Chapter 12

Infinite Series

Infinite sums can be used to represent numbers and functions.

The decimal expansion $\frac{1}{3} = 0.3333\ldots$ is a representation of $\frac{1}{3}$ as an infinite sum $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$. In this chapter, we will see how to represent numbers as infinite sums and to represent functions of x by infinite sums whose terms are monomials in x. For example, we will see that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

and

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \cdots.$$

Later in the chapter we shall use our knowledge of infinite series to study complex numbers and some differential equations. There are other important uses of series that are encountered in later courses. One of these is the topic of Fourier series; this enables one, for example, to decompose a complex sound into an infinite series of pure tones.

12.1 The Sum of an Infinite Series

The sum of infinitely many numbers may be finite.

An *infinite series* is a sequence of numbers whose terms are to be added up. If the resulting sum is finite, the series is said to be *convergent*. In this section, we define convergence in terms of limits, give the simplest examples, and present some basic tests. Along the way we discuss some further properties of the limits of sequences, but the reader should also review the basic facts about sequences from Section 11.4.

Our first example of the limit of a sequence was an expression for the number $\frac{1}{3}$:

$$\frac{1}{3} = \lim_{n \to \infty} \left(\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} \right).$$

This expression suggests that we may consider $\frac{1}{3}$ as the sum

$$\frac{3}{10}+\frac{3}{100}+\cdots+\frac{3}{10^n}+\cdots$$

of infinitely many terms. Of course, not every sum of infinitely many terms gives rise to a number (consider $1+1+1+\cdots$), so we must be precise about what we mean by adding together infinitely many numbers. Following the idea used in the theory of improper integrals (in Section 11.3), we will define the sum of an infinite series by taking finite sums and then passing to the limit as the sum includes more and more terms.

Convergence of Series

Let $a_1, a_2, \ldots$ be a sequence of numbers. The number $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the *nth partial sum* of the a_i's. If the sequence $S_1, S_2, \ldots$ of partial sums approaches a limit S as $n \to \infty$, we say that *the series* $a_1 + a_2 + \cdots = \sum_{i=1}^{\infty} a_i$ *converges*, and we write

$$\sum_{i=1}^{\infty} a_i = S;$$

that is,

$$\sum_{i=1}^{\infty} a_i \quad \text{is defined as} \quad \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

and is called the *sum* of the series.

If the series $\sum_{i=1}^{\infty} a_i$ does not converge, we say that it *diverges*. In this case, the series has no sum.

In summary:

$$\text{a series } \sum_{i=1}^{\infty} a_i \text{ converges if } \lim_{n\to\infty} \sum_{i=1}^{n} a_i \text{ exists (and is finite);}$$

$$\text{a series } \sum_{i=1}^{\infty} a_i \text{ diverges if } \lim_{n\to\infty} \sum_{i=1}^{n} a_i \text{ does not exist (or is infinite).}$$

Example 1 Write down the first four partial sums for each of the following series:

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots;$

(b) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots;$

(c) $1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots;$

(d) $\sum_{i=0}^{\infty} \frac{3}{2^{i+1}}$.

Solution (a) $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, $S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$, and $S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$.

(b) $S_1 = 1$, $S_2 = 1 - \frac{1}{2} = \frac{1}{2}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, and $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$.

(c) $S_1 = 1$, $S_2 = 1 + \frac{1}{5} = \frac{6}{5}$, $S_3 = 1 + \frac{1}{5} + \frac{1}{5^2} = \frac{31}{25}$, and $S_4 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = \frac{156}{125}$.

(d) $\sum_{i=0}^{\infty} \frac{3}{2^{i+1}} = \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \frac{3}{2^4} + \cdots$, so $S_1 = \frac{3}{2}$, $S_2 = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$, $S_3 = 3(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = \frac{21}{8}$, and $S_4 = 3(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}) = \frac{45}{16}$. ▲

Do not confuse a *sequence* with a *series*. A sequence is simply an infinite list of numbers (separated by commas): $a_1, a_2, a_3, \ldots$. A series is an infinite list of numbers (separated by plus signs) which are meant to be added together: $a_1 + a_2 + a_3 + \cdots$. Of course, the terms in an infinite series may themselves be considered as a sequence, but the most important sequence associated with the series $a_1 + a_2 + \cdots$ is its sequence of partial sums: $S_1, S_2, S_3, \ldots$—that is, the sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots .$$

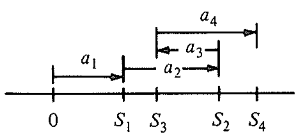


Figure 12.1.1. The term a_i of a series represents the "move" from the partial sum S_{i-1} to S_i. S_n is the cumulative result of the first n moves.

We may illustrate the difference between the a_i's and the S_n's pictorially. Think of $a_1, a_2, a_3, \ldots$ as describing a sequence of "moves" on the real number line, starting at 0. Then $S_n = a_1 + \cdots + a_n$ is the position reached after the nth move. (See Fig. 12.1.1.) Note that the term a_i can be recovered as the difference $S_i - S_{i-1}$.

To study the limits of partial sums, we will need to use some general properties of limits of sequences. The definition of convergence of a sequence was given in Section 11.4. The basic properties we need are proved and used in a manner similar to those for limits of functions (Section 11.1) and are summarized in the following display.

Properties of Limits of Sequences

Suppose that the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are convergent, and that c is a constant. Then:

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n$.
3. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
4. If $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all n, then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$
5. If f is continuous at $\lim_{n\to\infty} a_n$, then
$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$
6. $\lim_{n\to\infty} c = c$.
7. $\lim_{n\to\infty}(1/n) = 0$.
8. If $\lim_{x\to\infty} f(x) = l$, then $\lim_{n\to\infty} f(n) = l$.
9. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, and if $|r| > 1$ or $r = -1$, $\lim_{n\to\infty} r^n$ does not exist.

Here are a couple of examples of how the limit properties are used. We will see many more examples as we work with series.

Example 2 Find (a) $\lim_{n\to\infty} \frac{3+n}{2n+1}$ and (b) $\lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right)$.

Solution (a) $\lim_{n\to\infty} \frac{3+n}{2n+1} = \lim_{n\to\infty} \frac{3/n+1}{2+1/n} = \frac{3\lim_{n\to\infty} 1/n + \lim_{n\to\infty} 1}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} 1/n} = \frac{3\cdot 0+1}{2+0} = \frac{1}{2}$.

This solution used properties 1, 2, and 4 above, together with the facts that $\lim_{n\to\infty} 1/n = 0$ (property 7) and $\lim_{n\to\infty} c = c$ (property 6).
(b) Since $\sin x$ is a continuous function, we can use property 5 to get

$$\begin{aligned}\lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right) &= \sin\left[\lim_{n\to\infty}\left(\frac{\pi n}{2n+1}\right)\right]\\ &= \sin\left[\lim_{n\to\infty}\left(\frac{\pi}{2+1/n}\right)\right]\\ &= \sin\left(\frac{\pi}{2}\right) = 1. \blacktriangle\end{aligned}$$

We return now to infinite series. A simple but basic example is the geometric series

$$a + ar + ar^2 + \cdots$$

in which the ratio between each two successive terms is the same. To write a geometric series in summation notation, it is convenient to allow the index i to start at zero, so that $a_0 = a$, $a_1 = ar$, $a_2 = ar^2$, and so on. The general term is then $a_i = ar^i$, and the series is compactly expressed as $\sum_{i=0}^{\infty} ar^i$. In our notation $\sum_{i=1}^{\infty} a_i$ for a general series, the index i will start at 1, but in special examples we may start it wherever we wish. Also, we may replace the index i by any other letter; $\sum_{i=1}^{\infty} a_i = \sum_{j=1}^{\infty} a_j = \sum_{n=1}^{\infty} a_n$, and so forth.

To find the sum of a geometric series, we must first evaluate the partial sums $S_n = \sum_{i=0}^{n} ar^i$. We write

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^n,\\ rS_n &= \phantom{a +{}} ar + ar^2 + \cdots + ar^n + ar^{n+1}.\end{aligned}$$

Subtracting the second equation from the first and solving for S_n, we find

$$S_n = \frac{a(1-r^{n+1})}{1-r} \qquad (\text{if } r \neq 1). \tag{1}$$

The sum of the entire series is the limit

$$\begin{aligned}\sum_{i=0}^{\infty} ar^i &= \lim_{n\to\infty} S_n\\ &= \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r}\lim_{n\to\infty}(1-r^{n+1})\\ &= \frac{a}{1-r}\left(1 - \lim_{n\to\infty} r^{n+1}\right).\end{aligned}$$

(We used limit properties 1, 2, and 6.) If $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ (property 9), so in this case, $\sum_{i=1}^{\infty} ar^i$ is convergent, and its sum is $a/(1-r)$. If $|r| > 1$ or $r = -1$, $\lim_{n\to\infty} r^{n+1}$ does not exist (property 9), so if $a \neq 0$, the series diverges. Finally, if $r = 1$, then $S_n = a + ar + \cdots + ar^n = a(n+1)$, so if $a \neq 0$, the series diverges.

Geometric Series

If $|r| < 1$ and a is any number, then $a + ar + ar^2 + \cdots = \sum_{i=0}^{\infty} ar^i$ converges and the sum is $a/(1-r)$.

If $|r| \geqslant 1$ and $a \neq 0$, then $\sum_{i=0}^{\infty} ar^i$ diverges.

Example 3 Sum the series: (a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$, (b) $\sum_{n=0}^{\infty} \frac{1}{6^{n/2}}$, and (c) $\sum_{i=1}^{\infty} \frac{1}{5^i}$.

Solution (a) This is a geometric series with $r = \frac{1}{3}$ and $a = 1$. (Note that a is the first term and r is the ratio of any term to the preceding one.) Thus

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{1 - 1/3} = \frac{3}{2}.$$

(b) $\sum_{n=0}^{\infty}[1/(6^{n/2})] = 1 + (1/\sqrt{6}) + (1/\sqrt{6})^2 + \cdots = a/(1-r)$, where $a = 1$ and $r = 1/\sqrt{6}$, so the sum is $1/(1 - 1/\sqrt{6}) = (6 + \sqrt{6})/5$. (Note that the index here is n instead of i.)

(c) $\sum_{i=1}^{\infty} 1/5^i = 1/5 + 1/5^2 + \cdots = (1/5)/(1 - 1/5) = 1/4$. (We may also think of this as the series $\sum_{i=0}^{\infty} 1/5^i$ with the first term removed. The sum is thus $1/[1 - (1/5)] - 1 = 1/4$.) ▲

The following example shows how a geometric series may arise in a physical problem.

Example 4 A bouncing ball loses half of its energy on each bounce. The height reached on each bounce is proportional to the energy. Suppose that the ball is dropped vertically from a height of one meter. How far does it travel? (Fig. 12.1.2.)

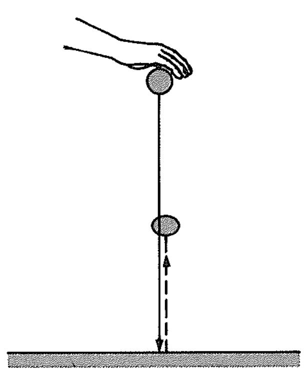


Figure 12.1.2. Find the total distance travelled by the bouncing ball.

Solution Each bounce is 1/2 as high as the previous one. After the ball falls from a height of 1 meter, it rises to 1/2 meter on the first bounce, $(1/2)^2 = 1/4$ meter on the second, and so forth. The total distance travelled, in meters, is $1 + 2(1/2) + 2(1/2)^2 + 2(1/2)^3 + \cdots$, which is

$$1 + 2\sum_{i=1}^{\infty}(1/2)^i = 1 + 2\left(\frac{1/2}{1 - 1/2}\right) = 3 \text{ meters.}$$ ▲

Two useful general rules for summing series are presented in the box on the following page. To prove the validity of these rules, one simply notes that the identities

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \quad \text{and} \quad \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

are satisfied by the partial sums. Taking limits as $n \to \infty$ and applying the sum and constant multiple rules for limits of sequences results in the rules in the box.

Algebraic Rules For Series

Sum rule

If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ converge, then $\sum_{i=1}^{\infty}(a_i + b_i)$ converges and

$$\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i .$$

Constant multiple rule

If $\sum_{i=1}^{\infty} a_i$ converges and c is any real number, then $\sum_{i=1}^{\infty} ca_i$ converges and

$$\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i .$$

Example 5 Sum the series $\displaystyle\sum_{i=0}^{\infty} \frac{3^i - 2^i}{6^i}$.

Solution We may write the ith term as

$$\frac{3^i}{6^i} - \frac{2^i}{6^i} = \left(\frac{1}{2}\right)^i - \left(\frac{1}{3}\right)^i = \left(\frac{1}{2}\right)^i + (-1)\left(\frac{1}{3}\right)^i.$$

Since the series $\sum_{i=0}^{\infty}(1/2)^i$ and $\sum_{i=0}^{\infty}(1/3)^i$ are convergent, with sums 2 and $\frac{3}{2}$ respectively, the algebraic rules imply that

$$\sum_{i=0}^{\infty} \frac{3^i - 2^i}{6^i} = \sum_{i=0}^{\infty}\left[\left(\frac{1}{2}\right)^i + (-1)\left(\frac{1}{3}\right)^i\right]$$

$$= \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i + (-1)\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = 2 - \frac{3}{2} = \frac{1}{2} . \blacktriangle$$

Example 6 Show that the series $1\frac{1}{2} + 3\frac{3}{4} + 7\frac{7}{8} + 15\frac{15}{16} + \cdots$ diverges. [*Hint*: Write it as the difference of a divergent and a convergent series.]

Solution The series is $\sum_{i=0}^{\infty}[2^i - (\frac{1}{2})^i]$. If it were convergent, we could add to it the convergent series $\sum_{i=0}^{\infty}(\frac{1}{2})^i$, and the result would have to converge by the sum rule; but the resulting series is $\sum_{i=0}^{\infty}[2^i - (\frac{1}{2})^i + (\frac{1}{2})^i] = \sum_{i=0}^{\infty} 2^i$, which diverges because $2 > 1$, so the original series must itself be divergent. $\blacktriangle$

The sum rule implies that *we may change (or remove—that is, change to zero) finitely many terms of a series without affecting its convergence*. In fact, changing finitely many terms of the series $\sum_{i=1}^{\infty} a_i$ is equivalent to adding to it a series whose terms are all zero beyond a certain point. Such a finite series is always convergent, so adding it to the convergent series produces a convergent result. Of course, the sum of the new series is *not* the same as that of the old one, but rather is the sum of the finite number of added terms plus the sum of the original series.

Example 7 Show that

$$1 + 2 + 3 + 4 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

is convergent and find its sum.

Solution The series $1/4 + 1/4^2 + 1/4^3 + 1/4^4 + \cdots$ is a geometric series with sum $(1/4)/(1 - 1/4) = 1/3$; thus the given series is convergent with sum $1 + 2 +$

$3 + 4 + 1/3 = 10\frac{1}{3}$. To use the sum rule as stated, one can write

$$\begin{aligned} &1 + 2 + 3 + 4 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots \\ &\quad = (1 + 2 + 3 + 4 + 0 + 0 + 0 + \cdots) \\ &\quad\quad + \left(0 + 0 + 0 + 0 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right). \blacktriangle \end{aligned}$$

We can obtain a simple necessary condition for convergence by recalling that $a_i = S_i - S_{i-1}$. If $\lim_{i\to\infty} S_i$ exists, then $\lim_{i\to\infty} S_{i-1}$ has the same value. Hence, using properties 1 and 2 of limits of sequences, we find $\lim_{i\to\infty} a_i = \lim_{i\to\infty} S_i - \lim_{i\to\infty} S_{i-1} = 0$. In other words, if the series $\sum_{i=1}^{\infty} a_i$ converges, then, the "move" from one partial sum to the next must approach zero (see Fig. 12.1.1).

The *i*th Term Test

If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i\to\infty} a_i = 0$.
If $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.
If $\lim_{i\to\infty} a_i = 0$, the test is inconclusive: the series could converge or diverge, and further analysis is necessary.

The *i*th-term test can be used to show that a series diverges, such as the one in Example 6, but it cannot be used to establish convergence.

Example 8 Test for convergence: (a) $\sum_{i=1}^{\infty} \frac{i}{1+i}$; (b) $\sum_{i=1}^{\infty} (-1)^i \frac{i}{\sqrt{1+i}}$; (c) $\sum_{i=1}^{\infty} \left(\frac{1}{i}\right)$.

Solution (a) Here $a_i = \frac{i}{1+i} = \frac{1}{1/i+1} \to 1$ as $i \to \infty$. Since a_i does not tend to zero, the series must diverge.

(b) Here $|a_i| = i/\sqrt{1+i} = \sqrt{i}/\sqrt{1/i+1} \to \infty$ as $i \to \infty$. Thus a_i does not tend to zero, so the series diverges.

(c) Here $a_i = 1/i$, which tends to zero as $i \to \infty$, so our test is inconclusive. $\blacktriangle$

As an example of the "further analysis" necessary when $\lim_{i\to\infty} a_i = 0$, we consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

from part (c) of Example 8, called the *harmonic series*. We show that the series diverges by noticing a pattern:

$$\begin{array}{lll} 1 & & \\ \frac{1}{2} & & \\ \frac{1}{3} + \frac{1}{4} & > \frac{1}{4} + \frac{1}{4} & = \frac{1}{2} \\ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} & > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} & = \frac{1}{2} \\ \frac{1}{9} + \cdots + \frac{1}{16} & > \frac{1}{16} + \cdots + \frac{1}{16} & = \frac{1}{2} \\ \frac{1}{17} + \cdots + \frac{1}{32} & > \frac{1}{32} + \cdots + \frac{1}{32} & = \frac{1}{2} \\ \vdots & \vdots \quad \vdots & \vdots \end{array}$$

and so on. Thus the partial sum S_4 is greater than $1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$, $S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$ and, in general $S_{2^n} > 1 + n/2$, which becomes arbitrarily large as n becomes large. Therefore, *the harmonic series diverges*.

Example 9 Show that the series (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$ and (b) $\sum_{i=1}^{\infty} 1/(1+i)$ diverge.

Solution (a) This series is $\sum_{i=1}^{\infty}(1/2i)$. If it converged, so would twice the series $\sum_{i=1}^{\infty} 2\cdot(1/2i)$, by the constant multiple rule; but $\sum_{i=1}^{\infty}(2\cdot 1/2i)=\sum_{i=1}^{\infty} 1/i$, which we have shown to diverge.
(b) This series is $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, which is the harmonic series with the first term missing; therefore this series diverges too. ▲

Supplement to Section 12.1: Zeno's Paradox

Zeno's paradox concerns a race between Achilles and a tortoise. The tortoise begins with a head start of 10 meters, and Achilles ought to overtake it. After a certain elapsed time from the start, Achilles reaches the point A where the tortoise started, but the tortoise has moved ahead to point B (Fig. 12.1.3).

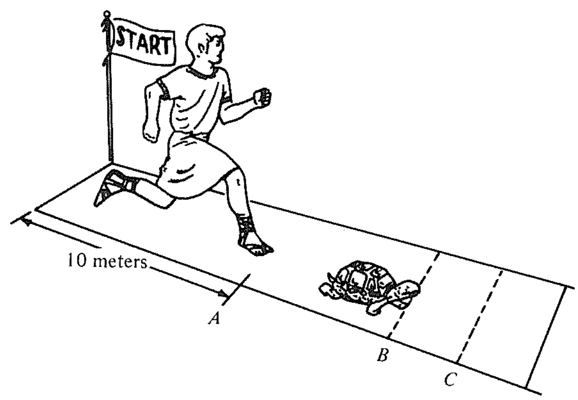


Figure 12.1.3. Will the runner overtake the tortoise?

After a certain further interval of time, Achilles reaches point B, but the tortoise has moved ahead to a point C, and so on forever. Zeno concludes from this argument that Achilles can never pass the tortoise. Where is the fallacy?

The resolution of the paradox is that although the number of time intervals being considered is infinite, the sum of their lengths is finite, so Achilles can overtake the tortoise in a finite time. The word *forever* in the sense of infinitely many terms is confused with "forever" in the sense of the time in the problem, resulting in the apparent paradox.

Exercises for Section 12.1

Write down the first four partial sums for the series in Exercises 1–4.

1. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$
2. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$
3. $\displaystyle\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^i$
4. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

Sum the series in Exercises 5–8.

5. $1+\dfrac{1}{7}+\dfrac{1}{7^2}+\dfrac{1}{7^3}+\cdots$
6. $2+\dfrac{2}{9}+\dfrac{2}{9^2}+\dfrac{2}{9^3}+\cdots$
7. $\displaystyle\sum_{i=1}^{\infty}\left(\frac{7}{8}\right)^i$
8. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{13}{15}\right)^n$

9. You wish to draw \$10,000 out of a Swiss bank account at age 65, and thereafter you want to draw $\frac{3}{4}$ as much each year as the preceding one. Assuming that the account earns no interest, how much money must you start with to be prepared for an arbitrarily large life span?
10. A decaying radioactive source emits $\frac{9}{10}$ as much radiation each year as the previous one. Assuming that 2000 roentgens are given off in the first year, what is the total emission over all time?

Sum the series (if they converge) in Exercises 11–20.

11. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{13^j}$
12. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k$
13. $\displaystyle\sum_{i=0}^{\infty} \frac{2^{3i+4}}{3^{2i+5}}$
14. $\displaystyle\sum_{l=0}^{\infty} \frac{4^{4l+2}}{5^{3l+80}}$
15. $\displaystyle\sum_{j=-3}^{\infty} \left(\frac{1}{3}\right)^j$
16. $\displaystyle\sum_{i=4}^{\infty} 5\left(\frac{1}{3}\right)^{i+1/2}$
17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$
18. $\displaystyle\sum_{k=1}^{\infty} \frac{3^{2k}+1}{27^k}$
19. $\displaystyle\sum_{n=5}^{\infty} \frac{2^{n+1}}{3^{n-2}}$
20. $\displaystyle\sum_{i=1}^{\infty} \left[\left(\frac{1}{2}\right)^i + \left(\frac{1}{3}\right)^{2i} + \left(\frac{1}{4}\right)^{3i+1}\right]$
21. Show that $\sum_{i=1}^{\infty}(1 + 1/2^i)$ diverges.
22. Show that $\sum_{i=0}^{\infty}(3^i + 1/3^i)$ diverges.
23. Sum $2 + 4 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.
24. Sum $1 + 1/2 + 1/3 + 1/3^2 + 1/3^3 + \cdots$.

Test the series in Exercises 25–30 for convergence.

25. $\displaystyle\sum_{i=1}^{\infty} \frac{i}{\sqrt{i}+1}$
26. $\displaystyle\sum_{i=1}^{\infty} \frac{\sqrt{i}+1}{\sqrt{i}+8}$
27. $\displaystyle\sum_{i=1}^{\infty} \frac{3}{5+5i}$
28. $\displaystyle\sum_{i=1}^{\infty} \frac{6}{7+7i}$
29. $1 + \underbrace{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4}}_{2} + \underbrace{\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}}_{4} + \underbrace{\tfrac{1}{16} + \cdots}_{8}$
30. $1 + \underbrace{\tfrac{1}{4} + \tfrac{1}{4}}_{2} + \underbrace{\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16}}_{4} + \underbrace{\tfrac{1}{64} + \tfrac{1}{64} + \cdots}_{8}$
31. Show that the series $\sum_{j=1}^{\infty}(1 - 2^{-j})/j$ diverges.
32. Show that the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$ diverges.
33. Give an example to show that $\sum_{i=1}^{\infty}(a_i + b_i)$ may converge while both $\sum_{i=1}^{\infty}a_i$ and $\sum_{i=1}^{\infty}b_i$ diverge.
34. Comment on the formula $1 + 2 + 4 + 8 + \cdots = 1/(1-2) = -1$.
35. A *telescoping series*, like a geometric series, can be summed. A series $\sum_{n=1}^{\infty}a_n$ is telescoping if its nth term a_n can be expressed as $a_n = b_{n+1} - b_n$ for some sequence b_n.
    (a) Verify that $a_1 + a_2 + a_3 + \cdots + a_n = b_{n+1} - b_1$; therefore the series converges exactly when $\lim_{n\to\infty} b_{n+1}$ exists, and $\sum_{n=1}^{\infty}a_n = \lim_{n\to\infty} b_{n+1} - b_1$.
    (b) Use partial fraction methods to write $a_n = 1/[n(n+1)]$ as $b_{n+1} - b_n$ for some sequence b_n. Then evaluate the sum of the series $\sum_{n=1}^{\infty}1/[n(n+1)]$.
36. An experiment is performed, during which time successive excursions of a deflected plate are recorded. Initially, the plate has amplitude b_0. The plate then deflects downward to form a "dish" of depth b_1, then a "dome" of height b_2, and so on. (See Fig. 12.1.4.) The a's and b's are related by $a_1 = b_0 - b_1, a_2 = b_1 - b_2, a_3 = b_2 - b_3 \ldots$. The value a_n measures the amplitude "lost" at the nth oscillation (due to friction, say).

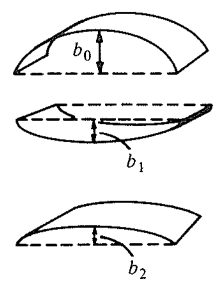


Figure 12.1.4. The deflecting plate in Exercise 36.

    (a) Find $\sum_{n=1}^{\infty}a_n$. Explain why $b_0 - \sum_{n=1}^{\infty}a_n$ is the "average height" of the oscillating plate after a large number of oscillations.
    (b) Suppose the "dishes" and "domes" decay to zero, that is, $\lim_{n\to\infty} b_{n+1} = 0$. Show that $\sum_{n=1}^{\infty}a_n = b_0$, and explain why this is physically obvious.
37. The joining of the transcontinental railroads occurred as follows. The East and West crews were setting track 12 miles apart, the East crew working at 5 miles per hour, the West crew working at 7 miles per hour. The official with the Golden Spike travelled feverishly by carriage back and forth between the crews until the rails joined. His speed was 20 miles per hour, and he started from the East.
    (a) Assume the carriage reversed direction with no waiting time at each encounter with an East or West crew. Let t_k be the carriage transit time for trip k. Verify that $t_{2n+2} = r^{n+1}\cdot(12/13)$, and $t_{2n+1} = r^n\cdot(12/27)$, where $r = (13/27)\cdot(15/25)$, $n = 0, 1, 2, 3, \ldots$.
    (b) Since the crews met in one hour, the total time for the carriage travel was one hour, i.e., $\lim_{n\to\infty}(t_1 + t_2 + t_3 + t_4 + \cdots + t_n) = 1$. Verify this formula using a geometric series.

12.2 The Comparison Test and Alternating Series

A series with positive terms converges if its terms approach zero quickly enough.

Most series, unlike the geometric series, cannot be summed explicitly. If we can prove that a given series converges, we can approximate its sum to any desired accuracy by adding up enough terms.

One way to tell whether a series converges or diverges is to compare it with a series which we already know to converge or diverge. As a fringe benefit of such a "comparison test," we sometimes get an estimate of the difference between the nth partial sum and the exact sum. Thus if we want to find the sum with a given accuracy, we know how many terms to take.

The comparison test for series is similar to that for integrals (Section 11.3). The test is simplest to understand for series with non-negative terms. Suppose that we are given series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ such that $0 \leqslant a_i \leqslant b_i$ for all i:

$$\text{if } \sum_{i=1}^{\infty} b_i \text{ converges, then so does } \sum_{i=1}^{\infty} a_i .$$

The reason for this is easy to see on an intuitive level. The partial sums $S_n = \sum_{i=1}^{n} a_n$ are moving to the right on the real number line since $a_i \geqslant 0$. They must either march off to ∞ or approach a limit. (The proof of this sentence requires a careful study of the real numbers, but we will take it for granted here. Consult the Supplement to this section and the theoretical references listed in the Preface.) However, $\sum_{i=1}^{n} a_i \leqslant \sum_{i=1}^{n} b_i \leqslant \sum_{i=1}^{\infty} b_i$, since $a_i \leqslant b_i$ and the partial sums $\sum_{i=1}^{n} b_i$ are marching to the right toward their limit. Hence all the S_n's are bounded by the fixed number $\sum_{i=1}^{\infty} b_i$, and so they cannot go to ∞.

Example 1 Show that $\displaystyle\sum_{i=1}^{\infty} \frac{3}{2^i + 4}$ converges.

Solution We know that $\sum_{i=1}^{\infty}(3/2^i)$ is convergent since it is a geometric series with $a = 3$ and $r = \frac{1}{2} < 1$; but

$$0 < \frac{3}{2^i + 4} < \frac{3}{2^i} ,$$

so the given series converges by the comparison test. ▲

For series $\sum_{i=1}^{\infty} a_i$ with terms that can be either positive or negative, we replace the condition $0 \leqslant a_i \leqslant b_i$ by $|a_i| \leqslant b_i$. Then if $\sum_{i=1}^{\infty} b_i$ converges, so must $\sum_{i=1}^{\infty} |a_i|$, by the test above. The following fact is true for any series:

$$\text{if } \sum_{i=1}^{\infty} |a_i| \text{ converges, so does } \sum_{i=1}^{\infty} a_i , \text{ and } \left|\sum_{i=1}^{\infty} a_i\right| \leqslant \sum_{i=1}^{\infty} |a_i| .$$

A careful proof of this fact is given at the end of this section; for now we simply observe that the convergence of $\sum_{i=1}^{\infty} |a_i|$ implies that the absolute values $|a_i|$ approach zero quickly, and the possibility of varying signs in the a_i's can only help in convergence. Therefore, if $0 \leqslant |a_i| \leqslant b_i$ and $\sum b_i$ converges, then $\sum |a_i|$ converges, and therefore so does $\sum a_i$. (We sometimes drop the "$i = 1$" and "∞" from $\sum$ if there is no danger of confusion.) This leads to the following test.

Comparison Test

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series such that $|a_i| \leqslant b_i$. If $\sum_{i=1}^{\infty} b_i$ is convergent, then so is $\sum_{i=1}^{\infty} a_i$.

Example 2 Show that $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i3^{i+1}}$ converges.

Solution We can compare the series with $\sum_{i=1}^{\infty} 1/3^i$. Let $a_i = (-1)^i/(i3^{i+1})$ and $b_i = 1/3^i$. Since $i3^{i+1} = (3i) \cdot 3^i > 3^i$, we have

$$|a_i| = \frac{1}{i3^{i+1}} < \frac{1}{3^i} = b_i .$$

Therefore, since $\sum_{i=1}^{\infty} b_i$ converges (it is a geometric series), so does $\sum_{i=1}^{\infty} a_i$. ▲

If the terms of two series $\sum a_i$ and $\sum b_i$ "resemble" one another, we may expect that one of the series converges if the other does. This is the case when the ratio a_i/b_i approaches a limit, as can be deduced from the comparison test. For instance, suppose that $\lim_{i\to\infty}(|a_i|/b_i) = M < \infty$, with all $b_i > 0$. Then for large enough i, we have $|a_i|/b_i < M + 1$, or $|a_i| < (M+1)b_i$. Now if $\sum b_i$ converges, so does $\sum (M+1)b_i$, by the constant multiple rule for series, and hence $\sum a_i$ converges by the comparison test.[1]

Example 3 Test for convergence: $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i - i}$.

