

Suppose that for $a$ fixed, $f$ is integrable on $[a,b]$ for all $b > a$. If the limit $\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists, we say that the *improper integral* $\int_a^\infty f(x)\,dx$ *is convergent*, and we define its value by

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Similarly, if, for fixed $b$, $f$ is integrable on $[a,b]$ for all $a < b$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$

if the limit exists.

Finally, if $f$ is integrable on $[a,b]$ for all $a < b$, we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx,$$

if the improper integrals on the right-hand side are *both* convergent.

If an improper integral is not convergent, it is called *divergent*.

**Example 2** For which values of the exponent $r$ is $\int_1^\infty x^r\,dx$ convergent?

**Solution** We have

$$\lim_{b\to\infty}\int_1^b x^r\,dx = \lim_{b\to\infty}\left.\frac{x^{r+1}}{r+1}\right|_1^b = \lim_{b\to\infty}\frac{b^{r+1}-1}{r+1} \qquad (r \neq -1).$$

If $r+1>0$ (that is, $r>-1$), the limit $\lim_{b\to\infty}b^{r+1}$ does not exist and the integral is divergent. If $r+1<0$ (that is, $r<-1$), we have $\lim_{b\to\infty}b^{r+1}=0$ and the integral is convergent—its value is $-1/(r+1)$. Finally, if $r=-1$ we have $\int_1^b x^{-1}\,dx = \ln b$, which does not converge as $b\to\infty$. We conclude that $\int_1^\infty x^r\,dx$ is convergent just for $r<-1$. ▲

**Example 3** Find $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

**Solution** We write $\int_{-\infty}^\infty (dx/(1+x^2)) = \int_{-\infty}^0 (dx/(1+x^2)) + \int_0^\infty (dx/(1+x^2))$. To evaluate these integrals, we use the formula $\int (dx/(1+x^2)) = \tan^{-1}x$. Then

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a)\\ &= 0 - \lim_{a\to-\infty}\tan^{-1}a = 0-\left(\frac{-\pi}{2}\right) = \frac{\pi}{2}.\end{aligned}$$

(See Fig. 5.4.5 for the horizontal asymptotes of $y=\tan^{-1}x$.) Similarly, we have

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0) = \frac{\pi}{2},$$

so $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$ ▲

Sometimes we wish to know that an improper integral converges, even though we cannot find its value explicitly. The following test is quite effective for this situation.

## Comparison Test

Suppose that $f$ and $g$ are functions such that

(i) $|f(x)| \leqslant g(x)$ for all $x \geqslant a$ and

(ii) $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist for every $b > a$.

Then

(1) If $\int_a^\infty g(x)\,dx$ is convergent, so is $\int_a^\infty f(x)\,dx$, and

(2) if $\int_a^\infty f(x)\,dx$ is divergent, so is $\int_a^\infty g(x)\,dx$.

Similar statements hold for integrals of the type

$$\int_{-\infty}^b f(x)\,dx \quad \text{and} \quad \int_{-\infty}^\infty f(x)\,dx.$$

Here we shall explain the idea behind the comparison test. A detailed proof is given at the end of this section.

If $f(x)$ and $g(x)$ are both positive functions (Fig. 11.3.3(a)), then the region under the graph of $f$ is contained in the region under the graph of $g$, so

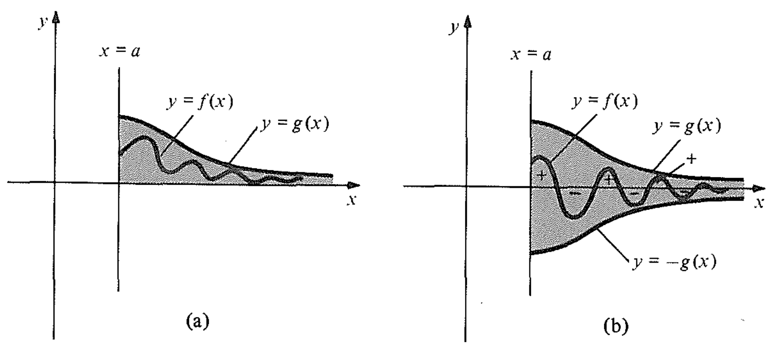



**Figure 11.3.3.** Illustrating the comparison test.

the integral $\int_a^b f(x)\,dx$ increases and remains bounded as $b \to \infty$. We expect, therefore, that it should converge to some limit. In the general case (Fig. 11.3.3(b)), the sums of the plus areas and the minus areas are both bounded by $\int_a^\infty g(x)\,dx$, and the cancellations can only help the integral to converge.

Note that in the event of convergence, the comparison test only gives the inequality $-\int_a^\infty g(x)\,dx \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$, but it does not give us the value of $\int_a^\infty f(x)\,dx$.

**Example 4** Show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{1+x^8}}$ is convergent, by comparison with $1/x^4$.

**Solution** We have $1/\sqrt{1+x^8} < 1/\sqrt{x^8} = 1/x^4$, so it is tempting to compare with $\int_0^\infty (dx/x^4)$. Unfortunately, the latter integral is not defined because $1/x^4$ is unbounded near zero. However, we can break the original integral in two parts:

$$\int_0^\infty \frac{dx}{\sqrt{1+x^8}} = \int_0^1 \frac{dx}{\sqrt{1+x^8}} + \int_1^\infty \frac{dx}{\sqrt{1+x^8}}$$

The first integral on the right-hand side exists because $1/\sqrt{1+x^8}$ is continuous on $[0,1]$. The second integral is convergent by the comparison test, taking $g(x) = 1/x^4$ and $f(x) = 1/\sqrt{1+x^8}$. Thus $\int_0^\infty (dx/\sqrt{1+x^8})$ is convergent. ▲

**Example 5** Show that $\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ converges (without attempting to evaluate).

**Solution** We may apply the comparison test by choosing $g(x) = 1/(1+x)^2$ and $f(x) = (\sin x)/(1+x)^2$, since $|\sin x| \leqslant 1$. To show that $\int_0^\infty (dx/(1+x)^2)$ is convergent, we can compare $1/(1+x)^2$ with $1/x^2$ on $[1, \infty)$, as in Example 4, or we can evaluate the integral explicitly:

$$\int_0^\infty \frac{dx}{(1+x)^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{(1+x)^2} = \lim_{b\to\infty} \left[ \frac{-1}{(1+x)} \right]\Bigg|_0^b$$
$$= \lim_{b\to\infty} \left[ 1 - \frac{1}{1+b} \right] = 1. \blacktriangle$$

**Example 6** Show that $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$ is divergent.

**Solution** We use the comparison test in the reverse direction, comparing $1/\sqrt{1+x^2}$ with $1/x$. In fact, for $x \geqslant 1$, we have $1/\sqrt{1+x^2} \geqslant 1/\sqrt{x^2+x^2} = 1/\sqrt{2}\,x$. But $\int_1^b (dx/\sqrt{2}\,x) = (1/\sqrt{2})\ln b$, and this diverges as $b \to \infty$. Therefore, by statement (2) in the comparison test, the given integral diverges. $\blacktriangle$

We shall now discuss the second type of improper integral. If the graph of a function $f$ has a vertical asymptote at one endpoint of the interval $[a, b]$, then the integral $\int_a^b f(x)\,dx$ is not defined in the usual sense, since the function $f$ is not bounded on the interval $[a, b]$. As with integrals of the form $\int_a^\infty f(x)\,dx$, we are dealing with areas of unbounded regions in the plane—this time the unboundedness is in the vertical rather than the horizontal direction. Following our earlier procedure, we can define the integrals of unbounded functions as limits, which are again called improper integrals.

## Integrals of Unbounded Functions

Suppose that the graph of $f$ has $x_0 = b$ as a vertical asymptote and that for $a$ fixed, $f$ is integrable on $[a, q]$ for all $q$ in $[a, b)$. If the limit $\lim_{q\to b-} \int_a^q f(x)\,dx$ exists, we shall say that the *improper integral* $\int_a^b f(x)\,dx$ *is convergent*, and we define

$$\int_a^b f(x)\,dx = \lim_{q\to b-} \int_a^q f(x)\,dx.$$

Similarly, if $x = a$ is a vertical asymptote, we define

$$\int_a^b f(x)\,dx = \lim_{p\to a+} \int_p^b f(x)\,dx,$$

if the limit exists. (See Fig. 11.3.4.)

If both $x = a$ and $x = b$ are vertical asymptotes, or if there are vertical asymptotes in the interior $(a, b)$, we may break up $[a, b]$ into subintervals such that the integral of $f$ on each subinterval is of the type considered in the preceding definition. If each part is convergent, we may add the results to get $\int_a^b f(x)\,dx$. The comparison test may be used to test each for convergence. (See Example 9 below.)

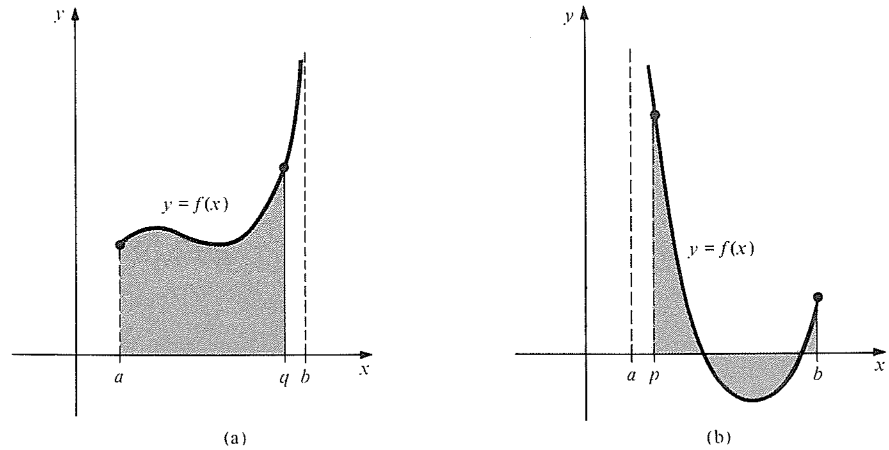



**Figure 11.3.4.** Improper integrals defined by (a) the limit $\lim_{q\to b-}\int_a^q f(x)\,dx$ and (b) the limit $\lim_{p\to a+}\int_p^b f(x)\,dx$.

**Example 7** For which values of $r$ is $\int_0^1 x^r\,dx$ convergent?

**Solution** If $r \geqslant 0$, $x^r$ is continuous on $[0, 1]$ and the integral exists in the ordinary sense. If $r < 0$, we have $\lim_{x\to 0+} x^r = \infty$, so we must take a limit. We have

$$\lim_{p\to 0+}\int_p^1 x^r\,dx = \lim_{p\to 0+}\left.\frac{x^{r+1}}{r+1}\right|_p^1 = \frac{1}{r+1}\left(1 - \lim_{p\to 0+} p^{r+1}\right),$$

provided $r \neq -1$. If $r + 1 > 0$ (that is, $r > -1$), we have $\lim_{p\to 0+} p^{r+1} = 0$, so the integral is convergent and equals $1/(r+1)$. If $r + 1 < 0$ (that is, $r < -1$), $\lim_{p\to 0+} p^{r+1} = \infty$, so the integral is divergent. Finally, if $r + 1 = 0$, we have $\lim_{p\to 0+}\int_p^1 x^r\,dx = \lim_{p\to 0+}(0 - \ln p) = \infty$. Thus the integral $\int_0^1 x^r\,dx$ converges just for $r > -1$. (Compare with Example 2.) ▲

**Example 8** Find $\int_0^1 \ln x\,dx$.

**Solution** We know that $\int \ln x\,dx = x\ln x - x + C$, so

$$\int_0^1 \ln x\,dx = \lim_{p\to 0+}(1\ln 1 - 1 - p\ln p + p)$$

$$= 0 - 1 - 0 + 0 = -1$$

($\lim_{p\to 0+} p\ln p = 0$ by Example 6, Section 11.2). ▲

**Example 9** Show that the improper integral $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent.

**Solution** This integral is improper at both ends; we may write it as $I_1 + I_2$, where

$$I_1 = \int_0^1 (e^{-x}/\sqrt{x})\,dx \quad \text{and} \quad I_2 = \int_1^\infty (e^{-x}/\sqrt{x})\,dx$$

and then we apply the comparison test to each term. On $[0, 1]$, we have $e^{-x} \leqslant 1$, so $e^{-x}/\sqrt{x} \leqslant 1/\sqrt{x}$. Since $\int_0^1 (dx/\sqrt{x})$ is convergent (Example 7), so is $I_1$. On $[1, \infty)$, we have $1/\sqrt{x} \leqslant 1$, so $e^{-x}/\sqrt{x} \leqslant e^{-x}$; but $\int_1^\infty e^{-x}\,dx$ is convergent because

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}(e^{-1} - e^{-b}) = e^{-1}$$

Thus $I_2$ is also convergent and so $\int_0^\infty (e^{-x}/\sqrt{x})\,dx$ is convergent. ▲

Improper integrals arise in arc length problems for graphs with vertical tangents.

**Example 10** Find the length of the curve $y=\sqrt{1-x^2}$ for $x$ in $[-1,1]$. Interpret your result geometrically.

**Solution** By formula (1), Section 10.3, the arc length is

$$\int_{-1}^{1}\sqrt{1+(dy/dx)^2}\,dx=\int_{-1}^{1}\sqrt{1+\left(-x/\sqrt{1-x^2}\right)^2}\,dx$$
$$=\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}.$$

The integral is improper at both ends, since

$$\lim_{x\to-1+}\frac{1}{\sqrt{1-x^2}}=\lim_{x\to1-}\frac{1}{\sqrt{1-x^2}}=\infty.$$

We break it up as

$$\int_{-1}^{0}\frac{dx}{\sqrt{1-x^2}}+\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$$
$$=\lim_{p\to-1+}\int_{p}^{0}\frac{dx}{\sqrt{1-x^2}}+\lim_{q\to1-}\int_{0}^{q}\frac{dx}{\sqrt{1-x^2}}$$
$$=\lim_{p\to-1+}(\sin^{-1}0-\sin^{-1}p)+\lim_{q\to1-}(\sin^{-1}q-\sin^{-1}0)$$
$$=0-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}-0=\pi.$$

Geometrically, the curve whose arc length we have just found is a semicircle of radius 1, so we recover the fact that the circumference of a circle of radius 1 is $2\pi$. ▲

**Example 11** Luke Skyrunner has just been knocked out in his spaceship by his archenemy, Captain Tralfamadore. The evil captain has set the controls to send the spaceship into the sun! His perverted mind insists on a slow death, so he sets the controls so that the ship makes a constant angle of 30° with the sun (Fig. 11.3.5). What path will Luke's ship follow? How long does Luke have to wake up if he is 10 million miles from the sun and his ship travels at a constant velocity of a million miles per hour?