Solution We cannot compare directly with $\sum_{i=1}^{\infty} 1/2^i$, since $1/(2^i - i)$ is *greater* than $1/2^i$. Instead, we look at the ratios a_i/b_i with $a_i = 1/(2^i - i)$ and $b_i = 1/2^i$. We have

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \lim_{i\to\infty} \frac{1}{1 - i/2^i} = \frac{1}{1-0} = 1$$

($\lim_{i\to\infty}(i/2^i) = 0$ by l'Hôpital's rule). Since $\sum_{i=1}^{\infty}(1/2^i)$ converges, so does $\sum_{i=1}^{\infty}[1/(2^i - i)]$. ▲

The following tests can both be similarly justified using the original comparison test.

Ratio Comparison Tests

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series, with $b_i > 0$ for all i.

If (1) $|a_i| \leqslant b_i$ for all i, or if $\lim_{i\to\infty}(|a_i|/b_i) < \infty$ and
(2) $\sum_{i=1}^{\infty} b_i$ is convergent, then $\sum_{i=1}^{\infty} a_i$ is convergent.

If (1) $a_i \geqslant b_i$ for all i, or if $\lim_{i\to\infty}(a_i/b_i) > 0$ and
(2) $\sum_{i=1}^{\infty} b_i$ is divergent, then $\sum_{i=1}^{\infty} a_i$ is divergent.

To choose b_i in applying the ratio comparison test, you should look for the "dominant terms" in the expression for a_i.

[1] Strictly speaking, to apply the comparison test we should have $|a_i| < (M+1)b_i$ for all i, not just sufficiently large i; but, as we saw earlier, the convergence or divergence of a series $\sum a_i$ is not affected by the values of its "early" terms, but only the behavior of a_i for large i. Of course, the *sum* of the series depends on all the terms.

Example 4 Show that $\sum_{i=1}^{\infty} \frac{2}{4+i}$ diverges.

Solution As $i \to \infty$, the dominant term in the denominator $4+i$ is i, that is, if i is very large (like 10^6), 4 is very small by comparison. Hence we are led to let $a_i = 2/(4+i)$, $b_i = 1/i$. Then

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \lim_{i\to\infty} \frac{2/(4+i)}{1/i} = \lim_{i\to\infty} \frac{2i}{4+i} = \lim_{i\to\infty} \frac{2}{(4/i)+1} = \frac{2}{0+1} = 2.$$

Since $2>0$, and $\sum_{i=1}^{\infty} 1/i$ is divergent, it follows that $\sum_{i=1}^{\infty}[2/(4+i)]$ is divergent as well. ▲

The next example illustrates how one may estimate the difference between a partial sum and the full series. We sometimes refer to this difference as a *tail* of the series; it is equal to the sum of all the terms not included in the partial sum.

Example 5 Find the partial sum $\sum_{i=1}^{3} \frac{(-1)^i}{i3^{i+1}}$ (see Example 2) and estimate the difference between this partial sum and the sum of the entire series.

Solution The sum of the first three terms is

$$-\frac{1}{3^2\cdot 1} + \frac{1}{3^3\cdot 2} - \frac{1}{3^4\cdot 3} = -\frac{1}{9} + \frac{1}{54} - \frac{1}{243} = -\frac{47}{486} \approx -0.0967.$$

The difference between the full sum of a series and the nth partial sum is given by $\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i = \sum_{i=n+1}^{\infty} a_i$. To estimate this tail in our example, we write

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \frac{(-1)^i}{3^{i+1}i} - (-0.0967)\right| &= \left|\sum_{i=4}^{\infty} \frac{(-1)^i}{3^{i+1}i}\right| \\ &\leqslant \sum_{i=4}^{\infty} \frac{1}{3^{i+1}i} \qquad \left(\text{since } \left|\sum a_i\right| \leqslant \sum |a_i|\right) \\ &\leqslant \sum_{i=4}^{\infty} \frac{1}{3^{i+1}} \qquad (\text{since } i>1) \\ &= \frac{1}{3^5}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right) \\ &= \frac{1}{3^5}\left(\frac{1}{1-1/3}\right) = \frac{1}{3^5}\cdot\frac{3}{2} = \frac{1}{162} \approxeq 0.0062.\end{aligned}$$

Thus the error is no more than 0.0062. We may therefore conclude that $\sum_{i=1}^{\infty}[(-1)^i/(3^{i+1}i)]$ lies in the interval $[-0.0967-0.0062, -0.0967+0.0062] = [-0.103, -0.090]$. ▲

The second kind of series which we will treat in this section is called an *alternating series*. To illustrate, recall that we saw in Section 12.1 that the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

is divergent even though $\lim_{i\to\infty}(1/i) = 0$. If we put a minus sign in front of every other term to obtain the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$

we might hope that the alternating positive and negative terms "neutralize" one another and cause the series to converge. The alternating series test will indeed guarantee convergence. First we need the following definition.

Alternating Series

A series $\sum_{i=1}^{\infty} a_i$ is called *alternating* if the terms a_i are alternately positive and negative and if the absolute values $|a_i|$ are decreasing to zero; that is, if:

1. $a_1 > 0$, $a_2 < 0$, $a_3 > 0$, $a_4 < 0$, and so on (or $a_1 < 0$, $a_2 > 0, \ldots$);
2. $|a_1| \geqslant |a_2| \geqslant |a_3| \geqslant \cdots$;
3. $\lim_{i\to\infty}|a_i| = 0$.

Conditions 1, 2, and 3 are often easy to verify.

Example 6 Is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ alternating?

Solution The terms alternate in sign, $+ - + - \cdots$, so condition 1 holds. Since the ith term $a_i = (-1)^{i+1}(1/i)$ has absolute value $1/i$, and $1/i > 1/(i+1)$, the terms are decreasing in absolute value, so condition 2 holds. Finally, since $\lim_{i\to\infty}|a_i| = \lim_{i\to\infty}(1/i) = 0$, condition 3 holds. Thus the series is alternating. ▲

Later in this section, we will prove that *every alternating series converges*. The proof is based on the idea that the partial sums $S_n = \sum_{i=1}^{n} a_i$ oscillate back and forth and get closer and closer together, so that they must close in on a limiting value S. This argument also shows that the sum S lies between any two successive partial sums, so that the tail corresponding to the partial sum S_n is less than $|a_{n+1}|$, the size of the first omitted term. (See Fig. 12.2.1.)

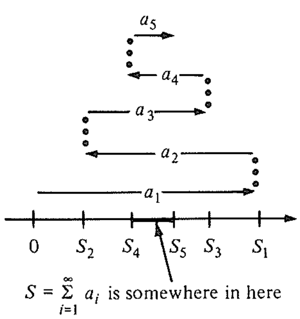


Figure 12.2.1. An alternating series converges, no matter how slowly the terms approach zero. The sum lies between each two successive partial sums.

Alternating Series Test

1. If $\sum_{i=1}^{\infty} a_i$ is a series such that the a_i alternate in sign, are decreasing in absolute value, and tend to zero, then it converges.
2. The error made in approximating the sum by $S_n = \sum_{i=1}^{n} a_i$ is not greater than $|a_{n+1}|$.

Example 7 Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges, and find its sum with an error of no more than 0.04.

Solution By Example 6, the series is alternating; therefore, by the alternating series test, it converges. To make the error at most $0.04 = \frac{1}{25}$, we must add up all the terms through $\frac{1}{24}$. Using a calculator, we find

$$S_{24} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots - \tfrac{1}{24} \approx 0.6727.$$

(Since the sum lies between S_{24} and $S_{25} \approx 0.7127$, an even better estimate is the midpoint $\frac{1}{2}(S_{24} + S_{25}) = 0.6927$, which can differ from the sum by at most 0.02.) ▲

Example 8 Test for convergence:

(a) $\sum_{i=1}^{\infty} \frac{(-1)^i}{(1+i)^2}$;

(b) $\frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \frac{2}{6} - \frac{1}{6} + \cdots$.

Solution (a) The terms alternate in sign since $(-1)^i = 1$ if i is even and $(-1)^i = -1$ if i is odd. The absolute values, $1/(1+i)^2$, are decreasing and converge to zero. Thus the series is alternating, so it converges.

(b) The terms alternate in sign and tend to zero, but the absolute values are not monotonically decreasing. Thus the series is not an alternating one and the alternating series test does not apply. If we group the terms by twos, we find that the series becomes

$$\left(\tfrac{2}{2} - \tfrac{1}{2}\right) + \left(\tfrac{2}{3} - \tfrac{1}{3}\right) + \left(\tfrac{2}{4} - \tfrac{1}{4}\right) + \left(\tfrac{2}{5} - \tfrac{1}{5}\right) + \cdots = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

which diverges. (Notice that the nth partial sum of the "grouped" series is the $2n$th partial sum of the original series.) ▲

We noted early in this section that a series $\sum_{i=1}^{\infty} a_i$ always converges if its terms go to zero quickly enough so that the series $\sum_{i=1}^{\infty} |a_i|$ of absolute values is convergent. Such a series $\sum_{i=1}^{\infty} a_i$ is said to be *absolutely convergent*. On the other hand, a series like $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, is convergent only due to the alternating signs of its terms; the series of absolute values, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, is divergent (it is the harmonic series). When $\sum_{i=1}^{\infty} a_i$ converges but $\sum_{i=1}^{\infty} |a_i|$ diverges, the series $\sum_{i=1}^{\infty} a_i$ is said to be *conditionally convergent*.

Example 9 Discuss the convergence of the series $\sum_{i=1}^{\infty} \frac{(-1)^i \sqrt{i}}{i+4}$.

Solution Let $a_i = (-1)^i \sqrt{i}/(i+4)$. We notice that for i large, $|a_i|$ appears to behave like $b_i = 1/\sqrt{i}$. The series $\sum_{i=1}^{\infty} b_i$ diverges by comparison with the harmonic series. To make the comparison between $|a_i|$ and b_i precise, look at the ratios: $\lim_{i\to\infty}(|a_i|/b_i) = \lim_{i\to\infty}[i/(i+4)] = 1$, so $\sum_{i=1}^{\infty} |a_i|$ diverges as well; hence our series is not *absolutely* convergent.

The series does look like it could be alternating: the terms alternate in sign and $\lim_{i\to\infty} a_i = 0$. To see whether the absolute values $|a_i|$ form a decreasing sequence, it is convenient to look at the function $f(x) = \sqrt{x}/(x+4)$. The derivative is

$$f'(x) = \frac{(1/2\sqrt{x})(x+4) - \sqrt{x}\cdot 1}{(x+4)^2} = \frac{2/\sqrt{x} - \sqrt{x}/2}{(x+4)^2} = \frac{4-x}{2\sqrt{x}\,(x+4)^2},$$

which is negative for $x > 4$, so $f(x)$ is decreasing for $x > 4$. Since $|a_i| = f(i)$, we have $|a_4| > |a_5| > |a_6| > \cdots$ which implies that our series $\sum a_i$, with its first three terms omitted, is alternating. It follows that the series is convergent; since it is not absolutely convergent, it is conditionally convergent. ▲

Absolute and Conditional Convergence

A series $\sum_{i=1}^{\infty} a_i$ is called *absolutely convergent* if $\sum_{i=1}^{\infty} |a_i|$ is convergent. Every absolutely convergent series converges.

A series may converge without being absolutely convergent; such a series is called *conditionally convergent*.

Supplement to Section 12.2: A Discussion of the Proofs of the Comparison and Alternating Series Tests

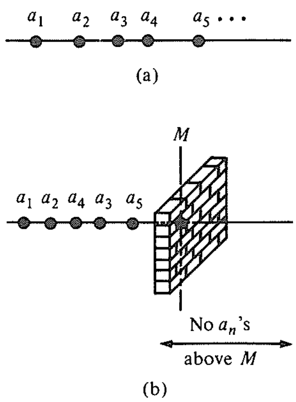


Figure 12.2.2. (a) An increasing sequence; (b) a sequence bounded above by M.

The key convergence property we need involves increasing sequences. It is similar to the existence of $\lim_{x\to\infty} f(x)$ if f is increasing and bounded above, which we used in Section 11.3 to establish the comparison test for integrals.

A sequence $a_1, a_2, \ldots$ of real numbers is called *increasing* in case $a_1 \leqslant a_2 \leqslant \cdots$. The sequence is said to be *bounded above* if there is a number M such that $a_n \leqslant M$ for all n. (See Fig. 12.2.2.)

For example, let $a_n = n/(n+1)$. Let us show that a_n is increasing and is bounded above by M if M is any number $\geqslant 1$. To prove that it is increasing, we must show that $a_n \leqslant a_{n+1}$—that is, that

$$\frac{n}{n+1} \leqslant \frac{n+1}{(n+1)+1} \quad \text{or} \quad n(n+2) \leqslant (n+1)^2$$

or

$$n^2 + 2n \leqslant n^2 + 2n + 1 \quad \text{or} \quad 0 \leqslant 1.$$

Reversing the steps gives a proof that $a_n \leqslant a_{n+1}$; i.e., the sequence is increasing. Since $n < n+1$, we have $a_n = n/(n+1) < 1$, so $a_n < M$ if $M \geqslant 1$.

We will accept without proof the following property of the real numbers (see the references listed in the Preface).

Increasing Sequence Property

If a_n is an increasing sequence which is bounded above, then a_n converges to some number a as $n \to \infty$. (Similarly, a decreasing sequence bounded below converges.)

The increasing sequence property expresses a simple idea: if the sequence is increasing, the numbers a_n increase, but they can never exceed M. What else could they do but converge? Of course, the limit a satisfies $a_n \leqslant a$ for all n.

For example, consider

$$a_1 = 0.3, \qquad a_2 = 0.33, \qquad a_3 = 0.333$$

and so forth. These a_n's are increasing (in fact, strictly increasing) and are bounded above by 0.4, so we know that they must converge. In fact, the increasing sequence property shows that any infinite decimal expansion converges and so represents a real number.

To prove the comparison test for series with positive terms, we apply the increasing sequence property to the sequence of partial sums. If $\sum_{i=1}^{\infty} a_i$ is a series with $a_i \geqslant 0$ for each i, then since the partial sums S_n satisfy $S_n - S_{n-1} = a_n \geqslant 0$, they must be an increasing sequence (see Fig. 12.2.3). If the partial sums are bounded above, the sequence must have a limit, and so the series must converge.

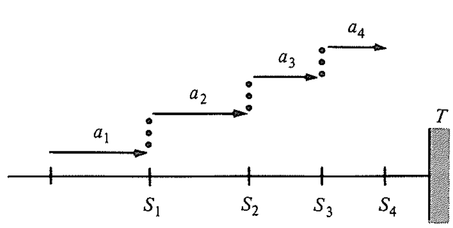


Figure 12.2.3. The partial sums of the series $\sum_{i=1}^{\infty} a_i$ are increasing and bounded above by T.

Now we may simply repeat the argument presented earlier in this section. If $0 \leqslant a_i \leqslant b_i$ for all i, and $T_n = \sum_{i=1}^{n} b_n$, then $S_n \leqslant T_n$. If the partial sums T_n approach a limit T, then they are bounded above by T, and so $S_n \leqslant T$ for all n. Thus $\lim_{n\to\infty} S_n$ exists and is less than or equal to T, i.e., $\sum_{i=1}^{\infty} a_n \leqslant \sum_{i=1}^{\infty} b_n$.

To complete the proof of the general comparison test, we must show that whenever $\sum_{i=1}^{\infty}|a_i|$ converges, so does $\sum_{i=1}^{\infty} a_i$; in other words, *every absolutely convergent series converges*. Suppose, then, that $\sum|a_i|$ converges.

We define two new series, $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$, by the formulas

$$b_i = \begin{Bmatrix} |a_i| & \text{if} \quad a_i \geqslant 0 \\ 0 & \text{if} \quad a_i < 0 \end{Bmatrix} = \begin{Bmatrix} a_i & \text{if} \quad a_i \geqslant 0 \\ 0 & \text{if} \quad a_i < 0 \end{Bmatrix},$$

$$c_i = \begin{Bmatrix} |a_i| & \text{if} \quad a_i \leqslant 0 \\ 0 & \text{if} \quad a_i > 0 \end{Bmatrix} = \begin{Bmatrix} -a_i & \text{if} \quad a_i \leqslant 0 \\ 0 & \text{if} \quad a_i > 0 \end{Bmatrix},$$

These are the "positive and negative parts" of the series $\sum_{i=1}^{\infty} a_i$. It is easy to check that $a_i = b_i - c_i$. The series $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$ are both convergent; in fact, since $b_i \leqslant |a_i|$, we have $\sum_{i=1}^{n} b_i \leqslant \sum_{i=1}^{n}|a_i| \leqslant \sum_{i=1}^{\infty}|a_i|$, which is finite since we assumed the series $\sum_{i=1}^{\infty} a_i$ to be absolutely convergent. Since $b_i \geqslant 0$ for all i, $\sum_{i=1}^{\infty} b_i$ is convergent. The same argument proves that $\sum_{i=1}^{\infty} c_i$ is convergent. The sum and constant multiple rules now apply to give the convergence of $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i - \sum_{i=1}^{\infty} c_i$.

Finally, we note that, by the triangle inequality,

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n}|a_i| \leqslant \sum_{i=1}^{\infty}|a_i|.$$

Since this is true for all n, and

$$\left|\sum_{i=1}^{\infty} a_i\right| = \left|\lim_{n\to\infty}\sum_{i=1}^{n} a_i\right| = \lim_{n\to\infty}\left|\sum_{i=1}^{n} a_i\right|$$

(the absolute value function is continuous), it follows that $|\sum_{i=1}^{\infty} a_i| \leqslant \sum_{i=1}^{\infty}|a_i|$. (Here we again used the fact that if $b_n \leqslant M$ for all n and b_n converges to b, then $b \leqslant M$).

We conclude this section with a proof that every alternating series converges.

Let $\sum_{i=1}^{\infty} a_i$ be an alternating series. If we let $b_i = (-1)^{i+1} a_i$, then all the b_i are positive, and our series is $b_1 - b_2 + b_3 - b_4 + b_5 \cdots$. In addition, we have $b_1 > b_2 > b_3 > \cdots$, and $\lim_{i\to\infty} b_i = 0$. Each even partial sum S_{2n} can be grouped as $(b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{n-1} - b_n)$, which is a series of positive terms, so we have $S_2 \leqslant S_4 \leqslant S_6 \leqslant \cdots$. On the other hand, the odd partial sums S_{2n+1} can be grouped as $b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n} - b_{2n+1})$, which is a sum of negative terms (except for the first), so we have $S_1 \geqslant S_3 \geqslant S_5 \geqslant \cdots$. Next, we note that $S_{2n+1} = S_{2n} + b_{2n+1} \geqslant S_{2n}$. Thus the even partial sums S_{2n} form an increasing sequence which is bounded above by any member of the decreasing sequence of odd partial sums. (See Fig. 12.2.1.) By the increasing sequence property, the sequence S_{2n} approaches a limit, S_{even}. Similarly, the decreasing sequence S_{2n+1} approaches a limit, S_{odd}.

Thus we have $S_2 \leqslant S_4 \leqslant S_6 \leqslant \cdots S_{2n} \leqslant \cdots \leqslant S_{\text{even}} \leqslant S_{\text{odd}} \cdots \leqslant S_{2n+1} \leqslant \cdots \leqslant S_3 \leqslant S_1$. Now $S_{2n+1} - S_{2n}$ is a_{2n+1}, which approaches zero as $n \to \infty$; the difference $S_{\text{odd}} - S_{\text{even}}$ is less than $S_{2n+1} - S_{2n}$, so it must be zero; i.e., $S_{\text{odd}} = S_{\text{even}}$. Call this common value S. Thus $|S_{2n} - S| \leqslant |S_{2n} - S_{2n+1}|$

$= b_{2n+1} = |a_{2n+1}|$ and $|S_{2n+1} - S| \leqslant |S_{2n+1} - S_{2n+2}| = b_{2n+2} = |a_{2n+2}|$, so each difference $|S_n - S|$ is less than $|a_{n+1}|$. Since $a_{n+1} \to 0$, we must have $S_n \to S$ as $n \to \infty$.

This argument also shows that each tail of an alternating series is no greater than the first term omitted from the partial sum.

Exercises for Section 12.2

Show that the series in Exercises 1–8 converge, using the comparison test for series with positive terms.

1. $\sum_{i=1}^{\infty} \frac{8}{3^i + 2}$
2. $\sum_{i=1}^{\infty} \frac{9}{4^i + 6}$
3. $\sum_{i=1}^{\infty} \frac{1}{3^i - 1}$
4. $\sum_{i=1}^{\infty} \frac{2}{4^i - 3}$
5. $\sum_{i=1}^{\infty} \frac{(-1)^i}{3^i + 2}$
6. $\sum_{i=1}^{\infty} \frac{(-1)^i}{4^i + 1}$
7. $\sum_{i=1}^{\infty} \frac{\sin i}{2^i - 1}$
8. $\sum_{i=1}^{\infty} \frac{\cos(\pi i)}{3^i - 1}$

Show that the series in Exercises 9–12 diverge, by using the comparison test.

9. $\sum_{i=1}^{\infty} \frac{3}{2 + i}$
10. $\sum_{i=1}^{\infty} \frac{8}{5 + 7i}$
11. $\sum_{i=1}^{\infty} \frac{8}{6i - 1}$
12. $\sum_{i=1}^{\infty} \frac{3}{2i - 1}$

Test the series in Exercises 13–34 for convergence.

13. $\sum_{n=1}^{\infty} \frac{3}{4^n + 2}$
14. $\sum_{n=1}^{\infty} \left(\frac{-4}{2^n + 3}\right)^n$
15. $\sum_{i=1}^{\infty} \frac{1}{2^i + 3^i}$
16. $\sum_{i=1}^{\infty} \frac{(1/2)^i}{i + 6}$
17. $\sum_{i=1}^{\infty} \frac{1}{3i + 1/i}$
18. $\sum_{i=1}^{\infty} \frac{2}{2i + 1}$
19. $\sum_{n=1}^{\infty} \frac{4^n + 5^n}{2^n 3^n}$
20. $\sum_{n=1}^{\infty} \frac{\sqrt{3 + n}}{4^n}$
21. $\sum_{i=1}^{\infty} \frac{1 + (-1)^i}{8i + 2^{i+1}}$
22. $\sum_{i=1}^{\infty} \frac{(-2)^i}{3^i + 1}$
23. $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i} + 2}$
24. $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i} + 1}$
25. $\sum_{i=1}^{\infty} \frac{3i}{2^i}$
26. $\sum_{j=1}^{\infty} \frac{2^j}{j}$
27. $\sum_{i=1}^{\infty} \left(\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3}\right)$
28. $\sum_{i=1}^{\infty} \frac{3}{1 + 3^i}$
29. $\sum_{j=1}^{\infty} \frac{\sin j}{2^j}$
30. $\sum_{n=1}^{\infty} e^{-n}$
31. $\sum_{i=2}^{\infty} \frac{1}{\ln i}$
32. $\sum_{i=1}^{\infty} \left(\frac{i}{i + 2}\right)^i$
33. $\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{33} + \frac{1}{65} + \cdots + \frac{1}{2^n + 1} + \cdots$
34. $1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots + \frac{1}{2^n - 1} + \cdots$

▦ Find the sum of the series in Exercises 35–38 with an error of no more than 0.01.

35. $\sum_{j=1}^{\infty} \frac{1}{j4^j}$
36. $\sum_{k=0}^{\infty} \frac{k}{2^k}$ [*Hint:* Compare with $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$.]
37. $\sum_{n=1}^{\infty} \frac{2^n - 1}{5^n + 1}$
38. $\sum_{p=1}^{\infty} \frac{(-1)^p}{2^p + p}$

Test the series in Exercises 39–50 for convergence and absolute convergence.

39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
40. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
41. $\sum_{k=1}^{\infty} \frac{k}{k + 1}$
42. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k + 1}$
43. $\sum_{i=1}^{\infty} \frac{\cos \pi i}{2^i}$
44. $\sum_{n=1}^{\infty} \frac{(-1)^n}{8n + 2}$
45. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$
46. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
47. $\sum_{i=1}^{\infty} (-1)^i \frac{i}{i^2 + 1}$
48. $\sum_{i=1}^{\infty} a_i$, where $a_i = 1/(2^i)$ if i is even and $a_i = 1/i$ if i is odd.
49. $\sum_{n=1}^{\infty} (-1)^n \ln[(n + 1)/n]$. [*Hint*: First prove that $\ln(1 + a) \geqslant a/2$ for small $a > 0$.]
50. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln[(n + 3)/n]$. (See the hint in 49.)

Estimate the sum of the series in Exercises 51–54 with an error of no more than that specified.

51. $\sum_{i=1}^{\infty} \frac{(-1)^i}{3^i + 1}$; 0.01
52. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^3 + 1}$; 0.005
53. $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{2n} + \frac{1}{5^n}\right)$; 0.02
54. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$; 0.02

55. Test for convergence: $\frac{1}{2}+\frac{1}{2}-\frac{1}{4}-\frac{1}{4}+\frac{1}{6}+\frac{1}{6}-\frac{1}{8}-\frac{1}{8}+\cdots$.

56. Does the series $\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\cdots$ converge?

Exercises 57 and 58 deal with an application of the increasing sequence test to inductively defined sequences. For example, let a_n be defined as follows:

$$a_0 = 0, \qquad a_1 = \sqrt{3}\,,$$

$$a_2 = \sqrt{3+a_1} = \sqrt{3+\sqrt{3}}\,,$$

$$a_3 = \sqrt{3+a_2} = \sqrt{3+\sqrt{3+\sqrt{3}}}\quad \ldots,$$

and, in general, $a_n = \sqrt{3+a_{n-1}}$. If we attempt to write out a_n "explicitly," we quickly find ourselves in a notational nightmare. However, numerical computation suggests that the sequence may be convergent:

$a_1 = 1.73205$	$a_2 = 2.17533$	$a_3 = 2.27493$
$a_4 = 2.29672$	$a_5 = 2.30146$	$a_6 = 2.30249$
$a_7 = 2.30271$	$a_8 = 2.30276$	$a_9 = 2.30277$
$a_{10} = 2.30278$	$a_{11} = 2.30278$	$a_{12} = 2.30278\ldots$

The sequence appears to be converging to a number $l \approx 2.30278\ldots$, but the numerical evidence only *suggests* that the sequence converges. The increasing sequence test enables us to prove this.

57. Let the sequence a_n be defined inductively by the rules $a_0 = 0$, $a_n = \sqrt{4+a_{n-1}}$.
   (a) Write out a_1, a_2, and a_3 in terms of square roots.
   (b) Calculate a_1 through a_{12} and guess the value of $\lim_{n\to\infty} a_n$ to four significant figures.

★58. (a) Prove by induction on n that for the sequence in Exercise 57, we have $a_n > a_{n-1}$ and $a_n < 5$.
   (b) Conclude that the limit $l = \lim_{n\to\infty} a_n$ exists.
   (c) Show that l must satisfy the equation $l = \sqrt{4+l}$.
   (d) Solve the equation in (c) for l and evaluate l to four significant figures. Compare the result with Exercise 57(b).

Show that the sequences in Exercises 59–62 are increasing (or decreasing) and bounded above (or below).

★59. $a_n = \dfrac{2n}{n+3}$

★60. $a_n = \dfrac{n}{n^2+1}$

★61. $a_n = \dfrac{1}{2n} - \dfrac{1}{n+1}$

★62. $b_n = n\sin\left(\dfrac{1}{n}\right)$

★63. Let $B > 0$ and $a_0 = 1$; $a_{n+1} = \frac{1}{2}(a_n + B/a_n)$. Show that $a_n \to \sqrt{B}$.

★64. Let $a_{n+1} = 3 - (1/a_n)$; $a_0 = 1$. Prove that the sequence is increasing and bounded above. What is $\lim_{n\to\infty} a_n$?

★65. Let $a_{n+1} = \frac{1}{2}a_n + \sqrt{a_n}$; $a_0 = 1$. Prove that a_n is increasing and bounded above. What is $\lim_{n\to\infty} a_n$?

★66. Let $a_{n+1} = \frac{1}{2}(1+a_n)$, and $a_0 = 1$. Show that $\lim_{n\to\infty} a_n = 1$.

★67. Give an alternative proof that $\lim_{n\to\infty} r^n = 0$ if $0 < r < 1$ as follows. Show that r^n decreases and is bounded below by zero. If the limit is l, show that $rl = l$ and conclude that $l = 0$. Why does the limit exist?

★68. Suppose that $a_0 = 1$, $a_{n+1} = 1 + 1/(1+a_n)$. Show that a_n converges and find the limit.

★69. The celebrated example due to Karl Weierstrass of a *nowhere differentiable continuous function* $f(x)$ in $-\infty < x < \infty$ is given by

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n x),$$

where $\phi(x+2) = \phi(x)$, and $\phi(x)$ on $0 \leqslant x \leqslant 2$ is the "triangle" through $(0,0), (1,1), (2,0)$. By construction, $0 \leqslant \phi(4^n x) \leqslant 1$. Verify by means of the comparison test that the series converges for any value of x. [See *Counterexamples in Analysis* by B. R. Gelbaum and J. M. H. Olmsted, Holden-Day, San Francisco (1964), p. 38 for the proof that f is nowhere differentiable.]

★70. Prove that $a_n = (1 + 1/n)^n$ is increasing and bounded above as follows:
   (a) If $0 \leqslant a \leqslant b$, prove that
   $$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1)b^n.$$
   That is, prove $b^n[(n+1)a - nb] < a^{n+1}$.
   (b) Let $a = 1 + [1/(n+1)]$ and $b = 1 + (1/n)$ and deduce that a_n is increasing.
   (c) Let $a = 1$ and $b = 1 + (1/2n)$ and deduce that $(1 + 1/2n)^{2n} < 4$.
   (d) Use parts (b) and (c) to show that $a_n < 4$. Conclude that a_n converges to some number (the number is e—see Section 6.3).

12.3 The Integral and Ratio Tests

The integral test establishes a connection between infinite series and improper integrals.

The sum of any infinite series may be thought of as an improper integral. Namely, given a series $\sum_{i=1}^{\infty} a_i$, we define a step function $g(x)$ on $[1, \infty)$ by the formulas:

$$g(x) = a_1 \qquad (1 \leqslant x < 2)$$
$$g(x) = a_2 \qquad (2 \leqslant x < 3)$$
$$\vdots$$
$$g(x) = a_i \qquad (i \leqslant x < i + 1)$$
$$\vdots$$

Since $\int_i^{i+1} g(x)\,dx = a_i$, the partial sum $\sum_{i=1}^{n} a_i$ is equal to $\int_1^{n+1} g(x)\,dx$, and the sum $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists if and only if the integral $\int_1^{\infty} g(x)\,dx = \lim_{b\to\infty} \int_1^{b} g(x)\,dx$ does.

By itself, this relation between series and integrals is not very useful. However, suppose now, as is often the case, that the formula which defines the term a_i as a function of i makes sense when i is a real number, not just an integer. In other words, suppose that there is a function $f(x)$, defined for all x satisfying $1 \leqslant x < \infty$, such that $f(i) = a_i$ when $i = 1, 2, 3, \ldots$. Suppose further that f satisfies these conditions:

1. $f(x) > 0$ for all x in $[1, \infty)$;
2. $f(x)$ is decreasing on $[1, \infty)$.

For example, if $a_i = 1/i$, the harmonic series, we may take $f(x) = 1/x$.

We may now compare $f(x)$ with the step function $g(x)$. When x satisfies $i \leqslant x < i + 1$, we have

$$0 \leqslant f(x) \leqslant f(i) = a_i = g(x).$$

Hence $0 \leqslant f(x) \leqslant g(x)$. (See Fig. 12.3.1.)

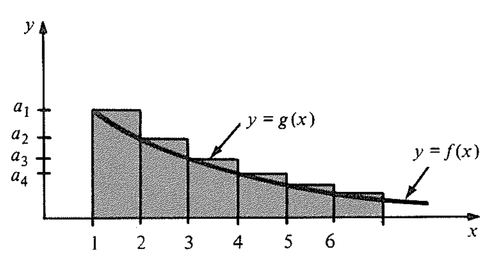


Figure 12.3.1. The area under the graph of f is less than the shaded area, so $0 \leqslant \int_1^{n+1} f(x)\,dx \leqslant \sum_{i=1}^{n} a_i$.

It follows that, for any n,

$$0 \leqslant \int_1^{n+1} f(x)\,dx \leqslant \int_1^{n+1} g(x)\,dx = \sum_{i=1}^{n} a_i . \tag{1}$$

We conclude that if the series $\sum_{i=1}^{\infty} a_i$ converges, then the integrals $\int_1^{n+1} f(x)\,dx$ are bounded above by the sum $\sum_{i=1}^{\infty} a_i$, so that the improper integral $\int_1^{\infty} f(x)\,dx$ converges (see Section 11.3).

In other words, if the integral $\int_1^{\infty} f(x)\,dx$ *diverges*, then so does the series $\sum_{i=1}^{\infty} a_i$.

Example 1 Show that

$$1+\frac{1}{2}+\cdots+\frac{1}{n} \geqslant \ln(n+1)$$

and so obtain a new proof that the harmonic series diverges.

Solution We take our function $f(x)$ to be $1/x$. Then, from formula (1) above, we get

$$1+\frac{1}{2}+\cdots+\frac{1}{n}=\sum_{i=1}^{n}\frac{1}{i} \geqslant \int_1^{n+1}\frac{1}{x}\,dx=\ln(n+1).$$

Since $\lim_{n\to\infty}\ln(n+1)=\infty$, the integral $\int_1^\infty(1/x)\,dx$ diverges; hence the series $\sum_{i=1}^\infty(1/i)$ diverges, too. ▲

We would like to turn around the preceding argument to show that if $\int_1^\infty f(x)\,dx$ converges, then $\sum_{i=1}^\infty a_i$ converges as well. To do so, we draw the rectangles with height a_i to the *left* of $x=i$ rather than the right; see Fig. 12.3.2. This procedure defines a step function $h(x)$ on $[1,\infty)$ defined by

$$h(x)=a_{i+1} \qquad (i \leqslant x < i+1).$$

Now we have $\int_i^{i+1}h(x)\,dx=a_{i+1}$, so $\sum_{i=2}^n a_i=\int_1^n h(x)\,dx$. If x satisfies $i \leqslant x < i+1$, we have

$$f(x) \geqslant f(i+1)=a_{i+1}=h(x) \geqslant 0.$$

Hence $f(x) \geqslant h(x) \geqslant 0$. (See Fig. 12.3.2.) Thus

$$\int_1^n f(x)\,dx \geqslant \int_1^n h(x)\,dx=\sum_{i=2}^n a_i \geqslant 0. \qquad (2)$$

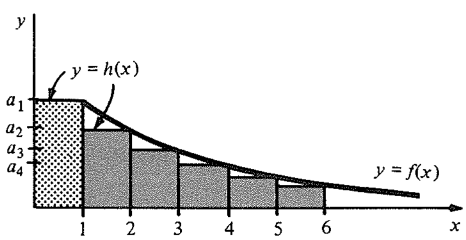


Figure 12.3.2. The area under the graph of f is greater than the shaded area, so $0 \leqslant \sum_{i=2}^n a_i \leqslant \int_1^n f(x)\,dx$.

If the integral $\int_1^\infty f(x)\,dx$ converges, then the partial sums $\sum_{i=1}^n a_i=a_1+\sum_{i=2}^n a_i$ are bounded above by $a_1+\int_1^\infty f(x)\,dx$, and therefore the series $\sum_{i=1}^\infty a_i$ is convergent (see the Supplement to Section 12.2).

Integral Test

To test the convergence of a series $\sum_{i=1}^\infty a_i$ of positive decreasing terms, find a positive, decreasing function $f(x)$ on $[1,\infty)$ such that $f(i)=a_i$.

If $\int_1^\infty f(x)\,dx$ converges, so does $\sum_{i=1}^\infty a_i$.

If $\int_1^\infty f(x)\,dx$ diverges, so does $\sum_{i=1}^\infty a_i$.

Example 2 Show that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ converges.

Solution This series is $\sum_{i=1}^\infty(1/i^2)$. We let $f(x)=1/x^2$; then

$$\int_1^\infty\frac{1}{x^2}\,dx=\lim_{b\to\infty}\int_1^b\frac{1}{x^2}\,dx=\lim_{b\to\infty}\left(1-\frac{1}{b}\right)=1.$$

The indefinite integral converges, so the series does, too. ▲

Example 3 Show that $\sum_{m=2}^{\infty} \frac{1}{m\sqrt{\ln m}}$ diverges, but $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$ converges.

Solution Note that the series start at $m = 2$ rather than $m = 1$. We consider the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \int_2^b (\ln x)^{-p} \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left.\frac{(\ln x)^{-p+1}}{-p+1}\right|_2^b$$

$$= \frac{1}{-p+1} \lim_{b\to\infty} \left[(\ln b)^{-p+1} - (\ln 2)^{-p+1} \right].$$

The limit is finite if $p = 2$ and infinite if $p = \frac{1}{2}$, so the integral converges if $p = 2$ and diverges if $p = \frac{1}{2}$. It follows that $\sum_{m=2}^{\infty}[1/(m\sqrt{\ln m})]$ diverges and $\sum_{m=2}^{\infty}[1/m(\ln m)^2]$ converges. ▲

Examples 1 and 2 are special cases of a result called the p-series test, which arises from the integral test with $f(x) = 1/x^p$. We recall that $\int_1^{\infty} x^n\,dx$ converges if $n < -1$ and diverges if $n \geqslant -1$ (see Example 2, Section 11.3). Thus we arrive at the test in the following box.

p-Series

If $p \leqslant 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^p}$ diverges.

If $p > 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges.

The p-series are often useful in conjunction with the comparison test.

Example 4 Test for convergence:

$$\text{(a)}\ \sum_{i=1}^{\infty} \frac{1}{1+i^2}; \qquad \text{(b)}\ \sum_{j=1}^{\infty} \frac{j^2+2j}{j^4-3j^2+10}; \qquad \text{(c)}\ \sum_{n=1}^{\infty} \frac{3n+\sqrt{n}}{2n^{3/2}+2}.$$

Solution (a) We compare the given series with the convergent p series $\sum_{i=1}^{\infty} 1/i^2$. Let $a_i = 1/(1+i^2)$ and $b_i = 1/i^2$. Then $0 < a_i < b_i$ and $\sum_{i=1}^{\infty} b_i$ converges, so $\sum_{i=1}^{\infty} a_i$ does, too.
(b) Let $a_j = (j^2+2j)/(j^4-3j^2+10)$ and $b_j = j^2/j^4 = 1/j^2$. Then

$$\lim_{j\to\infty} \left| \frac{a_j}{b_j} \right| = \lim_{j\to\infty} \left| \frac{1+2/j}{1-3/j^2+10/j^4} \right| = 1.$$

Since $\sum_{j=1}^{\infty} b_j$ converges, so does $\sum_{j=1}^{\infty} a_j$, by the ratio comparison test.
(c) Take $a_n = (3n+\sqrt{n})/(2n^{3/2}+2)$ and $b_n = n/(n^{3/2}) = 1/\sqrt{n}$. Then

$$\lim_{n\to\infty} \frac{3+(1/\sqrt{n})}{2+(2/n^{3/2})} = \frac{3}{2}.$$

Since $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} a_n$. ▲

What is the error in approximating a p-series by a partial sum? Let us show that $\sum_{n=1}^{N}(1/n^p)$ approximates $\sum_{n=1}^{\infty}(1/n^p)$ with error which does not exceed $1/[(p-1)N^{p-1}]$.

Indeed, just as in the proof of formula (2), we have

$$\sum_{n=N+1}^{\infty} \frac{1}{n^p} \leqslant \int_N^{\infty} \frac{1}{x^p}\,dx.$$

The left-hand side is the error:

$$\sum_{n=N+1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} - \sum_{n=1}^{N} \frac{1}{n^p} \leqslant \int_N^{\infty} \frac{1}{x^p}\,dx = \frac{1}{(p-1)N^{p-1}}.$$

$$\text{Thus, error} \leqslant \frac{1}{(p-1)N^{p-1}}. \tag{3}$$

Example 5 It is known that $\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$. Use this equation[2] to calculate $\pi^2/6$ with error less than 0.05.

Solution By equation (3), the error in stopping at N terms is at most $1/N$. To have error $< 0.05 = \frac{1}{20}$, we must take 20 terms (note that 100 terms are needed to get two decimal places!). We find:

$$\begin{aligned} 1 &= 1, \\ 1 + \tfrac{1}{4} &= 1.25, \\ 1 + \tfrac{1}{4} + \tfrac{1}{9} &= 1.36, \\ 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} &= 1.42, \\ 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} &= 1.46, \end{aligned}$$

and so forth, obtaining 1.49, 1.51, 1.53, 1.54, 1.55, 1.56, ... Finally, $1 + \frac{1}{4} + \cdots + \frac{1}{400} = 1.596 \ldots$. (Notice the "slowness" of the convergence.) We may compare this with the exact value $\pi^2/6 = 1.6449 \ldots$. ▲

The idea used in the preceding example can be used to estimate the tail of a series whenever convergence is proven by the integral test. (See Exercise 11.)

Another important test for convergence is called the *ratio test*. This test provides a general way to compare a series with a geometric series, but it formulates the hypotheses in a way which is particularly convenient, since no explicit comparison is needed. Here is the test.

Ratio Test

Let $\sum_{i=1}^{\infty} a_i$ *be a series. Suppose that* $\lim_{i\to\infty}\left|\frac{a_i}{a_{i-1}}\right|$ *exists.*

1. *If* $\lim_{i\to\infty}\left|\frac{a_i}{a_{i-1}}\right| < 1$, *then the series converges (absolutely).*
2. *If* $\lim_{i\to\infty}\left|\frac{a_i}{a_{i-1}}\right| > 1$, *then the series diverges.*
3. *If* $\lim_{i\to\infty}\left|\frac{a_i}{a_{i-1}}\right| = 1$, *the test is inconclusive.*

[2] For a proof using only elementary calculus, see Y. Matsuoka, "An Elementary Proof of the Formula $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$," *American Mathematical Monthly* **68**(1961): 485–487 (reprinted in T. M. Apostol (ed.), *Selected Papers on Calculus*, Math. Assn. of America (1969), p. 372). The formula may also be proved using Fourier series; see for instance J. Marsden, Elementary Classical Analysis, Freeman (1974), Ch. 10.

Do not confuse this test, in which ratios of successive terms in the *same* series are considered, with the ratio comparison test in Section 12.2, where we took the ratios of terms in two *different* series.

Proof of the ratio test By definition of the limit, $|a_i/a_{i-1}|$ will be close to its limit l for i large. To prove part 1, let $l < 1$ and let $r = (l+1)/2$ be the midpoint between l and 1, so that $l < r < 1$. Thus there is an N such that

$$\left|\frac{a_i}{a_{i-1}}\right| < r \qquad \text{if} \quad i > N.$$

We will show this implies that the given series converges.

We have $|a_{N+1}/a_N| < r$ so $|a_{N+1}| < |a_N|r$, $|a_{N+2}/a_{N+1}| < r$; hence $|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$ and, in general, $|a_{N+j}| < |a_N|r^j$; but $\sum_{j=1}^{\infty}|a_N|r^j = |a_N|\sum_{j=1}^{\infty}r^j$ is a convergent geometric series since $r < 1$. Hence, by the comparison test, $\sum_{j=1}^{\infty}|a_{N+j}|$ converges. Since we have omitted only $|a_1|$, $|a_2|, \ldots, |a_N|$, the series $\sum_{j=1}^{\infty}|a_j|$ converges as well and part 1 is proved.

For part 2 we find, as in part 1, that $|a_{N+j}| > |a_N|r^j$, where $r = (l+1)/2$ is now greater than 1. As $j \to \infty$, $r^j \to \infty$, so $|a_{N+j}| \to \infty$. Thus the series cannot converge, since its terms do not converge to zero.

To prove part 3, we consider the p-series with $a_i = i^p$. The ratio is $|a_i/a_{i-1}| = [i/(i-1)]^p$, and $\lim_{i\to\infty}[i/(i-1)]^p = [\lim_{i\to\infty}(i/(i-1))]^p = 1^p = 1$ for all $p > 0$; but the p-series is convergent if $p > 1$ and divergent if $p \leqslant 1$, so the ratio test does not give any useful information for these series. ■

Example 6 Test for convergence: $2 + \frac{2^2}{2^8} + \frac{2^3}{3^8} + \frac{2^4}{4^8} + \cdots = 2 + \frac{1}{64} + \frac{8}{6561} + \frac{1}{4096} + \cdots$.

Solution We have $a_i = 2^i/i^8$. The ratio a_i/a_{i-1} is

$$\frac{2^i}{i^8} \cdot \frac{(i-1)^8}{2^{i-1}} = 2 \cdot \left(\frac{i-1}{i}\right)^8,$$

so

$$\lim_{i\to\infty} \frac{a_i}{a_{i-1}} = 2\left[\lim_{i\to\infty}\left(\frac{i-1}{i}\right)\right]^8 = 2 \cdot 1^8 = 2$$

which is greater than 1, and so the series diverges. ▲

Example 7 Test for convergence:

(a) $\sum_{n=1}^{\infty}\frac{1}{n!}$, where $n! = n(n-1)\cdots 3\cdot 2\cdot 1$

(b) $\sum_{j=1}^{\infty}\frac{b^j}{j!}$, b any constant

Solution (a) Here $a_n = 1/n!$, so

$$\frac{a_n}{a_{n-1}} = \frac{1/n(n-1)\cdots 3\cdot 2\cdot 1}{1/(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = \frac{1}{n}.$$

Thus $|a_n/a_{n-1}| = 1/n \to 0 < 1$, so we have convergence.

(b) Here $a_j = b^j/j!$, so

$$\frac{a_j}{a_{j-1}} = \frac{b^j/j!}{b^{j-1}/(j-1)!} = \frac{b}{j}.$$

Thus $|a_j/a_{j-1}| = b/j \to 0$, so we have convergence. In this example, note that the numerator b^j and the denominator $j!$ tend to infinity, but the denominator does so much faster. In fact, since the series converges, $b^j/j! \to 0$ as $j \to \infty$. ▲

Let us show that if $|a_n/a_{n-1}| < r < 1$ for $n > N$, then the error made in approximating $\sum_{n=1}^{\infty} a_n$ by $\sum_{n=1}^{N} a_n$ is no greater than $|a_N| r/(1-r)$. In short,

$$\text{error} \leqslant \frac{|a_N| r}{1-r}. \tag{4}$$

Indeed, $\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n$. As in the proof of the ratio test, $|a_{N+1}| < |a_N| r$, and, in general, $|a_{N+j}| < |a_N| r^j$, so $\sum_{j=1}^{\infty} |a_{N+j}| \leqslant |a_N| r/(1-r)$ by the formula for the sum of a geometric series and the comparison test. Hence the error is no greater than $|a_N| r/(1-r)$.

Example 8 What is the error made in approximating $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ by $\displaystyle\sum_{n=1}^{4} \frac{1}{n!}$?

Solution Here $|a_n/a_{n-1}| = 1/n$, which is $< \frac{1}{5}$ if $n > 4 = N$. By inequality (4), the error is no more than $a_4/5(1-1/5) = 1/4 \cdot 4! = 1/96 < 0.0105$. The error becomes small very quickly if N is increased. ▲

Our final test is similar in spirit to the ratio test, in that it is also proved by comparison with a geometric series.

Root Test

Let $\displaystyle\sum_{i=1}^{\infty} a_i$ be a given series, and suppose that $\displaystyle\lim_{i\to\infty} |a_i|^{1/i}$ exists.

1. If $\displaystyle\lim_{i\to\infty} |a_i|^{1/i} < 1$, then $\displaystyle\sum_{i=1}^{\infty} a_i$ converges absolutely.
2. If $\displaystyle\lim_{i\to\infty} |a_i|^{1/i} > 1$, then $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges.
3. If $\displaystyle\lim_{i\to\infty} |a_i|^{1/i} = 1$, the test is inconclusive.

To prove 1, let $l = \lim_{n\to\infty}(|a_n|^{1/n})$ and let $r = (1+l)/2$ be the midpoint of 1 and l, so $l < r < 1$. From the definition of the limit, there is an N such that $|a_n|^{1/n} < r < 1$ if $n > N$. Hence $|a_n| < r^n$ if $n > N$. Thus, by direct comparison of $\sum_{n=N+1}^{\infty} |a_n|$ with the geometric series $\sum_{n=N+1}^{\infty} r^n$, which converges since $r < 1$, $\sum_{n=N+1}^{\infty} |a_n|$ converges. Since we have neglected only finitely many terms, the given series converges.

Cases 2 and 3 are left as exercises (see Exercises 37 and 38).

Example 9 Test for convergence: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ and (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$.

Solution (a) Here $a_n = 1/n^n$, so $|a_n|^{1/n} = 1/n$. Thus $\lim_{n\to\infty} |a_n|^{1/n} = 0 < 1$. Thus, by the root test (with i replaced by n), the series converges (absolutely). [This example can also be done by the comparison test: $1/n^n < 1/n^2$ for $n \geqslant 2$.]
(b) Here $a_n = 3^n/n^2$, so $|a_n|^{1/n} = 3/n^{2/n}$; but $\lim_{n\to\infty} n^{2/n} = 1$, since $\ln(n^{2/n}) = 2(\ln n)/n \to 0$ as $n \to \infty$ (by l'Hôpital's rule). Thus $\lim_{n\to\infty} |a_n|^{1/n} = 3 > 1$, so the series diverges. ▲

The tests we have covered enable us to deal with a wide variety of series. Of course, if the series is geometric, it may be summed. Otherwise, either the ratio test, the root test, comparison with a p-series, the integral test, or the alternating series test will usually work.

Example 10 Test for convergence: (a) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ and (b) $\sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n}$.

Solution (a) We use the ratio test. Here, $a_n = n^n/n!$, so

$$\left|\frac{a_n}{a_{n-1}}\right| = \frac{n^n}{(n-1)^{n-1}} \cdot \frac{(n-1)!}{n!} = \frac{n \cdot n^{n-1}}{(n-1)^{n-1}} \cdot \frac{1}{n} = \left(\frac{n}{n-1}\right)^{n-1}$$

$$= \frac{1}{(1-1/n)^{n-1}} = \frac{1-1/n}{(1-1/n)^n}.$$

The numerator approaches 1 while the denominator approaches e^{-1} (see Section 6.4), so $\lim_{n\to\infty}|a_n/a_{n-1}| = e > 1$, and the series diverges.

(b) We expect the series to behave like $\sum_{n=1}^{\infty}(1/n^2)$, so we use the ratio comparison test, with $a_i = 1/(i^2 - \ln i)$ and $b_i = 1/i^2$. The ratio between the terms in the two series is

$$\frac{a_i}{b_i} = \frac{i^2}{i^2 - \ln i} = \frac{1}{1 - \ln i/i^2}.$$

Since $\lim_{i\to\infty}[(\ln i)/i^2] = 0$ (by l'Hôpital's rule), $\lim_{i\to\infty} a_i/b_i = 1$. The p-series $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty}(1/i^2)$ converges, so the series

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{1}{i^2 - \ln i} \text{ converges, too. } \blacktriangle$$

Exercises for Section 12.3

Use the integral test to determine the convergence or divergence of the series in Exercises 1–4.

1. $\sum_{i=1}^{\infty} \frac{i}{i^2+1}$

2. $\sum_{i=1}^{\infty} \frac{1}{i^2+4}$

3. $\sum_{i=2}^{\infty} \frac{1}{i(\ln i)^{3/2}}$

4. $\sum_{i=2}^{\infty} \frac{1}{i(\ln i)^{2/3}}$

Use the p-series test and a comparison test to test the series in Exercises 5–8 for convergence or divergence.

5. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

Estimate the sums in Exercises 9 and 10 to within 0.05.

9. $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

10. $\sum_{n=1}^{\infty} \frac{\sin n}{n^4}$

11. Let $f(x)$ be a positive decreasing function on $[1, \infty)$ such that $\int_1^{\infty} f(x)\,dx$ converges. Show that

$$\left|\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n)\right| \leqslant \int_N^{\infty} f(x)\,dx.$$

12. Estimate $\sum_{n=1}^{\infty}[(1+n^2)/(1+n^8)]$ to within 0.02. (Use the comparison test *and* the integral test.)

Use the ratio test to determine the convergence or divergence of the series in Exercises 13–16.

13. $\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{3^n}$

14. $\sum_{n=1}^{\infty} \frac{3^n}{2\sqrt{n}}$

15. $\sum_{i=1}^{\infty} \frac{i^3 \cdot 3^i}{i!}$

16. $\sum_{n=1}^{\infty} \frac{2n^2 + n!}{n^5 + (3n)!}$

Estimate the sums in Exercises 17 and 18 to within 0.05.

17. $\sum_{n=0}^{\infty} \frac{\pi^{2n+1}}{(2n+1)!}$

18. $\sum_{n=0}^{\infty} \frac{(\pi/2)^{2n+1}}{(2n+1)!}$

19. Estimate $\sum_{n=1}^{\infty}(1/n!)$: (a) To within 0.05. (b) To within 0.005. (c) How many terms would you need to calculate to get an accuracy of five decimal places?

20. (a) Show that $\sum_{n=1}^{\infty} \frac{\sin(\pi n/2)}{n!}$ converges.
(b) Estimate the sum to within 0.01.

Use the root test to determine the convergence or divergence of the series in Exercises 21–24.

21. $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

22. $\sum_{n=1}^{\infty} \frac{n^n}{2^n}$

23. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

24. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Test for convergence in Exercises 25–36.

25. $\sum_{i=1}^{\infty} \frac{1}{i^4}$

26. $\sum_{j=3}^{\infty} \frac{j^2 + \cos j}{j^4 + \sin j}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$

28. $\sum_{m=1}^{\infty} \frac{(-1)^m(m+1)}{m^2+1}$

29. $\sum_{k=2}^{\infty} \frac{\cos k\pi}{\ln k}$

30. $\sum_{j=1}^{\infty} (-1)^j \sin\left(\frac{\pi}{4j}\right)$

31. $\sum_{j=1}^{\infty} \frac{(j+1)^{100}}{j!}$

32. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + n + n^{3/2}}{\sqrt{n} + n + n^{5/2} + n^3}$

33. $\sum_{r=0}^{\infty} \frac{2^r}{2^r + 3^r}$

34. $\sum_{s=1}^{\infty} \frac{s - \ln s}{s^2 + \ln s}$

35. $\sum_{t=2}^{\infty} \frac{(-1)^t}{(\ln t)^{1/2}}$

36. $\sum_{t=1}^{\infty} \frac{(-1)^t}{t^{1/4}}$

In Exercises 37 and 38, complete the proof of the root test by showing the following.

★37. If $\lim_{n\to\infty}|a_n|^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

★38. If $\lim_{n\to\infty}|a_n|^{1/n} = 1$, the test is inconclusive. (You may use the fact that $\lim_{n\to\infty} n^{1/n} = 1$.)

★39. For which values of p does $\sum_{i=1}^{\infty}[\sin(1/i)]^p$ converge?

★40. For which values of p does $\sum_{n=2}^{\infty}[1/n(\ln n)^p]$ converge?

★41. For which p does $\sum_{n=2}^{\infty}(1/n^p \ln n)$ converge?

★42. For which values of p and q is the series $\sum_{n=2}^{\infty} 1/[n^p(\ln n)^q]$ convergent?

★43. (a) Let $f(x)$ be positive and decreasing on $[1,\infty)$, and suppose that $f(i) = a_i$ for $i = 1, 2, 3, \ldots$. Show that

$$S - \frac{1}{2} f(n) \leqslant \sum_{i=1}^{\infty} a_i \leqslant S + \frac{1}{2} f(n),$$

where

$$S = \sum_{i=1}^{n} f(i) + \frac{1}{2}\int_{n}^{n+1} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx.$$

[*Hint:* Look at the proof of the integral test; show that $\int_{n+1}^{\infty} f(x)\,dx \leqslant \sum_{i=n+1}^{\infty} a_i \leqslant \int_{n}^{\infty} f(x)\,dx$.]

(b) Estimate $\sum_{n=1}^{\infty} 1/n^4$ to within 0.0001. How many terms did you use? How much work do you save by using the method of part (a) instead of the formula: error $< 1/(p-1)N^{p-1}$?

★44. Using Fourier analysis, it is possible to show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots.$$

(a) Show directly that the series on the right is convergent, by means of the integral test.

(b) Determine how many terms are needed to compute $\pi^4/96$ accurate to 20 digits.

★45. A bar of length L is loaded by a weight W at its midpoint. At $t = 0$ the load is removed. The deflection $y(t)$ at the midpoint, measured from the straight profile $y = 0$, is given by

$$y(t) = \frac{2WL^2}{\pi^4 EI}\left[\cos r + \frac{\cos(9r)}{3^4} + \frac{\cos(25r)}{5^4} + \cdots\right],$$

where $r = \left(\frac{\pi^2}{L^2}\sqrt{\frac{EIg}{\gamma\Omega}}\right)t$. The numbers $E, I, g, \gamma, \Omega, L$ are positive constants.

(a) Show by substitution that the bracketed terms are the first three terms of the infinite series

$$\sum_{n=0}^{\infty} \frac{\cos\left[(2n+1)^2 r\right]}{(2n+1)^4}.$$

(b) Make accurate graphs of the first three partial sums

$$S_1(r) = \cos(r),$$

$$S_2(r) = \cos(r) + \frac{1}{3^4}\cos(9r),$$

$$S_3(r) = \cos(r) + \frac{1}{3^4}\cos(9r) + \frac{1}{5^4}\cos(25r).$$

Up to a magnification factor, these graphs approximate the motion of the midpoint of the bar.

(c) Using the integral test and the comparison test, show that the series converges.

12.4 Power Series

Many functions can be expressed as "polynomials with infinitely many terms."

A series of the form $\sum_{i=0}^{\infty} a_i(x - x_0)^i$, where the a_i's and x_0 are constants and x is a variable, is called a *power series* (since we are summing the powers of $(x - x_0)$). In this section, we show how a power series may be considered as a function of x, defined on a certain interval. In the next section, we begin with an arbitrary function and show how to find the power series which represents it (if there is such a series).

We first consider power series in which $x_0 = 0$; that is, those of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i,$$

where the a_i are given constants. The domain of f can be taken to consist of those x for which the series converges.

If there is an integer N such that $a_i = 0$ for all $i > N$, then the power series is equal to a finite sum, $\sum_{i=0}^{N} a_i x^i$, which is just a polynomial of degree N. In general, we may think of a power series as a polynomial of "infinite degree"; we will see that as long as they converge, power series may be manipulated (added, subtracted, multiplied, divided, differentiated) just like ordinary polynomials.

The simplest power series, after a polynomial, is the geometric series

$$f(x) = 1 + x + x^2 + \cdots,$$

which converges when $|x| < 1$; the sum is the function $1/(1-x)$. Thus we have written $1/(1-x)$ as a power series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{if} \quad |x| < 1.$$

Convergence of general power series may often be determined by a test similar to the ratio test.

Ratio Test for Power Series

Let $\sum_{i=0}^{\infty} a_i x^i$ be a power series. Assume that

$$\lim_{i\to\infty} \left| \frac{a_i}{a_{i-1}} \right| = l$$

exists. Let $R = 1/l$; if $l = 0$, let $R = \infty$, and if $l = \infty$, let $R = 0$. Then:

1. If $|x| < R$, the power series converges absolutely.
2. If $|x| > R$, the power series diverges.
3. If $x = \pm R$, the power series could converge or diverge.

To prove part 1, we use the ratio test for series of numbers; the ratio of successive terms for $\sum_{i=0}^{\infty} a_i x^i$ is

$$\left| \frac{a_i x^i}{a_{i-1} x^{i-1}} \right| = \left| \frac{a_i}{a_{i-1}} \right| |x|.$$

By hypothesis, this converges to $l \cdot |x| < l \cdot R = 1$. Hence, by the ratio test, the series converges absolutely when $|x| < R$. The proof of part 2 is similar, and the examples below will show that at $x = \pm R$, either convergence or divergence can occur.

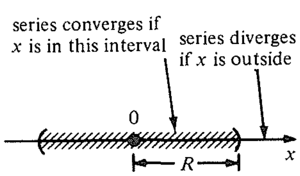


Figure 12.4.1. R is the radius of convergence of $\sum_{i=0}^{\infty} a_i x^i$.

The number R in this test is called the *radius of convergence* of the series (see Fig. 12.4.1). One can show that a number R (possibly infinity) with the three properties in the preceding box exists for any power series, even if $\lim_{i\to\infty} |a_i / a_{i-1}|$ does not exist.

Example 1 For which x does $\sum_{i=0}^{\infty} \frac{i}{i+1} x^i$ converge?

Solution Here $a_i = i/(i+1)$. Then

$$\frac{a_i}{a_{i-1}} = \frac{i/(i+1)}{(i-1)/i} = \frac{i^2}{(i+1)(i-1)}$$
$$= \frac{1}{(1+1/i)(1-1/i)} \to 1 \qquad \text{as} \quad i \to \infty.$$

Hence $l = 1$. Thus the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, then $\lim_{i\to\infty}[i/(i+1)]x^i = 1$, so the series diverges at $x = 1$ since the terms do not go to zero. If $x = -1$, $\lim_{i\to\infty}|[i/(i+1)]x^i| = \lim_{i\to\infty}[i/(i+1)] = 1$, so again the series diverges. ▲

Example 2 Determine the radius of convergence of $\sum_{k=0}^{\infty} \frac{k^5}{(k+1)!} x^k$.