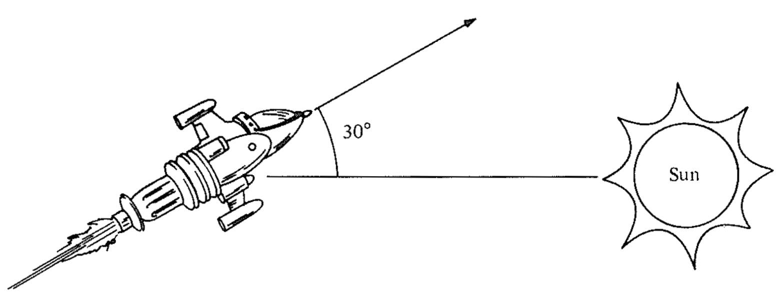



**Figure 11.3.5.** Luke Skyrunner's ill-fated ship.

**Solution** We use polar coordinates to describe a curve $(r(t),\theta(t))$ such that the radius makes a constant *angle* $\alpha$ with the tangent ($\alpha=30°$ in the problem). To find $dr/d\theta$, we observe, from Fig. 11.3.6(a), that

$$\Delta r\approx\frac{r\,\Delta\theta}{\tan\alpha}\quad\text{so}\quad\frac{dr}{d\theta}=\frac{r}{\tan\alpha}.\tag{1}$$

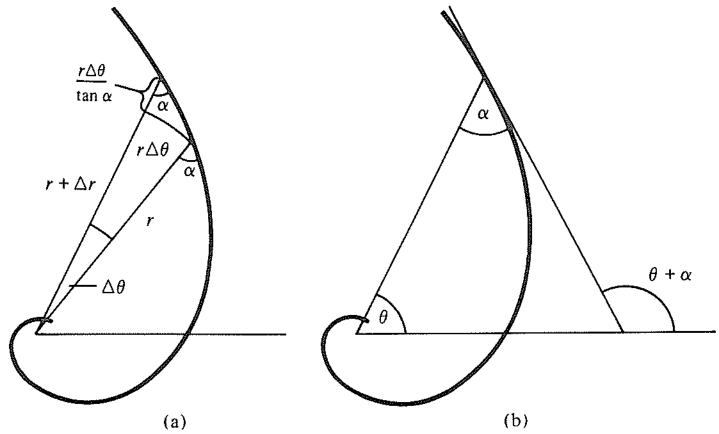



**Figure 11.3.6.** The geometry of Luke's path.

We can derive formula (1) rigorously, but also more laboriously, by calculating the slope of the tangent line in polar coordinates and setting it equal to $\tan(\theta+\alpha)$ as in Fig. 11.3.6(b). This approach gives

$$\frac{\tan\theta(dr/d\theta)+r}{dr/d\theta-r\tan\theta}=\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha},$$

so that again

$$\frac{dr}{d\theta}=\frac{r}{\tan\alpha}.$$

The solution of equation (1) is[3]

$$r(\theta)=r(0)e^{\theta/\tan\alpha}. \tag{2}$$

For this solution to be valid, we must regard $\theta$ as a continuous variable ranging from $-\infty$ to $\infty$, not as being between zero and $2\pi$. As $\theta\to\infty$, $r(\theta)\to\infty$ and as $\theta\to-\infty$, $r(\theta)\to 0$, so the curve spirals outward as $\theta$ increases and inward as $\theta$ decreases (if $0<\alpha<\pi/2$). This answers the first question: Luke follows the logarithmic spiral given by equation (2), where $\theta=0$ is chosen as the starting point.

From Section 10.6, the distance Luke has to travel is the arc length of equation (2) from $\theta=0$ to $\theta=-\infty$, namely, the improper integral

$$\begin{aligned}\int_{-\infty}^{0}\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta&=\int_{-\infty}^{0}\sqrt{\frac{r^2}{\tan^2\alpha}+r^2}\,d\theta=\int_{-\infty}^{0}r\sqrt{\cot^2\alpha+1}\,d\theta\\&=\int_{-\infty}^{0}r(0)e^{\theta/\tan\alpha}\frac{1}{\sin\alpha}\,d\theta\\&=\frac{r(0)}{\cos\alpha}e^{\theta/\tan\alpha}\Big|_{-\infty}^{0}=\frac{r(0)}{\cos\alpha}.\end{aligned}$$

With velocity $=10^6$, $r(0)=10^7$, and $\cos\alpha=\cos 30^\circ=\sqrt{3}/2$, the time needed to travel the distance is

$$\text{time}=\frac{\text{distance}}{\text{velocity}}=\frac{10^7}{\sqrt{3}/2}\times\frac{1}{10^6}=\frac{20}{\sqrt{3}}\approx 11.547\text{ hours}$$

Thus Luke has less than 11.547 hours to wake up. ▲

[3] See Section 8.2. If you have not read Chapter 8, you may simply check directly that equation (2) is a solution of (1).

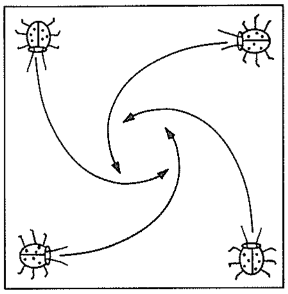


Figure 11.3.7. These love bugs follow logarithmic spirals.

The logarithmic spiral turns up in another interesting situation. Place four love bugs at the corners of a square (Fig. 11.3.7). Each bug, being in love, walks directly toward the bug in front of it, at constant top bug speed. The result is that the bugs all spiral in to the center of the square following logarithmic spirals. The time required for the bugs to reach the center can be calculated as in Example 11 (see Exercise 46).

We conclude this section with a proof of the comparison test. The proof is based on the following principle:

*Let $F$ be a function defined on $[a, \infty)$ such that*

(i) *$F$ is nondecreasing; i.e., $F(x_1) \leqslant F(x_2)$ whenever $x_1 < x_2$;*

(ii) *$F$ is bounded above: there is a number $M$ such that $F(x) \leqslant M$ for all $x$.*

*Then $\lim_{x\to\infty} F(x)$ exists and is at most $M$.*

The principle is quite plausible, since the graph of $F$ never descends and never crosses the line $y = M$, so that we expect it to have a horizontal asymptote as $x \to \infty$. (See Fig. 11.3.8).

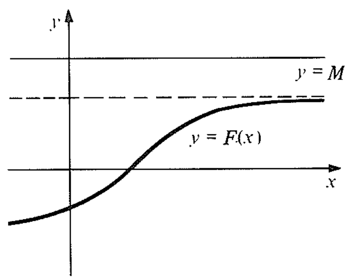



Figure 11.3.8. The graph of a nondecreasing function lying below the line $y = M$ has a horizontal asymptote.

A rigorous proof of the principle requires a careful study of the real numbers,[4] so we shall simply take the principle for granted, just as we did for some basic facts in Chapter 3. A similar principle holds for nonincreasing functions which are bounded below.

Now we are ready to prove statement (1) in the comparison test as stated in the box on p. 530. (Statement (2) follows from (1), for if $\int_a^\infty g(x)\,dx$ converged, so would $\int_a^\infty f(x)\,dx$.)

Let

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geqslant 0 \end{cases}$$

be the positive and negative parts of $f$, respectively. (See Fig. 11.3.9.)

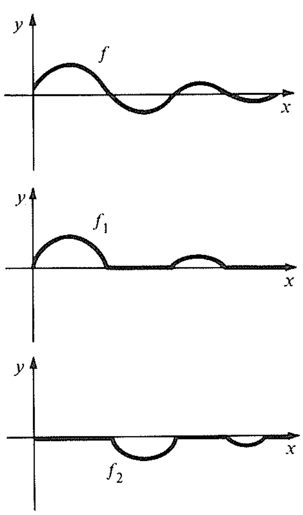



Figure 11.3.9. $f_1$ and $f_2$ are the positive and negative parts of $f$.

Notice that $f = f_1 + f_2$. Let $F_1(x) = \int_a^x f_1(t)\,dt$ and $F_2(x) = \int_a^x f_2(t)\,dt$. Since $f_1$ is always non-negative, $F_1(x)$ is increasing. Moreover, by the assumptions of the comparison test,

$$F_1(x) \leqslant \int_a^x |f(t)|\,dt \leqslant \int_a^x g(t)\,dt \leqslant \int_a^\infty g(t)\,dt,$$

so $F_1$ is bounded above by $\int_a^\infty g(t)\,dt$. Thus, $F_1$ has a limit as $x \to \infty$. Likewise,

[4] See the theoretical references listed in the preface.

$F_2$ has a limit since $F_2$ is decreasing and bounded below. Since

$$\int_a^x f(t)\,dt = F_1(x) + F_2(x),$$

it, too, has a limit as $x \to \infty$. ■

## Exercises for Section 11.3

Evaluate the improper integrals in Exercises 1–8.

1. $\int_1^{\infty} \frac{3}{x^2}\,dx$
2. $\int_2^{\infty} \frac{dx}{x^2}$
3. $\int_1^{\infty} e^{-5x}\,dx$
4. $\int_0^{\infty} e^{-3x}\,dx$
5. $\int_2^{\infty} \frac{dx}{x^2-1}$
6. $\int_0^{\infty} \frac{1}{(1+x)^2}\,dx$
7. $\int_{-\infty}^{\infty} \frac{dx}{4+x^2}$
8. $\int_{-\infty}^{\infty} \frac{dx}{9+x^2}$

Show, using the comparison test, that the integrals in Exercises 9–12 are convergent.

9. $\int_0^{\infty} \frac{dx}{3+x^3}$
10. $\int_0^{\infty} \frac{\sin x\,dx}{\sqrt{1+x^4}}$
11. $\int_0^{\infty} \frac{e^{-x}}{1+x}\,dx$
12. $\int_1^{\infty} \frac{e^{-x}}{1+\ln x}\,dx$

Show, using the comparison test, that the integrals in Exercises 13–16 are divergent.

13. $\int_0^{\infty} \frac{dx}{\sqrt{2+x^2}}$
14. $\int_0^{\infty} \frac{dx}{8+x+1/x}$
15. $\int_1^{\infty} \frac{(2+\sin x)\,dx}{1+x}$
16. $\int_1^{\infty} \frac{(3-\cos x)}{\sqrt{1+x^2}}\,dx$

Evaluate the improper integrals in Exercises 17–20.

17. $\int_0^{10} \frac{dx}{x^{2/3}}$
18. $\int_0^{1} \frac{dx}{x^{3/4}}$
19. $\int_0^{1} \frac{dx}{\sqrt{1-x}}$
20. $\int_0^{1} \frac{dx}{(1-x)^{2/3}}$

Using the comparison test, determine the convergence of the improper integrals in Exercises 21–24.

21. $\int_{-1}^{1} \frac{dx}{x^2+x}$
22. $\int_{-1}^{1} \frac{dx}{(x^2+x)^{1/3}}$
23. $\int_{-\infty}^{\infty} e^{-|x|}\,dx$
24. $\int_0^{\infty} \frac{dx}{(1+x^3)^{1/3}}$

Determine the convergence or divergence of the integrals in Exercises 25–40.

25. $\int_{-1}^{\infty} \frac{\tan^{-1}x}{(2+x)^3}\,dx$
26. $\int_0^{\infty} \frac{\sin x}{1+x^2}\,dx$
27. $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^{3/2}}\,dx$
28. $\int_2^{\infty} \frac{1}{t^2-1}\,dt$
29. $\int_1^{\infty} \frac{1}{(5x^2+1)^{2/3}}\,dx$
30. $\int_1^{\infty} \frac{1}{x^2}\left(1-\frac{1}{x}\right)dx$
31. $\int_1^{\infty} \frac{1}{(4x-3)^{1/3}}\,dx$
32. $\int_0^{\infty} \left[\cos x + \frac{1}{(x+1)^2}\right]dx$
33. $\int_{-\infty}^{-2} \left(\frac{1}{x^{6/5}} - \frac{1}{x^{4/3}}\right)dx$
34. $\int_0^{1} \frac{e^{-t}}{\sqrt[3]{t^2}}\,dt$
35. $\int_1^{2} \frac{1}{\sqrt{t-1}}\,dt.$
36. $\int_{-\infty}^{\infty} \frac{\cos(x^2+1)}{x^2}\,dx.$ [*Hint*: Use the comparison test on a small interval.]
37. $\int_{-\infty}^{2} \left(\frac{1}{x^{5/3}} - \frac{1}{x^{4/3}}\right)dx$
38. $\int_{-4}^{10} \left[\frac{1}{(x+4)^{2/3}} + \frac{1}{(x-10)^{2/3}}\right]dx$
39. $\int_2^{\infty} \frac{dx}{x\ln x}$
40. $\int_1^{\infty} e^{-x}\ln x\,dx$
41. Consider the spirals defined in polar coordinates by the parametric equations $\theta = t$, $r = t^{-k}$. For which values of $k$ does the spiral have finite arc length for $\pi/2 \leqslant t < \infty$? (Use the comparison test.)
42. Does the spiral $\theta = t$, $r = e^{-\sqrt{t}}$ have finite arc length for $\pi \leqslant t < \infty$?
43. Find the area under the graph of the function $f(x) = (3x+5)/(x^3-1)$ from $x = 2$ to $x = \infty$.
44. Find the area between the graphs $y = x^{-4/3}$ and $y = x^{-5/3}$ on $[1, \infty)$.
45. In Example 11, suppose that Luke's airhoses melt down when he is $10^6$ miles form the sun. Now how long does he have to wake up?
46. Let $\alpha$ in Fig. 11.3.7 be 60° ($\alpha$ is defined in Example 11). Find the time required for the bugs to reach the center in terms of their speed and their initial distance from the center.
47. The region under curve $y = e^{-x}$ is rotated about the $x$ axis to form a solid of revolution. Find the volume obtained by discarding the portion on $-\infty < x \leqslant 10$ (after slicing the solid at $x = 10$).