Solution To use the ratio test, we look at

$$l = \lim_{k\to\infty} \left| \frac{a_k}{a_{k-1}} \right|.$$

Here $a_k = k^5/(k+1)!$, so

$$l = \lim_{k\to\infty} \left| \frac{k^5}{(k+1)!} \cdot \frac{k!}{(k-1)^5} \right|$$
$$= \lim_{k\to\infty} \left(\frac{k}{k-1} \right)^5 \cdot \lim_{k\to\infty} \frac{1}{k+1} = 1 \cdot 0 = 0.$$

Thus $l = 0$, so $R = \infty$ and the radius of convergence is infinite (that is, the series converges for all x). ▲

Example 3 For which x do the following series converge? (a) $\sum_{i=1}^{\infty} \frac{x^i}{i}$ (b) $\sum_{i=1}^{\infty} \frac{x^i}{i^2}$ (c) $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ (By convention, we define $0! = 1$.)

Solution (a) We have $a_i = 1/i$, so

$$l = \lim_{i\to\infty} \left| \frac{a_i}{a_{i-1}} \right| = \lim_{i\to\infty} \left(\frac{i-1}{i} \right) = 1;$$

the series therefore converges for $|x| < 1$ and diverges for $|x| > 1$. When $x = 1$, $\sum_{i=1}^{\infty} x^i/i$ is the divergent harmonic series; for $x = -1$, the series is alternating, so it converges.

(b) We have $a_i = 1/i^2$, so

$$l = \lim_{i\to\infty} \frac{(i-1)^2}{i^2} = 1$$

and the radius of convergence is again 1. This time, when $x = 1$, we get the p-series $\sum_{i=1}^{\infty}(1/i^2)$, which converges since $p = 2 > 1$. The series for $x = -1$, $\sum_{i=1}^{\infty}[(-1)^i/i^2]$, converges absolutely, so is also convergent.

(c) Here $a_i = 1/i!$, so $|a_i/a_{i-1}| = (i-1)!/i! = 1/i \to 0$ as $i \to \infty$. Thus $l = 0$, so the series converges for all x. ▲

Series of the form $\sum_{i=0}^{\infty} a_i(x - x_0)^i$ are also called power series; their theory is essentially the same as for the case $x_0 = 0$ already studied, because $\sum_{i=0}^{\infty} a_i(x - x_0)^i$ may be written as $\sum_{i=0}^{\infty} a_i w^i$, where $w = x - x_0$.

Example 4 For which x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{\sqrt{2n+5}}(x+5)^n$ converge?

Solution This series is of the form $\sum_{i=0}^{\infty} a_i(x - x_0)^i$, with $a_i = 4^i/\sqrt{2i+5}$ and $x_0 = -5$. We have

$$l = \lim_{i\to\infty} \frac{a_i}{a_{i-1}} = \lim_{i\to\infty} \frac{4^i}{\sqrt{2i+5}} \cdot \frac{\sqrt{2(i-1)+5}}{4^{i-1}} = \lim_{i\to\infty} 4\sqrt{\frac{2i+3}{2i+5}} = 4,$$

so the radius of convergence is $\frac{1}{4}$. Thus the series converges for $|x+5| < \frac{1}{4}$ and diverges for $|x+5| > \frac{1}{4}$. When $x = -5\frac{1}{4}$, the series becomes $\sum_{i=0}^{\infty}[(-1)^i/\sqrt{2i+5}\,]$, which converges because it is alternating. When $x = -4\frac{3}{4}$, the series is $\sum_{i=0}^{\infty}[1/\sqrt{2i+5}\,]$, which diverges by the ratio comparison test with $\sum_{i=1}^{\infty}(1/\sqrt{2i}\,)$ (or by the integral test). Thus our power series converges when $-5\frac{1}{4} \leqslant x < -4\frac{3}{4}$. ▲

In place of the ratio test, one can sometimes use the root test in the same way.

Root Test for Power Series

Let $\displaystyle\sum_{i=0}^{\infty} a_i x^i$ be a given power series. Assume that $\lim_{i\to\infty} |a_i|^{1/i} = \rho$ exists. Then the radius of convergence is $R = 1/\rho$.

Indeed, if $|x| < R$, $\lim_{i\to\infty}|a_i x^i|^{1/i} = \lim_{i\to\infty}|a_i|^{1/i}|x| = \rho|x| < \rho R = 1$, so the power series converges by the root test.

Example 5 Find the radius of convergence of the series $\displaystyle\sum_{i=1}^{\infty} \frac{x^i}{(2+1/i)^i}$.

Solution $\rho = \lim_{i\to\infty}|a_i|^{1/i} = \lim_{i\to\infty}(1/(2+1/i)^i)^{1/i} = \lim_{i\to\infty}\{1/[2+(1/i)]\} = \frac{1}{2}$, so the radius of convergence is $R = 2$. ▲

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, defined where the series converges. By analogy with ordinary polynomials, we might guess that

$$f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$$

and that

$$\int f(x)\,dx = \sum_{i=0}^{\infty} \frac{a_i x^{i+1}}{i+1} + C.$$

In fact, this is true. The proof is contained in (the moderately difficult) Exercises 41–45 at the end of the section.

Example 6 If $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, show that $f'(x) = f(x)$. Conclude that $f(x) = e^x$.

Solution By Example 3(c), the series for $f(x)$ converges for all x. Then $f'(x) = \sum_{i=1}^{\infty}(ix^{i-1}/i!) = \sum_{i=1}^{\infty}[x^{i-1}/(i-1)!] = \sum_{i=0}^{\infty}(x^i/i!) = f(x)$. By the uniqueness of the solution of the differential equation $f'(x) = f(x)$ (see Section 8.2), $f(x)$ must be ce^x for some c. Since $f(0) = 1$, c must be 1, and so $f(x) = e^x$. ▲

Differentiation and Integration of Power Series

To differentiate or integrate a power series within its radius of convergence R, differentiate or integrate it term by term: if $|x - x_0| < R$,

$$\frac{d}{dx}\sum_{i=0}^{\infty} a_i(x - x_0)^i = \sum_{i=1}^{\infty} ia_i(x - x_0)^{i-1},$$

$$\text{and} \quad \int\left[\sum_{i=0}^{\infty} a_i(x - x_0)^i\right]dx = \sum_{i=0}^{\infty} \frac{a_i}{i+1}(x - x_0)^{i+1} + C.$$

(The resulting series converge if $|x - x_0| < R$.)

Example 7 Let $f(x) = \sum_{i=0}^{\infty} \frac{i}{i+1} x^i$. Find a series expression for $f'(x)$. Where is it valid?

Solution By Example 1, $f(x)$ converges for $|x| < 1$. Thus $f'(x)$ also converges if $|x| < 1$, and we may differentiate term by term:

$$f'(x) = \sum_{i=0}^{\infty} \frac{i^2}{i+1} x^{i-1}, \qquad |x| < 1 \text{ (this series diverges at } x = \pm 1).$$

(Notice that $f'(x)$ is again a power series, so it too can be differentiated. Since this can be repeated, we conclude that f can be differentiated as many times as we please. We say that f is *infinitely differentiable*.) ▲

Example 8 Write down power series for $x/(1 + x^2)$ and $\ln(1 + x^2)$. Where do they converge?

Solution First, we expand $1/(1 + x^2)$ as a geometric series using the general formula $1/(1 - r) = 1 + r + r^2 + \cdots$, with r replaced by $-x^2$, obtaining $1 - x^2 + x^4 - \cdots$. Multiplying by x gives $x/(1 + x^2) = x - x^3 + x^5 - \cdots$, which converges for $|x| < 1$. (It diverges for $x = \pm 1$.)

Now we observe that $(d/dx)\ln(1 + x^2) = 2x/(1 + x^2)$, so

$$\begin{aligned}\ln(1 + x^2) &= 2\int \frac{x}{1 + x^2}\,dx = 2\int(x - x^3 + x^5 - \cdots)\,dx \\ &= 2\left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots\right) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots.\end{aligned}$$

(The integration constant was dropped because $\ln(1 + 0^2) = 0$.) This series converges for $|x| < 1$, and also for $x = \pm 1$, because there it is alternating. ▲

The operations of addition and multiplication by a constant may be performed term by term on power series, just as on polynomials. This may be proved using the limit theorems. The operations of multiplication and division proceed by the same methods one uses for polynomials, but are more subtle to justify. We state the results in the following box.

Algebraic Operations on Power Series

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, with radius of convergence R.

Let $g(x) = \sum_{i=0}^{\infty} b_i x^i$, with radius of convergence S.

If T is the smaller of R and S, then

$$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \qquad \text{for} \quad |x| < T;$$

$$cf(x) = \sum_{i=0}^{\infty} (ca_i) x^i \qquad \text{for} \quad |x| < R;$$

$$f(x)g(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i \qquad \text{for} \quad |x| < T.$$

If $b_0 \neq 0$, then $f(x)/g(x) = \sum_{i=0}^{\infty} c_i x^i$ for x near zero, where the c_i's may be determined by long division. The determination of the radius of convergence of f/g requires further analysis.

Example 9 Write down power series of the form $\sum_{i=0}^{\infty} a_i x^i$ for $2/(3-x)$, $5/(4-x)$, and $(23-7x)/[(3-x)(4-x)]$. What are their radii of convergence?

Solution We may write

$$\frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-x/3}\right) = \frac{2}{3}\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i = \sum_{i=0}^{\infty} \frac{2}{3^{i+1}} x^i.$$

The ratio of successive coefficients is $(1/3^{i+1})/(1/3^i) = 1/3$, so the radius of convergence is 3.

Similarly,

$$\frac{5}{4-x} = \sum_{i=0}^{\infty} \frac{5}{4^{i+1}} x^i$$

with radius of convergence 4. Finally, we may use partial fractions (Section 10.2) to write $(23-7x)/[(3-x)(4-x)] = 2/(3-x) + 5/(4-x)$, so we have

$$\frac{23-7x}{(3-x)(4-x)} = \sum_{i=0}^{\infty}\left(\frac{2}{3^{i+1}} + \frac{5}{4^{i+1}}\right)x^i.$$

By the preceding box, the radius of convergence of this series is at least 3. In fact, a limit computation shows that the ratio of successive coefficients approaches $\frac{1}{3}$, so the radius of convergence is exactly 3. ▲

In practice, we do not use the formula for $f(x)g(x)$ in the box above, but merely multiply the series for f and g term by term; in the product, we collect the terms involving each power of x.

Example 10 Write down the terms through x^4 in the series for $e^x/(1-x)$.

Solution We have $e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + \cdots$ (from Example 6) and $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$. We multiply terms in the first series by terms in the second series, in all possible ways.

	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\frac{x^4}{24}$	$\cdots$
1	1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\frac{x^4}{24}$	$\cdots$
x	x	x^2	$\frac{x^3}{2}$	$\frac{x^4}{6}$	$\cdots$	
x^2	x^2	x^3	$\frac{x^4}{2}$	$\cdots$		
x^3	x^3	x^4	$\cdots$			
x^4	x^4	$\cdots$				

(Since we want the product series only through x^4, we may neglect the terms in higher powers of x.) Reading along diagonals from lower left to upper right, we collect the powers of x to get

$$\begin{aligned}\frac{e^x}{1-x} &= 1 + (x+x) + \left(x^2 + x^2 + \frac{x^2}{2}\right) + \left(x^3 + x^3 + \frac{x^3}{2} + \frac{x^3}{6}\right)\\ &\quad + \left(x^4 + x^4 + \frac{x^4}{2} + \frac{x^4}{6} + \frac{x^4}{24}\right) + \cdots\\ &= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \cdots. \blacktriangle\end{aligned}$$

Exercises for Section 12.4

For which x do the series in Exercises 1–10 converge?

1. $\sum_{i=0}^{\infty} \frac{2}{i+1} x^i$
2. $\sum_{i=0}^{\infty} (2i+1)x^i$
3. $\sum_{n=1}^{\infty} \frac{3}{n^2} x^n$
4. $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n(n+1)} x^n$
5. $\sum_{i=1}^{\infty} \frac{5i+1}{i}(x-1)^i$
6. $\sum_{r=0}^{\infty} \frac{r!}{3^{2r}}(x+2)^r$
7. $\sum_{n=2}^{\infty} \frac{1}{n!\sin(\pi/n)} x^n$
8. $\sum_{i=14}^{\infty} \frac{i(i+3)}{i^3-4i+7} x^i$
9. $\sum_{n=1}^{\infty} \frac{x^n}{2^n+4^n}$
10. $\sum_{s=1}^{\infty} \left(\frac{2^s+1}{8s^7}\right)^{3/2} x^s$

Find the radius of convergence of the series in Exercises 11–14.

11. $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
12. $1 + \frac{x}{2} + \frac{2!}{4!}x^2 + \frac{3!}{6!}x^3 + \cdots$
13. $\frac{5x}{2} + \frac{10x^2}{4} + \frac{15x^3}{8} + \frac{20x^4}{16} + \cdots$
14. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$

Find the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n x^n$ in Exercises 15–18 for the given choices of a_n. Discuss convergence at $\pm R$.

15. $a_n = 1/(n+1)^n$
16. $a_n = (-1)^n/(n+1)$
17. $a_n = (n^2+n^3)/(1+n)^5$
18. $a_n = n$

Use the root test to determine the radius of convergence of the series in Exercises 19–22.

19. $\sum_{n=1}^{\infty} \frac{x^n}{(3+1/n)^n}$
20. $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^n}$
21. $\sum_{n=1}^{\infty} (-1)^n n^n x^n$
22. $\sum_{n=1}^{\infty} \frac{2x^n}{1+5^n}$
23. Let $f(x) = x - x^3/3! + x^5/5! - \cdots$. Show that f is defined and is differentiable for all x. Show that $f''(x) + f(x) = 0$. Use the uniqueness of solutions of this equation (Section 8.1) to show that $f(x) = \sin x$.
24. By differentiating the result of Exercise 23, find a series representation for $\cos x$.
25. Let $f(x) = \sum_{i=1}^{\infty}(i+1)x^i$.
    (a) Find the radius of convergence of this series.
    (b) Find the series for $\int_0^x f(t)\,dt$.
    (c) Use the result of part (b) to sum the series $f(x)$.
    (d) Sum the series $\frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$.
26. (a) Write a power series representing the integral of $1/(1-x)$ for $|x| < 1$. (b) Write a power series for $\ln x = \int(dx/x)$ in powers of $1-x$. Where is it valid?

Write power series representations for the functions in Exercises 27–30.

27. e^{-x^2}. (Use Example 6.)
28. $(d/dx)e^{-x^2}$

29. $\tan^{-1}x$ and its derivative. [*Hint:* Do the derivative first.]
30. The second derivative of $1/(1-x)$.
31. Find the series for $1/[(1-x)(2-x)]$ by writing
$$\frac{1}{(1-x)(2-x)}=\frac{A}{1-x}+\frac{B}{2-x}$$
and adding the resulting geometric series.
32. Find the series for $x/(x^2-4x+3)$. (See Exercise 31).
33. Using the result of Exercise 23, write the terms through x^6 in a power series expansion of $\sin^2 x$.
34. Find the terms through x^6 in the series for $\sin^3 x/x$.
35. Find series $f(x)$ and $g(x)$ such that the series $f(x)+g(x)$ is not identically zero but has a larger radius of convergence than either $f(x)$ or $g(x)$.
36. Find series $f(x)$ and $g(x)$, each of them having radius of convergence 2, such that $f(x)+g(x)$ has radius of convergence 3.
37. (a) By dividing the series for $\sin x$ by that for $\cos x$, find the terms through x^5 in the series for $\tan x$.
    (b) Find the terms through x^4 in the series for $\sec^2 x=(d/dx)\tan x$.
    (c) Using the result of part (b), find the terms through x^4 in the series for $1/\sec^2 x$.
38. Find the terms through x^5 in the series for
$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$
39. Find a power series which converges just when $-1<x\leqslant 1$.
40. Why can't $x^{1/3}$ be represented in the form of a series $\sum_{i=0}^{\infty}a_i x^i$, convergent near $x=0$?

Exercises 41–45 contain the proof of the results on the differentiation and integration of power series. For simplicity, we consider only the case $x_0=0$. Refer to the following theorem.

Theorem *Suppose that $\sum_{i=0}^{\infty}a_i x^i$ converges for some particular value of x, say $x=x_0$. Then:*

1. *There is an integer N such that $\sqrt[i]{|a_i|}<1/|x_0|$ for all $i\geqslant N$.*
2. *If $|y|<|x_0|$, then $\sum_{i=0}^{\infty}a_i y^i$ converges absolutely.*

Proof For part 1, suppose that $\sqrt[i]{|a_i|}\geqslant 1/|x_0|$ for arbitrarily large values of i. Then for these values of i we have $|a_i|\geqslant 1/|x_0|^i$, and $|a_i x_0^i|\geqslant 1$; but then we could not have $a_i x_0^i\to 0$, as is required for convergence.

For part 2, let $r=|y|/|x_0|$, so that $|r|<1$. By part 1, $|a_i y^i|=|a_i|\,|x_0|^i r^i<r^i$ for all $i\geqslant N$. By the comparison test, the series $\sum_{i=N}^{\infty}a_i y^i$ converges absolutely; it follows that the entire series converges absolutely as well. ■

★41. Prove that the series $f(x)=\sum_{i=0}^{\infty}a_i x^i$, $g(x)=\sum_{i=1}^{\infty}ia_i x^{i-1}$, and $h(x)=\sum_{i=0}^{\infty}[a_i/(i+1)]x^{i+1}$ all have the same radius of convergence. [*Hint:* Use the theorem above and the definition of radius of convergence on p. 587.]
★42. Prove that if $0<R_1<R$, where R is the radius of convergence of $f(x)=\sum_{i=0}^{\infty}a_i x^i$, then given any $\varepsilon>0$, there is a positive number M such that, for every number N greater than M, the difference $|f(x)-\sum_{i=0}^{N}a_i x^i|$ is less than ε for all x in the interval $[-R_1,R_1]$. [*Hint:* Compare $\sum_{i=N+1}^{\infty}a_i x^i$ with a geometric series, using the theorem above.]
★43. Prove that if $|x_0|<R$, where R is the radius of convergence of $f(x)=\sum_{i=0}^{\infty}a_i x^i$, then f is continuous at x_0. [*Hint:* Use Exercise 42, together with the fact that the polynomial $\sum_{i=0}^{N}a_i x^i$ is continuous. Given $\varepsilon>0$, write $f(x)-f(x_0)$ as a sum of terms, each of which is less than $\varepsilon/3$, by choosing N large enough and $|x-x_0|$ less than some δ.]
★44. Prove that if $|x|<R$, where R is the radius of convergence of $f(x)=\sum_{i=0}^{\infty}a_i x^i$, then the integral $\int_0^x f(t)\,dt$ (which exists by Exercise 43) is equal to $\sum_{i=0}^{\infty}[a_i/(i+1)]x^{i+1}$. [*Hint:* Use the result of Exercise 42 to show that the difference $|\int_0^x f(t)\,dt-\sum_{i=0}^{\infty}[a_i x^i/(i+1)]|$ is less than any positive number ε.]
★45. Prove that if $f(x)=\sum_{i=0}^{\infty}a_i x^i$ and $g(x)=\sum_{i=1}^{\infty}ia_i x^{i-1}$ have radius of convergence R, then $f'(x)=g(x)$ on $(-R,R)$. [*Hint:* Apply the result of Exercise 44 to $\int_0^x g(t)\,dt$; then use the alternative version of the fundamental theorem of calculus.]

12.5 Taylor's Formula

The power series which represents a function is determined by the derivatives of the function at a single point.

Up until now, we have used various makeshift methods to find power series expansions for specific functions. In this section, we shall see how to do this systematically. The idea is to assume the existence of a power series and to identify the coefficients one by one.

If $f(x) = \sum_{i=0}^{\infty} a_i(x - x_0)^i$ is convergent for $x - x_0$ small enough, we can find the coefficient a_0 simply by setting $x = x_0$: $f(x_0) = \sum_{i=0}^{\infty} a_i(x_0 - x_0)^i = a_0$. Differentiating and *then* substituting $x = x_0$, we can find a_1. Writing out the series explicitly will clarify the procedure:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots, \text{ so } f(x_0) = a_0;$$

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \cdots,$$
$$\text{so} \quad f'(x_0) = a_1.$$

Similarly, by taking more and more derivatives before we substitute, we find

$$\begin{aligned} f''(x) &= 2a_2 + 3 \cdot 2a_3(x - x_0) \\ &\qquad + 4 \cdot 3a_4(x - x_0)^2 + \cdots & \text{so} \quad & f''(x_0) = 2a_2; \\ f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x - x_0) + \cdots & \text{so} \quad & f'''(x_0) = 3 \cdot 2a_3; \\ f''''(x) &= 4 \cdot 3 \cdot 2a_4 + \cdots & \text{so} \quad & f''''(x_0) = 4 \cdot 3 \cdot 2a_4; \end{aligned}$$

etc.

Solving for the a_i's, we have $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = f''(x_0)/2$, $a_3 = f'''(x_0)/2 \cdot 3$, and, in general, $a_i = f^{(i)}(x_0)/i!$. Here $f^{(i)}$ denotes the ith derivative of f, and we recall that $i! = i \cdot (i-1) \cdots 3 \cdot 2 \cdot 1$, read "$i$ factorial." (We use the conventions that $f^{(0)} = f$ and $0! = 1$.)

This argument shows that if a function $f(x)$ can be written as a power series in $(x - x_0)$, then this series *must* be

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

For any f, this series is called the *Taylor series* of f about the point $x = x_0$. (This formula is responsible for the factorials which appear in so many important power series.)

The point x_0 is often chosen to be zero, in which case the series becomes

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

and is called the *Maclaurin*[3] *series* of f.

[3] Brook Taylor (1685–1731) and Colin Maclaurin (1698–1746) participated in the development of calculus following Newton and Leibniz. According to the *Guinness Book of World Records*, Maclaurin has the distinction of being the youngest full professor of all time at age 19 in 1717. He was recommended by Newton. Another mathematician-physicist, Lord Kelvin, holds the record for the youngest and fastest graduation from college—between October 1834 and November 1834, at age 10.

Taylor and Maclaurin Series

If f is infinitely differentiable on some interval containing x_0, the series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$$

is called the *Taylor series* of f at x_0.

When $x_0 = 0$, the series has the simpler form

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}x^i$$

and is called the *Maclaurin series* of f.

Example 1 Write down the Maclaurin series for $\sin x$.

Solution We have

$$\begin{aligned}
f(x) &= \sin x, & f(0) &= 0;\\
f'(x) &= \cos x, & f'(0) &= 1;\\
f''(x) &= -\sin x, & f''(0) &= 0;\\
f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1;\\
f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0;
\end{aligned}$$

and the pattern repeats from here on. Hence the Maclaurin series is

$$\frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \blacktriangle$$

Example 2 Find the terms through cubic order in the Taylor series for $1/(1+x^2)$ at $x_0 = 1$.

Solution *Method* 1. We differentiate $f(x)$ three times:

$$\begin{aligned}
f(x) &= \frac{1}{1+x^2}, & f(1) &= \frac{1}{2}, & a_0 &= f(1) = \frac{1}{2};\\
f'(x) &= \frac{-2x}{(1+x^2)^2}, & f'(1) &= -\frac{1}{2}, & a_1 &= f'(1) = -\frac{1}{2};\\
f''(x) &= \frac{6x^2-2}{(1+x^2)^3}, & f''(1) &= \frac{1}{2}, & a_2 &= \frac{f''(1)}{2!} = \frac{1}{4};\\
f'''(x) &= \frac{-24x^3+24x}{(1+x^2)^4}, & f'''(1) &= 0, & a_3 &= \frac{f'''(1)}{3!} = 0;
\end{aligned}$$

so the Taylor series begins

$$\frac{1}{1+x^2} = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 + 0\cdot(x-1)^3 + \cdots.$$

Method 2. Write

$$\frac{1}{1+x^2} = \frac{1}{1+\left[(x-1)+1\right]^2} = \frac{1}{2+2(x-1)+(x-1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{1+(x-1)+\frac{1}{2}(x-1)^2}\right]$$

$$=\frac{1}{2}\left[1-\left((x-1)+\frac{1}{2}(x-1)^2\right)\right.$$
$$\left.+\left((x-1)+\frac{(x-1)^2}{2}\right)^2-\left((x-1)+\frac{(x-1)^2}{2}\right)^3+\cdots\right] \quad \text{(geometric series)}$$
$$=\frac{1}{2}\left[1-(x-1)-\frac{1}{2}(x-1)^2+(x-1)^2+(x-1)^3-(x-1)^3+\cdots\right]$$
$$=\frac{1}{2}-\frac{1}{2}(x-1)+\frac{1}{4}(x-1)^2+0\cdot(x-1)^3+\cdots. \blacktriangle$$

Notice that we can write the Taylor series for any function which can be differentiated infinitely often, but we do not yet know whether the series converges to the given function. To understand when this convergence takes place, we proceed as follows. Using the fundamental theorem of calculus, write

$$f(x)=f(x_0)+\int_{x_0}^{x} f'(t)\,dt. \tag{1}$$

We now use integration by parts with $u=f'(t)$ and $v=x-t$. The result is

$$\int_{x_0}^{x} f'(t)\,dt=-\int_{x_0}^{x} u\,dv=-\left(uv\Big|_{x_0}^{x}-\int_{x_0}^{x} v\,du\right)$$
$$=f'(x_0)(x-x_0)+\int_{x_0}^{x}(x-t)f''(t)\,dt.$$

Thus we have proved the identity

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\int_{x_0}^{x}(x-t)f''(t)\,dt. \tag{2}$$

Note that the first two terms on the right-hand side of formula (2) equal the first two terms in the Taylor series of f. If we integrate by parts again with

$$u=f''(t) \quad \text{and} \quad v=\frac{(x-t)^2}{2},$$

we get

$$\int_{x_0}^{x}(x-t)f''(t)\,dt=-\int_{x_0}^{x} u\,dv=-uv\Big|_{x_0}^{x}+\int_{x_0}^{x} v\,du$$
$$=\frac{f''(x_0)}{2}(x-x_0)^2+\int_{x_0}^{x}\frac{(x-t)^2}{2}f'''(t)\,dt;$$

so, substituting into formula (2),

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\int_{x_0}^{x}\frac{(x-t)^2}{2}f'''(t)\,dt. \tag{3}$$

Repeating the procedure n times, we obtain the formula

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\cdots$$
$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\int_{x_0}^{x}\frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt \tag{4}$$

which is called *Taylor's formula with remainder in integral form.* The expression

$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \tag{5}$$

is called the *remainder*, and formula (4) may be written in the form

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + R_n(x). \tag{6}$$

By the second mean value theorem of integral calculus (Review Exercise 40, Chapter 9), we can write

$$R_n(x) = f^{(n+1)}(c)\left[\int_{x_0}^{x} \frac{(x-t)^n}{n!}\, dt\right] = f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)!} \tag{7}$$

for some point c between x_0 and x. Substituting formula (7) into formula (6), we have

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \tag{8}$$

Formula (8), which is called *Taylor's formula with remainder in derivative form,* reduces to the usual mean value theorem when we take $n=0$; that is,

$$f(x) = f(x_0) + f'(c)(x-x_0)$$

for some c between x_0 and x.

If $R_n(x) \to 0$ as $n \to \infty$, then formula (6) tells us that the Taylor series of f will converge to f.

The following box summarizes our discussion of Taylor series.

Convergence of Taylor Series

1. If $f(x) = \sum_{i=0}^{\infty} a_i(x-x_0)^i$ is a convergent power series on an open interval I centered at x_0, then f is infinitely differentiable and $a_i = f^{(i)}(x_0)/i!$, so

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i.$$

2. If f is infinitely differentiable on an open interval I centered at x_0, and if $R_n(x) \to 0$ as $n \to \infty$ for x in I, where $R_n(x)$ is defined by formula (5), then the Taylor series of f converges on I and equals f:

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i.$$

Example 3 (a) Expand the function $f(x) = 1/(1+x^2)$ in a Maclaurin series.
(b) Use part (a) to find $f'''''(0)$ and $f''''''(0)$ without calculating derivatives of f directly.
(c) Integrate the series in part (a) to prove that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for } |x| < 1.$$

★(d) Justify the formula of Euler:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Solution (a) We expand $1/(1 + x^2)$ as a geometric series:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$
$$= 1 - x^2 + x^4 - x^6 + \cdots$$

which is valid if $|-x^2| < 1$; that is, if $|x| < 1$. By the box above this is the Maclaurin series of $f(x) = 1/(1 + x^2)$.

(b) We find that $f'''''(0)/5!$ is the coefficient of x^5. Hence, as this coefficient is zero, $f'''''(0) = 0$. Likewise, $f''''''(0)/6!$ is the coefficient of x^6; thus $f''''''(0) = -6!$. This is *much* easier than calculating the sixth derivative of $f(x)$.

(c) Integrating from zero to x (justified in Section 12.4) gives

$$\int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots;$$

but we know that the integral of $1/(1 + t^2)$ is $\tan^{-1}t$, so

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for} \quad |x| < 1.$$

(d) If we set $x = 1$ and use $\tan^{-1}1 = \pi/4$, we get Euler's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots;$$

but this is not quite justified, since the series for $\tan^{-1}x$ is valid only for $|x| < 1$. (It is plausible, though, since $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, being an alternating series, converges.) To justify Euler's formula, we may use the finite form of the geometric series expansion:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 + \cdots + (-1)^n t^{2n} + (-1)^{n+1}\frac{t^{2n+2}}{1+t^2}.$$

Integrating from 0 to 1, we have

$$\frac{\pi}{4} = \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + (-1)^{n+1}\int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt.$$

We will be finished if we can show that the last term goes to zero as $n \to \infty$. We have

$$0 \leqslant \int_0^1 \frac{t^{2n+2}}{1+t^2}\,dt \leqslant \int_0^1 t^{2n+2}\,dt = \frac{1}{2n+3}.$$

Since $\lim_{n\to\infty}[1/(2n+3)] = 0$, the limit of

$$(-1)^{n+1}\int_0^1 \frac{t^{n+2}}{1+t^2}\,dt$$

is zero as well (by the comparison test on p. 543). ▲

There is a simple test which guarantees that the remainder of a Taylor series tends to zero.

Taylor Series Test

To prove that a function $f(x)$ equals its Taylor series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i \qquad \text{on} \quad I,$$

it is sufficient to show:

1. f is infinitely differentiable on I;
2. the derivatives of f grow no faster than a constant C times the powers of a constant M; that is, for x in I,

$$|f^{(n)}(x)| \leqslant CM^n, \qquad n = 0, 1, 2, 3, \ldots .$$

To justify this, we must show that $R_n(x) \to 0$. By formula (7),

$$|R_n(x)| = \left| f^{(n+1)}(c)\frac{(x - x_0)^{n+1}}{(n+1)!} \right| \leqslant \frac{CM^{n+1}|x - x_0|^{n+1}}{(n+1)!} .$$

For any number b, however, $b^n/n! \to 0$, since $\sum_{i=0}^{\infty}(b^i/i!)$ converges by Example 7, Section 12.3. Choosing $b = M|x - x_0|$, we can conclude that $R_n(x) \to 0$, so the Taylor series converges to f.

Example 4 Prove that:

(a) $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \cdots$ for all x.

(b) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ for x in $(-\infty, \infty)$.