48. Determine the lateral surface area of the surface of revolution obtained by revolving $y = e^{-x}$, $0 \leqslant x < \infty$, about the $x$ axis.
49. Show that $\lim_{A \to 0^+} [\int_{-3}^{-A}(dx/x) + \int_A^2(dx/x)]$ exists and determine its value.
50. Discuss the following "calculations":

(a) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1 + (-1) = -2;$

(b) $$\int_{\pi/2}^{5\pi/2} \frac{\cos x}{(1+\sin x)^3}\,dx$$
$$= -\frac{1}{2} \cdot \frac{1}{(1+\sin x)^2}\bigg|_{\pi/2}^{5\pi/2} = 0.$$

51. You can simulate the logarithmic spiral yourself as follows: Stand in an open field containing a lone tree and lock your neck muscles so that your head is pointed at a fixed angle $\alpha$ to your body. Walk forward in such a way that you are always looking at the tree. Prove that you will walk along a logarithmic spiral.
52. The probability $P$ that a phonograph needle will last in excess of 150 hours is given by the formula $P = \int_{150}^{\infty} \frac{1}{100} e^{-t/100}\,dt$. Find the value of $P$.
53. The probability $p$ that the score on a reading comprehension test is no greater than the value $a$ is
$$p = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\tau-\mu)^2/2\sigma^2}\,d\tau;$$
$\sigma$, $\mu$ are constants.
(a) Let $x = (\tau - \mu)/\sigma$ and $x_1 = (a - \mu)/\sigma$. Show that
$$p = \int_{-\infty}^{x_1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.$$
(b) Show that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx < \infty$.

★54. Pearson and Lee studied the inheritance of physical characteristics in families in 1903. One law that resulted from these studies is
$$P = \int_{-\infty}^{(\tau-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$
for the probability $P$ that a mother's height is not greater than $\tau$ inches. The estimated values of $\mu$ and $\sigma$ are $\mu = 62.484$ inches, $\sigma^2 = 5.7140$ square inches.
(a) Determine the value of $P$ by appeal to integral tables for
$$\int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$
using $\tau = 63$ inches. Look in a mathematical table under *probability functions* or *normal distribution*.
(b) According to the study, how many mothers out of 100 are likely to have height not exceeding 63 inches?

★55. (a) Evaluate $\displaystyle\int_0^{\infty} \frac{du}{u^{1/2} + u^{3/2}}$.
(b) For what $p$ and $q$ is
$$\int_0^{\infty} \frac{dx}{x^p + x^q}$$
convergent?

★56. Consider the surface of revolution obtained by revolving the graph of $f(x) = 1/x$ on the interval $[1, \infty)$ about the $x$ axis.
(a) Show that the area of this surface is infinite.
(b) Show that the volume of the solid of revolution bounded by this surface is finite.
(c) The results of parts (a) and (b) suggest that one could fill the solid with a finite amount of paint, but it would take an infinite amount of paint to paint the surface. Explain this paradox.

Next consider the surface of revolution obtained by revolving the curve $y = 1/x^r$ for $x$ in $[1, \infty)$ about the $x$ axis.
(d) For which values of $r$ does this surface have finite area?
(e) For which values of $r$ does the solid surrounded by this surface have finite volume? Compute the volume for these values of $r$.

★57. Show that if $0 < f'(x) < 1/x^2$ for all $x$ in $[0, \infty)$, then $\lim_{x \to \infty} f(x)$ exists.

## 11.4 Limits of Sequences and Newton's Method

*Solutions of equations can often be found as the limits of sequences.*

This section begins with a discussion of sequences and their limits. The topic will be taken up again in Section 12.1 when we study infinite series. A *sequence* is just an "infinite list" of numbers: $a_1, a_2, a_3, \ldots,$ with one $a_n$ for each natural number $n$. A number $l$ is called the *limit* of this sequence if, roughly speaking, $a_n$ comes and remains arbitrarily close to $l$ as $n$ increases.

Perhaps the most familiar example of a sequence with a limit is that of an infinite decimal expansion. Consider, for instance, the equation

$$\tfrac{1}{3} = 0.333\ldots \qquad (1)$$

in which the dots on the right-hand side are taken to stand for "infinitely many 3's." We can interpret equation (1) without recourse to any metaphysical notion of infinity: the finite decimals 0.3, 0.33, 0.333, and so on are approximations to $\frac{1}{3}$, and we can make the approximation as good as we wish by taking enough 3's. Our sequence $a_1, a_2, \ldots$ is defined in this case by $a_n = 0.33\ldots3$, with $n$ 3's (here the three dots stand for only finitely many 3's). In other words,

$$a_n = \frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}. \qquad (2)$$

We can estimate the difference between $a_n$ and $\frac{1}{3}$ by using some algebra. Multiplying equation (2) by 10 gives

$$10a_n = 3 + \frac{3}{10} + \cdots + \frac{3}{10^{n-1}}, \qquad (3)$$

and subtracting equation (2) from equation (3) gives

$$9a_n = 3 - \frac{3}{10^n},$$

$$a_n = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{10^n}\right).$$

Finally,

$$\frac{1}{3} - a_n = \frac{1}{3}\left(\frac{1}{10^n}\right). \qquad (4)$$

As $n$ is taken larger and larger, the denominator $10^n$ becomes larger and larger, and so the difference $\frac{1}{3} - a_n$ becomes smaller and smaller. In fact, if $n$ is chosen large enough, we can make $\frac{1}{3} - a_n$ as small as we please. (See Fig. 11.4.1.)

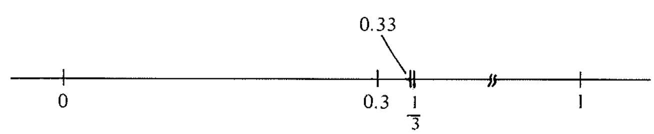



**Figure 11.4.1.** The decimal approximations to $\frac{1}{3}$ form a sequence converging to $\frac{1}{3}$.

**Example 1** How large must $n$ be for the error $\frac{1}{3} - a_n$ to be less than 1 part in 1 million?

**Solution** By equation (4), we must have

$$\frac{1}{3}\left(\frac{1}{10^n}\right) < 10^{-6}$$

or $10^{-n} < 3 \cdot 10^{-6}$. It suffices to have $n \geqslant 6$, so the finite decimal 0.333333 approximates $\frac{1}{3}$ to within 1 part in a million. So do the longer decimals 0.3333333, 0.33333333, and so on. ▲

There is nothing special about the number $10^{-6}$ in Example 1. Given *any* positive number $\varepsilon$, we will always be able to make $\frac{1}{3} - a_n = \frac{1}{3}(1/10^n)$ less than $\varepsilon$ by letting $n$ be sufficiently large. We express this fact by saying that $\frac{1}{3}$ is the *limit* of the numbers

$$a_n = \frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}$$

as $n$ becomes arbitrarily large, or

$$\lim_{n\to\infty}\left(\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}\right) = \frac{1}{3}.$$

We may think of a sequence $a_1, a_2, a_3, \ldots$ as a function whose domain consists of the natural numbers $1, 2, 3, \ldots$ (Occasionally, we allow the domain to start at zero or some other integer.) Thus we may represent a sequence graphically in two ways—either by plotting the points $a_1, a_2, \ldots$ on a number line or by plotting the pairs $(n, a_n)$ in the plane.

**Example 2** (a) Write the first six terms of the sequence $a_n = n/(n+1)$, $n = 1, 2, 3, \ldots$. Represent the sequence graphically in two ways. Find the value of $\lim_{n\to\infty}[n/(n+1)]$. (b) Repeat for $a_n = (-1)^n/n$. (c) Repeat for $a_n = (-1)^n n/(n+1)$.

**Solution** (a) We obtain the terms $a_1$ through $a_6$ by substituting $n = 1, 2, 3, \ldots, 6$ into the formula for $a_n$, giving $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$. These values are plotted in Fig. 11.4.2. As $n$ gets larger, the fraction $n/(n+1)$ gets larger and larger but never exceeds 1; we may guess that the limit is equal to 1.

To verify this guess, we look at the difference $1 - n/(n+1)$. We have

$$1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1},$$

which does indeed become arbitrarily small as $n$ increases, so

$$\lim_{n\to\infty}\frac{n}{(n+1)} = 1.$$

(b) The terms $a_1$ through $a_6$ are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$. They are plotted in Fig. 11.4.3. As $n$ gets larger, the number $(-1)^n/n$ seems to get closer to zero. Therefore we guess that $\lim_{n\to\infty}[(-1)^n/n] = 0$.

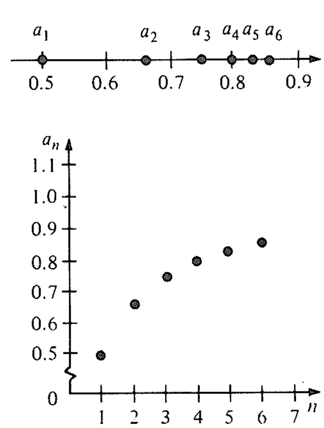



**Figure 11.4.2.** The sequence $a_n = n/(n+1)$ represented graphically in two different ways.

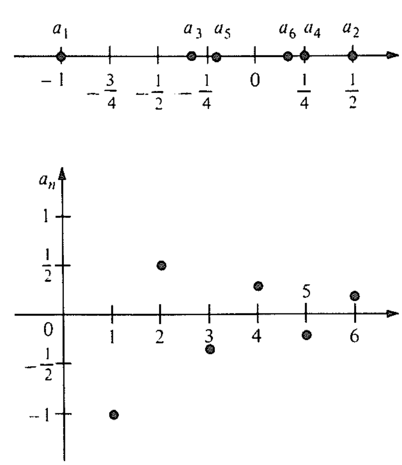



**Figure 11.4.3.** The sequence $a_n = (-1)^n/n$ plotted in two ways.

(c) We have, for $a_1$ through $a_6$, $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$. They are plotted in Fig. 11.4.4. In this case, the numbers $a_n$ do not approach any particular number. (Some of them are approaching 1, others $-1$.) We guess that the sequence does not have a limit. ▲

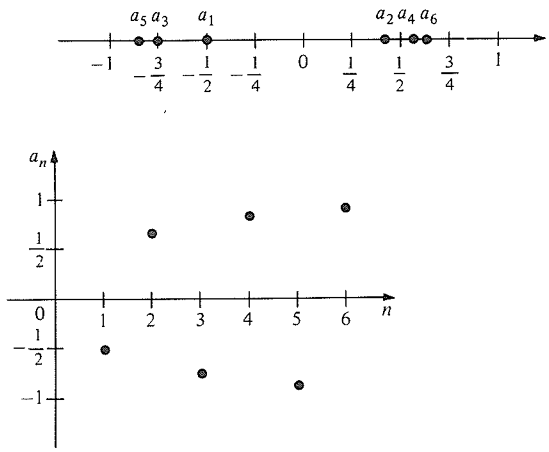



**Figure 11.4.4.** The sequence $a_n = (-1)^n n/(n+1)$ plotted in two ways.

Just as with the $\varepsilon$-$\delta$ definition for limits of functions, there is an $\varepsilon$-$N$ definition for limits of sequences which makes the preceding ideas precise.

## Limits of Sequences

The sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ approaches $l$ as a limit if $a_n$ gets close to and remains arbitrarily close to $l$ as $n$ becomes large. In this case, we write

$$\lim_{n\to\infty} a_n = l.$$

In precise terms, $\lim_{n\to\infty} a_n = l$ if, for every $\varepsilon > 0$, there is an $N$ such that $|a_n - l| < \varepsilon$ for all $n \geqslant N$.

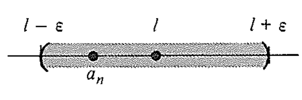



**Figure 11.4.5.** The relationship between $a_n$, $l$, and $\varepsilon$ in the definition of the limit of a sequence.

It is useful to think of the number $\varepsilon$ in this definition as a *tolerance*, or allowable error. The definition specifies that if $l$ is to be the limit of the sequence $a_n$, then, given any tolerance, all the terms of the sequence beyond a certain point should be within that tolerance of $l$. Of course, as the tolerance is made smaller, it will usually be necessary to go farther out in the sequence to bring the terms within tolerance of the limit. (See Fig. 11.4.5.)

The purpose of the $\varepsilon$-$N$ definition is to lay a framework for a precise discussion of limits of sequences and their properties—just as the definitions in Section 11.1.

Let us check the limit of a simple sequence using the $\varepsilon$-$N$ definition.

**Example 3** Prove that $\lim_{n\to\infty}(1/n) = 0$, using the $\varepsilon$-$N$ definition.

**Solution** To show that the definition is satisfied, we must show that for any $\varepsilon > 0$ there is a number $N$ such that $|1/n - 0| < \varepsilon$ if $n > N$. If we choose $N \geqslant 1/\varepsilon$, we get, for $n > N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} \leqslant \varepsilon.$$

Thus the assertion is proved. ▲

## Calculator Discussion

Limits of sequences can sometimes be visualized on a calculator. Consider the sequence obtained by taking successive square roots of a given positive number $a$:

$$a_0 = a, \quad a_1 = \sqrt{a}\,, \quad a_2 = \sqrt{\sqrt{a}}\,, \quad a_3 = \sqrt{\sqrt{\sqrt{a}}}\,,$$

and so forth. (See Fig. 11.4.6.)

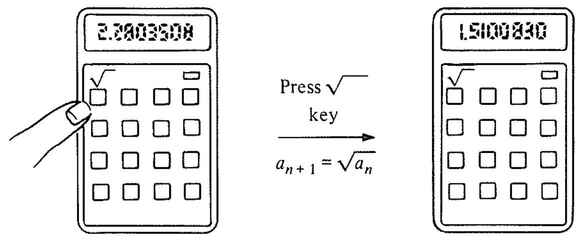



**Figure 11.4.6.** For a recursively defined sequence $a_{n+1} = f(a_n)$, the next member in the sequence is obtained by depressing the "$f$" key. Here $f = \sqrt{\ }$.

For instance, if we start by entering $a = 5.2$, we get

$$a_0 = 5.2,$$
$$a_1 = \sqrt{5.2} = 2.2803508,$$
$$a_2 = \sqrt{2.2803508} = 1.5100830,$$
$$a_3 = \sqrt{1.5100830} = 1.2288544,$$

and so on. After pressing the $\sqrt{\ }$ repeatedly you will see the numbers getting closer and closer to 1 until roundoff error causes the number 1 to appear and then stay forever. This sequence has 1 as a limit. (Of course, the calculation does not *prove* this fact, but does suggest it.) Observe that the sequence is defined *recursively*—that is, each member of the sequence is obtained from the previous one by some specific process. The sequence $1, 2, 4, 8, 16, 32, \ldots$ is another example; each term is twice the previous one: $a_{n+1} = 2a_n$. ▲

Limits of sequences are closely related to limits of functions. For example, if $f(x)$ is defined for $x \geqslant 0$, then $a_n = f(n)$ is a sequence. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n$ exists as well and these limits are equal. This fact can sometimes be used to evaluate some limits. For instance,

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1+1/x} = \frac{1}{1+\lim\limits_{x\to\infty}(1/x)} = \frac{1}{1+0} = 1,$$

and so $\lim_{n\to\infty}[n/(n+1)] = 1$, confirming our calculations in Example 2(a).

Limits of sequences also obey rules similar to those for functions.[5] We illustrate:

**Example 4** Find (a) $\lim\limits_{n\to\infty} \left( \dfrac{n^2+1}{3n^2+n} \right)$

and (b) $\lim\limits_{n\to\infty} \left( 1 - \dfrac{3}{n} + \dfrac{n}{n+1} \right)$.

**Solution** (a) Write

$$\lim_{n\to\infty} \left( \frac{n^2+1}{3n^2+n} \right) = \lim_{n\to\infty} \left( \frac{1+1/n^2}{3+1/n} \right) \qquad \text{(dividing numerator and denominator by } n^2\text{)}$$

[5] These are written out formally in Section 12.1.

$$= \frac{1 + \lim_{n\to\infty}(1/n^2)}{3 + \lim_{n\to\infty}(1/n)} \qquad \text{(quotient and sum rules)}$$

$$= \frac{1+0}{3+0} = \frac{1}{3}.$$

(b) $\lim_{n\to\infty}\left(1 - \frac{3}{n} + \frac{n}{n+1}\right) = 1 - 3\left(\lim_{n\to\infty}\frac{1}{n}\right) + \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)$

$$= 1 - 3\cdot 0 + 1 = 2. \ \blacktriangle$$

The connection with limits of functions allows us to use l'Hôpital's rule to find limits of sequences.

**Example 5** (a) Using numerical calculations, guess the value of $\lim_{n\to\infty}\sqrt[n]{n}$. (b) Use l'Hôpital's rule to verify the result in (a).

**Solution** (a) Using a calculator we find:

| $n$ | $\sqrt[n]{n}$ |
|---|---|
| 1 | 1 |
| 5 | 1.37973 |
| 10 | 1.25893 |
| 50 | 1.08138 |
| 100 | 1.04713 |
| 500 | 1.01251 |
| 1000 | 1.00693 |
| 5000 | 1.00170 |
| 10,000 | 1.00092 |

Thus it appears that $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

(b) To verify this, we use l'Hôpital's rule to show that $\lim_{x\to\infty}x^{1/x} = 1$. The limit is in $\infty^0$ form, so we use logarithms:

$$x^{1/x} = e^{(\ln x)/x}.$$

Now $\lim_{x\to\infty}(\ln x/x)$ is in $\frac{\infty}{\infty}$ form, and l'Hôpital's rule gives

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Hence

$$\lim_{x\to\infty}x^{1/x} = \exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)$$
$$= \exp(0) = 1,$$

confirming our numerical calculations. $\blacktriangle$

When we introduced limits of sequences in Example 1, we implicitly used the fact that $\lim_{n\to\infty}(1/10^n) = 0$. The following general fact is useful.

## Limits of Powers

$$\lim_{n\to\infty} r^n = \begin{cases} \infty, & \text{if } r > 1, \\ 1, & \text{if } r = 1, \\ 0, & \text{if } 0 \leqslant r < 1. \end{cases}$$

To see this, first consider the case $r > 1$. We write $r$ as $1 + s$ where $s > 0$. If we expand $r^n = (1+s)^n$, we get $r^n = 1 + ns + $ (other positive terms.) Therefore, $r^n \geqslant 1 + ns$, which goes to $\infty$ as $n \to \infty$. Second, if $r = 1$, then $r^n = 1$ for all $n$, so $\lim_{n\to\infty} r^n = 1$. Finally, if $0 \leqslant r < 1$, then excluding the easy case $r = 0$, we let $\rho = 1/r$ so $\rho > 1$, and so $\lim_{n\to\infty}\rho^n = \infty$. Therefore, $\lim_{n\to\infty} r^n = \lim_{n\to\infty}(1/\rho^n) = 0$ (compare the reciprocal test for limits of functions in Section 11.1).

**Example 6** Evaluate (a) $\lim_{n\to\infty} 3^n$, (b) $\lim_{n\to\infty} e^{-n}$, (c) $\lim_{n\to\infty}(e + (\frac{2}{3})^n)^4$.

**Solution** (a) Here $r = 3 > 1$, so $\lim_{n\to\infty} 3^n = \infty$.
(b) $e^{-n} = (1/e)^n$, and $1/e < 1$, so $\lim_{n\to\infty} e^{-n} = 0$.
(c) $\lim_{n\to\infty}[e + (\frac{2}{3})^n]^4 = [e + \lim_{n\to\infty}(\frac{2}{3})^n]^4 = [e + 0]^4 = e^4$. ▲

Another useful test is the comparison test: it says that if $\lim_{n\to\infty} a_n = 0$ and if $|b_n| \leqslant |a_n|$, then $\lim_{n\to\infty} b_n = 0$ as well. This is plausible since $b_n$ is squeezed between $-|a_n|$ and $|a_n|$ which are tending to zero. We ask the reader to supply the proof in Exercise 56.

## Comparison Test

If $\lim_{n\to\infty} a_n = 0$ and $|b_n| \leqslant |a_n|$ then $\lim_{n\to\infty} b_n = 0$.

**Example 7** Find (a) $\displaystyle\lim_{n\to\infty}\frac{\sin n}{n}$ and (b) $\displaystyle\lim_{n\to\infty}\frac{(-1)^n + n}{n}$.

**Solution** (a) If $a_n = 1/n$ and $b_n = (\sin n)/n$, then $a_n \to 0$ and $|b_n| \leqslant |a_n|$, so by the comparison test, $\lim_{n\to\infty}(\sin n)/n = 0$.
(b) $|(-1)^n/n| \leqslant 1/n \to 0$, so $(-1)^n/n \to 0$ by the comparison test. Thus

$$\lim_{n\to\infty}\left(\frac{(-1)^n + n}{n}\right) = \lim_{n\to\infty}\left(\frac{(-1)^n}{n}\right) + \lim_{n\to\infty}\frac{n}{n} = 0 + 1 = 1. \ ▲$$

Many questions in mathematics and its applications lead to the problem of solving an equation of the form

$$f(x) = 0, \tag{5}$$

where $f$ is some function. The solutions of equation (5) are called the *roots* or *zeros* of $f$. If $f$ is a polynomial of degree at most 4, one can find the roots of $f$ by substituting the coefficients of $f$ into a general formula (see pp. 17 and 173). On the other hand, if $f$ is a polynomial of degree 5 or greater, or a function involving the trigonometric or exponential functions, there may be no explicit formula for the roots of $f$, and one may have to search for the solution numerically.

Newton's method uses linear approximations to produce a sequence $x_0, x_1, x_2, \ldots$ which converges to a solution of $f(x) = 0$. Let $x_0$ be a first guess. We seek to correct this guess by an amount $\Delta x$ so that $f(x_0 + \Delta x) = 0$. Solving this equation for $\Delta x$ is no easier than solving the original equation (5), so we manufacture an easier problem, replacing $f$ by its first-order approximation at $x_0$; that is, we replace $f(x_0 + \Delta x)$ by $f(x_0) + f'(x_0)\Delta x$. If $f(x_0)$ is not equal to zero, we can solve the equation $f(x_0) + f'(x_0)\Delta x = 0$ to obtain $\Delta x = -f(x_0)/f'(x_0)$, so that our new guess is

$$x_1 = x_0 + \Delta x = x_0 - f(x_0)/f'(x_0).$$

Geometrically, we have found $x_1$ by following the tangent line to the graph of $f$ at $(x_0, f(x_0))$ until it meets the $x$ axis; the point where it meets is $(x_1, 0)$ (see Fig. 11.4.7).

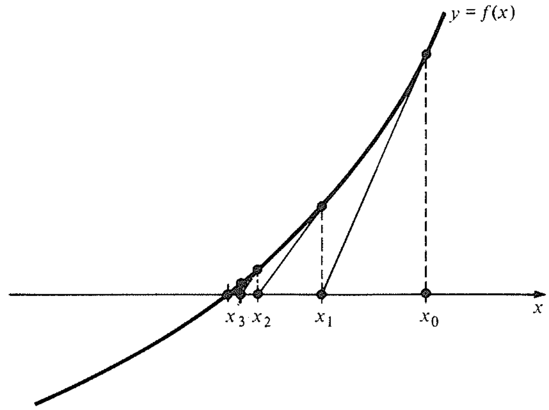



**Figure 11.4.7.** The geometry of Newton's method.

Now we find a new guess $x_2$ by repeating the procedure with $x_1$ in place of $x_0$; that is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, once we have found $x_n$, we define $x_{n+1}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6}$$

Let us see how the method works in a case where we know the answer in advance. (This iteration procedure is particularly easy to use on a programmable calculator.)

**Example 8** Use Newton's method to find the first few approximations to a solution of the equation $x^2 = 4$, taking $x_0 = 1$.

**Solution** To put the equation $x^2 = 4$ in the form $f(x) = 0$, we let $f(x) = x^2 - 4$. Then $f'(x) = 2x$, so the iteration rule (6) becomes $x_{n+1} = x_n - (x_n^2 - 4)/2x_n$, which may be simplified to $x_{n+1} = \frac{1}{2}(x_n + 4/x_n)$. Applying this formula repeatedly, with $x_0 = 1$, we get (to the limits of our calculator's accuracy)

$$\begin{aligned} x_1 &= 2.5 \\ x_2 &= 2.05 \\ x_3 &= 2.000609756 \\ x_4 &= 2.000000093 \\ x_5 &= 2 \\ x_6 &= 2 \\ &\vdots \end{aligned}$$

and so on forever. The number 2 is, of course, precisely the positive root of our equation $x^2 = 4$. ▲

**Example 9** Use Newton's method to locate a root of $x^5 - x^4 - x + 2 = 0$. Compare what happens with various starting values of $x_0$ and attempt to explain the phenomenon.

**Solution** The iteration formula is

$$x_{n+1} = x_n - \frac{x_n^5 - x_n^4 - x_n + 2}{5x_n^4 - 4x_n^3 - 1} = \frac{4x_n^5 - 3x_n^4 - 2}{5x_n^4 - 4x_n^3 - 1}.$$

For the purpose of convenient calculation, we may write this as

$$x_{n+1} = \frac{(4x_n - 3)x_n^4 - 2}{(5x_n - 4)x_n^3 - 1}.$$

Starting at $x_0 = 1$, we find that the denominator is undefined, so we can go no further. (Can you interpret this difficulty geometrically?)

Starting at $x_0 = 2$, we get

$$\begin{aligned}
x_1 &= 1.659574468,\\
x_2 &= 1.372968569,\\
x_3 &= 1.068606737,\\
x_4 &= -0.5293374382,\\
x_5 &= 169.5250382.
\end{aligned}$$

The iteration process seems to have sent us out on a wild goose chase. To see what has gone wrong, we look at the graph of $f(x) = x^5 - x^4 - x + 2$. (See Fig. 11.4.8.) There is a "bowl" near $x_0 = 2$; Newton's method attempts to take us down to a nonexistent root. (Only after many iterations does one converge to the root—see Exercise 59 and Example 10.)

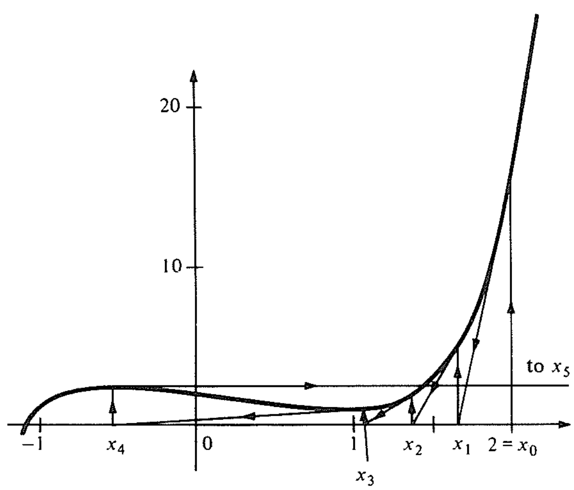



**Figure 11.4.8.** Newton's method does not always work.

Finally, we start with $x_0 = -2$. The iteration gives

$$\begin{aligned}
x_0 &= -2, & f(x_0) &= -44;\\
x_1 &= -1.603603604, & f(x_1) &= -13.61361361;\\
x_2 &= -1.323252501, & f(x_2) &= -3.799819057;\\
x_3 &= -1.162229582, & f(x_3) &= -0.782974790;\\
x_4 &= -1.107866357, & f(x_4) &= -0.067490713;\\
x_5 &= -1.102228599, & f(x_5) &= -0.000663267;\\
x_6 &= -1.102172085, & f(x_6) &= -0.000000061;\\
x_7 &= -1.102172080, & f(x_7) &= -0.000000003.
\end{aligned}$$

Since the numbers in the $f(x)$ column appear to be converging to zero and those in the $x$ column are converging, we obtain a root to be (approximately) $-1.10217208$. Since $f(x)$ is negative at this value (where $f(x) = -0.000000003$) and positive at $-1.10217207$ (where $f(x) = 0.000000115$), we can conclude, by the intermediate value theorem, that the root is between these two values. ▲

Example 9 illustrates several important features of Newton's method. First of all, it is important to start with an initial guess which is reasonably close to a root—graphing is a help in making such a guess. Second, we notice that once we get near a root, then convergence becomes very rapid—in fact, the number of correct decimal places is approximately *doubled* with each iteration. Finally, we notice that the process for passing from $x_n$ to $x_{n+1}$ is the same for each value of $n$; this feature makes Newton's method particularly attractive for use with a programmable calculator or a computer. Human intelligence still comes into play in the choice of the first guess, however.