(c) $1 = \dfrac{\pi}{2} - \dfrac{\pi^3}{2^3 \cdot 3!} + \dfrac{\pi^5}{2^5 \cdot 5!} - \dfrac{\pi^7}{2^7 \cdot 7!} + \cdots$.

Solution (a) Let $f(x) = e^x$. Since $f^{(n)}(x) = e^x$, f is infinitely differentiable. Since all the derivatives at $x_0 = 0$ are 1, the Maclaurin series of e^x is $\sum_{n=0}^{\infty}(x^n/n!)$. To establish equality, it suffices to show $|f^{(n)}(x)| \leqslant CM^n$ on any finite interval I; but $f^{(n)}(x) = e^x$, independent of n, so in fact we can choose $M = 1$ and C the maximum of e^x on I.

(b) Since $f'(x) = \cos x$, $f''(x) = -\sin x, \ldots,$ we see that f is infinitely differentiable. Notice that $f^{(n)}(x)$ is $\pm\cos x$ or $\pm\sin x$, so $|f^{(n)}(x)| \leqslant 1$. Thus we can choose $C = 1$, $M = 1$. Hence $\sin x$ equals its Maclaurin series, which was shown in Example 1 to be $x - x^3/3! + x^5/5! - \cdots$.

(c) Let $x = \pi/2$ in part (b). ▲

Some discussion of the limitations of Taylor series is in order. Consider, for example, the function $f(x) = 1/(1 + x^2)$, whose Maclaurin series is $1 - x^2 + x^4 - x^6 + \cdots$. Even though the function f is infinitely differentiable on the whole real line, its Maclaurin series converges only for $|x| < 1$. If we wish to represent $f(x)$ for x near 1 by a series, we may use a Taylor series with $x_0 = 1$ (see Example 2).

Another instructive example is the function $g(x) = e^{-1/x^2}$, where $g(0) = 0$. This function is infinitely differentiable, but all of its derivatives at $x = 0$ are equal to zero (see Review Exercise 123). Thus the Maclaurin series of g is $\sum_{i=0}^{\infty} 0 \cdot x^i$, which converges (it is zero) for all x, but not to the function g. There also exist infinitely differentiable functions with Taylor series having radius of convergence zero.[4] In each of these examples, the hypothesis that $R_n(x) \to 0$ as $n \to \infty$ fails, so the assertion in the box above is not contradicted. It simply does not apply. (Functions which satisfy $R_n(x) \to 0$, and so equal their Taylor series for x close to x_0, are important objects of study; these functions are called *analytic*).

The following box contains the most basic series expansions. They are worth memorizing.

Some Important Taylor and Maclaurin Series

Geometric: $$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i, \qquad R = 1.$$

Binomial: $$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$
$$= \sum_{i=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}x^i, \qquad R = 1.$$

Sine: $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}, \qquad R = \infty.$$

Cosine: $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \qquad R = \infty.$$

Exponential: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \qquad R = \infty.$$

Logarithm: $$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$
$$= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(x-1)^i}{i}, \qquad R = 1.$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \qquad R = 1.$$

The only formula in the box which has not yet been justified is the binomial series. It may be proved by evaluating the derivatives of $f(x) = (1+x)^{\alpha}$ at $x = 0$ and verifying convergence by the *method* of the test in the box entitled Taylor series test. (See Review Exercise 124.) If $\alpha = n$ is a positive integer, the series terminates and we get the binomial formula

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

is the number of ways of choosing k objects from a collection of n objects.

[4] See B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Holden-Day, San Francisco (1964), p. 68.

Example 5 Expand $\sqrt{1+x^2}$ about $x_0 = 0$.

Solution The binomial series, with $\alpha = \frac{1}{2}$ and x^2 in place of x, gives

$$(1+x^2)^{1/2} = 1 + \tfrac{1}{2}x^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}x^4 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \cdots$$
$$= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots, \text{ valid for } |x| < 1. \blacktriangle$$

Taylor's formula with remainder,

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + R_n(x)$$

can be used to obtain approximations to $f(x)$; we can estimate the accuracy of these approximations using the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

(for some c between x and x_0) and estimating $f^{(n+1)}$ on the interval between x and x_0. The partial sum of the Taylor series,

$$\sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i$$

is a polynomial of degree n in x called the nth *Taylor (or Maclaurin* if $x_0 = 0$) *polynomial for f at x_0*, or the nth-order approximation to f at x_0. The first Taylor polynomial,

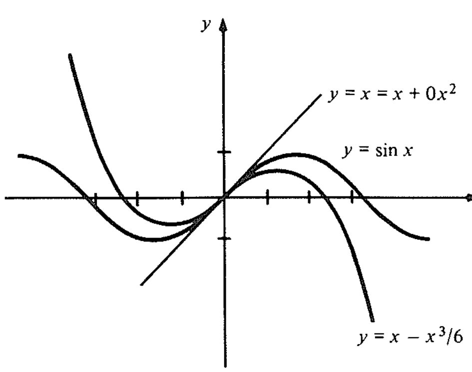
$$f(x_0) + f'(x_0)(x-x_0)$$

is just the *linear approximation* to $f(x)$ at x_0; the formula for the remainder $R_1(x) = [f''(c)/2](x-x_0)^2$ shows that we can estimate the error in the first-order approximation in terms of the size of the second derivative f'' on the interval between x and x_0.

A useful consequence of Taylor's theorem is that for many functions we can improve upon the linear approximation by using Taylor polynomials of higher order.

Example 6 Sketch the graph of $\sin x$ along with the graphs of its Maclaurin polynomials of degree 1, 2, and 3. Evaluate the polynomials at $x = 0.02$, 0.2, and 2, and compare with the exact value of $\sin x$.

Solution The Maclaurin polynomials of order 1, 2, and 3 are x, $x + 0x^2$, and $x - x^3/6$. They are sketched in Fig. 12.5.1. Evaluating at $x = 0.02$, 0.2, 2, and 20 gives the results shown in the table below.



x	$x - x^3/6$	$\sin x$
0.02	0.0199986667	0.0199986667
0.2	0.1986666	0.1986693
2	0.666666	0.909
20	−1313	0.912

$\blacktriangle$

Figure 12.5.1. The first- and third-order approximations to $\sin x$.

The Maclaurin polynomials through degree 71 for $\sin x$ are shown in Fig. 12.5.2.[5] Notice that as n increases, the interval on which the nth Taylor polynomial is a good approximation to $\sin x$ becomes larger and larger; if we go beyond this interval, however, the polynomials of higher degree "blow up" more quickly than the lower ones.

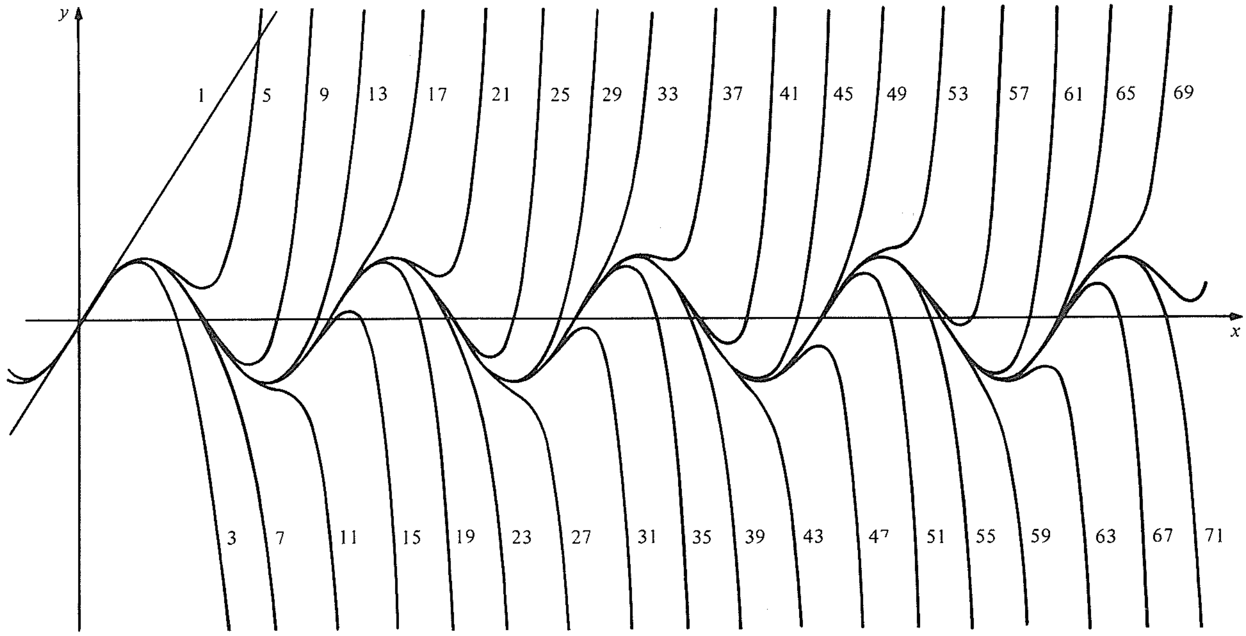


Figure 12.5.2. The Maclaurin polynomials for $\sin x$ through order 71. (The graphs to the left of the y axis are obtained by rotating the figure through 180°.)

The following example shows how errors may be estimated.

Example 7 Write down the Taylor polynomials of degrees 1 and 2 for $\sqrt[3]{x}$ at $x_0 = 27$. Use these polynomials to approximate $\sqrt[3]{28}$, and estimate the error in the second-order approximation by using the formula for $R_2(x)$.

Solution Let $f(x) = x^{1/3}$, $x_0 = 27$, $x = 28$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, and $f'''(x) = \frac{10}{27}x^{-8/3}$. Thus $f(27) = 3$, $f'(27) = \frac{1}{27}$, and $f''(27) = -(2/3^7)$, so the Taylor polynomials of degree 1 and 2 are, respectively,

$$3 + \frac{1}{27}(x - 27) \quad \text{and} \quad 3 + \frac{1}{27}(x - 27) - \frac{1}{3^7}(x - 27)^2.$$

Evaluating these at $x = 28$ gives 3.0370 . . . and 3.0365798 . . . for the first- and second-order approximations. The error in the second-order approximation is at most $1/3!$ times the largest value of $(10/27)x^{-8/3}$ on $[27, 28]$, which is

$$\frac{1}{6}\,\frac{10}{27}\,\frac{1}{3^8} = \frac{5}{3^{12}} \leqslant 0.00001. \qquad (\text{Actually, } \sqrt[3]{28} = 3.0365889\ldots.) \ \blacktriangle$$

[5] We thank H. Ferguson for providing us with this computer-generated figure.

Example 8 By integrating a series for e^{-x^2}, calculate $\int_0^1 e^{-x^2}\,dx$ to within 0.001.

Solution Substituting $-x^2$ for x in the series for e^x gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots .$$

Integrating term by term gives

$$\int_0^x e^{-t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots ,$$

and so

$$\int_0^1 e^{-x^2}\,dx = 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \tfrac{1}{1320} + \cdots .$$

This is an alternating series, so the error is no greater than the first omitted term. To have accuracy 0.001, we should include $\frac{1}{216}$. Thus, within 0.001,

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.747.$$

This method has an advantage over the methods in Section 11.5: to increase accuracy, we need only add on another term. Rules like Simpson's, on the other hand, require us to start over. (See Review Exercise 84 for Chapter 11.) Of course, if we have numerical data, or a function with an unknown or complicated series, using Simpson's rule may be necessary. ▲

Example 9 Calculate $\sin(\pi/4 + 0.06)$ to within 0.0001 by using the Taylor series about $x_0 = \pi/4$. How many terms would have been necessary if you had used the Maclaurin series?

Solution With $f(x) = \sin x$, and $x_0 = \pi/4$, we have

$$\begin{aligned}
f(x) &= \sin x, & f(x_0) &= \frac{1}{\sqrt{2}}\,;\\
f'(x) &= \cos x, & f'(x_0) &= \frac{1}{\sqrt{2}}\,;\\
f''(x) &= -\sin x, & f''(x_0) &= -\frac{1}{\sqrt{2}}\,;\\
f'''(x) &= -\cos x, & f'''(x_0) &= -\frac{1}{\sqrt{2}}\,;\\
f''''(x) &= \sin x, & f''''(x_0) &= \frac{1}{\sqrt{2}}\,;
\end{aligned}$$

and so on. We have

$$R_n(x) = \frac{f^{(n+1)}(c)(x - x_0)^{n+1}}{(n+1)!}$$

for c between $\pi/4$ and $\pi/4 + 0.06$. Since $f^{(n+1)}(c)$ has absolute value less than 1, we have $|R_n(x)| \leqslant (0.06)^{n+1}/(n+1)!$. To make $|R_n(x)|$ less than 0.0001, it suffices to choose $n = 2$. The second-order approximation to $\sin x$ is

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2.$$

Evaluating at $x = \pi/4 + 0.06$ gives 0.7483.

If we had used the Maclaurin polynomial of degree n, the error estimate would have been $|R_n(x)| \leqslant (\pi/4 + 0.06)^{n+1}/(n+1)!$. To make $|R_n(x)|$ less than 0.0001 would have required $n = 6$. ▲

Finally, we show how Taylor series can be used to evaluate limits in indeterminate form. The method illustrated below is sometimes more efficient than l'Hôpital's rule when that rule must be applied several times.

Example 10 Evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ using a Maclaurin series.

Solution Since $\sin x = x - x^3/3! + x^5/5! - \cdots$, $\sin x - x = -x^3/3! + x^5/5! - \cdots$, and so $(\sin x - x)/x^3 = -1/6 + x^2/5! - \cdots$. Since this power series converges, it is continuous at $x = 0$, and so

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}. \blacktriangle$$

Example 11 Use Taylor series to evaluate

$$\text{(a)} \quad \lim_{x\to 0} \frac{\sin x - x}{\tan x - x}$$

(compare Example 4, Section 11.2) and

$$\text{(b)} \quad \lim_{x\to 1} \frac{\ln x}{e^x - e}.$$

Solution (a)

$$\frac{\sin x - x}{\tan x - x} = \frac{(\sin x)(\cos x) - x\cos x}{\sin x - x\cos x}$$

$$= \frac{(x - x^3/6 + \cdots)(1 - x^2/2 + \cdots) - x(1 - x^2/2 + \cdots)}{(x - x^3/6 + \cdots) - x(1 - x^2/2 + \cdots)}$$

$$= \frac{x - x^3/2 - x^3/6 + \cdots - x + x^3/2 + \cdots}{x - x^3/6 + \cdots - x + x^3/2 + \cdots}$$

$$= \frac{-x^3/6 + \cdots}{(1/3)x^3 + \cdots} = \frac{-1/6 + \cdots}{1/3 + \cdots} \qquad \text{(dividing by } x^3\text{)}.$$

Since the terms denoted "$+ \cdots$" tend to zero as $x \to 0$, we get

$$\lim_{x\to 0} \frac{\sin x - x}{\tan x - x} = -\frac{1/6}{1/3} = -\frac{1}{2}.$$

(b)

$$\lim_{x\to 1} \frac{\ln x}{e^x - e} = \lim_{x\to 1} \frac{\ln x}{e(e^{x-1} - 1)}$$

$$= \frac{1}{e} \lim_{x\to 1} \frac{(x-1) - (1/2)(x-1)^2 + \cdots}{1 + (x-1) + (1/2)(x-1)^2 + \cdots - 1}$$

$$= \frac{1}{e} \lim_{x\to 1} \frac{1 - (1/2)(x-1) + \cdots}{1 + (1/2)(x-1) + \cdots}$$

$$= \frac{1}{e} \cdot \frac{1}{1} = \frac{1}{e}. \blacktriangle$$

For the last example, l'Hôpital's rule would have been a little easier to use.

Exercises for Section 12.5

Write down the Maclaurin series for the functions in Exercises 1–4.

1. $\sin 3x$
2. $\cos 4x$
3. $\cos x + e^{-2x}$
4. $\sin 2x - e^{-4x}$

Find the terms through x^3 in the Taylor series at $x_0 = 1$ for the functions in Exercises 5–8.

5. $1/(1 + x^2 + x^4)$
6. $1/\sqrt{2 - x^2}$
7. e^x
8. $\tan(\pi x/4)$

9. (a) Expand $f(x) = 1/(1 + x^2 + x^4)$ in a Maclaurin series through the terms in x^6, using a geometric series. (b) Use (a) to calculate $f''''''(0)$.
10. Expand $g(x) = e^{x^2}$ in a Maclaurin series as far as necessary to calculate $g^{(8)}(0)$ and $g^{(9)}(0)$.

Establish the equalities in Exercises 11–14 for a suitable domain in x.

11. $\ln(1 + x) = x - x^2/2 + x^3/3 - \cdots$.
12. $e^{1+x} = e + ex + \dfrac{ex^2}{2!} + \dfrac{ex^3}{3!} + \cdots$.
13. $\sqrt{x} = 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3 - \cdots$.
14. $$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{(x - \pi/4)^2}{2!} - \frac{(x - \pi/4)^3}{3!} + \cdots\right].$$
15. (a) Write out the Maclaurin series for the function $1/\sqrt{1 + x^2}$.(Use the binomial series.) (b) What is $(d^{20}/dx^{20})(1/\sqrt{1 + x^2})|_{x=0}$?
16. (a) Using the binomial series, write out the Maclaurin series for $g(x) = \sqrt{1 + x} + \sqrt{1 - x}$. (b) Find $g^{(20)}(0)$ and $g^{(2001)}(0)$.
17. Sketch the graphs of the Maclaurin polynomials through degree 4 for $\cos x$.
18. Sketch the graphs of the Maclaurin polynomials through degree 4 for $\tan x$.
19. Calculate $\ln(1.1)$ to within 0.001 by using a power series.
20. Calculate $e^{\ln 2 + 0.02}$ to within 0.0001 using a Taylor series about $x_0 = \ln 2$. How many terms would have been necessary if you had used the Maclaurin series?
21. Use the power series for $\ln(1 + x)$ to calculate $\ln 2\frac{1}{2}$, correct to within 0.1. [*Hint:* $2\frac{1}{2} = \frac{3}{2}\cdot\frac{5}{3}$.]
22. Continue the work of Example 7 by finding the third-, fourth-, fifth- (and so on) order approximations to $\sqrt[3]{28}$. Stop when the round-off errors on your calculator become greater than the remainder of the series.
23. Using the Maclaurin expansion for $1/(1 + x)$, approximate $\int_0^{1/2}[dx/(1 + x)]$ to within 0.01.
24. Use a binomial expansion to approximate $\int_0^{1/4}\sqrt{1 + x^3}\,dx$ to within 0.01.
25. (a) Use the second-order approximation at x_0 to derive the approximation
$$\int_{x_0 - R}^{x_0 + R} f(x)\,dx \approx 2Rf(x_0) + \frac{2f''(x_0)}{3!}R^3.$$
Find an estimate for the error.
(b) Using the formula given in part (a), find an approximate value for $\int_{-1/2}^{1/2}(dx/\sqrt{1 + x^2})$. Compare the answer with that obtained from Simpson's rule with $n = 4$.
26. (a) Can we use the binomial expansion of $\sqrt{1 + x}$ to obtain a convergent series for $\sqrt{2}$. Why or why not?
(b) Writing $2 = \frac{9}{4}\cdot\frac{8}{9}$, we have $\sqrt{2} = \frac{3}{2}\sqrt{8/9}$. Use this equation, together with the binomial expansion, to obtain an approximation to $\sqrt{2}$ correct to two decimal places.
(c) Use the method of part (b) to obtain an approximation to $\sqrt{3}$ correct to two decimal places.

Evaluate the limits in Exercises 27–30 using Maclaurin series.

27. $\lim_{x\to 0}\dfrac{\sin 2x - 2x}{x^3}$
28. $\lim_{x\to 0}\dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$
29. $\lim_{x\to 0}\left(\dfrac{1}{x\sin x} - \dfrac{1}{x^2}\right)$ (use a common denominator).
30. $\lim_{x\to\pi}\dfrac{1 + \cos x}{(x - \pi)^2}$

Expand each of the functions in Exercises 31–36 as a Maclaurin series and determine for what x it is valid.

31. $\dfrac{1}{1 - x}$
32. $\dfrac{1}{1 + x}$
33. $\dfrac{1}{1 - x} - \dfrac{1}{1 + x}$
34. $\dfrac{1}{2}\left(\dfrac{1}{1 - x} + \dfrac{1}{1 + x}\right)$
35. $\dfrac{1}{1 - x^2}$
36. $\dfrac{1}{1 - x^2} - \dfrac{1}{1 + x}$

37. Find the Maclaurin series for $f(x) = (1 + x^2)^2$ in two ways:
(a) by multiplying out the polynomial;
(b) by taking successive derivatives and evaluating them at $x = 0$ (without multiplying out).
38. Write down the Taylor series for $\ln x$ at $x_0 = 2$.
39. Find a power series expansion for $\int_1^x \ln t\,dt$. Compare this with the expansion for $x\ln x$. What is your conclusion?
40. Using the Taylor series for $\sin x$ and $\cos x$, find the terms through x^6 in the series for $(\sin x)^2 + (\cos x)^2$.

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Find a_0, a_1, a_2, and a_3 for each of the functions in Exercises 41–44.

41. $\sec x$
42. $\sqrt{1-x^2}$
43. $(d/dx)\sqrt{1-x^2}$
44. e^{1+x}

Find Maclaurin expansions through the term in x^5 for each of the functions in Exercises 45–48.

45. $(1-\cos x)/x^2$
46. $\dfrac{x - \sin 3x}{x^3}$
47. $\dfrac{1-x}{1+x}$
48. $\dfrac{d^2}{dx^2}\dfrac{1}{\sqrt{1+x^2}}$

49. Find the Taylor polynomial of degree 4 for $\ln x$ at: (a) $x_0 = 1$; (b) $x_0 = e$; (c) $x_0 = 2$.

50. (a) Find a power series expansion for a function $f(x)$ such that $f(0) = 0$ and $f'(x) - f(x) = x$. (Write $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ and solve for the a_i's one after another.) (b) Find a formula for the function whose series you found in part (a).

Find the first four nonvanishing terms in the power series expansion for the functions in Exercises 51–54.

51. $\ln(1+e^x)$
52. e^{x^2+x}
53. $\sin(e^x)$
54. $e^x\cos x$

55. An engineer is about to compute sin(36°), when the batteries in her hand calculator give out. She quickly grabs a backup unit, only to find it is made for statistics and does not have a "sin" key. Unperturbed, she enters 3.1415926, divides by 5, and enters the result into the memory, called "x" hereafter. Then she computes $x(1 - x^2/6)$ and uses it for the value of sin(36°).
(a) What was her answer?
(b) How good was it?
(c) Explain what she did in the language of Taylor series expansions.
(d) Describe a similar method for computing tan(10°).

56. An automobile travels on a straight highway. At noon it is 20 miles from the next town, travelling at 50 miles per hour, with its acceleration kept between 20 miles per hour per hour and -10 miles per hour per hour. Use the formula $x(t) = x(0) + x'(0)t + \int_0^t (t-s)x''(s)\,ds$ to estimate the auto's distance from the town 15 minutes later.

★57. (a) Let
$$f(x) = \begin{cases} (\sin x)/x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$
Find $f'(0)$, $f''(0)$, and $f'''(0)$.
(b) Find the Maclaurin expansion for $(\sin x)/x$.

★58. Using Taylor's formula, prove the following inequalities:
(a) $e^x - 1 \geqslant x$ for $x \geqslant 0$.
(b) $6x - x^3 + x^5/20 \geqslant 6\sin x \geqslant 6x - x^3$ for $x \geqslant 0$.
(c) $x^2 - x^4/12 \leqslant 2 - 2\cos x \leqslant x^2$ for $x \geqslant 0$.

★59. Prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

★60. (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1}x$. Where do they converge?
(b) Find the terms through x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ in the series for $\sin^{-1}x$; that is, if $\sin^{-1}x = a_0 + a_1x + a_2x^2 + \cdots$, then
$$\sin^{-1}(\sin x) = a_0 + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + a_2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \cdots.$$
(c) Use the substitution method of part (b) to *obtain* the first five terms of the series for $\sin^{-1}x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 through a_5.
(d) Find the terms through x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. (Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g.)

12.6 Complex Numbers

Complex numbers provide a square root for -1.

This section is a brief introduction to the algebra and geometry of complex numbers; i.e., numbers of the form $a+b\sqrt{-1}$. We show the utility of complex numbers by comparing the series expansions for $\sin x$, $\cos x$, and e^x derived in the preceding section. This leads directly to Euler's formula relating the numbers 0, 1, e, π, and $\sqrt{-1}$: $e^{\pi\sqrt{-1}}+1=0$. Applications of complex numbers to second-order differential equations are given in the next section. Section 12.8, on series solutions, can, however, be read before this one.

If we compare the three power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \tag{1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{3}$$

it looks as if $\sin x$ and $\cos x$ are almost the "odd and even parts" of e^x. If we write the series

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \tag{4}$$

subtract equation (4) from equation (3) and divide by 2, we get

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots. \tag{5}$$

Similarly, adding equations (3) and (4) and dividing by 2, gives

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots. \tag{6}$$

These are the Maclaurin series of the hyperbolic functions $\sinh x$ and $\cosh x$; they are just missing the alternating signs in the series for $\sin x$ and $\cos x$.

Can we get the right signs by an appropriate substitution other than changing x to $-x$? Let us try changing x to ax, where a is some constant. We have, for example,

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2} = 1 + a^2\frac{x^2}{2!} + a^4\frac{x^4}{4!} + a^6\frac{x^6}{6!} + \cdots.$$

This would become the series for $\cos x$ if we had $a^2 = a^6 = a^{10} = \cdots = -1$ and $a^4 = a^8 = a^{12} = \cdots = 1$. In fact, all these equations would follow from the one relation $a^2 = -1$.

We know that the square of any real number is positive, so that the equation $a^2 = -1$ has no real solutions. Nevertheless, let us pretend that there is a solution, which we will denote by the letter i, for "imaginary." Then we would have $\cosh ix = \cos x$.

Example 1 What is the relation between $\sinh ix$ and $\sin x$?

Solution Since $i^2 = -1$, we have $i^3 = -i$, $i^4 = (-i)\cdot i = 1$, $i^5 = i$, $i^6 = -1$, etc., so substituting ix for x in (5) gives

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{2} = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots.$$

Comparing this with equation (1), we find that $\sinh ix = i \sin x$. ▲

The sum of the two series (5) and (6) is the series (3), i.e., $e^x = \cosh x + \sinh x$. Substituting ix for x, we find

$$e^{ix} = \cosh ix + \sinh ix$$

or

$$e^{ix} = \cos x + i \sin x. \tag{7}$$

Formula (7) is called *Euler's formula*. Substituting π for x, we find that

$$e^{i\pi} = -1,$$

and adding 1 to both sides gives

$$e^{i\pi} + 1 = 0, \tag{8}$$

a formula composed of seven of the most important symbols in mathematics: 0, 1, $+$, $=$, e, i, and π.

Example 2 Using formula (7), express the sine and cosine functions in terms of exponentials.

Solution Substituting $-x$ for x in equation (7) and using the symmetry properties of cosine and sine, we obtain

$$e^{-ix} = \cos x - i \sin x.$$

Adding this equation to (7) and dividing by 2 gives

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

while subtracting the equations and dividing by $2i$ gives

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \ ▲$$

Example 3 Find $e^{i(\pi/2)}$ and $e^{2\pi i}$.

Solution Using formula (7), we have

$$e^{i(\pi/2)} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

and

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1. \ ▲$$

Since there is no real number having the property $i^2 = -1$, all of the calculations above belong so far to mathematical "science fiction." In the following paragraphs, we will see how to construct a number system in which -1 does have a square root; in this new system, all the calculations which we have done above will be completely justified.

When they were first introduced, square roots of negative numbers were deemed merely to be symbols on paper with no real existence (whatever that means) and therefore "imaginary." These imaginary numbers were not taken seriously until the cubic and quartic equations were solved in the sixteenth century (in the formula in the Supplement to Section 3.4 for the roots of a cubic equation, the symbol $\sqrt{-3}$ appears and must be contended with, even if all the roots of the equation are real.) A proper way to define square roots of negative numbers was finally obtained through the work of Girolamo Cardano around 1545 and Bombelli in 1572, but it was only with the work of L. Euler, around 1747, that their importance was realized. A way to understand imaginaries in terms of real numbers was discovered by Wallis, Wessel, Argand, Gauss, Hamilton, and others in the early nineteenth century.

To define a number system which contains $i = \sqrt{-1}$, we note that such a system ought to contain all expressions of the form $a + b\sqrt{-1} = a + bi$, where a and b are ordinary real numbers. Such expressions should obey the laws

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Thus the sum and product of two of these expressions are expressions of the same type.

All the data in the "number" $a + bi$ is carried by the *pair* (a,b) of real numbers, which may be considered a point in the xy plane. Thus we define our new number system, the complex numbers, by imposing the desired operations on pairs of real numbers.

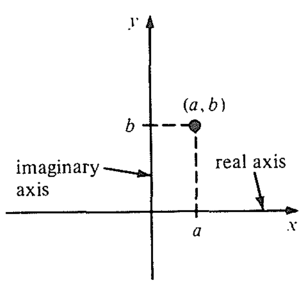


Figure 12.6.1. A complex number is just a point (a,b) in the plane.

Complex Numbers

A *complex number* is a point (a,b) in the xy plane. Complex numbers are added and multiplied as follows:

$$(a,b) + (c,d) = (a + c, b + d),$$

$$(a,b)(c,d) = (ac - bd, ad + bc).$$

The point $(0,1)$ is denoted by the symbol i, so that $i^2 = (-1,0)$ (using $a = 0$, $c = 0$, $b = 1$, $d = 1$ in the definition of muliplication). The x axis is called the *real axis* and the y axis is the *imaginary axis*. (See Fig. 12.6.1.)

It is convenient to denote the point $(a,0)$ just by a since we are thinking of points on the real axis as ordinary real numbers. Thus, in this notation, $i^2 = -1$. Also,

$$(a,b) = (a,0) + (0,b) = (a,0) + (b,0)(0,1)$$

as is seen from the definition of multiplication. Replacing $(a,0)$ and $(b,0)$ by a and b, and $(0,1)$ by i, we see that

$$(a,b) = a + bi.$$

Since two points in the plane are equal if and only if their coordinates are equal, we see that

$$a + ib = c + id \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d.$$

Thus, if $a + ib = 0$, both a and b must be zero.

We now see that sense can indeed be made of the symbol $a + ib$, where $i^2 = -1$. The notation $a + ib$ is much easier to work with than ordered pairs, so we now revert to the old notation $a + ib$ and dispense with ordered pairs in our calculations. However, the geometric picture of plotting $a + ib$ as the point (a,b) in the plane is very useful and will be retained.

It can be verified, although we shall not do it, that the usual laws of algebra hold for complex numbers. For example, if we denote complex numbers by single letters such as $z = a + ib$, $w = c + id$, and $u = e + if$, we have

$$z(w+u) = zw + zu,$$

$$z(wu) = (zw)u,$$

etc.

Example 4 (a) Plot the complex number $8 - 6i$. (b) Simplify $(3 + 4i)(8 + 2i)$. (c) Factor $x^2 + x + 3$. (d) Find $\sqrt{i}$.