## Newton's Method

To find a root of the equation $f(x) = 0$, where $f$ is a differentiable function such that $f'$ is continuous, start with a guess $x_0$ which is reasonably close to a root. Then produce the sequence $x_0, x_1, x_2, \dots$ by the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $\lim_{n\to\infty} x_n = \bar{x}$, then $f(\bar{x}) = 0$.

To justify the last statement in the box above, we suppose that $\lim_{n\to\infty} x_n = \bar{x}$. Taking limits on both sides of the equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$, we obtain $\bar{x} = \bar{x} - \lim_{n\to\infty}[f(x_n)/f'(x_n)]$, or $\lim_{n\to\infty}[f(x_n)/f'(x_n)] = 0$. Now let $a_n = f(x_n)/f'(x_n)$. Then we have $\lim_{n\to\infty} a_n = 0$, while $f(x_n) = a_n f'(x_n)$. Taking limits as $n \to \infty$ and using the continuity of $f$ and $f'$, we find

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} f'(x_n), \quad \text{so}$$

$$f(\bar{x}) = 0 \cdot f'(\bar{x}) = 0.$$

Newton's method, applied with care, can also be used to solve equations involving trigonometric or exponential functions.

**Example 10** Use Newton's method to find a positive number $x$ such that $\sin x = x/2$.

**Solution** With $f(x) = \sin x - x/2$, the iteration formula becomes

$$x_{n+1} = x_n - \frac{\sin x_n - x_n/2}{\cos x_n - 1/2} = \frac{2(x_n \cos x_n - \sin x_n)}{2\cos x_n - 1}.$$

Taking $x_0 = 0$ as our first guess, we get $x_1 = 0$, $x_2 = 0$, and so forth, since zero is already a root of our equation. To find a positive root, we try another guess, say $x_0 = 6$. We get

$$
\begin{aligned}
x_1 &= 13.12652598 & x_5 &= 266.0803351 \\
x_2 &= 30.50101246 & x_6 &= 143.3754278 \\
x_3 &= 176.5342378 & &\ \vdots \\
x_4 &= 448.4888306 & x_9 &= -759.1194553 \\
& & x_{10} &= 3{,}572.554623
\end{aligned}
$$

We do not seem to be getting anywhere. To see what might be wrong, we draw a sketch (Fig. 11.4.9). The many places where the graph of $\sin x - x/2$

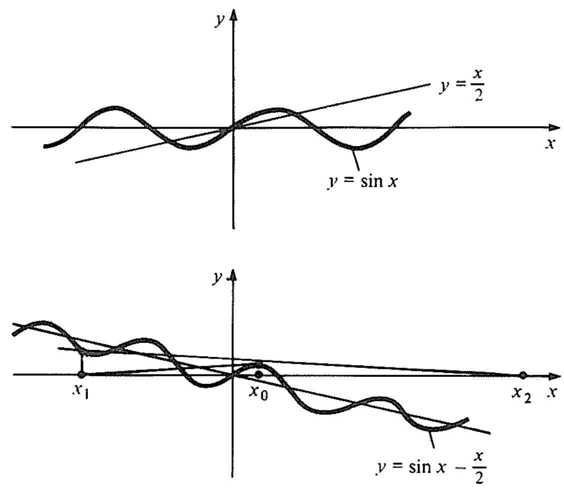



**Figure 11.4.9.** Newton's method goes awry.

has a horizontal or nearly horizontal tangent causes the Newton sequence to make wild excursions.[6] We need to make a better first guess; we try $x_0 = 3$. This gives

$$\begin{aligned} x_1 &= 2.08799541, & x_5 &= 1.89549427,\\ x_2 &= 1.91222926, & x_6 &= 1.89549427,\\ x_3 &= 1.89565263, & x_7 &= 1.89549427,\\ x_4 &= 1.89549428, & x_8 &= 1.89549427. \end{aligned}$$

We conclude that our root is somewhere near 1.89549427. Substituting this value for $x$ in $\sin x - x/2$ gives $1.0 \times 10^{-11}$. There may be further doubt about the last figure, due to internal roundoff errors in the calculator; we are probably safe to announce our result as 1.8954943. ▲

You may find it amusing to try other starting values for $x_0$ in Example 10. For instance, the values 6.99, 7, and 7.01 seem to lead to totally different results. (This was on a HP 15C hand calculator. Numerical errors may be crucial in a calculation such as this.) Recently, the study of sequences defined by iteration has become important as a model for the long-time behavior of dynamical systems. For instance, sequences defined by simple rules of the form $x_{n+1} = ax_n(1 - x_n)$ display very different behavior according to the value of the constant $a$. (See the supplement to this section and Exercise 59.)

## Supplement to Section 11.4
## Newton's Method and Chaos

The sequences generated by Newton's method may exhibit several types of strange behavior if the starting guess is not close to a root:

(a) the sequence $x_0, x_1, x_2, \ldots$ may wander back and forth over the real line for some time before converging to a root;

[6] Try these calculations and those in Example 9 on your calculator and see if you converge to the root after many iterations. You will undoubtedly get different numbers from ours, probably due to roundoff errors, computer inaccuracies and the extreme sensitivity of the calculations. We got four different sets of numbers with four calculators. (The ones here were found on an HP 15C which also has a SOLVE algorithm which cleverly avoids many difficulties.)

(b) slightly different choices of $x_0$ or the use of different calculators may lead to very different sequences;
(c) the sequence $x_0, x_1, x_2, \ldots$ may eventually cycle between two or more values, none of which is a root of the equation we are trying to solve;
(d) the sequence $x_0, x_1, x_2, \ldots$ may wander "forever" without ever settling into a regular pattern.

Recent research in pure and applied mathematics has shown that the type of erratic behavior just described is the rule rather than the exception for many mathematical operations and the physical processes which they model (see Exercise 59 for a simple example). Indeed, "chaotic" behavior is observed in fluid flow, chemical reactions, and biological systems, and is responsible for the inherent unpredictability of the weather.

Some references on this work on "chaos", aimed at the nonexpert reader, are:

M. J. Feigenbaum, "Universal behavior in nonlinear systems," Los Alamos Science **1** (Summer 1980), 4–27.
D. R. Hofstadter, "Metamagical themas," Scientific American **245** (November 1981), 22–43.
L. P. Kadanoff, "Roads to chaos," Physics Today **36** (December 1983), 46–53.
D. Ruelle, "Strange attractors," Math. Intelligencer **2** (1980), 126–137.
D. G. Saari and J. B. Urenko, "Newton's method, circle maps, and chaotic motion," American Mathematical Monthly **91** (1984), 3–17; see also **92** (1985) 157–158.

## Exercises for Section 11.4

1. If $a_n = 1/10 + 1/100 + \cdots + 1/10^n$, how large must $n$ be for $\frac{1}{9} - a_n$ to be less than $10^{-6}$?
2. If $a_n = 7/10 + 7/100 + \cdots + 7/10^n$, how large must $n$ be for $\frac{7}{9} - a_n$ to be less than $10^{-8}$?

Find the limits of the sequences in Exercises 3 and 4.

3. $a_n = 1 + 1/2 + 1/4 + \cdots + 1/2^n$.
4. $a_n = \sin(n\pi/2)$.

Write down the first six terms of the sequences in Exercises 5–10.

5. $k_n = n^2 - 2\sqrt{n}$; $n = 0, 1, 2, \ldots$.
6. $a_n = (-1)^{n+1}[(n-1)/n!]$; $n = 0, 1, 2, \ldots$ $(0! = 1,\ n! = n(n-1)\cdots 3 \cdot 2 \cdot 1.)$
7. $b_n = nb_{n-1}/(1+n)$; $b_0 = \frac{1}{7}$.
8. $c_{n+1} = -c_n/[2n(4n+1)]$; $c_1 = 2$.
9. $a_{n+1} = [1/(n+1)]\sum_{i=0}^{n} a_i$; $a_0 = \frac{1}{2}$.
10. $k_n = \sqrt{3n^2 + 2n}$; $n = 1, 2, 3, \ldots$.

Establish the limits in Exercises 11–14 using the $\varepsilon$-$N$ definition.

11. $\lim_{n\to\infty} \frac{3}{n} = 0$
12. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$
13. $\lim_{n\to\infty} \frac{3}{2n+1} = 0$
14. $\lim_{n\to\infty} \frac{2}{2n+5} = 0$

Evaluate the limits in Exercises 15–24.

15. $\lim_{n\to\infty} \frac{3n}{n+1}$
16. $\lim_{n\to\infty} \frac{2n}{8n-1}$
17. $\lim_{n\to\infty} \frac{n-3n^2}{n^2+1}$
18. $\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$
19. $\lim_{n\to\infty} \left[\frac{3n^2-2n+1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)}\right]^2$
20. $\lim_{n\to\infty} \frac{2+1/n}{(n^2-2)/(n^2+1)}$
21. $\lim_{n\to\infty} \frac{(\sin n)^2}{n+2}$
22. $\lim_{n\to\infty} \frac{(1+n)\cos(n+1)}{n^2+1}$
23. $\lim_{n\to\infty} \frac{(-1)^n \cdot 2}{n+1}$
24. $\lim_{n\to\infty} \frac{\cos(\pi\sqrt{n})}{n^2}$

Using numerical calculations, guess the limit as $n \to \infty$ of the sequences in Exercises 25–28. Verify your answers using l'Hôpital's rule.

25. $\sqrt[n]{n/2}$
26. $\sqrt[n]{n/3}$
27. $\sqrt[n]{n(n+1)/4}$
28. $\sqrt[n]{n(n+3)/7}$

Find the limits in Exercises 29–34.

29. $\lim_{n\to\infty} \frac{1}{8^n}$
30. $\lim_{n\to\infty} \pi^n$
31. $\lim_{n\to\infty} \frac{n+(3/4)^n}{n^2+2}$
32. Find $\lim_{n\to\infty} (\pi + (\frac{2}{3})^n)^3$
33. $\lim_{n\to\infty} \left[\frac{3b+(1/2)^{2n}}{n^2-1}\right]^3$; $b$ constant
34. $\lim_{n\to\infty} \left(\frac{2a+e^{-2n}}{n-1}\right)^2$; $a$ constant

35. (a) Use Newton's method to find a solution of $x^3 - 8x^2 + 2x + 1 = 0$. (b) Use division and the quadratic formula to find the other two roots.[7]

36. Use Newton's method to find all real roots of $x^3 - x + \frac{1}{10}$.

37. Use Newton's method to locate a root of $f(x) = x^5 + x^2 - 3$ with starting values $x_0 = 0$, $x_0 = 2$.

38. Use Newton's method to locate a zero for $f(x) = x^4 - 2x^3 - 1$. Use $x_0 = 2$, 3, and $-1$ as starting values and compare the results.

39. Use Newton's method to locate a root of $\tan x = x$ in $[\pi/2, 3\pi/2]$.

40. Use Newton's method to find the following numbers: (a) $\sqrt{2}$; (b) $\sqrt[3]{2}$

41. The equation $\tan x = \alpha x$ appears in heat conduction problems to determine values $\lambda_1, \lambda_2, \lambda_3, \ldots$ that appear in the expression for the temperature distribution. The numbers $\lambda_1, \lambda_2, \ldots$ are the positive solutions of $\tan x = \alpha x$, listed in increasing order. Find the numbers $\lambda_1, \lambda_2, \lambda_3$ for $\alpha = 2$, 3, 5, by Newton's method. Display your answers in a table.

42. (a) Use Newton's method to solve the equation $x^2 - 2 = 0$ to 8 decimal places of accuracy, using the initial guess $x_0 = 2$.
★(b) Find a constant $C$ such that $|x_n - \sqrt{2}| \leqslant C|x_{n-1} - \sqrt{2}|^2$ for $n = 1, 2, 3$, and 4. (See Review Exercise 101 for the theory of the rapid convergence of Newton's method.)

43. Experiment with Newton's method for evaluation of the root $1/e$ of the equation $e^{-ex} = 1/e$.

44. Enter the display value 1.0000000 on your calculator and repeatedly press the "sin" key using the "radian mode". This process generates display numbers $a_1 = 1.0000000$, $a_2 = 0.84147$, $a_3 = 0.74562, \ldots$.
(a) Write a formula for $a_n$, using function notation.
(b) Conjecture the value of $\lim_{n\to\infty} a_n$. Explain with a graph.

45. Display the number 2 on your calculator. Repeatedly press the "$x^2$" key. You should get the numbers 2, 4, 16, 256, 65536, .... Express the display value $a_n$ after $n$ repetitions by a formula.

46. Let $f(x) = 1 + 1/x$. Equipped with a calculator with a reciprocal function, complete the following:
(a) Write out $f(f(f(f(f(f(2))))))$ as a division problem, and calculate the value. We abbreviate this as $f^{(6)}(2)$, meaning to display the value 2, press the "$1/x$" key and add 1, successively six times.
(b) Experiment to determine $\lim_{n\to\infty}[1/f^{(n)}(2)]$ to five decimal places.

47. Suppose that $\lim_{n\to\infty} a_n = a$ and that $a > 0$. Prove that there is a positive integer $N$ such that $a_n > 0$ for all $n > N$.

48. Let $a_n = 1$ if $n$ is even and $-1$ if $n$ is odd. Does $\lim_{n\to\infty} a_n$ exist?

49. If a radioactive substance has a half-life of $T$, so that half of it decays after time $T$, write a sequence $a_n$ showing the fraction remaining after time $nT$. What is $\lim_{n\to\infty} a_n$?

50. Evaluate:
(a) $\lim_{n\to\infty}\left[\frac{(-1)^n}{3} + (-1)^{n+1}\left(\frac{1}{3} + \frac{2}{n} + \frac{1}{n^2}\right)\right]$;
(b) $\lim_{n\to\infty}\left\{\frac{(-1)^n}{2} + n\left[\frac{1 + (-1)^{n+1}(3n+1)}{6n^2 - 5n + 2}\right]\right\}$.

Find the limit or prove that the limit does not exist in Exercises 51–54.

51. $\lim_{n\to\infty}\left(\frac{1}{2} + (-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)\right)$.

52. $\lim_{n\to\infty}\left(\frac{(-1)^n \sin(n\pi/2)}{n}\right)\left(\frac{n^2+1}{n+1}\right)$.

53. $\lim_{n\to\infty}\left(\frac{3n}{4n+1} + \frac{(-1)^n \sin n}{n+1}\right)$.

54. $\lim_{n\to\infty}\frac{(1+n)\cos n}{n}$.

★55. (a) Give an $A$-$N$ definition of what $\lim_{n\to\infty} a_n = \infty$ means. (b) Prove, using your definition in part (a), that $\lim_{n\to\infty}[(1+n^2)/(1+8n)] = \infty$.

★56. If $a_n \to 0$ and $|b_n| \leqslant |a_n|$, show that $b_n \to 0$.

★57. Suppose that $a_n$, $b_n$, and $c_n$, $n = 1, 2, 3, \ldots$, are sequences of numbers such that for each $n$, we have $a_n < b_n < c_n$.
(a) If $\lim_{n\to\infty} a_n = L$ and if $\lim_{n\to\infty} b_n$ exists, show that $\lim_{n\to\infty} b_n \geqslant L$. [*Hint:* Suppose not!]
(b) If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, prove that $\lim_{n\to\infty} b_n = L$.

★58. A rubber ball is released from a height $h$. Each time it strikes the floor, it rebounds with two-thirds of its previous velocity.
(a) How far does the ball rise on each bounce? (Use the fact that the height $y$ of the ball at time $t$ from the beginning of each bounce is of the form $y = vt - \frac{1}{2}gt^2$ during the bounce. The constant $g$ is the acceleration of gravity.)
(b) How long does each bounce take?
(c) Show that the ball stops bouncing after a finite time has passed.
(d) How far has the ball travelled when it stops bouncing?
(e) How would the results differ if this experiment were done on the moon?

[7] For a computer, this method is preferable to using the formula for the roots of a cubic!

★59. (Research Problems)

(a) Experiment with the recursion relation $x_{n+1} = ax_n(1 - x_n)$ for various values of the parameter $a$ where $0 < a \leqslant 4$ and $x_0$ is in $[0, 1]$. How does the behavior of the sequences change when $a$ varies?

(b) Study the bizarre behavior of Newton's method in Example 9 for various starting values $x_0$. Can you see a pattern? Does $x_n$ always converge?

(c) Study the bizarre behavior of Newton's method in Example 10.

## 11.5 Numerical Integration

*Integrals can be approximated by sequences which can be computed numerically.*

The fundamental theorem of calculus does not solve all our integration problems. The antiderivative of a given integrand may not be easy or even possible to find. The integrand might be given, not by a formula, but by a table of values; for example, we can imagine being given power readings from an energy cell and asked to find the energy stored. In either case, it is necessary to use a method of numerical integration to find an approximate value for the integral.

In using a numerical method, it is important to estimate errors so that the final answer can be said, with confidence, to be correct to so many significant figures. The possible errors include errors in the method, roundoff errors, and roundoff errors in arithmetic operations. The task of keeping careful track of possible errors is a complicated and fascinating one, of which we can give only some simple examples.[8]

The simplest method of numerical integration is based upon the fact that the integral is a limit of Riemann sums (see Section 4.3). Suppose we are given $f(x)$ on $[a,b]$, and divide $[a,b]$ into subintervals $a = x_0 < x_1 < \cdots < x_n = b$. Then $\int_a^b f(x)\,dx$ is approximated by $\sum_{i=1}^n f(c_i)\,\Delta x_i$, where $c_i$ lies in $[x_{i-1}, x_i]$. Usually, the points $x_i$ are taken to be equally spaced, so $\Delta x_i = (b-a)/n$ and $x_i = a + i(b-a)/n$. Choosing $c_i = x_i$ or $x_{i+1}$ gives the method in the following box.

### Riemann Sums

To calculate an approximation to $\int_a^b f(x)\,dx$, let $x_i = a + i(b-a)/n$ and form the sum

$$\frac{b-a}{n}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right] \tag{1a}$$

or

$$\frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]. \tag{1b}$$

[8] For a further discussion of error analysis in numerical integration, see, for example, P. J. Davis, *Interpolation and Approximation*, Wiley, New York (1963).

**Example 1** Let $f(x) = \cos x$. Evaluate $\int_0^{\pi/2} \cos x\, dx$ by the method of Riemann sums, taking 10 equally spaced points: $x_0 = 0$, $x_1 = \pi/20$, $x_2 = 2\pi/20, \ldots, x_{10} = 10\pi/20 = \pi/2$, and $c_i = x_i$. Compare the answer with the actual value.

**Solution** Formula (1a) gives

$$\begin{aligned}\int_0^{\pi/2} \cos x\, dx &\approx \frac{\pi}{20}\left(1 + \cos\frac{\pi}{20} + \cos\frac{2\pi}{20} + \cdots + \cos\frac{9\pi}{20}\right)\\ &= \frac{\pi}{20}(1 + 0.98769 + 0.95106 + \cdots + 0.15643)\\ &= \frac{\pi}{20}(6.85310) = 1.07648.\end{aligned}$$

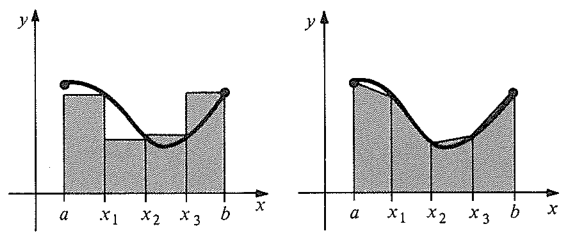
The actual value is $\sin(\pi/2) - \sin(0) = 1$, so our estimate is about 7.6% off. ▲

Unfortunately, this method is inefficient, because many points $x_i$ are needed to get an accurate estimate of the integral. For this reason we will seek alternatives to the method of Riemann sums.

To get a better method, we estimate the area in each interval $[x_{i-1}, x_i]$ more accurately by replacing the rectangular approximation by a trapezoidal one. (See Fig. 11.5.1.) We join the points $(x_i, f(x_i))$ by straight line segments to obtain a set of approximating trapezoids. The area of the trapezoid between $x_{i-1}$ and $x_i$ is

$$A_i = \tfrac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]\Delta x_i$$

since the area of a trapezoid is its average height times its width.



(a) Riemann sums method (b) Trapezoidal method

**Figure 11.5.1.** Comparing two methods of numerical integration.

The approximation to $\int_a^b f(x)\,dx$ given by the trapezoidal rule is $\sum_{i=1}^n \frac{1}{2}[f(x_{i-1}) + f(x_i)]\Delta x_i$. This becomes simpler if the points $x_i$ are equally spaced. Then $\Delta x_i = (b-a)/n$, $x_i = a + i(b-a)/n$, and the sum is

$$\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

which can be rewritten as

$$\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

since every term occurs twice except those from the endpoints. Although we used areas to obtain this formula, we may apply it even if $f(x)$ takes negative values.

## Trapezoidal Rule

To calculate an approximation to $\int_a^b f(x)\,dx$, let $x_i = a + i(b-a)/n$ and form the sum

$$\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]. \tag{2}$$

Formula (2) turns out to be much more accurate than the method of Riemann sums, even though it is just the average of the Riemann sums (1a) and (1b). Using results of Section 12.5, one can show that the error in the *method* (apart from other roundoff or cumulative errors) is $\leqslant [(b-a)/12]M_2(\Delta x)^2$, where $M_2$ is the maximum of $|f''(x)|$ on $[a,b]$. Of course, if we are given only numerical data, we have no way of estimating $M_2$, but if a formula for $f$ is given, $M_2$ can be determined. Note, however, that the error depends on $(\Delta x)^2$, so if we divide $[a,b]$ into $k$ times as many divisions, the error goes down by a factor of $1/k^2$. The error in the Riemann sums method, on the other hand, is $\leqslant (b-a)M_1(\Delta x)$, where $M_1$ is the maximum of $|f'(x)|$ on $[a,b]$. Here $\Delta x$ occurs only to the first power. Thus even if we do not know how large $M_1$ and $M_2$ are, if $n$ is taken large enough, the error in the trapezoidal rule will eventually be much smaller than that in the Riemann sums method.

**Example 2** Repeat Example 1 by using the trapezoidal rule. Compare the answer with the true value.

**Solution** Now formula (2) becomes

$$\frac{\pi/2}{2\cdot 10}\left(\cos 0 + 2\cos\frac{\pi}{20} + \cdots + 2\cos\frac{9\pi}{20} + \cos\frac{\pi}{2}\right)$$

$$\approx \frac{\pi}{40}\left[1 + 2(0.9877 + 0.9511 + \cdots + 0.1564) + 0\right] \approx 0.9979.$$

The answer is correct to within about 0.2%, much better than the accuracy in Example 1. ▲

**Example 3** Use the trapezoidal rule with $n = 10$ to estimate numerically the area of the surface obtained by revolving the graph of $y = x/(1+x^2)$ about the $x$ axis, $0 \leqslant x \leqslant 1$.

**Solution** The area is given by formula (2) on p. 483:

$$A = 2\pi\int_0^1\left(\frac{x}{1+x^2}\right)\sqrt{1 + \left[\frac{d}{dx}\left(\frac{x}{1+x^2}\right)\right]^2}\,dx$$

$$= 2\pi\int_0^1 \frac{x\sqrt{(1+x^2)^4 + (1-x^2)^2}}{(1+x^2)^3}\,dx.$$

There is little hope of carrying out this integration, so a numerical approach seems appropriate. We use the trapezoidal rule with the following values:

| $x_i$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y_i = f(x_i)$ | 0 | 0.13797 | 0.25713 | 0.34668 | 0.40650 | 0.44369 | 0.46684 | 0.48204 | 0.49216 | 0.49807 | 0.50000 |

where $f(x) = x\sqrt{(1+x^2)^4 + (1-x^2)^2} \,/\, (1+x^2)^3$. Inserting these data in the formula

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

with $x_i = a + [i(b-a)/n]$, $a = 0$ and $b = 1$, gives

$$\int_0^1 \frac{x\sqrt{(1+x^2)^4 + (1-x^2)^2}}{(1+x^2)^3}\,dx \approx 0.37811,$$

so the area is $A \approx (2\pi)(0.37811) = 2.3757$. Of course, we cannot be sure how many decimal places in this result are correct without an error analysis (see Exercise 17).[9] ▲

There is a yet more powerful method of numerical integration called Simpson's rule,[10] which is based on approximating the graph by parabolas rather than straight lines. To determine a parabola we need to specify three points through which it passes; we will choose the adjacent points

$$(x_{i-1}, f(x_{i-1})), \qquad (x_i, f(x_i)), \qquad (x_{i+1}, f(x_{i+1})).$$

It is easily proved (see Exercise 11) that the integral from $x_{i-1}$ to $x_{i+1}$ of the quadratic function whose graph passes through these three points is

$$A_i = \frac{\Delta x}{3}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right],$$

where $\Delta x = x_i - x_{i-1} = x_{i+1} - x_i$ (equally spaced points). See Fig. 11.5.2.

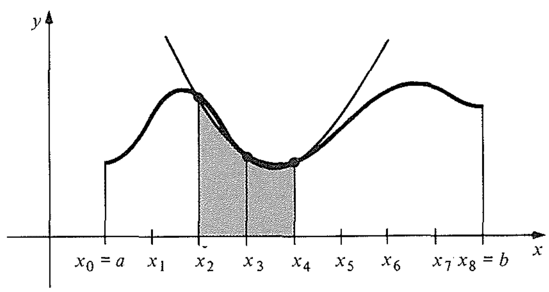



**Figure 11.5.2.** Illustrating Simpson's rule.

If we do this for every set of three adjacent points, starting at the left endpoint $a$—that is, for $\{x_0, x_1, x_2\}$, then $\{x_2, x_3, x_4\}$, then $\{x_4, x_5, x_6\}$, and so on—we will get an approximate formula for the area. In order for the points to fill the interval exactly, $n$ should be even, say $n = 2m$.

As in the trapezoidal rule, the contributions from endpoints $a$ and $b$ are counted only once, as are those from the center points of triples $\{x_{i-1}, x_i, x_{i+1}\}$ (that is, $x_i$ for $i$ odd), while the others are counted twice. Thus we are led to Simpson's rule, stated in the box on the next page.

This method is very accurate; the error in using formula (3) does not exceed $[(b-a)/180]M_4(\Delta x)^4$, where $M_4$ is the maximum of the fourth derivative of $f(x)$ on $[a,b]$. As $\Delta x$ is taken smaller and smaller, this error decreases much faster than in the other two methods. It is remarkable that juggling the

[9] The HP 15C has a clever integration program that is careful about errors. It gives 2.3832 for this integral in a few minutes.

[10] It was discussed by Thomas Simpson in his book, *Mathematical Dissertations on Physical and Analytical Subjects* (1743).

coefficients to give formula (3) in place of formula (1) or formula (2) can increase the accuracy so much.

## Simpson's Rule

To calculate an approximation to $\int_a^b f(x)\,dx$, let $n = 2m$ be even and $x_i = a + i(b-a)/n$. Form the sum

$$\frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]. \tag{3}$$

**Example 4** Repeat Example 1 using Simpson's rule. Compare the answer with the true value.

**Solution** Using a calculator, we can evaluate formula (3) by

$$\begin{aligned}&\frac{\pi/2}{3\cdot 10}\Big(\cos 0 + 4\cos\frac{\pi}{20} + 2\cos\frac{2\pi}{20} + 4\cos\frac{3\pi}{20} + 2\cos\frac{4\pi}{20}\\ &\quad + 4\cos\frac{5\pi}{20} + 2\cos\frac{6\pi}{20} + 4\cos\frac{7\pi}{20} + 2\cos\frac{8\pi}{20} + 4\cos\frac{9\pi}{20} + \cos\frac{\pi}{2}\Big)\\ &\approx \frac{\pi}{60}(1 + 3.9507534 + \cdots + 0) = \frac{\pi}{60}\cdot 19.098658\\ &\approx 1.0000034.\end{aligned}$$

The error is less than four parts in a million. ▲

**Example 5** Suppose that you are given the following table of data:

| | | |
|---|---|---|
| $f(0) = 0.846$ | $f(0.4) = 1.121$ | $f(0.8) = 2.321$ |
| $f(0.1) = 0.928$ | $f(0.5) = 1.221$ | $f(0.9) = 3.101$ |
| $f(0.2) = 0.882$ | $f(0.6) = 1.661$ | $f(1.0) = 3.010$ |
| $f(0.3) = 0.953$ | $f(0.7) = 2.101$ | |

Evaluate $\int_0^1 f(x)\,dx$ by Simpson's rule.