Solution (a) $8 - 6i$ corresponds to the point $(8, -6)$, plotted in Fig. 12.6.2.

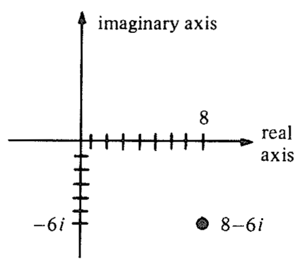


Figure 12.6.2. The point $8-6i$ plotted in the xy plane.

(b)
$$\begin{aligned}(3+4i)(8+2i) &= 3\cdot 8 + 3\cdot 2i + 4\cdot 8i + 2\cdot 4i^2\\ &= 24 + 6i + 32i - 8\\ &= 16 + 38i.\end{aligned}$$

(c) By the quadratic formula, the roots of $x^2 + x + 3 = 0$ are given by $(-1 \pm \sqrt{1-12})/2 = (-1/2) \pm (\sqrt{11}/2)i$. We may factor using these two roots: $x^2 + x + 3 = [x + (1/2) - (\sqrt{11}/2)i][x + (1/2) + (\sqrt{11}/2)i]$. (You may check by multiplying out.)

(d) We seek a number $z = a + ib$ such that $z^2 = i$; now $z^2 = a^2 - b^2 + 2abi$, so we must solve $a^2 - b^2 = 0$ and $2ab = 1$. Hence $a = \pm b$, so $b = \pm(1/\sqrt{2})$. Thus there are two numbers whose square is i, namely, $\pm[(1/\sqrt{2}) + (i/\sqrt{2})]$, i.e., $\sqrt{i} = \pm(1/\sqrt{2})(1+i) = \pm(\sqrt{2}/2)(1+i)$. Although for positive real numbers, there is a "preferred" square root (the positive one), this is not the case for a general complex number. ▲

Example 5 (a) Show that if $z = a + ib \neq 0$, then

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

is a complex number whose product with z equals 1; thus, $1/z$ is the inverse of z, and we can divide by nonzero complex numbers.

(b) Write $1/(3 + 4i)$ in the form $a + bi$.

Solution (a)
$$\begin{aligned}\left(\frac{a-ib}{a^2+b^2}\right)(a+ib) &= \left(\frac{1}{a^2+b^2}\right)(a-ib)(a+ib)\\ &= \left(\frac{1}{a^2+b^2}\right)(a^2 + aib - iba - b^2i^2)\\ &= \left(\frac{1}{a^2+b^2}\right)(a^2+b^2) = 1.\end{aligned}$$

Hence $z\left(\dfrac{a-ib}{a^2+b^2}\right) = 1$, so $(a - ib)/(a^2 + b^2)$ can be denoted $1/z$. Note that $z \neq 0$ means that not both a and b are zero, so $a^2 + b^2 \neq 0$ and division by the real number $a^2 + b^2$ is legitimate.

(b) $1/(3 + 4i) = (3 - 4i)/(3^2 + 4^2) = (3/25) - (4/25)i$ by the formula in (a). ▲

Terminology for Complex Numbers

If $z = a + ib$ is a complex number, then:

(i) a is called the *real part* of z;
(ii) b is called the *imaginary part* of z (note that the imaginary part is itself a real number);
(iii) $a - ib$ is called the *complex conjugate* of z and is denoted $\bar{z}$;
(iv) $r = \sqrt{a^2 + b^2}$ is called the *length* or *absolute value* of z and is denoted $|z|$;
(v) θ defined by $a = r\cos\theta$ and $b = r\sin\theta$ is called the *argument* of z.

The notions in the box above are illustrated in Fig. 12.6.3. Note that the real and imaginary parts are simply the x and y coordinates, the complex conjugate is the reflection in the x axis, and the absolute value is (by Pythagoras' theorem) the length of the line joining the origin and z. The argument of z is the angle this line makes with the x axis. Thus, (r, θ) are simply the polar coordinates of the point (a, b).

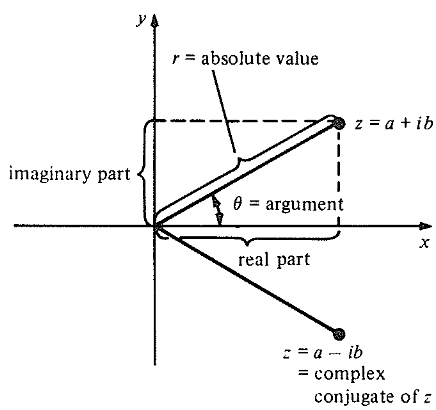


Figure 12.6.3. Illustrating various quantities attached to a complex number.

The terminology and notation above simplify manipulations with complex numbers. For example, notice that

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2,$$

so that $1/z = \bar{z}/|z|^2$ which reproduces the result of Example 5(a). Notice that we can remember this by:

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

Example 6 (a) Find the absolute value and argument of $1 + i$.
(b) Find the real parts of $1/i$, $1/(1+i)$, and $(8+2i)/(1-i)$.

Solution (a) The real part is 1, and the imaginary part is 1. Thus the absolute value is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the argument is $\tan^{-1}(1/1) = \pi/4$.
(b) $1/i = (1/i)(-i/-i) = -i/1 = -i$, so the real part of $1/i = -i$ is zero. $1/(1+i) = (1-i)/(1+i)(1-i) = (1-i)/2$, so the real part of $1/(1+i)$ is $1/2$. Finally,

$$\frac{8+2i}{1-i} = \frac{(8+2i)}{(1-i)}\,\frac{1+i}{1+i} = \frac{8+10i-2}{2} = \frac{6+10i}{2} = 3 + 5i,$$

so the real part of $(8+2i)/(1-i)$ is 3. ▲

Properties of Complex Numbers

(i) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$, $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$;
(ii) z is real if and only if $z = \bar{z}$;
(iii) $|z_1 z_2| = |z_1| \cdot |z_2|$, $|z_1/z_2| = |z_1|/|z_2|$; and
(iv) $|z_1 + z_2| \leqslant |z_1| + |z_2|$ (triangle inequality).

The proofs of these properties are left to the examples and exercises.

Example 7 (a) Prove property (i) of complex numbers.
(b) Express $\overline{(1+i)^{100}}$ without a bar.

Solution (a) Let $z_1 = a + ib$ and $z_2 = c + id$, so $\bar{z}_1 = a - ib$, $\bar{z}_2 = c - id$. From $z_1 z_2 = (ac - bd) + (ad + bc)i$, we get $\overline{z_1 z_2} = (ac - bd) - (ad + bc)i$; we also have $\bar{z}_1 \cdot \bar{z}_2 = (a - ib)(c - id) = (ac - bd) - ibc - aid = \overline{z_1 z_2}$. For the quotient, write $z_2 \cdot z_1/z_2 = z_1$ so by the rule just proved, $\bar{z}_2 \cdot \overline{(z_1/z_2)} = \bar{z}_1$. Dividing by $\bar{z}_2$ gives the result.
(b) Since the complex conjugate of a product is the product of the complex conjugates (proved in (a)), we similarly have $\overline{z_1 z_2 z_3} = \overline{z_1 z_2}\bar{z}_3 = \bar{z}_1 \bar{z}_2 \bar{z}_3$ and so on for any number of factors. Thus $\overline{z^n} = \bar{z}^n$, and hence $\overline{(1+i)^{100}} = (\overline{1+i})^{100} = (1-i)^{100}$. ▲

Example 8 Given $z = a + ib$, construct iz geometrically and discuss.

Solution If $z = a + ib$, $iz = ai - b = -b + ia$. Thus in the plane, $z = (a,b)$ and $iz = (-b,a)$. This point $(-b,a)$ is on the line perpendicular to the line Oz since the slopes are negative reciprocals. See Fig. 12.6.4. Since iz has the same length as z, we can say that iz is obtained from z by a rotation through 90°. ▲

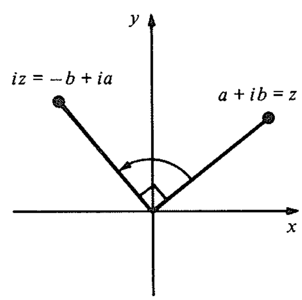


Figure 12.6.4. The number iz is obtained from z by a 90° rotation about the origin.

Using the algebra of complex numbers, we can define $f(z)$ when f is a rational function and z is a complex number.

Example 9 If $f(z) = (1 + z)/(1 - z)$ and $z = 1 + i$, express $f(z)$ in the form $a + bi$.

Solution Substituting $1 + i$ for z, we have

$$f(1+i) = \frac{1+1+i}{1-(1+i)} = \frac{2+i}{-i} = -1 - \frac{2}{i} = -1 + 2i. \ \blacktriangle$$

How can we define more general functions of complex numbers, like e^z? One way is to use power series, writing

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

To make sense of this, we would have to define the limit of a sequence of complex numbers so that the sum of the infinite series could be taken as the limit of its sequence of partial sums. Fortunately, this is possible, and in fact the whole theory of infinite series carries over to the complex numbers. This approach would take us too far afield,[6] though, and we prefer to take the approach of *defining* the particular function e^{ix}, for x real, by Euler's formula

$$e^{ix} = \cos x + i \sin x. \tag{9}$$

Since $e^{x+y} = e^x e^y$, we expect a similar law to hold for e^{ix}.

Example 10 (a) Show that

$$e^{i(x+y)} = e^{ix} e^{iy} \tag{10}$$

(b) Give a definition of e^z for $z = x + iy$.

Solution (a) The right-hand side of equation (10) is

$$\begin{aligned}(\cos x + i \sin x)&(\cos y + i \sin y)\\ &= \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x)\\ &= \cos(x + y) + i \sin(x + y) = e^{i(x+y)}\end{aligned}$$

by equation (9) and the addition formulae for sin and cos.
(b) We would like to have $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for any complex numbers, so we should define $e^{x+iy} = e^x \cdot e^{iy}$, i.e., $e^{x+iy} = e^x(\cos y + i \sin y)$. [With this definition, the law $e^{z_1+z_2} = e^{z_1} e^{z_2}$ can then be proved for all z_1 and z_2.] ▲

Equation (10) contains all the information in the trigonometric addition formulas. This is why the use of e^{ix} is so convenient: the laws of exponents are easier to manipulate than the trigonometric identities.

Example 11 (a) Calculate $\overline{e^{i\theta}}$ and $|e^{i\theta}|$. (b) Calculate $e^{i\pi/2}$ and $e^{i\pi}$. (c) Prove that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{1}{2}\left(\frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}\right)$$

by considering $1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}$.

Solution (a) $e^{i\theta} = \cos\theta + i \sin\theta$, so by definition of the complex conjugate we should change the sign of the imaginary part:

$$\overline{e^{i\theta}} = \cos\theta - i \sin\theta = \cos(-\theta) + i \sin(-\theta) = e^{-i\theta},$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Thus $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ using the general definition $|z| = \sqrt{a^2 + b^2}$, where $z = a + ib$.
(b) $e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = i$ and $e^{i\pi} = \cos\pi + i \sin\pi = -1$.
(c) Since $\cos n\theta$ is the real part of $e^{in\theta}$, we are led to consider $1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta}$. Recalling that $1 + r + \cdots + r^n = (1 - r^{n+1})/(1 - r)$, we get

$$\begin{aligned}1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\\ &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \cdot \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}}\end{aligned}$$

[6] See a text on complex variables such as J. Marsden, *Basic Complex Analysis*, Freeman, New York (1972) for a thorough treatment of complex series.

$$= \frac{1 - e^{-i\theta} - e^{i(n+1)\theta} + e^{in\theta}}{2 - (e^{i\theta} + e^{-i\theta})}$$

$$= \frac{1 - e^{-i\theta} - e^{i(n+1)\theta} + e^{in\theta}}{2(1 - \cos\theta)}.$$

Taking the real part of both sides gives the result. ▲

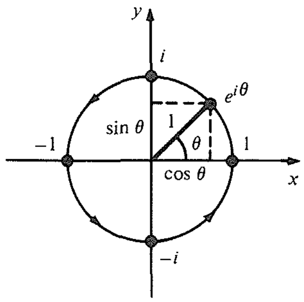


Figure 12.6.5. As θ goes from 0 to 2π, the point $e^{i\theta}$ goes once around the unit circle in the complex plane.

Let us push our analysis of e^{ix} a little further. Notice that $e^{i\theta} = \cos\theta + i\sin\theta$ represents a point on the unit circle with argument θ. As θ ranges from 0 to 2π, this point moves once around the circle (Fig. 12.6.5). (This is the same basic geometric picture we used to introduce the trigonometric functions in Section 5.1).

Recall that if $z = a + ib$, and r, θ are the polar coordinates of (a, b), then $a = r\cos\theta$ and $b = r\sin\theta$. Thus

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Hence we arrive at the following.

Polar Representation of Complex Numbers

If $z = a + ib$ and if (r, θ) are the polar coordinates of (a, b), i.e., the absolute value and argument of z, then

$$z = re^{i\theta}.$$

This representation is very convenient for algebraic manipulations. For example,

$$\text{if} \quad z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \qquad \text{then} \quad z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)},$$

which shows how the absolute value and arguments behave when we take products; i.e., it shows that $|z_1 z_2| = |z_1||z_2|$ and that the argument of $z_1 z_2$ is the sum of the arguments of z_1 and z_2.

Let us also note that if $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$. Thus if we wish to solve $z^n = w$ where $w = \rho e^{i\phi}$, we must have $r^n = \rho$, i.e., $r = \sqrt[n]{\rho}$ (remember that r, ρ are non-negative) and $e^{in\theta} = e^{i\phi}$, i.e., $e^{i(n\theta - \phi)} = 1$, i.e., $n\theta = \phi + 2\pi k$ for an integer k (this is because $e^{it} = 1$ exactly when t is a multiple of 2π—see Fig. 12.6.5). Thus $\theta = \phi/n + 2\pi k/n$. When $k = n$, $\theta = \phi/n + 2\pi$, so $e^{i\theta} = e^{i\phi/n}$. Thus we get the same value for $e^{i\theta}$ when $k = 0$ and $k = n$, and we need take only $k = 0, 1, 2, \ldots, n-1$. Hence we get the following formula for the nth roots of a complex number.

De Moivre's Formula[7]

The numbers z such that $z^n = w = \rho e^{i\phi}$, i.e., the nth roots of w, are given by

$$\sqrt[n]{\rho}\, e^{i(\phi/n + 2\pi k/n)}, \qquad k = 0, 1, 2, \ldots, n-1.$$

[7] Abraham DeMoivre (1667–1754), of French descent, worked in England around the time of Newton.

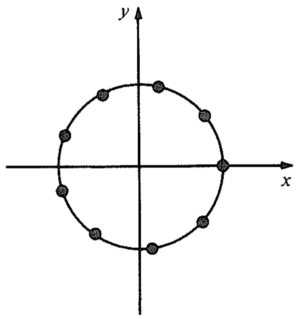


Figure 12.6.6. The ninth roots of 1.

For example, the ninth roots of 1 are the complex numbers $e^{i2\pi k/9}$, for $k = 0, 1, \ldots, 8$, which are 9 points equally spaced around the unit circle. See Fig. 12.6.6.

It is shown in more advanced books that any nth degree polynomial $a_0 + a_1 z + \cdots + a_n z^n$ has at least one complex root[8] z_1 and, as a consequence, that the polynomial can be completely factored:

$$a_0 + a_1 z + \cdots + a_n z^n = (z - z_1) \cdots (z - z_n).$$

For example,

$$z^2 + z + 1 = \left(z + \frac{1 + \sqrt{3}\,i}{2}\right)\left(z + \frac{1 - \sqrt{3}\,i}{2}\right),$$

although $z^2 + z + 1$ cannot be factored using only real numbers.

Example 12 (a) Redo Example 8 using the polar representation. (b) Give a geometric interpretation of multiplication by $(1 + i)$.

Solution (a) Since $i = e^{i\pi/2}$, $iz = re^{i(\theta + \pi/2)}$ if $z = re^{i\theta}$. Thus iz has the same magnitude as z but its argument is increased by $\pi/2$. Hence iz is z rotated by 90°, in agreement with the solution to Example 8.

(b) Since $(1 + i) = \sqrt{2}\, e^{i\pi/4}$, multiplication of a complex number z by $(1 + i)$ rotates z through an angle $\pi/4 = 45°$ and multiplies its length by $\sqrt{2}$. ▲

Example 13 Find the 4th roots of $1 + i$.

Solution $1 + i = \sqrt{2}\, e^{i\pi/4}$, since $1 + i$ has $r = \sqrt{2}$ and $\theta = \pi/4$. Hence the fourth roots are, according to DeMoivre's formula,

$$\sqrt[8]{2}\, e^{i((\pi/16) + (\pi k/2))}, \qquad k = 0, 1, 2, 3,$$

i.e.,

$$\sqrt[8]{2}\, e^{i\pi/16}, \quad \sqrt[8]{2}\, e^{i\pi 9/16}, \quad \sqrt[8]{2}\, e^{i\pi \cdot 17/16}, \quad \text{and} \quad \sqrt[8]{2}\, e^{i\pi \cdot 25/16}. \ ▲$$

Exercises for Section 12.6

Express the quantities in Exercises 1–4 in the form $a + bi$.

1. $e^{-\pi i/2}$
2. $e^{\pi i/4}$
3. $e^{(3\pi/2)i}$
4. $e^{-i\pi}$

Plot the complex numbers in Exercises 5–12 as points in the xy plane.

5. $4 + 2i$
6. $-1 + i$
7. $3i$
8. $-(2 + i)$
9. $-\frac{2}{3}i$
10. $3 + 7i$
11. $0.1 + 0.2i$
12. $0 + 1.5i$

Simplify the expressions in Exercises 13–20.

13. $(1 + 2i) - 3(5 - 2i)$
14. $(4 - 3i)(8 + i) + (5 - i)$
15. $(2 + i)^2$
16. $\dfrac{1}{(3 + i)}$
17. $\dfrac{1}{5 - 3i}$
18. $\dfrac{2i}{1 - i}$
19. $\dfrac{(1 + i)(3 - 2i)}{8 + i}$
20. $\dfrac{(2 + 2i) + 6i}{(1 + 2i)(-4i)}$

Write the solutions of the equations in Exercises 21–26 in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

21. $z^2 + 3 = 0$
22. $z^2 - 2z + 5 = 0$
23. $z^2 + \frac{1}{3}z + \frac{1}{2} = 0$
24. $z^3 + 2z^2 + 2z + 1 = 0$ [*Hint:* factor]
25. $z^2 - 7z - 1 = 0$
26. $z^3 - 3z^2 + 3z - 1 = 0$ [*Hint:* factor]

Using the method of Example 4(d), find the quantities in Exercises 27–30.

27. $\sqrt{8i}$
28. $\sqrt{9i}$
29. $\sqrt{-16i}$
30. $\sqrt{\sqrt{i}}$

[8] See any text in complex variables, such as J. Marsden, *op. cit.* The theorem referred to is called the "fundamental theorem of algebra." It was first proved by Gauss in his doctoral thesis in 1799.

Find the imaginary part of the complex numbers in Exercises 31–36.

31. $\dfrac{1+i}{i}$
32. $\dfrac{2-3i}{1+3i}$
33. $\dfrac{10+5i}{(1+2i)^2}$
34. $(1-8i)\left(2+\frac{1}{4}i\right)^{-1}$
35. $\dfrac{1/2+(3/5)i}{7/8-i}$
36. $\dfrac{(3/4)i}{9/4+(1/5)i}$

Find the complex conjugate of the complex numbers in Exercises 37–46.

37. $5+2i$
38. $1-bi$
39. $\sqrt{3}+\frac{1}{2}i$
40. $1/i$
41. $\dfrac{2-i}{3i}$
42. $i(1+i)$
43. $\dfrac{3-5i}{4+8i}$
44. $\dfrac{1}{2i}\left(\dfrac{1+i}{1-i}\right)$
45. 3
46. $\dfrac{10+i}{7+4i}$

Find the absolute value and argument of the complex numbers in Exercises 47–58. Plot.

47. $-1-i$
48. $7+2i$
49. 2
50. $4i$
51. $\frac{1}{2}-\frac{2}{3}i$
52. $3-2i$
53. $-5+7i$
54. $-10+\frac{1}{2}i$
55. $-8-2i$
56. $5+5i$
57. $1.2+0.7i$
58. $50+10i$

59. Prove property (iii) of complex numbers.
60. Prove property (iv) of complex numbers.
61. Express $\overline{(8-3i)^4}$ without a bar.
62. Express $\overline{(2+3i)^2(8-i)^3}$ without a bar.

In Exercises 63–66, draw an illustration of the addition of the pairs of complex numbers, i.e., plot both along with their sums.

63. $1+\frac{1}{2}i, 3-i$
64. $-8-2i, 5-i$
65. $-3+4i, 6i$
66. $7, 4i$

67. Find $|(1+i)(2-i)(\sqrt{2}\,i)|$.
68. If $z=x+iy$, express x and y in terms of z and $\bar{z}$.
69. If $z=x+iy$ with x and y real, what is $|e^z|$ and the argument of e^z?
70. Find the real and imaginary parts of $(x+iy)^3$ as polynomials in x and y.

Write the numbers in Exercises 71–76 in the form $a+bi$.

71. $e^{i\pi/3}$
72. $e^{-\pi i/3}$
73. $e^{1-\pi i/2}$
74. e^{1+2i}
75. $e^{1+\pi i/2}$
76. $e^{(1-\pi/6)i}$

77. If $f(z)=1/z^2$, express $f(2+i)$ in the form $a+bi$.
78. Express $f(i)$ in the form $a+bi$, if $f(z)=z^2+2z+1$.
79. (a) Using a trigonometric identity, show that $e^{ix}e^{-ix}=1$. (b) Show that $e^{-z}=1/e^z$ for all complex numbers z.
80. Show that $e^{3z}=(e^z)^3$ for all complex z.
81. Prove that $e^{i(\theta+3\pi/2)}=-ie^{i\theta}$.
82. Prove that
$$\sin\theta+\sin 2\theta+\cdots+\sin n\theta = \left(\cot\frac{\theta}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\left(\frac{\sin n\theta-\sin(n+1)\theta}{\sin\theta}\right)\right).$$
83. Prove that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, if n is an integer.
84. Use Exercise 83 to find the real part of $\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)^3$ and the imaginary part of $\left(\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^9$.

Find the polar representation (i.e., $z=re^{i\theta}$) of the complex numbers in Exercises 85–94.

85. $1+i$
86. $\dfrac{1}{i}$
87. $(2+i)^{-1}$
88. $\sqrt{3}$
89. $7-3i$
90. $4+i^3$
91. $-\frac{1}{2}-3i$
92. $\dfrac{(2+5i)}{(1-i)}$
93. $(3+4i)^2$
94. $-1+\frac{1}{3}i$

95. Find the fifth roots of $\frac{1}{2}-\frac{1}{2}\sqrt{3}\,i$ and $1+2i$. Sketch.
96. Find the fourth roots of i and $\sqrt{i}$. Sketch.
97. Find the sixth roots of $\sqrt{5}+3i$ and $3+\sqrt{5}\,i$. Sketch.
98. Find the third roots of $1/7$ and $i/7$. Sketch.
99. Give a geometric interpretation of division by $1-i$.
100. (a) Give a geometric interpretation of multiplication by an arbitrary complex number $z=re^{i\theta}$.
(b) What happens if we divide?
101. Prove that if $z^6=1$ and $z^{10}=1$, then $z=\pm 1$.
102. Suppose we know that $z^7=1$ and $z^{41}=1$. What can we say about z?
103. Let $z=re^{i\theta}$. Prove that $\bar{z}=re^{-i\theta}$.
104. (a) Let $f(z)=az^3+bz^2+cz+d$, where a, b, c, and d are real numbers. Prove that $f(\bar{z})=\overline{f(z)}$.
(b) Does equality still hold if a, b, c, and d are allowed to be arbitrary complex numbers?

Factor the polynomials in Exercises 105–108, where z is complex. [*Hint:* Find the roots.]

105. z^2+2z+i
106. $z^2+2iz-4$
107. $z^2+2iz-4-4i$
108. $3z^2+z-e^{i\pi/3}$

109. (a) Write $\tan i\theta$ in the form $a+bi$ where a and b are real functions of θ.
(b) Write $\tan i\theta$ in the form $re^{i\phi}$.
110. Let $z=f(t)$ be a *complex valued* function of the *real* variable t. If $z=x+iy=g(t)+ih(t)$, where g and h are real valued, we *define* $dz/dt=f'(t)$ to be $(dx/dt)+i(dy/dt)=g'(t)+ih'(t)$.
(a) Show that $(d/dt)(Ce^{i\omega t})=i\omega Ce^{i\omega t}$, if C is

any complex number and ω is any real number.

(b) Show that $z = Ce^{i\omega t}$ satisfies the *spring equation* (see Section 8.1): $z'' + \omega^2 z = 0$.

(c) Show that $z = De^{-i\omega t}$ also satisfies the spring equation.

(d) Find C and D such that $Ce^{i\omega t} + De^{-i\omega t} = f(t)$ satisfies $f(0) = A$, $f'(0) = B$. Express the resulting function $f(t)$in terms of sines and cosines.

(e) Compare the result of (d) with the results in Section 8.1.

111. Let z_1 and z_2 be nonzero complex numbers. Find an algebraic relation between z_1 and z_2 which is equivalent to the fact that the lines from the origin through z_1 and z_2 are perpendicular.

112. Let $w = f(z) = (1 + (z/2))/(1 - (z/2))$.
(a) Show that if the real part of z is 0, then $|w| = 1$.
(b) Are all points on the circle $|w| = 1$ in the range of f? [*Hint:* Solve for z in terms of w.]

113. (a) Show that, if $z^n = 1$, n a positive integer, then either $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$ or $z = 1$.
(b) Show that, if $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$, then $z^n = 1$.
(c) Find all the roots of the equation $z^3 + z^2 + z + 1 = 0$.

114. Describe the motion in the complex plane, as the real number t goes from $-\infty$ to ∞, of the point $z = e^{i\omega t}$, when
(a) $\omega = i$, (b) $\omega = 1 + i$,
(c) $\omega = -i$, (d) $\omega = -1 - i$,
(e) $\omega = 0$, (f) $\omega = 1$,
(g) $\omega = -1$.

115. Describe the motion in the complex plane, as the real number t varies. of the point given by $z = 93{,}000{,}000\ e^{2\pi i(t/365)} + 1{,}000{,}000 e^{2\pi i(t/29)}$. What astronomical phenomenon does this represent?

116. What is the relation between e^z and $e^{\bar{z}}$?

★117. (a) Find *all* complex numbers z for which $e^z = -1$. (b) How might you define $\ln(-1)$? What is the difficulty here?

★118. (a) Find λ such that the function $x = e^{\lambda t}$ satisfies the equation $x'' - 2x' + 2x = 0$; $x' = dx/dt$.
(b) Express the function $e^{\lambda t} + e^{-\lambda t}$ in terms of sines, cosines, and *real* exponents.
(c) Show that the function in (b) satisfies the differential equation in (a).

12.7 Second-Order Linear Differential Equations

The nature of the solutions of $ay'' + by' + cy = 0$ depends on whether the roots of $ar^2 + br + c = 0$ are real or complex.

We shall now use complex numbers to study second-order differential equations more general than the spring equation discussed in Section 8.1.

We begin by studying the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where y is an unknown function of x, $y' = dy/dx$, $y'' = d^2y/dx^2$, and a, b, c are constants. We assume that $a \neq 0$; otherwise equation (1) would be a first-order equation, which we have already studied in Sections 8.2 and 8.6.

We look for solutions of equation (1) in the form

$$y = e^{rx}, \qquad r \text{ a constant.} \tag{2}$$

Substituting equation (2) into equation (1) gives

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0,$$

which is equivalent to

$$ar^2 + br + c = 0, \tag{3}$$

since $e^{rx} \neq 0$. Equation (3) is called the *characteristic equation* of equation (1). By the quadratic formula, it has roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which we shall denote by r_1 and r_2. Thus, $y = e^{r_1 x}$ and $y = e^{r_2 x}$ are solutions of equation (1).

By analogy with the spring equation, we expect the general solution of equation (1) to involve two arbitrary constants. In fact, $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is a solution of equation (1) for constants c_1 and c_2; indeed, note that if y_1 and y_2 solve equation (1), so does $c_1 y_1 + c_2 y_2$ since

$$\begin{aligned} a(c_1 y_1 + c_2 y_2)'' + b(c_1 y_1 + c_2 y_2)' + c(c_1 y_1 + c_2 y_2) \\ = c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) = 0. \end{aligned}$$

If r_1 and r_2 are distinct, then one can show that $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the *general solution*; i.e., any solution has this form for particular values of c_1 and c_2. (See the Supplement to this section for the proof.)

Second-Order Equations: Distinct Roots

If $ar^2 + br + c = 0$ has distinct roots r_1 and r_2, then the general solution of

$$ay'' + by' + cy = 0$$

is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \qquad c_1, c_2 \text{ constants.}$$

Example 1 Consider the equation $2y'' - 3y' + y = 0$. (a) Find the general solution, and (b) Find the particular solution satisfying $y(0) = 1$, $y'(0) = 0$.

Solution (a) The characteristic equation is $2r^2 - 3r + 1 = 0$, which factors: $(2r - 1)(r - 1) = 0$. Thus $r_1 = 1$ and $r_2 = \frac{1}{2}$ are the roots, and so

$$y = c_1 e^x + c_2 e^{x/2}$$

is the general solution.

(b) Substituting $y(0) = 1$ and $y'(0) = 0$ in the preceding formula for y gives

$$\begin{aligned} c_1 + c_2 &= 1, \\ c_1 + \tfrac{1}{2}c_2 &= 0. \end{aligned}$$

Subtracting gives $\frac{1}{2}c_2 = 1$, so $c_2 = 2$ and hence $c_1 = -1$. Thus

$$y = 2e^{x/2} - e^x$$

is the particular solution sought. ▲

If the roots of the characteristic equation are distinct but complex, we can convert the solution to sines and cosines using the relation $e^{ix} = \cos x + i \sin x$, which was established in Section 12.6. Differentiating a complex valued function is carried out by differentiating the real and imaginary parts separately. One finds that $(d/dt)Ce^{rt} = Cre^{rt}$ for any complex numbers C and r (see Exercise 110 in Section 12.6). Thus, the results in the above box still work if r_1, r_2, C_1 and C_2 are complex.

Example 2 Find the general solution of $y'' + 2y' + 2y = 0$.

Solution The characteristic equation is $r^2 + 2r + 2 = 0$, whose roots are

$$r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i.$$

Thus

$$\begin{aligned}
y &= c_1e^{(-1+i)x} + c_2e^{(-1-i)x} \\
&= c_1e^{-x}e^{ix} + c_2e^{-x}e^{-ix} \\
&= e^{-x}\left[c_1(\cos x + i\sin x) + c_2(\cos x - i\sin x)\right] \\
&= e^{-x}(C_1\cos x + C_2\sin x),
\end{aligned}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$. If we desire a real (as opposed to complex) solution, C_1 and C_2 should be real. (Although we used complex numbers as a helpful tool in our computations, the final answer involves only real numbers and can be verified directly.) ▲

For the spring equation $y'' + \omega^2 y = 0$, the characteristic equation is $r^2 + \omega^2 = 0$, which has roots $r = \pm i\omega$, so the general solution is

$$\begin{aligned}
y &= c_1e^{i\omega x} + c_2e^{-i\omega x} \\
&= C_1\cos\omega x + C_2\sin\omega x,
\end{aligned}$$

where C_1 and C_2 are as in Example 2. Thus we recover the same general solution that we found in Section 8.1.