**Solution** By formula (3),

$$\int_0^1 f(x)\,dx \approx \frac{1}{30}\left[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1.0)\right].$$

Inserting the given values of $f$ and evaluating on a calculator, we get

$$\int_0^1 f(x)\,dx = \frac{1}{30}(49.042) = 1.635.$$

This should be quite accurate unless the fourth derivative of $f$ is very large. ▲

**Example 6** How small must we take $\Delta x$ in the trapezoidal rule to evaluate $\int_2^4 e^{-x^2}\,dx$ to within $10^{-6}$? For Simpson's rule?

**Solution** Let $f(x) = e^{-x^2}$, $a = 2$, and $b = 4$. The error in the trapezoidal rule is no more than $[(b-a)/12]M_2(\Delta x)^2$, where $M_2$ is the maximum of $|f''(x)|$ on $[a,b]$. We find

$$f'(x) = -2xe^{-x^2}, \quad \text{and} \quad f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}.$$

Now $f'''(x) = (12x - 8x^3)e^{-x^2} = 4x(3 - 2x^2)e^{-x^2} < 0$ on $[2,4]$, so $f''(x)$ is decreasing. Also, $f''(x) > 0$ on $[2,4]$, so $|f''(x)| = f''(x) \leqslant f''(2) = 14e^{-4} = M_2$, so the error is at most

$$\frac{b-a}{12} M_2(\Delta x)^2 = \frac{1}{6} \cdot 14e^{-4}(\Delta x)^2.$$

To make this less than $10^{-6}$, we should choose $\Delta x$ so that

$$\tfrac{1}{6} \cdot 14e^{-4}(\Delta x)^2 < 10^{-6},$$
$$(\Delta x)^2 < e^4 10^{-6} \cdot \tfrac{3}{7} = 0.0000234,$$
$$\Delta x < 0.0048.$$

That is, we should take at least $n = (b-a)/\Delta x = 416$ divisions.

For Simpson's rule, the error is at most $[(b-a)/180]M_4(\Delta x)^4$. Here

$$f''''(x) = 4(4x^4 - 12x^2 + 3)e^{-x^2}.$$

On $[2,4]$, we find that $4x^4 - 12x^2 + 3$ is increasing and $e^{-x^2}$ is decreasing, so

$$|f''''(x)| \leqslant 4(4 \cdot 4^4 - 12 \cdot 4^2 + 3)e^{-4}$$
$$= 61.17 = M_4.$$

Thus $[(b-a)/180]M_4(\Delta x)^4 = \frac{1}{90} \cdot 61.17(\Delta x)^4 = 0.68(\Delta x)^4$. Hence if we are to have error less than $10^{-6}$, it suffices to have

$$0.68(\Delta x)^4 \leqslant 10^{-6},$$
$$\Delta x \leqslant 0.035.$$

Thus we should take at least $n = (b-a)/\Delta x = 57$ divisions. ▲

## Exercises for Section 11.5

Use the indicated numerical method(s) to approximate the integrals in Exercises 1–4.

1. $\int_{-1}^{1}(x^2+1)\,dx$. Use Riemann sums with $n = 10$ (that is, divide $[-1,1]$ into 10 subintervals of equal length). Compare with the actual value.
2. $\int_0^{\pi/2}(x + \sin x)\,dx$. Use Riemann sums and the trapezoidal rule with $n = 8$. Compare these two approximate values with the actual value.
3. $\int_1^3[(\sin \pi x/2)/(x^2 + 2x - 1)]\,dx$. Use the trapezoidal rule and Simpson's rule with $n = 12$.
4. $\int_0^2(1/\sqrt{x^3+1})\,dx$. Use the trapezoidal rule and Simpson's rule with $n = 20$.
5. Use Simpson's rule with $n = 10$ to find an approximate value for $\int_0^1(x/\sqrt{x^3+2})\,dx$.
6. Estimate the value of $\int_1^3 e^{\sqrt{x}}\,dx$, using Simpson's rule with $n = 4$. Check your answer using $x = u^2$, $dx = 2u\,du$.
7. Suppose you are given the following table of data:

| | | |
|---|---|---|
| $f(0) = 1.384$ | $f(0.4) = 0.915$ | $f(0.8) = 0.935$ |
| $f(0.1) = 1.179$ | $f(0.5) = 0.768$ | $f(0.9) = 1.262$ |
| $f(0.2) = 0.973$ | $f(0.6) = 0.511$ | $f(1.0) = 1.425$ |
| $f(0.3) = 1.000$ | $f(0.7) = 0.693$ | |

Numerically evaluate $\int_0^1(x + f(x))\,dx$ by the trapezoidal rule.

8. Numerically evaluate $\int_0^1 2f(x)\,dx$ by Simpson's rule, where $f(x)$ is the function in Exercise 7.
9. Suppose that you are given the following table of data:

| | |
|---|---|
| $f(0.0) = 2.037$ | $f(1.3) = 0.819$ |
| $f(0.2) = 1.980$ | $f(1.4) = 1.026$ |
| $f(0.4) = 1.843$ | $f(1.5) = 0.799$ |
| $f(0.6) = 1.372$ | $f(1.6) = 0.662$ |
| $f(0.8) = 1.196$ | $f(1.7) = 0.538$ |
| $f(1.0) = 0.977$ | $f(1.8) = 0.555$ |
| $f(1.2) = 0.685$ | |

Numerically evaluate $\int_0^{1.8} f(x)\,dx$ by using Simpson's rule. [*Hint*: Watch out for the spacing of the points.]

10. Numerically evaluate $\int_0^{1.8} f(x)\,dx$ by using the trapezoidal rule, where $f(x)$ is the function in Exercise 9.

11. Evaluate $\int_a^b (px^2 + qx + r)\,dx$. Verify that Simpson's rule with $n = 2$ gives the exact answer. What happens if you use the trapezoidal rule? Discuss.

12. Evaluate $\int_a^b (px^3 + qx^2 + rx + s)\,dx$ by Simpson's rule with $n = 2$ and compare the result with the exact integral.

13. How large must $n$ be taken in the trapezoidal rule to guarantee an accuracy of $10^{-5}$ in the evaluation of the integral in Exercise 2? Answer the same question for Simpson's rule.

14. *Gaussian quadrature* is an approximation method based on interpolation. The formula for integration on the interval $[-1, 1]$ is $\int_{-1}^{1} f(x)\,dx = f(1/\sqrt{3}) + f(-1/\sqrt{3}) + R$, where the remainder $R$ satisfies $|R| \leqslant M/135$, $M$ being the largest value of $f^{(4)}(x)$ on $-1 \leqslant x \leqslant 1$.
    (a) The remainder $R$ is zero for cubic polynomials. Check it for $x^3, x^3 - 1, x^3 + x + 1$.
    (b) Find $\int_{-1}^{1}[x^2/(1 + x^4)]\,dx$ to two places.
    (c) What is $R$ for $\int_{-1}^{1} x^6\,dx$? Why is it so large?

15. A tank 15 meters by 60 meters is filled to a depth of 3.2 meters above the bottom. The time $T$ it takes to empty half the tank through an orifice 0.5 meters wide by 0.2 meters high placed 0.1 meters from the bottom is given by

$$\frac{(2.2)T}{10^4} = \frac{1}{\sqrt{19.6}} \int_{130}^{190} \frac{dx}{(x + 20)^{3/2} - x^{3/2}}.$$

Compute $T$ from Simpson's rule with $n = 6$.

★16. A metropolitan sports and special events complex is circular in shape with an irregular roof that appears from a distance to be almost hemispherical (Fig. 11.5.3).

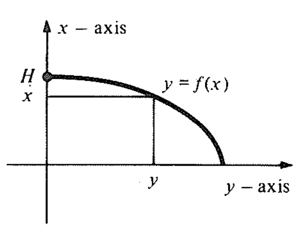



**Figure 11.5.3.** The profile of the roof of a sports complex.

A summer storm severely damaged the roof, requiring a roof replacement to go out for bid. Responding contractors were supplied with plans of the complex from which to determine an estimate. Estimators had to find the roof profile $y = f(x)$, $0 \leqslant x \leqslant H$, which generates the roof by revolution about the $x$ axis ($x$ and $y$ in feet, $x$ vertical, $y$ horizontal).
    (a) Find the square footage of the roof via a surface area formula. This number determines the amount of roofing material required.
    (b) To check against construction errors, a tape measure is tossed over the roof and the measurement recorded. Give a formula for this measurement using the arc length formula.
    (c) Suppose the curve $f$ is not given explicitly in the plans, but instead $f(0), f(4), f(8), f(12), \ldots, f(H)$ are given (complex center-to-ceiling distances every 4 feet). Discuss how to use this information to numerically evaluate the integrals in (a), (b) above, using Example 3 as a guide.
    (d) Find an expression which approximates the surface area of the roof by assuming it is a *conoid* produced by a piecewise linear function constructed from the numbers $f(0), f(4), f(8), \ldots, f(H)$.

★17. How many digits in the approximate value $A \approx 2.3757$ in Example 3 can be justified by an error analysis?

★18. (*Another numerical integration method*)
    (a) Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be $n$ points in the plane such that all the $x_i$'s are different. Show that the polynomial of degree no more than $n - 1$ whose graph passes through the given points is

$$P(x) = y_1L_1(x) + y_2L_2(x) + \cdots + y_nL_n(x),$$

where $L_i(x) = A_i(x)/A'(x_i)$,

$$A(x) = (x - x_1)(x - x_2)\ldots(x - x_n),$$
$$A_i(x) = A(x)/(x - x_i), \qquad i = 1, 2, \ldots, n.$$

($P$ is called the *Lagrange interpolation polynomial*.)
    (b) Suppose that you are given the following data for an unknown function $f(x)$:

$f(0) = 0.01$, $f(0.3) = 1.18$,
$f(0.1) = 0.12$, $f(0.4) = 0.91$.
$f(0.2) = 0.82$,

Estimate the value of $f(0.16)$ by using the Lagrange interpolation formula.
    (c) Estimate $\int_0^{0.4} f(x)\,dx$ (1) by using the trapezoidal rule, (2) by using Simpson's rule, and (3) by integrating the Lagrange interpolation polynomial.
    (d) Estimate $\int_0^{\pi/2} \cos x\,dx$ by using a Lagrange interpolation polynomial with $n = 4$. Compare your result with those obtained by the trapezoidal and Simpson's rules in Examples 2 and 4.

## Review Exercises for Chapter 11

Verify the limits in Exercises 1–4 using the $\varepsilon$-$\delta$ definition.

1. $\lim_{x\to 1}(x^2+x-1)=1$
2. $\lim_{x\to 0}(x^3+3x+2)=2$
3. $\lim_{x\to 2}(x^2-8x+8)=-4$
4. $\lim_{x\to 5}(x^2-25)=0$

Calculate the limits in Exercises 5–16.

5. $\lim_{x\to 0}\tan\left(\frac{x+1}{x-1}\right)$
6. $\lim_{x\to 1}\cos\left[\left(\frac{x+1}{x+2}\right)\frac{3\pi}{2}\right]$
7. $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)$
8. $\lim_{x\to\infty}\left(\frac{x^2+2}{3x^2+2x+1}\right)$
9. $\lim_{x\to\infty}(\sqrt{2x^2+1}-\sqrt{2}\,x)$
10. $\lim_{x\to\infty}(\sqrt{4x^2+1}-2x)$
11. $\lim_{x\to 1-}\frac{1}{\sqrt{1-x}}$
12. $\lim_{x\to 2+}\frac{\sin\sqrt{x-2}}{\sqrt{x-2}}$
13. $\lim_{x\to 0}[x\sin(3/x^2)]$
14. $\lim_{x\to 0}\sin(\sqrt{x^2+2}-x)$
15. $\lim_{x\to 0}\frac{x^4+8x}{3x^4+2}$
16. $\lim_{x\to 0}\frac{x+3}{3x+8}$
17. Find the horizontal asymptotes of the graph $y=\tan^{-1}(3x+2)$. Sketch.
18. Find the vertical asymptotes for the graph of $y=1/(x^2-3x-10)^2$. Sketch.

Find the horizontal and vertical asymptotes of the functions in Exercises 19 and 20, and sketch.

19. $f(x)=\frac{x-1}{x^2+1}$
20. $f(x)=\frac{2x+3}{3x+5}$

Find the limits if they exist, using l'Hôpital's rule, in Exercises 21–44.

21. $\lim_{x\to\infty}\frac{x^3+8x+9}{4x^3-9x^2+10}$
22. $\lim_{x\to\infty}\frac{x}{x+2}$
23. $\lim_{x\to 0}\frac{1-\cos x}{3^x-2^x}$
24. $\lim_{x\to 1}\frac{x^x-1}{x-1}$
25. $\lim_{x\to 0}\frac{(\sqrt{x^2+9}-3)}{\sin x}$
26. $\lim_{x\to 0}\frac{\sqrt[3]{x^3+27}-3}{x}$
27. $\lim_{x\to 0}\frac{\sin 5x}{x}$
28. $\lim_{x\to 0}\frac{\tan^2 x}{x^2}$
29. $\lim_{x\to 2}\frac{\sin(x-2)-x+2}{(x-2)^3}$
30. $\lim_{x\to 0}\frac{24\cos x-24+12x^2-x^4}{x^5}$
31. $\lim_{x\to 0}\frac{\tan(x+3)-\tan 3}{x}$
32. $\lim_{x\to\pi^2}\frac{\cos\sqrt{x}+1}{x-\pi^2}$
33. $\lim_{x\to 0}x\cot x$
34. $\lim_{x\to 0+}\frac{\cot x}{\ln x}$
35. $\lim_{x\to 0+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$
36. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}-\frac{1}{2}\right)$
37. $\lim_{x\to\infty}x^2e^{-x}$
38. $\lim_{x\to 0+}x^3(\ln x)^2$
39. $\lim_{x\to 0+}x^{\sin x}$
40. $\lim_{x\to\infty}(\sin e^{-x})^{1/\sqrt{x}}$
41. $\lim_{x\to 0+}(1+\sin 2x)^{1/x}$
42. $\lim_{x\to 0+}(\cos 2x)^{1/x^2}$
43. $\lim_{x\to\infty}\frac{(\ln x)^2}{x}$
44. $\lim_{x\to 2}\frac{x^2+x-6}{x^2+2x-8}$

Decide which improper integrals in Exercises 45–54 are convergent. Evaluate when possible.