If the roots of the characteristic equation are equal ($r_1 = r_2$), then we have so far only the solution $y = c_1e^{r_1x}$, where c_1 is an arbitrary constant. We still expect another solution, since the general solution of a second-order equation should involve two arbitrary constants. To find the second solution, we may use either of two methods.

Method 1. Reduction of Order. We seek another solution of the form

$$y = ve^{r_1x}. \tag{4}$$

where v is now a function rather than a constant. To see what equation is satisfied by v, we substitute equation (4) into equation (1). Noting that

$$y' = v'e^{r_1x} + r_1ve^{r_1x},$$

and

$$y'' = v''e^{r_1x} + 2r_1v'e^{r_1x} + r_1^2ve^{r_1x},$$

substitution into (1) gives

$$a(v'' + 2r_1v' + r_1^2v)e^{r_1x} + b(v' + r_1v)e^{r_1x} + cve^{r_1x} = 0;$$

but $e^{r_1x} \neq 0$, $ar_1^2 + br_1 + c = 0$, and $2ar_1 + b = 0$ (since r_1 is a repeated root), so this reduces to $av'' = 0$. Hence $v = c_1 + c_2x$, so equation (4) becomes

$$y = (c_1 + c_2x)e^{r_1x}. \tag{5}$$

This argument actually proves that equation (5) is the *general solution* to equation (1) in the case of a repeated root. (The reason for the name "reduction of order" is that for more general equations $y'' + b(x)y' + c(x)y = 0$, if one solution $y_1(x)$ is known, one can find another one of the form $v(x)y_1(x)$, where $v'(x)$ satisfies a *first* order equation—see Exercise 48.)

Method 2. Root Splitting. If $ay'' + by' + cy = 0$ has a repeated root r_1, the characteristic equation is $(r - r_1)(r - r_1) = 0$. Now consider the new equation $(r - r_1)(r - (r_1 + \varepsilon)) = 0$ which has distinct roots r_1 and $r_2 = r_1 + \varepsilon$ if $\varepsilon \neq 0$. The corresponding differential equation has solutions e^{r_1x} and $e^{(r_1+\varepsilon)x}$. Hence $(1/\varepsilon)(e^{(r_1+\varepsilon)x} - e^{r_1x})$ is also a solution. Letting $\varepsilon \to 0$, we get the solution $(d/dr)e^{rx}|_{r=r_1} = xe^{r_1x}$ for the given equation. (If you are suspicious of this reasoning, you may verify directly that xe^{r_1x} satisfies the given equation).

Second-Order Equations: Repeated Roots

If $ar^2 + br + c = 0$ has a repeated root $r_1 = r_2$, then the general solution of

$$ay'' + by' + cy = 0$$

is

$$y = (c_1 + c_2 x)e^{r_1 x}, \tag{5}$$

where c_1 and c_2 are constants.

Example 3 Find the solution of $y'' - 4y' + 4y = 0$ satisfying $y'(0) = -1$ and $y(0) = 3$.

Solution The characteristic equation is $r^2 - 4r + 4 = 0$, or $(r-2)^2 = 0$, so $r_1 = 2$ is a repeated root. Thus the general solution is given by equation (5):

$$y = (c_1 + c_2 x)e^{2x}.$$

Thus $y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$. The data $y(0) = 3$, $y'(0) = -1$ give

$$c_1 = 3 \quad \text{and} \quad 2c_1 + c_2 = -1,$$

so $c_1 = 3$ and $c_2 = -7$. Thus $y = (3 - 7x)e^{2x}$. ▲

Now we shall apply the preceding methods to study damped harmonic motion. In Figure 12.7.1 we show a weight hanging from a spring; recall from

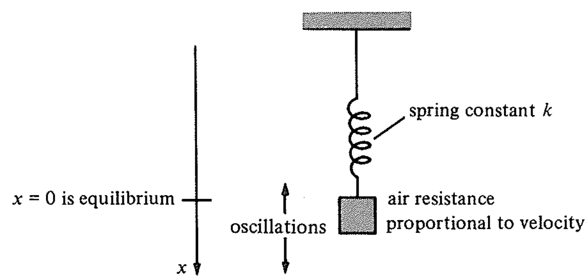


Figure 12.7.1. The physical set up for damped harmonic motion.

Section 8.1 that the equation of motion of the spring is $m(d^2x/dt^2) = F$, where F is the total force acting on the weight. The force due to the spring is $-kx$, just as in Section 8.1. (The force of gravity determines the equilibrium position, which we have called $x = 0$; see Exercise 51.) We also suppose that the force of air resistance is proportional to the velocity. Thus $F = -kx - \gamma(dx/dt)$, so the equation of motion becomes

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}, \tag{6}$$

where $\gamma > 0$ is a constant. (Can you see why there is a minus sign before γ?). If we rewrite equation (6) as

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \omega^2 x = 0, \tag{7}$$

where $\beta = \gamma/m$ and $\omega^2 = k/m$, it has the form of equation (1) with $a = 1$, $b = \beta$, and $c = \omega^2$. To solve it, we look at the characteristic equation

$$r^2 + \beta r + \omega^2 = 0 \qquad \text{which has roots} \qquad r = \frac{-\beta \pm \sqrt{\beta^2 - 4\omega^2}}{2}.$$

If $\beta^2 > 4\omega^2$ (i.e., $\beta > 2\omega$), then there are two real roots and so the solution is $x = c_1e^{r_1t} + c_2e^{r_2t}$, where r_1 and r_2 are the two roots $\frac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4\omega^2})$. Note that r_1 and r_2 are both negative, so the solution tends to zero as $t \to \infty$, although it will cross the t axis once if c_1 and c_2 have opposite signs; this case is called the *overdamped case*. A possible solution is sketched in Fig. 12.7.2.

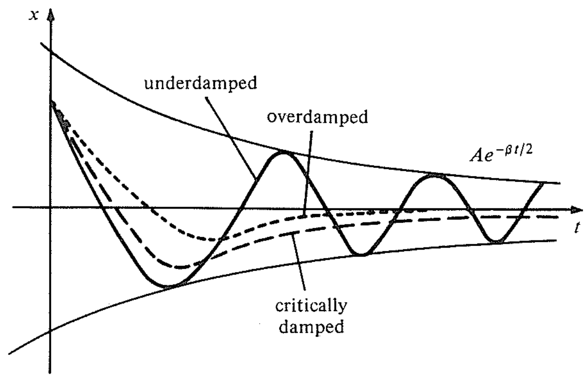


Figure 12.7.2. Damped harmonic motion.

If $\beta^2 = 4\omega^2$, there is a repeated root $r_1 = -\beta/2$, so the solution is $x = (c_1 + c_2t)e^{-\beta t/2}$. This case is called *critically damped*. Here the solution also tends to zero as $t \to \infty$, although it may cross the t axis once if c_1 and c_2 have opposite signs (this depends on the initial conditions). A possible trajectory is given in Figure 12.7.2.

Finally, if $\beta^2 < 4\omega^2$, then the roots are complex. If we let $\bar{\omega} = \frac{1}{2}\sqrt{4\omega^2 - \beta^2} = \omega\sqrt{1 - \beta^2/4\omega^2}$, then the solution is

$$x = e^{-\beta t/2}(c_1\cos\bar{\omega}t + c_2\sin\bar{\omega}t)$$

which represents *underdamped oscillations* with frequency $\bar{\omega}$. (Air resistance slows down the motion so the frequency $\bar{\omega}$ is lower than ω.) These solutions may be graphed by utilizing the techniques of Section 8.1; write $x = Ae^{-\beta t/2}\cos(\bar{\omega}t - \theta)$, where (A, θ) are the polar coordinates of c_1 and c_2. A typical graph is shown in Fig. 12.7.2.At $t = 0$, $0 = x = c_1$.

Example 4 Consider a spring with$\beta = \pi/4$ and $\omega = \pi/6$.

(a) Is it over, under, or critically damped?

(b) Find and sketch the solution with $x(0) = 0$ and $x'(0) = 1$, for $t \geqslant 0$.

(c) Find and sketch the solution with the same initial conditions but with $\beta = \pi/2$.

Solution (a) Here $\beta^2 - 4\omega^2 = \pi^2/16 - 4\pi^2/36 = -7\pi^2/36 < 0$, so the spring is underdamped.

(b) The effective frequency is $\bar{\omega} = \omega\sqrt{1 - \beta^2/4\omega^2} = (\pi/6)\sqrt{1 - 9/16} = \pi\sqrt{7}/24$, so the general solution is

$$x = e^{-\pi t/8}\left(c_1\cos\left(\frac{\pi\sqrt{7}\,t}{24}\right) + c_2\sin\left(\frac{\pi\sqrt{7}\,t}{24}\right)\right).$$

At $t = 0$, $0 = x = c_1$. Thus,

$$x = c_2e^{-\pi t/8}\sin\left(\frac{\pi\sqrt{7}}{24}t\right).$$

Hence

$$x' = c_2\left[\left(-\frac{\pi}{8}\right)e^{-\pi t/8}\sin\left(\frac{\pi\sqrt{7}}{24}t\right) + \frac{\pi\sqrt{7}}{24}e^{-\pi t/8}\cos\left(\frac{\pi\sqrt{7}}{24}t\right)\right].$$

At $t = 0$, $x' = 1$, so $1 = c_2[\pi\sqrt{7}/24]$, and hence $c_2 = 24/\pi\sqrt{7}$. Thus the solution is

$$x = \frac{24}{\pi\sqrt{7}}e^{-\pi t/8}\sin\left(\frac{\pi\sqrt{7}}{24}t\right).$$

This is a sine wave multiplied by the decaying factor $e^{-\pi t/8}$; it is sketched in Fig. 12.7.3. The first maximum occurs when $x' = 0$; i.e., when $\tan((\pi\sqrt{7}/24)t) = 8\sqrt{7}/24$, or $t \approx 2.09$, at which point $x \approx 0.84$.

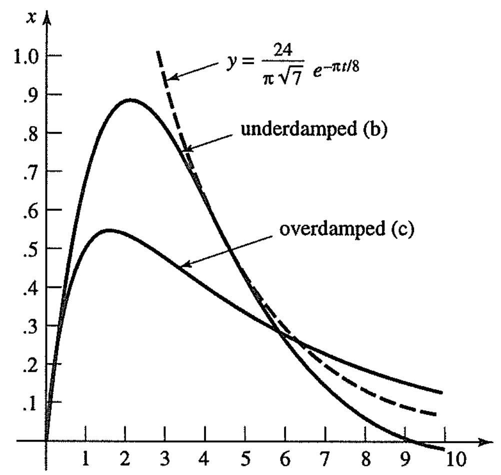


Figure 12.7.3. Graph of the solution to Example 4.

(c) For $\beta = \pi/2$, we have $\beta^2 - 4\omega^2 = \pi^2/4 - 4\pi^2/36 = 5\pi^2/36 > 0$, so the spring is overdamped. The roots r_1 and r_2 are

$$\frac{1}{2}\left(-\beta \pm \sqrt{\beta^2 - 4\omega^2}\right) = \frac{1}{2}\left(\frac{-\pi}{2} \pm \frac{\sqrt{5}}{6}\right) = \frac{\pi}{4}\left(-1 \pm \frac{\sqrt{5}}{3}\right),$$

so the solution is of the form $x = c_1e^{(\pi/4)(-1+\sqrt{5}/3)t} + c_2e^{(\pi/4)(-1-\sqrt{5}/3)t}$. At $t = 0$, $x = 0$, so $c_1 + c_2 = 0$ or $c_1 = -c_2$. Also, at $t = 0$,

$$1 = x' = c_1\frac{\pi}{4}\left(-1 + \frac{\sqrt{5}}{3}\right) + c_2\frac{\pi}{4}\left(-1 - \frac{\sqrt{5}}{3}\right) = c_1\frac{\pi\sqrt{5}}{6},$$

so $c_1 = 6/\pi\sqrt{5}$ and $c_2 = -6/\pi\sqrt{5}$. Thus our solution is

$$x = c_1(e^{r_1t} - e^{r_2t}) = \frac{6}{\pi\sqrt{5}}e^{(\pi/4)(-1+\sqrt{5}/3)t} - \frac{6}{\pi\sqrt{5}}e^{(\pi/4)(-1-\sqrt{5}/3)t}$$

$$= (0.8541)(e^{-0.200t} - e^{-1.375t})$$

The derivative is $x' = c_1(r_1e^{r_1t} - r_2e^{r_2t}) = c_1(r_1 - r_2e^{(r_2-r_1)t})e^{r_1t}$ which vanishes when

$$\frac{r_1}{r_2} = e^{(r_2-r_1)t} \quad \text{or} \quad \frac{-1+\sqrt{5}/3}{-1-\sqrt{5}/3} = e^{-(\pi\sqrt{5}/6)t}, \quad \text{or} \quad t \approx 1.64;$$

at this point, $x \approx 0.731$. See Fig. 12.7.3. ▲

In the preceding discussion we have seen how to solve the equation (1): $ay'' + by' + cy = 0$. Let us now study the problem of solving

$$ay'' + by' + cy = F(x), \tag{8}$$

where $F(x)$ is a given function of x. We call equation (1) the *homogeneous equation*, while equation (8) is called the *nonhomogeneous equation*. Using what we know about equation (1), we can find the general solution to equation (8) provided we can find just one particular solution.

Nonhomogeneous Equations: Particular and General Solutions

If $y_h = c_1y_1 + c_2y_2$ is the general solution to the homogeneous equation (1) and if y_p is a particular solution to the nonhomogeneous equation (8), then

$$y = y_p + c_1y_1 + c_2y_2 = y_p + y_h \tag{9}$$

is the general solution to the nonhomogeneous equation.

To see that equation (9) is a solution of equation (8), note that

$$\begin{aligned} &a(y_p + y_h)'' + b(y_p + y_h)' + c(y_p + y_h) \\ &\quad = (ay_p'' + by_p' + cy_p) + (ay_h'' + by_h' + cy_h) \\ &\quad = F(x) + 0 = F(x). \end{aligned}$$

To see that equation (9) is the *general* solution of equation (8), note that if $\tilde{y}$ is any solution to equation (9), then $\tilde{y} - y_p$ solves equation (1) by a calculation similar to the one just given. Hence $\tilde{y} - y_p$ must equal y_h for suitable c_1 and c_2 since y_h is the general solution to equation (1). Thus $\tilde{y}$ has the form of equation (9), so equation (9) is the general solution.

Sometimes equation (8) can be solved by inspection; for example, if $F(x) = F_0$ is a constant and if $c \neq 0$, then $y = F_0/c$ is a particular solution.

Example 5 (a) Solve $2y'' - 3y' + y = 10$. (b) Solve $2x'' - 3x' + x = 8\cos(t/2)$ (where x is a function of t). (c) Solve $2y'' - 3y' + y = 2e^{-x}$.

Solution (a) Here a particular solution is $y = 10$. From Example 1,

$$y_h = c_1e^x + c_2e^{x/2}.$$

Thus the general solution, given by equation (9), is

$$y = c_1e^x + c_2e^{x/2} + 10.$$

(b) When the right-hand side is a trigonometric function, we can try to find a particular solution which is a combination of sines and cosines of the same frequency, since they reproduce linear combinations of each other when differentiated. In this case, $8\cos(t/2)$ appears, so we try

$$x_p = A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right),$$

where A and B are constants, called *undetermined coefficients*. Then

$$\begin{aligned}2x_p'' - 3x_p' + x_p &= 2\left[-\frac{A}{4}\cos\left(\frac{t}{2}\right) - \frac{B}{4}\sin\left(\frac{t}{2}\right)\right]\\ &\quad - 3\left[-\frac{A}{2}\sin\left(\frac{t}{2}\right) + \frac{B}{2}\cos\left(\frac{t}{2}\right)\right]\\ &\quad + \left[A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right]\\ &= \left(\frac{1}{2}A - \frac{3}{2}B\right)\cos\left(\frac{t}{2}\right) + \left(\frac{3}{2}A + \frac{1}{2}B\right)\sin\left(\frac{t}{2}\right).\end{aligned}$$

For this to equal $8\cos(t/2)$, we choose A and B such that

$$\tfrac{1}{2}A - \tfrac{3}{2}B = 8, \quad \text{and} \quad \tfrac{3}{2}A + \tfrac{1}{2}B = 0.$$

The second equation gives $B = -3A$ which, upon substitution in the first, gives $\frac{1}{2}A + \frac{9}{2}A = 8$. Thus $A = \frac{8}{5}$ and $B = -\frac{24}{5}$, so

$$x_p = \frac{8}{5}\left[\cos\left(\frac{t}{2}\right) - 3\sin\left(\frac{t}{2}\right)\right],$$

and the general solution is

$$x = c_1e^t + c_2e^{t/2} + \frac{8}{5}\left[\cos\left(\frac{t}{2}\right) - 3\sin\left(\frac{t}{2}\right)\right].$$

A good way to check your arithmetic is to substitute this solution into the original differential equation.

(c) Here we try $y_p = Ae^{-x}$ since e^{-x} reproduces itself, up to a factor, when differentiated. Then

$$2y_p'' - 3y_p' + y_p = 2Ae^{-x} + 3Ae^{-x} + Ae^{-x}.$$

For this to match $2e^x$, we require $6A = 2$ or $A = \frac{1}{3}$. Thus $y_p = \frac{1}{3}e^{-x}$ is a particular solution, and so the general solution is

$$y = c_1e^x + c_2e^{x/2} + \tfrac{1}{3}e^{-x}. \quad \blacktriangle$$

The technique used in parts (b) and (c) of this example is called the *method of undetermined coefficients*. This method works whenever the right-hand side of equation (8) is a polynomial, an exponential, sums of sines and cosines (of the same frequency), or products of these functions.

There is another method called *variation of parameters* or *variation of constants* which always enables us to find a particular solution of equation (8) in terms of integrals. This method proceeds as follows. We seek a solution of the form

$$y = v_1y_1 + v_2y_2 \tag{10}$$

where y_1 and y_2 are solutions of the homogeneous equation (1) and v_1 and v_2 are functions of x to be found. Note that equation (10) is obtained by replacing the *constants* (or *parameters*) c_1 and c_2 in the general solution to the homogeneous equation by *functions*. This is the reason for the name "variation of parameters." (Note that a similar procedure was used in the method of reduction of order—see equation (4).) Differentiating v_1y_1 using the product rule gives

$$(v_1y_1)' = v_1'y_1 + v_1y_1',$$

and

$$(v_1y_1)'' = v_1''y_1 + 2v_1'y_1' + v_1y_1'',$$

and similarly for v_2y_2. Substituting these expressions into equation (8) gives

$$a\left[(v_1''y_1 + 2v_1'y_1' + v_1y_1'') + (v_2''y_2 + 2v_2'y_2' + v_2y_2'')\right] \\ + b\left[(v_1'y_1 + v_1y_1') + (v_2'y_2 + v_2y_2')\right] + c(v_1y_1 + v_2y_2) = F.$$

Simplifying, using (1) for y_1 and y_2, we get

$$a\left[v_1''y_1 + 2v_1'y_1' + v_2''y_2 + 2v_2'y_2')\right] + b\left[v_1'y_1 + v_2'y_2\right] = F. \tag{11}$$

This is only one condition on the two functions v_1 and v_2, so we are free to impose a second condition; we shall do so to make things as simple as possible. As our second condition, we require that the coefficient of b vanish (identically, as a function of x):

$$v_1'y_1 + v_2'y_2 = 0. \tag{12}$$

This implies, on differentiation, that $v_1''y_1 + v_1'y_1' + v_2''y_2 + v_2'y_2' = 0$, so equation (11) simplifies to

$$v_1'y_1' + v_2'y_2' = \frac{F}{a}. \tag{13}$$

Equations (12) and (13) can now be solved algebraically for v_1' and v_2' and the resulting expressions integrated to give v_1 and v_2. (Even if the resulting integrals cannot be evaluated, we have succeeded in expressing our solution in terms of integrals; the problem is then generally regarded as "solved").

Variation of Parameters

A particular solution of the nonhomogeneous equation (8) is given by

$$y_p = v_1y_1 + v_2y_2,$$

where y_1 and y_2 are solutions of the homogeneous equation and where v_1 and v_2 are found by solving equations (12) and (13) algebraically for v_1' and v_2' and then integrating.

Combining the two preceding boxes, one has a recipe for finding the general solution to the nonhomogeneous equation.

Example 6 (a) Find the general solution of $2y'' - 3y' + y = e^{2x} + e^{-2x}$ using variation of parameters. (b) Find the general solution of $2y'' - 3y' + y = 1/(1 + x^2)$ (express your answer in terms of integrals).

Solution (a) Here $y_1 = e^x$ and $y_2 = e^{x/2}$ from Example 1. Thus, equations (12) and (13) become

$$v_1'e^x + v_2'e^{x/2} = 0,$$

$$v_1'e^x + \frac{1}{2}v_2'e^{x/2} = \frac{e^{2x} + e^{-2x}}{2},$$

respectively. Subtracting,

$$\frac{1}{2}v_2'e^{x/2} = -\left(\frac{e^{2x} + e^{-2x}}{2}\right),$$

and so

$$v_2' = -e^{-x/2}(e^{2x} + e^{-2x}) = -e^{3x/2} - e^{-5x/2}.$$

Similarly,

$$v_1' = -e^{-x/2}v_2' = e^x + e^{-3x},$$

and so integrating, dropping the constants of integration,

$$v_2 = -\tfrac{2}{3}e^{3x/2} + \tfrac{2}{5}e^{-5x/2},$$
$$v_1 = e^x - \tfrac{1}{3}e^{-3x}.$$

Hence a particular solution is

$$y_p = v_1y_1 + v_2y_2 = e^{2x} - \tfrac{1}{3}e^{-2x} - \tfrac{2}{3}e^{2x} + \tfrac{2}{5}e^{-2x} = \tfrac{1}{3}e^{2x} + \tfrac{1}{15}e^{-2x},$$

and so the general solution is

$$y = c_1e^x + c_2e^{x/2} + \tfrac{1}{3}e^{2x} + \tfrac{1}{15}e^{-2x}.$$

The reader can check that the method of undetermined coefficients gives the same answer.

(b) Here equations (12) and (13) become

$$v_1'e^x + v_2'e^{x/2} = 0,$$
$$v_1'e^x + \frac{1}{2}v_2'e^{x/2} = \frac{1}{2}\frac{1}{1+x^2},$$

respectively. Solving,

$$v_2' = -\frac{e^{-x/2}}{1+x^2}, \qquad \text{so} \quad v_2 = -\int\frac{e^{-x/2}}{1+x^2}\,dx$$

and

$$v_1' = \frac{e^{-x}}{1+x^2}, \qquad \text{so} \quad v_1 = \int\frac{e^{-x}}{1+x^2}\,dx.$$

Thus the general solution is

$$y = c_1e^x + c_2e^{-x} + e^x\int\frac{e^{-x}}{1+x^2}\,dx - e^{x/2}\int\frac{e^{-x/2}}{1+x^2}\,dx. \ \blacktriangle$$

Let us now apply the above method to the problem of *forced oscillations*. Imagine that our weight on a spring is subject to a periodic external force $F_0\cos\Omega t$; the spring equation (6) then becomes

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + F_0\cos\Omega t. \tag{14}$$

A periodic force can be directly applied to our bobbing weight by, for example, an oscillating magnetic field. In many engineering situations, equation (14) is used to model the phenomenon of *resonance*; the response of a ship to a periodic swell in the ocean and the response of a bridge to the periodic steps of a marching army are examples of this phenomenon. When the forcing frequency is close to the natural frequency, large oscillations can set in—this is resonance.[9] We shall see this emerge in the subsequent development.

Let us first study the case in which there is no damping: $\gamma = 0$, so that $m(d^2x/dt^2) + kx = F_0\cos\Omega t$. This is called a *forced oscillator equation*. A particular solution can be found by trying $x_p = C\cos\Omega t$ and solving for C. We find $x_p = [F_0/m(\omega^2 - \Omega^2)]\cos\Omega t$, where $\omega = \sqrt{k/m}$ is the frequency of the unforced oscillator. Thus the general solution is

[9] For further information on resonance and how it was involved in the Tacoma bridge disaster of 1940, see M. Braun, *Differential Equations and their Applications*, Second Edition, 1981, Springer-Verlag, New York, Section 2.6.1.

$$x = A\cos\omega t + B\sin\omega t + \frac{F_0}{m(\omega^2-\Omega^2)}\cos\Omega t, \tag{15}$$

where A and B are determined by the initial conditions.

Example 7 Find the solution of $d^2x/dt^2 + 9x = 5\cos 2t$ with $x(0)=0$, $x'(0)=0$, and sketch its graph.

Solution We try a particular solution of the form $x = C\cos 2t$; substituting into the equation gives $-4C\cos 2t + 9C\cos 2t = 5\cos 2t$, so C must be 1. On the other hand, the solution of the homogeneous equation $d^2x/dt^2 + 9x = 0$ is $A\cos 3t + B\sin 3t$, and therefore the general solution of the given equation is $x(t) = A\cos 3t + B\sin 3t + \cos 2t$. For this solution, $x(0) = A + 1$ and $x'(0) = 3B$, so if $x(0)=0$ and $x'(0)=0$, we must have $A=-1$ and $B=0$. Thus, our solution is $x(t) = -\cos 3t + \cos 2t$.

To graph this function, we will use the product formula

$$\sin Rt \sin St = \tfrac{1}{2}\left[\cos(R-S)t - \cos(R+S)t\right].$$

To recover $-\cos 3t + \cos 2t$, we must have $R+S=3$ and $R-S=2$, so $R=\frac{5}{2}$ and $S=\frac{1}{2}$. Thus

$$x(t) = 2\sin(\tfrac{5}{2}t)\sin(\tfrac{1}{2}t).$$

We may think of this as a rapid oscillation, $\sin\frac{5}{2}t$, with variable amplitude $2\sin\frac{1}{2}t$, as illustrated in Fig. 12.7.4. The function is periodic with period 2π, with a big peak coming at π, 3π, 5π, etc., when $-\cos 3t$ and $\cos 2t$ are simultaneously equal to 1. ▲

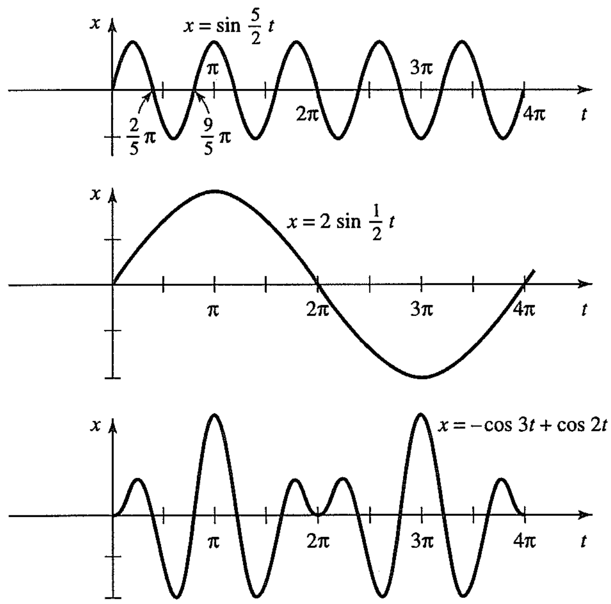


Figure 12.7.4. $x(t) = -\cos 3t + \cos 2t = 2\sin\frac{1}{2}t\sin\frac{5}{2}t$.

If in equation (15), $x(0)=0$ and $x'(0)=0$, then we find, as in Example 7, that

$$\begin{aligned} x(t) &= \frac{F_0}{m(\omega^2-\Omega^2)}(\cos\Omega t - \cos\omega t) \\ &= \frac{2F_0}{m(\omega^2-\Omega^2)}\left[\sin\left(\frac{(\omega-\Omega)t}{2}\right)\right]\left[\sin\left(\frac{(\omega+\Omega)t}{2}\right)\right]. \end{aligned}$$

If $\omega - \Omega$ is small, then this is the product of a relatively rapidly oscillating function $[\sin((\omega + \Omega)t/2)]$ with a slowly oscillating one $[\sin((\omega - \Omega)t/2)]$. The slowly oscillating function "modulates" the rapidly oscillating one as shown in Fig. 12.7.5. The slow rise and fall in the amplitude of the fast oscillation is the phenomenon of *beats*. It occurs, for example, when two musical instruments are played slightly out of tune with one another.

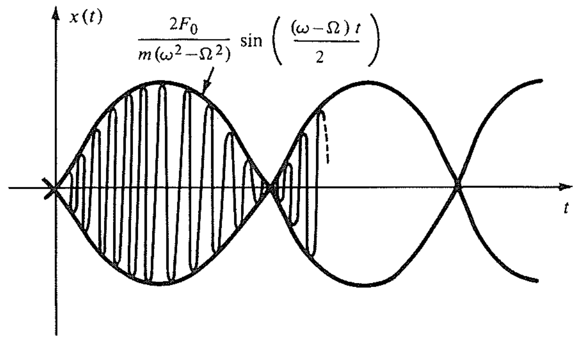


Figure 12.7.5. Beats.

The function (15) is the solution to equation (14) in the case where $\gamma = 0$ (no damping). The general case ($\gamma \neq 0$) is solved similarly. The method of undetermined coefficients yields a particular solution of the form $x_p = \alpha\cos\Omega t + \beta\sin\Omega t$, which is then added to the general solution of the homogeneous equation found by the method of Example 4. We state the result of such a calculation in the following box and ask the reader to verify it in Review Exercise 110.

Damped Forced Oscillations

The solution of

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + F_0\cos\Omega t$$

is

$$x(t) = c_1e^{r_1t} + c_2e^{r_2t} + \frac{F_0}{\sqrt{m^2(\omega^2-\Omega^2)^2 + \gamma^2\omega^2}}\cos(\Omega t - \delta), \qquad (16)$$

where c_1 and c_2 are constants determined by the initial conditions, $\omega = \sqrt{k/m}$, r_1 and r_2 are roots of the characteristic equation $mr^2 + \gamma r + k = 0$ [if r_1 is a repeated root, replace $c_1e^{r_1t} + c_2e^{r_2t}$ by $(c_1 + c_2t)e^{r_1t}$], and

$$\delta = \tan^{-1}\left(\frac{\gamma\Omega}{m(\omega^2-\Omega^2)}\right),$$

$$\cos\delta = \frac{m(\omega^2-\Omega^2)}{\sqrt{m^2(\omega^2-\Omega^2)^2+\gamma^2\Omega^2}}, \qquad \sin\delta = \frac{\gamma\Omega}{\sqrt{m^2(\omega^2-\Omega^2)^2+\gamma^2\Omega^2}}$$

In equation (16), as $t \to \infty$, the solution $c_1e^{r_1t} + c_2e^{r_2t}$ tends to zero (if $\gamma > 0$) as we have seen. This is called the *transient part*; the solution thus approaches the *oscillatory part*,

$$\frac{F_0}{\sqrt{m^2(\omega^2-\Omega^2)^2+\gamma^2\Omega^2}}\cos(\Omega t-\delta),$$

which oscillates with a modified amplitude at the forcing frequency Ω and with the *phase shift* δ. If we vary ω, the amplitude is largest when $\omega=\Omega$; this is the *resonance* phenomenon.