45. $\int_1^\infty\frac{1}{x^2}\,dx.$
46. $\int_{-\infty}^\infty\frac{\sin x}{x^2+3}\,dx.$
47. $\int_2^\infty\frac{dx}{\ln x}$. [*Hint*: Prove $\ln x\leqslant x$ for $x\geqslant 2$.]
48. $\int_1^\infty\frac{dx}{\sqrt{x^2+8x+12}}.$
49. $\int_1^2\frac{1}{\sqrt{x-1}}\,dx.$
50. $\int_{-1}^0\frac{x+1}{\sqrt{1-x^2}}\,dx.$
51. $\int_0^1\frac{dx}{(1-x)^{2/5}}.$
52. $\int_1^\infty x^2e^{-x}\,dx.$
53. $\int_0^1 x\ln x\,dx.$
54. $\int_0^\infty(x+2)e^{-(x^2+4x)}\,dx.$

Evaluate the limits in Exercises 55 and 56.

55. $\lim_{x\to\infty}\int_0^x\frac{dt}{t^2+t+1}$
56. $\lim_{x\to 0+}\int_x^1\frac{dt}{\sqrt{t}}$
57. The region under the curve $y=xe^{-x}$ on $[0,\infty)$ is revolved about the $x$ axis. Find the volume of the resulting solid.
58. The curve $y=\sin x/x^2$ on $[1,\infty)$ is revolved around the $x$ axis. Determine whether the resulting surface has finite area.

Evaluate the limits of the sequences in Exercises 59–72.

59. $\lim_{n\to\infty}\left(8+\left(\frac{2}{3}\right)^n\right)^5$
60. $\lim_{n\to\infty}\sqrt[3]{\frac{8n^3-2n+1}{n^3+1}}$
61. $\lim_{n\to\infty}\left(1+\frac{8}{n}\right)^n$
62. $\lim_{n\to\infty}\left(\frac{n-3}{n}\right)^{-2n}$
63. $\lim_{n\to\infty}(n^2+3n+1)e^{-n}$
64. $\lim_{n\to\infty}\frac{2^n}{n^2}$
65. $\lim_{n\to\infty}\frac{n}{n+2}$
66. $\lim_{n\to\infty}\frac{n^2+2n}{3n^2+1}$
67. $\lim_{n\to\infty}\tan\left[\frac{3n}{n+8}\right]$
68. $\lim_{n\to\infty}[\ln(n^2+1)-\ln(3n^2+5)]$

69. $\lim_{n\to\infty} \dfrac{\sin(\pi n/2)}{n^{-3}}$ 70. $\lim_{n\to\infty} \dfrac{n\cos 4n\pi}{2n+1}$

71. $\lim_{n\to\infty} \dfrac{8-2n}{5n}$ 72. $\lim_{n\to\infty}\left(1-\dfrac{2+n}{3n+1}\right)$

Using l'Hôpital's rule if necessary, evaluate the limits of the sequences in Exercises 73–76.

73. $\lim_{n\to\infty} \sqrt[2n]{3n}$ 74. $\lim_{m\to\infty} m\log_{10}(2^{-m})$

75. $\lim_{n\to\infty}\left[\dfrac{1}{n^2}-\dfrac{1}{(n+1)^2}\right]$ 76. $\lim_{j\to\infty}\dfrac{2e^{3j}}{e^{3j}+j^5}$

Use Newton's method for Exercises 77–80.

77. Locate the roots of $x^3-3x^2+8=0$.
78. Find the cube root of 21.
79. Solve the equation $e^x=2+x$.
80. Find two numbers, each of whose square is ten times its natural logarithm.
81. Evaluate $\int_2^3(x^2\,dx/\sqrt{x^2+1})$ by the trapezoidal rule with $n=10$.
82. Evaluate the integral in Exercise 81 by Simpson's rule with $n=10$.
83. Use Simpson's rule with $n=10$ to calculate the volume obtained by revolving the curve $y=f(x)$ on $[1,3]$ about the $x$ axis, given the data:

| | |
|---|---|
| $f(1)=2.03$ | $f(2.2)=3.16$ |
| $f(1.2)=2.08$ | $f(2.4)=3.01$ |
| $f(1.4)=2.16$ | $f(2.6)=2.87$ |
| $f(1.6)=2.34$ | $f(2.8)=2.15$ |
| $f(1.8)=2.82$ | $f(3)=1.96$ |
| $f(2)=3.01$ | |

84. (a) Evaluate $(2/\sqrt{\pi})\int_0^1 e^{-t^2}dt$ by using Simpson's rule with 10 subdivisions.
(b) Given an upper bound for the error in part (a). (See Example 6 of Section 11.5.)
(c) What does Simpson's rule with 10 subdivisions give for $(2/\sqrt{\pi})\int_0^{10}e^{-t^2}dt$?
(d) The function $(2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ is denoted $\operatorname{erf}(x)$ and is called the error function. Its values are tabulated. (For example: *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, June 1964, pp. 310–311.) Compare your results with the tabulated results. *Note*: $\lim_{x\to\infty}\operatorname{erf}(x)=1$, and $\operatorname{erf}(10)$ is so close to 1 that it probably won't be listed in the tables. Explain your result in part (c).

85. Let $f(x)=\cos x$ for $x\geqslant 0$ and $f(x)=1$ for $x<0$. Decide whether or not $f$ is continuous or differentiable or both.
86. Let $f(x)=x^{1/\sin(x-1)}$. How should $f(1)$ be defined in order to make $f$ continuous?
87. Find a function on $[0,1]$ which is integrable (as an improper integral) but whose square is not.
88. Show that $\int_0^\infty[(\sin x)/(1+x)]\,dx$ is convergent. [*Hint*: Integrate by parts.]
89. (a) Show that
$$f''(x_0)=\lim_{h\to0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$$
if $f''$ is continuous at $x_0$. [*Hint*: Use l'Hôpital's rule.]
★(b) Find a similar formula for $f'''(x_0)$.
90. Show that
$$f''(x_0)=\lim_{\Delta x\to0}\frac{f(x_0+2\Delta x)+f(x_0)-2f(x_0+\Delta x)}{(\Delta x)^2}$$
if $f''$ is continuous at $x_0$.
91. Use Riemann sums to evaluate
$$\lim_{n\to\infty}\sum_{i=1}^{n}(\ln n-\ln i)/n.$$
92. Let
$$S_n=\sum_{i=1}^{2n}\left[2-\cos\left(\frac{i}{n}\pi\right)\right]\frac{1}{n}.$$
Prove that $\lim_{n\to\infty}S_n=4$ using Riemann sums.
93. Let
$$S_n=\sum_{i=1}^{n}\left(\frac{i}{n}+\frac{i^2}{n^2}\right)\frac{1}{n}.$$
Prove that $S_n\to\frac{5}{6}$ as $n\to\infty$ by using Riemann sums.
94. Expressing the following sums as Riemann sums, show that:
(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[\sqrt{\frac{i}{n}}-\left(\frac{i}{n}\right)^{3/2}\right]\frac{1}{n}=\frac{4}{15}$;
(b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3n}{(2n+i)^2}=\frac{1}{2}$.
95. $P$ dollars is deposited in an account each day for a year. The account earns interest at an annual rate $r$ (e.g., $r=0.05$ means 5%) compounded continuously. Use Riemann sums to show that the amount in the account at the end of the year is approximately
$$(365P/r)\,(e^r-1).$$
★96. Evaluate:
$$\lim_{x\to\pi^2}\left[\frac{\sin\sqrt{x}}{(\sqrt{x}-\pi)(\sqrt{x}+\pi)}+\tan\sqrt{x}\right].$$
★97. Limits can sometimes be evaluated by geometric techniques. An important instance occurs when the curve $y=f(x)$ is trapped between the two intersecting lines through $(a,L)$ with slopes $m$ and $-m$, $0<|x-a|\leqslant h$. Then $\lim_{x\to a}f(x)=L$, because points approaching $y=f(x)$ from the left or right are forced into a vertex, and therefore to the point $(a,L)$.

(a) The equations of the two lines are $y = L + m(x - a)$, $y = L - m(x - a)$. Draw these on a figure and insert a representative graph for $f$ which stays between the lines.

(b) Show that the algebraic condition that $f$ stay between the two straight lines is

$$\left|\frac{f(x) - L}{x - a}\right| \leqslant m, \qquad 0 < |x - a| \leqslant h.$$

This is called a *Lipschitz condition*.

(c) Argue that a Lipschitz condition implies $\lim_{x\to a} f(x) = L$, by appeal to the definition of limit.

★98. Another geometric technique for evaluation of limits is obtained by requiring that $y = f(x)$ be trapped on $0 \leqslant |x - a| \leqslant h$ between two power curves

$$y = L + m(x - a)^\alpha, \qquad y = L - m(x - a)^\alpha,$$

where $\alpha > 0$, $m > 0$. The resulting algebraic condition is called a *Hölder condition*:

$$\frac{|f(x) - L|}{|x - a|^\alpha} \leqslant m, \qquad 0 < |x - a| \leqslant h.$$

(a) Verify that the described geometry leads to the Hölder condition.

(b) Argue geometrically that, in the presence of a Hölder condition, $\lim_{x\to a} f(x) = L$.

(c) Prove the contention in (b) by appeal to the definition of limit.

★99. Prove the chain rule for differentiable functions, $(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$, as follows:

(a) Let $y = g(x)$ and $z = f(y)$, and write $\Delta y = g'(x_0)\Delta x + \rho(x)$. Show that

$$\lim_{\Delta x\to 0} \frac{\rho(x)}{\Delta x} = 0$$

Also write $\Delta z = f'(y_0)\Delta y + \sigma(y)$, where $y_0 = g(x_0)$ and show that

$$\lim_{\Delta y\to 0} \frac{\sigma(y)}{\Delta y} = 0.$$

(b) Show that

$$\Delta z = f'(y_0)\, g'(x_0)\,\Delta x + f'(y_0)\rho(x) + \sigma(g(x)).$$

(c) Note that $\sigma(g(x)) = 0$ if $\Delta y = 0$. Thus show that

$$\frac{\sigma(g(x))}{\Delta x} = \begin{cases} \dfrac{\sigma(g(x))}{\Delta y}\,\dfrac{\Delta y}{\Delta x} & \text{if } \Delta y \neq 0 \\ 0 & \text{if } \Delta y = 0 \end{cases} \to 0$$

as $\Delta x \to 0$.

(d) Use parts (a), (b), and (c) above to show that $\lim_{\Delta x\to 0}[\Delta z/\Delta x] = f'(y_0)g'(x_0)$. (This proof avoids the problem of division by zero mentioned on p. 113.)

★100. An alternative to Newton's method for finding solutions of the equation $f(x) = 0$ is the iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)},$$

sometimes known as *Picard's method*. Notice that this method requires evaluating $f'$ only at the initial guess $x_0$ and so requires less computation at each step.

(a) Show that, if the sequence $x_0, x_1, x_2, \ldots$ converges, then $\lim_{n\to\infty} x_n$ is a solution of $f(x) = 0$.

(b) Compare Picard's method and Newton's method on the problem $x^5 = x + 1$, using the initial guess $x_0 = 1$ in each case and iterating until the solution is found to six decimal places of accuracy.

(c) Suppose that $f(q) = 0$ and in addition that $0 < \frac{1}{2}f'(x_0) < f'(x) < \frac{3}{2}f'(x_0)$ for all $x$ in the interval $I = (q - a, q + a)$. Prove that if $x_0$ is any initial guess in $I$, then $|x_{n+1} - q| < \frac{1}{2}|x_n - q|$, and so $\lim_{n\to\infty} x_n = q$. [*Hint*: $x_{n+1} = P(x_n)$, where $P(x) = x - f(x)/f'(x_0)$. Differentiate $P(x)$ and apply the mean value theorem.] (A similar analysis for Newton's method is presented in the following Review Exercise.)

★101. Newton's method for solving $f(x) = 0$ can be described by saying that $x_{n+1} = N(x_n)$, where the *Newton iteration* function $N$ is defined by $N(x) = x - f(x)/f'(x)$ for all $x$ such that $f'(x) \neq 0$.

(a) Show that $N(x) = x$ if and only if $f(x) = 0$

(b) Show that $N'(x) = f(x)f''(x)/[f'(x)]^2$

(c) Suppose that $\bar{x}$ is a root of $f$, that $[a, b]$ is an interval containing $\bar{x}$, and that there are numbers $p$, $q$ and $M$ such that

$$0 < p \leqslant f'(x) \leqslant q \quad \text{and} \quad |f''(x)| \leqslant M$$

for all $x$ in $[a, b]$. Show that there is a constant $C$ such that

$$|N(x) - \bar{x}| \leqslant C|x - \bar{x}|^2$$

for all $x$ in $[a, b]$. Express $C$ in terms of $p$, $q$ and $M$.

This establishes the "quadratic convergence" of $x_0, x_1, x_2, \ldots$ to $\bar{x}$ as soon as some $x_i$ is in $[a, b]$. [*Hint*: Apply the mean value theorem to $N$ to conclude $N(x) - \bar{x} = N'(\xi)(x - \bar{x})$ for some $\xi$ between $x$ and $\bar{x}$. Use the mean value theorem again to show that $|N'(\xi)| \leqslant D|\xi|$ for a constant $D > 0$.]

(d) How many iterations are needed to solve $x^2 - 2 = 0$ to within 20 decimal places, assuming an initial guess in the interval [1.4. 1.5]?