Example 8 Consider the equation

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+25x=2\cos t.$$

(a) Write down the solution with $x(0)=0$, $x'(0)=0$.
(b) Discuss the behavior of the solution for large t.

Solution (a) The characteristic equation is

$$r^2+8r+25=0$$

which has roots $r=(-8\pm\sqrt{64-100})/2=-4\pm 3i$. Also, $m=1$, $\omega=5$, $\Omega=1$, $F_0=2$, and $\gamma=8$; so

$$\frac{F_0}{\sqrt{m^2(\omega^2-\Omega^2)^2+\gamma^2\Omega^2}}=\frac{2}{\sqrt{576+64}}=\frac{2}{\sqrt{640}}=\frac{1}{4\sqrt{10}}\approx 0.079$$

and $\delta=\tan^{-1}(\frac{8}{24})=\tan^{-1}(\frac{1}{3})\approx 0.322$. The general solution is given by equation (16); writing sines and cosines in place of the complex exponentials, we get

$$x(t)=e^{-4t}(A\cos 3t+B\sin 3t)+\frac{1}{4\sqrt{10}}\cos(t-\delta).$$

At $t=0$ we get

$$\begin{aligned}0=x(0)&=A+\frac{1}{4\sqrt{10}}\cos\delta\\&=A+\frac{1}{4\sqrt{10}}\cdot\frac{24}{\sqrt{640}}\\&=A+\frac{3}{40};\end{aligned}$$

so $A=-\frac{3}{40}$. Computing $x'(t)$ and substituting $t=0$ gives

$$\begin{aligned}0=x'(0)&=-4A+3B+\frac{1}{4\sqrt{10}}\sin\delta\\&=-4A+3B+\frac{1}{4\sqrt{10}}\cdot\frac{8}{\sqrt{640}}\\&=-4A+3B+\frac{1}{40}=\frac{12}{40}+3B+\frac{1}{40}.\end{aligned}$$

Thus $B=-\frac{13}{120}$, and our solution becomes

$$x(t)=-\frac{e^{-4t}}{120}(9\cos 3t+13\sin 3t)+0.079\cos(t-0.322).$$

(b) As $t\to\infty$ the transient part disappears and we get the oscillatory part $0.079\cos(t-0.322)$. ▲

Supplement to Section 12.7: Wronskians

In this section we have shown how to find solutions to equation (1): $ay'' + by' + cy = 0$; whether the roots of the characteristic equation $ar^2 + br + c = 0$ are real, complex, or coincident, we found two solutions y_1 and y_2. We then asserted that the *linear combination* $c_1 y_1 + c_2 y_2$ represents the *general* solution. In this supplement we shall prove this.

Suppose that y_1 and y_2 are solutions of equation (1); our goal is to show that every solution y of equation (1) can be written as $y = c_1 y_1 + c_2 y_2$. To do so, we try to find c_1 and c_2 by matching initial conditions at x_0:

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0),$$
$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0).$$

We can solve these equations for c_1 by multiplying the first equation by $y_2'(x_0)$, the second by $y_2(x_0)$, and subtracting:

$$c_1 = \frac{y(x_0)y_2'(x_0) - y_2(x_0)y'(x_0)}{y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)}. \tag{17a}$$

Similarly,

$$c_2 = \frac{y(x_0)y_1'(x_0) - y_1(x_0)y'(x_0)}{y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)}. \tag{17b}$$

These are valid as long as $y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$. The expression

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x). \tag{18}$$

is called the *Wronskian* of y_1 and y_2 [named after the Polish mathematician Count Hoëné Wronski (1778–1853)]. (The expression (18) is a determinant—see Exercise 43, Section 13.6).

Two solutions y_1 and y_2 are said to be a *fundamental set* if their Wronskian does not vanish. It is an important fact that $W(x)$ *is either everywhere zero or nowhere zero*. To see this, we compute the derivative of W:

$$\begin{aligned} W'(x) &= \left[y_1'(x)y_2'(x) + y_1(x)y_2''(x) \right] - \left[y_2'(x)y_1'(x) + y_2(x)y_1''(x) \right] \\ &= y_1(x)y_2''(x) - y_2(x)y_1''(x). \end{aligned}$$

If y_1 and y_2 are solutions, we can substitute $-(b/a)y_1' - (c/a)y_1$ for y_1'' and similarly for y_2'' to get

$$\begin{aligned} W'(x) &= y_1(x)\left[-\frac{b}{a}y_2'(x) - \frac{c}{2}y_2(x) \right] - y_2(x)\left[-\frac{b}{a}y_1'(x) - \frac{c}{a}y_1(x) \right] \\ &= -\frac{b}{a}\left[y_1(x)y_2'(x) - y_2(x)y_1'(x) \right]. \end{aligned}$$

Thus

$$W'(x) = -\frac{b}{a}W(x).$$

Therefore, from Section 8.2, $W(x) = Ke^{-(b/a)x}$ for some constant K. We note that W is nowhere zero unless $K = 0$, in which case it is identically zero.

If y_1 and y_2 are a fundamental set, then equation (17) makes sense, and so we can find c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ attains any given initial conditions. Such a specification of initial conditions gives a unique solution and determines y uniquely; therefore $y = c_1 y_1 + c_2 y_2$. In fact, the proof of uniqueness of a solution given its initial conditions also follows fairly easily from what we have done; see Exercise 46 for a special case and Exercise 47 for the general case. Thus, in summary, we have proved:

If y_1 and y_2 are a fundamental set of solutions for $ay'' + by' + cy = 0$, then $y = c_1 y_1 + c_2 y_2$ is the general solution.

To complete the justification of the claims about general solutions made earlier in this section, we need only check that in each specific case, the solutions form a fundamental set. For example, suppose that r_1 and r_2 are distinct roots of $ar^2 + br + c = 0$. We know that $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions. To check that they form a fundamental set, we compute

$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\ &= e^{r_1 x} \cdot r_2 e^{r_2 x} - e^{r_2 x} \cdot r_1 e^{r_1 x} \\ &= (r_2 - r_1)e^{(r_1 + r_2)x}. \end{aligned}$$

This is nonzero since $r_2 \neq r_1$, so we have a fundamental set. One can similarly check the case of a repeated root (Exercise 45).

Exercises for Section 12.7

Find the general solution of the differential equations in Exercises 1–4.

1. $y'' - 4y' + 3y = 0.$
2. $2y'' - y = 0.$
3. $3y'' - 4y' + y = 0.$
4. $y'' - y' - 2y = 0.$

Find the particular solutions of the stated equations in Exercises 5–8 satisfying the given conditions.

5. $y'' - 4y' + 3y = 0, y(0) = 0, y'(0) = 1.$
6. $2y'' - y = 0, y(1) = 0, y'(1) = 1.$
7. $3y'' - 4y' + y = 0, y(0) = 1, y'(0) = 1.$
8. $y'' - y' - 2y = 0, y(1) = 0, y'(1) = 2.$

Find the general solution of the differential equations in Exercises 9–12.

9. $y'' - 4y' + 5y = 0.$
10. $y'' + 2y' + 5y = 0.$
11. $y'' - 6y' + 13y = 0.$
12. $y'' + 2y' + 26y = 0.$

Find the solution of the equations in Exercises 13–16 satisfying the given conditions.

13. $y'' - 6y' + 9y = 0, y(0) = 0, y'(0) = 1.$
14. $y'' - 8y' + 16y = 0, y(0) = -3, y'(0) = 0.$
15. $y'' - 2\sqrt{2}\,y' + 2y = 0, y(1) = 0, y'(1) = 1.$
16. $y'' - 2\sqrt{3}\,y' + 3y = 0, y(0) = 0, y'(0) = -1.$

In Exercises 17–20 consider a spring with β, ω, $x(0)$, and $x'(0)$ as given. (a) Determine if the spring is over, under, or critically damped. (b) Find and sketch the solution.

17. $\beta = \pi/16, \omega = \pi/2, x(0) = 0, x'(0) = 1.$
18. $\beta = 1, \omega = \pi/8, x(0) = 1, x'(0) = 0.$
19. $\beta = \pi/3, \omega = \pi/6, x(0) = 0, x'(0) = 1.$
20. $\beta = 0.03, \omega = \pi/2, x(0) = 1, x'(0) = 1.$

In Exercises 21–28, find the general solution to the given equation (y is a function of x or x is a function of t as appropriate).

21. $y'' - 4y' + 3y = 6x + 10.$
22. $y'' - 4y' + 3y = 2e^x.$
23. $3x'' - 4x' + x = 2\sin t.$
24. $3x'' - 4x' + x = e^t + e^{-t}.$
25. $y'' - 4y' + 5y = x + x^2.$
26. $y'' - 4y' + 5y = 10 + e^{-x}.$
27. $y'' - 2\sqrt{2}\,y' + 2y = \cos x + \sin x.$
28. $y'' - 2\sqrt{2}\,y' + 2y = \cos x - e^{-x}.$

In Exercises 29–32 find the general solution to the given equation using the method of variation of parameters.

29. $y'' - 4y' + 3y = 6x + 10.$
30. $y'' - 4y' + 3y = 2e^x.$
31. $3x'' - 4x' + x = 2\sin t.$
32. $3x'' - 4x' + x = e^t + e^{-t}.$

In Exercises 33–36 find the general solution to the given equation. Express your answer in terms of integrals if necessary.

33. $y'' - 4y' + 3y = \tan x.$
34. $y'' - 4y' + 3y = \dfrac{1}{x^2 + 2}.$
35. $y'' - 4y' + 5y = \dfrac{1}{1 + \cos^2 x}.$
36. $y'' - 4y' + 5y = \dfrac{e^x}{1 + x^2}.$

In Exercises 37–40, find the solution of the given forced oscillator equation satisfying the given initial conditions.

37. $x'' + 4x = 3\cos t, x(0) = 0, x'(0) = 0.$
38. $x'' + 9x = 4\sin 4t, x(0) = 0, x'(0) = 0.$
39. $x'' + 25x = \cos t, x(0) = 0, x'(0) = 1.$
40. $x'' + 25x = \cos 6t, x(0) = 1, x'(0) = 0.$

In Exercises 41–44, (a) write down the solution of the given equation with the stated initial conditions and (b) discuss the behavior of the solution for large t.

41. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 25x = 2\cos 2t, \; x(0) = 0, \; x'(0) = 0.$
42. $\dfrac{dx^2}{dt^2} + 2\dfrac{dx}{dt} + 36x = 4\cos 3t, \; x(0) = 0, \; x'(0) = 0.$
43. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 4x = \cos t, x(0) = 1, x'(0) = 0.$
44. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 9x = \cos 4t, x(0) = 1, x'(0) = 0.$
45. If r_1 is a repeated root of the characteristic equation, use the Supplement to this section to show that $y_1 = e^{r_1 x}$ and $y_2 = xe^{r_1 x}$ form a fundamental set and hence conclude that $y = c_1 y_1 + c_2 y_2$ is the general solution to equation (1).

★46. Suppose that in (1), $a > 0$ and $b^2 - 4ac < 0$. If y satisfies equation (1), prove $w(x) = e^{(b/2a)x}y(x)$ satisfies $w'' + [(4ac - b^2)/4a^2]w = 0$, which is a spring equation. Use this observation to do the following.
(a) Derive the general form of the solution to equation (1) if the roots are complex.
(b) Use existence and uniqueness results for the spring equation proved in Section 8.1 to prove corresponding results for equation (1) if the roots are complex.

★47. If we know that equation (1) admits a fundamental set y_1, y_2, show uniqueness of solutions to equation (1) with given values of $y(x_0)$ and $y'(x_0)$ as follows.
(a) Demonstrate that it is enough to show that if $y(x_0) = 0$ and $y'(x_0) = 0$, then $y(x) \equiv 0$.
(b) Use facts above the Wronskian proved in the Supplement in order to show that $y(x)y_1'(x) - y'(x)y_1(x) = 0$ and that $y(x)y_2'(x) - y'(x)y_2(x) = 0$.
(c) Solve the equations in (b) to show that $y(x) = 0$.

★48. (a) Generalize the method of reduction of order so it applies to the differential equation $a(x)y'' + b(x)y' + c(x)y = 0$, $a(x) \neq 0$. Thus, given one solution, develop a method for finding a second one.
(b) Show that x^r is a solution of *Euler's* equation $x^2y'' + \alpha xy' + \beta y = 0$ if $r^2 + (\alpha - 1)r + \beta = 0$.
(c) Use (a) to show that if $(\alpha - 1)^2 = 4\beta$, then $(\ln x)x^{(1-\alpha)/2}$ is a second solution.

★49. (a) Show that the basic facts about Wronskians, fundamental sets, and general solutions proved in the Supplement also apply to the equation in Exercise 48(a).
(b) Show that the solutions of Euler's equations found in Exercises 48(b) and 48(c) form a fundamental set. Write down the general solution in each case.

★50. (a) Generalize the method of variation of parameters to the equation $a(x)y'' + b(x)y' + c(x)y = F(x)$.
(b) Find the general solution to the equation $x^2y'' + 5xy' + 3y = xe^x$ (see Exercise 48; express your answer in terms of integrals if necessary).

★51. In Fig. 12.7.1, consider the motion relative to an arbitrarily placed x axis pointing downward.
(a) Taking all forces, including the constant force g of gravity into account, show that the equation of motion is

$$m\frac{d^2y}{dt^2} = -k(y - y_e) + g - \gamma\frac{dy}{dt},$$

where y_e is the equilibrium position of the spring in the absence of the mass.
(b) Make a change of variables $x = y + c$ to reduce this equation to equation (6).

★52. Show that solutions of equation (15) exhibit beats, without assuming that $x(0) = 0$ and $x'(0) = 0$.

★53. Find the general solution of $y'''' + y = 0$.

★54. Find the general solution of $y'''' - y = 0$.

★55. Find the general solution of $y'''' + y = e^x$.

★56. Find the general solution of $y'''' - y = \cos x$.

12.8 Series Solutions of Differential Equations

Power series solutions of differential equations can often be found by the method of undetermined coefficients.

Many differential equations cannot be solved by means of explicit formulas. One way of attacking such equations is by the numerical methods discussed in Section 8.5. In this section, we show how to use infinite series in a systematic way for solving differential equations.

Many equations arising in engineering and mathematical physics can be treated by this method. We shall concentrate on equations of the form $a(x)y'' + b(x)y' + c(x)y = f(x)$, which are similar to equation (1) in Section 12.7, except that a, b, and c are now functions of x rather than constants. The basic idea in the power series method is to consider the a_i's in a sum $y = \sum_{i=0}^{\infty} a_i x^i$ as undetermined coefficients and to solve for them in successive order.

Example 1 Find a power series solution of $y'' + xy' + y = 0$.

Solution If a solution has a convergent series of the form $y = a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$, we may use the results of Section 12.4 to write

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{i=1}^{\infty} ia_ix^{i-1} \quad \text{and}$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \cdots = \sum_{i=2}^{\infty} i(i-1)a_ix^{i-2}.$$

Therefore

$$y'' + xy' + y = \sum_{i=2}^{\infty} i(i-1)a_ix^{i-2} + \sum_{i=1}^{\infty} ia_ix^i + \sum_{i=0}^{\infty} a_ix^i = 0.$$

In performing manipulations with series, it is important to keep careful track of the summation index; writing out the first few terms explicitly usually helps. Thus,

$$\begin{aligned} y'' &+ xy' + y \\ &= \left(2a_2 + 6a_3x + 12a_4x^2 + \cdots\right) + \left(a_1x + 2a_2x^2 + \cdots\right) \\ &\quad + \left(a_0 + a_1x + a_2x^2 + \cdots\right). \end{aligned}$$

To write this in summation notation, we shift the summation index so all x's appear with the same exponent:

$$y'' + xy' + y = \sum_{i=0}^{\infty} (i+2)(i+1)a_{i+2}x^i + \sum_{i=1}^{\infty} ia_ix^i + \sum_{i=0}^{\infty} a_ix^i = 0.$$

(Check the first few terms from the explicit expression.) Now we set the coefficient of each x^i equal to zero in an effort to determine the a_i. The first two conditions are

$$\begin{aligned} 2a_2 + a_0 &= 0 \qquad \text{(constant term)}, \\ 6a_3 + 2a_1 &= 0 \qquad \text{(coefficient of } x). \end{aligned}$$

Note that this determines a_2 and a_3 in terms of a_0 and a_1: $a_2 = -\frac{1}{2}a_0$, $a_3 = -\frac{1}{3}a_1$. For $i \geqslant 1$, equating the coefficient of x^i to zero gives

$$(i+2)(i+1)a_{i+2} + (i+1)a_i = 0$$

or

$$a_{i+2} = -\frac{1}{(i+2)}a_i.$$

Thus,

$$\begin{aligned} a_2 &= -\frac{1}{2}a_0 & a_3 &= -\frac{1}{3}a_1 \\ a_4 &= -\frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0 & a_5 &= -\frac{1}{5}a_3 = \frac{1}{5\cdot 3}a_1 \\ a_6 &= -\frac{1}{6}a_4 = -\frac{1}{6\cdot 4\cdot 2}a_0 & a_7 &= -\frac{1}{7}a_5 = \frac{-1}{7\cdot 5\cdot 3}a_1 \\ &\vdots \end{aligned}$$

Hence

$$a_{2n} = \frac{(-1)^n}{2n\cdot(2n-2)\cdot(2n-4)\ldots 4\cdot 2}a_0$$

$$= \frac{(-1)^n}{2 \ldots 2 \cdot n(n-1) \ldots 2 \cdot 1} a_0 = \frac{(-1)^n}{2^n n!} a_0$$

and

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)(2n-1)(2n-3) \ldots 5 \cdot 3} a_1 = \frac{(-1)^n 2^n n!}{(2n+1)!} a_1 .$$

Thus, we get (using $0! = 1$),

$$y = a_0\left(\sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{2^i i!} \right) + a_1\left(\sum_{i=0}^{\infty} (-1)^i \frac{2^i i!\, x^{2i+1}}{(2i+1)!} \right). \tag{1}$$

What we have shown so far is that any convergent series solution must be of the form of equation (1). To show that equation (1) *really is* a solution, we must show that the given series converges; but this convergence follows from the ratio test. ▲

The constants a_0 and a_1 found in Example 1 are arbitrary and play the same role as the two arbitrary constants we found for the solutions of constant coefficient equations in the preceding section.

Example 2 Find the first four nonzero terms in the power series solution of $y'' + x^2y = 0$ satisfying $y(0) = 0$, $y'(0) = 1$.

Solution Let $y = a_0 + a_1x + a_2x^2 + \cdots$. The initial conditions $y(0) = 0$ and $y'(0) = 1$ can be put in immediately if we set $a_0 = 0$ and $a_1 = 1$, so that $y = x + a_2x^2 + \cdots$. Then

$$y'' = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots + (i+1)ia_{i+1}x^{i-1} + \cdots$$

and so

$$x^2y = x^3 + a_2x^4 + a_3x^5 + \cdots + a_{i-3}x^{i-1} + \cdots .$$

Setting $y'' + x^2y = 0$ gives

$$\begin{aligned}
&a_2 = 0 && \text{(constant term)},\\
&a_3 = 0 && \text{(coefficient of } x),\\
&a_4 = 0 && \text{(coefficient of } x^2),\\
&a_5 = -\frac{1}{5 \cdot 4} && \text{(coefficient of } x^3),\\
&a_6 = 0 = a_7 = a_8 && \text{(coefficients of } x^4, x^5, x^6),\\
&a_9 = -\frac{1}{9 \cdot 8} a_5 = \frac{1}{9 \cdot 8 \cdot 5 \cdot 4} && \text{(coefficient of } x^7), \text{ etc.}
\end{aligned}$$

Thus, the first four nonzero terms are

$$y = x - \frac{1}{5 \cdot 4} x^5 + \frac{1}{9 \cdot 8 \cdot 5 \cdot 4} x^9 - \frac{1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{13} + \cdots .$$

[The recursion relation is

$$a_{i+1} = -\frac{1}{i(i+1)} a_{i-3}$$

and the general term is

$$(-1)^j \frac{1}{(4j+1)(4j) \ldots 9 \cdot 8 \cdot 5 \cdot 4} x^{4j+1}.$$

The ratio test shows that this series converges.] ▲

Example 3 **(Legendre's equation)**[10] Find the recursion relation and the first few terms for the solution of $(1-x^2)y'' - 2xy' + \lambda y = 0$ as a power series.

Solution We write $y = \sum_{i=0}^{\infty} a_i x^i$ and get

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + \cdots + a_i x^i + \cdots, \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + i a_i x^{i-1} + \cdots, \\
-2xy' &= -2a_1 x - 2\cdot 2a_2 x^2 - 2\cdot 3a_3 x^3 - \cdots - 2i a_i x^i - \cdots, \\
y'' &= 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots + i(i-1)a_i x^{i-2} + \cdots, \\
-x^2 y'' &= -2a_2 x^2 - 3\cdot 2a_3 x^3 - \cdots - i(i-1)a_i x^i - \cdots.
\end{aligned}$$

Thus, setting $(1-x^2)y'' - 2xy' + \lambda y = 0$ gives

$$\begin{aligned}
2a_2 + \lambda a_0 &= 0 && \text{(constant term)}, \\
3\cdot 2a_3 - 2a_1 + \lambda a_1 &= 0 && \text{(coefficient of } x), \\
4\cdot 3a_4 - 2a_2 - 4a_2 + \lambda a_2 &= 0 && \text{(coefficient of } x^2), \\
5\cdot 4a_5 - 3\cdot 2a_3 - 2\cdot 3a_3 + \lambda a_3 &= 0 && \text{(coefficient of } x^3), \\
&\vdots
\end{aligned}$$

Solving,

$$\begin{aligned}
a_2 &= -\frac{\lambda}{2}a_0, \qquad a_3 = \frac{2-\lambda}{3\cdot 2}a_1, \\
a_4 &= \frac{6-\lambda}{4\cdot 3}a_2 = -\frac{6-\lambda}{4\cdot 3}\,\frac{\lambda}{2}a_0, \\
a_5 &= \frac{12-\lambda}{5\cdot 4}a_3 = \frac{12-\lambda}{5\cdot 4}\cdot\frac{2-\lambda}{3\cdot 2}a_1, \text{ etc.}
\end{aligned}$$

Thus, the solution is

$$\begin{aligned}
y = a_0\left(1 - \frac{\lambda}{2}x^2 - \frac{(6-\lambda)}{4\cdot 3\cdot 2}x^4 + \cdots\right) \\
+ a_1\left(x + \frac{2-\lambda}{3\cdot 2}x^3 + \frac{(12-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}x^5 + \cdots\right).
\end{aligned}$$

The recursion relation comes from setting the coefficient of x^i equal to zero:

$$(i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + \lambda a_i = 0,$$

so

$$a_{i+2} = \frac{i(i+1)-\lambda}{(i+2)(i+1)}a_i.$$

From the ratio test one sees that the series solution has a radius of convergence of at least 1. It is exactly 1 unless there is a nonnegative integer n such that $\lambda = n(n+1)$, in which case the series can terminate: if n is even, set $a_1 = 0$; if n is odd, set $a_0 = 0$. Then the solution is a polynomial of degree n called *Legendre's polynomial*; it is denoted $P_n(x)$. The constant is fixed by demanding $P_n(1) = 1$. ▲

[10] This equation occurs in the study of wave phenomena and quantum mechanics using spherical coordinates (see Section 14.5).

Example 4 **(Hermite's equation)**[11] Find the recursion relation and the first few terms for the solution of $y'' - 2xy' + \lambda y = 0$ as a power series.

Solution Again write $y = a_0 + a_1x + a_2x^2 + \cdots + a_ix^i + \cdots$, so

$$\lambda y = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_i x^i + \cdots,$$

$$-2xy' = -2a_1x - 4a_2x^2 - 2\cdot 3a_3x^3 - \cdots - 2ia_ix^i - \cdots,$$

and

$$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \cdots + i(i-1)a_ix^{i-2} + \cdots.$$

Setting the coefficients of powers of x to zero in $y'' - 2xy' + y = 0$, we get

$$\begin{aligned} 2a_2 + \lambda a_0 &= 0 &&\text{(constant term)},\\ 3\cdot 2a_3 - 2a_1 + \lambda a_1 &= 0 &&\text{(coefficient of } x),\\ 4\cdot 3a_4 - 4a_2 + \lambda a_2 &= 0 &&\text{(coefficient of } x^2), \end{aligned}$$

and in general

$$(i+2)(i+1)a_{i+2} - 2ia_i + \lambda a_i = 0.$$

Thus

$$a_2 = -\frac{\lambda}{2}a_0, \qquad a_3 = \frac{2-\lambda}{3\cdot 2}a_1,$$

$$a_4 = \frac{4-\lambda}{4\cdot 3}a_2 = -\frac{\lambda(4-\lambda)}{4\cdot 3\cdot 2}a_0,$$

and in general,

$$a_{i+2} = \frac{2i-\lambda}{(i+2)(i+1)}a_i.$$

Thus

$$a_5 = \frac{6-\lambda}{5\cdot 4}a_3 = \frac{(6-\lambda)(2-\lambda)}{5!}a_1,$$

etc., and so

$$y = a_0\left(1 - \frac{\lambda}{2}x^2 - \frac{(4-\lambda)\lambda}{4!}x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!}x^6 - \cdots\right)$$
$$+ a_1\left(x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7!} + \cdots\right.$$

This series converges for all x. If λ is an even integer, one of the series, depending on whether or not λ is a multiple of 4, terminates, and so we get a polynomial solution (called a Hermite polynomial). ▲

Sometimes the power series method runs into trouble—it may lead to only one solution, or the solution may not converge (see below and Exercise 23 for examples). To motivate the method that follows, which is due to Georg Frobenius (1849–1917), we consider Euler's equation:

$$x^2y'' + \alpha xy' + \beta y = 0.$$

Here we could try $y = a_0 + a_1x + a_2x^2 + \cdots$ as before, but as we will now show, this leads nowhere. To be specific, we choose $\alpha = \beta = 1$. Write

[11] This equation arises in the quantum mechanics of a harmonic oscillator.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

$$xy' = a_1x + 2a_2x^2 + 3a_3x^3 + \cdots,$$

and

$$x^2y'' = 2a_2x^2 + 3\cdot 2a_3x^3 + \cdots.$$

Setting $x^2y'' + xy' + y = 0$, we get

$$\begin{aligned} a_0 &= 0 && \text{(constant term)},\\ 2a_1 &= 0 && \text{(coefficient of } x),\\ 5a_2 &= 0 && \text{(coefficient of } x^2),\\ 10a_3 &= 0 && \text{(coefficient of } x^3)\text{, etc.,} \end{aligned}$$

and so all of the a_i's are zero and we get only a trivial solution.

The difficulty can be traced to the fact that the coefficient of y'' vanishes at the point $x = 0$ about which we are expanding our solution. One can, however, try to find a solution of the form x^r. Letting $y = x^r$, where r need not be an integer, we get

$$y' = rx^{r-1} \quad \text{so} \quad \alpha xy' = \alpha rx^r$$

and

$$y'' = r(r-1)x^{r-2} \quad \text{so} \quad x^2y'' = r(r-1)x^r.$$

Thus, Euler's equation is satisfied if

$$r(r-1) + \alpha r + \beta = 0$$

which is a quadratic equation for r with, in general, two solutions. (See Exercise 48 of Section 12.7 for the case when the roots are coincident.)

Frobenius' idea is that, by analogy with the Euler equation, we should look for solutions of the form $y = x^r\sum_{i=0}^{\infty}a_ix^i$ whenever the coefficient of y'' in a second-order equation vanishes at $x = 0$. Of course, r is generally not an integer; otherwise we would be dealing with ordinary power series.

Example 5 Find the first few terms in the general solution of $4xy'' - 2y' + y = 0$ using the Frobenius method.

Solution We write

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots,$$

so $-2y' = -2ra_0x^{r-1} - 2(r+1)a_1x^r - 2(r+2)a_2x^{r+1} - \cdots$ and

$$4xy'' = 4r(r-1)a_0x^{r-1} + 4(r+1)ra_1x^r + 4(r+2)(r+1)a_2x^{r+1} + \cdots.$$

Thus to make $4xy'' - 2y' + y = 0$, we set

$$a_0\left[4r(r-1) - 2r\right] = 0 \qquad \text{(coefficient of } x^{r-1}).$$

If a_0 is to be allowed to be nonzero (which we desire, to avoid the difficulty encountered in our discussion of Euler's equation), we set $4r(r-1) - 2r = 0$. Thus $r(4r-6) = 0$, so $r = 0$ or $r = \frac{3}{2}$. First, we take the case $r = 0$:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

$$-2y' = -2a_1 - 2\cdot 2a_2x - 3\cdot 2a_3x^2 - \cdots,$$

$$4xy'' = 4\cdot 2a_2x + 4\cdot 6a_3x^2 + 4\cdot 12a_4x^3 + \cdots.$$

Then $4xy'' - 2y' + y = 0$ gives

$$a_0 - 2a_1 = 0 \qquad \text{(constant term)},$$
$$4 \cdot 2a_2 - 2 \cdot 2a_2 + a_1 = 0 \qquad \text{(coefficient of } x\text{)},$$
$$4 \cdot 6a_3 - 3 \cdot 2a_3 + a_2 = 0 \qquad \text{(coefficient of } x^2\text{)};$$

so $a_1 = \frac{1}{2}a_0$, $a_2 = -\frac{1}{4}a_1 = -\frac{1}{8}a_0$, and $a_3 = -\frac{1}{18}a_2 = \frac{1}{144}a_0$.

Thus

$$y = a_0\left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{144}x^3 - \cdots\right).$$

For the case $r = \frac{3}{2}$, we have

$$y = a_0x^{3/2} + a_1x^{5/2} + a_2x^{7/2} + a_3^{9/2} + \cdots,$$
$$-2y' = -3a_0x^{1/2} - 5a_1x^{3/2} - 7a_2x^{5/2} - 9a_3x^{7/2} - \cdots,$$

and

$$4xy'' = 3a_0x^{1/2} + 5 \cdot 3a_1x^{3/2} + 7 \cdot 5a_2x^{5/2} + 9 \cdot 7a_3x^{7/2} + \cdots.$$

Equating coefficients of $4x'' - 2y' + y = 0$ to zero gives

$$3a_0 - 3a_0 = 0 \qquad \text{(coefficient of } x^{1/2}\text{)},$$
$$5 \cdot 3a_1 - 5a_1 + a_0 = 0 \qquad \text{(coefficient of } x^{3/2}\text{)},$$
$$7 \cdot 5a_2 - 7a_2 + a_1 = 0 \qquad \text{(coefficient of } x^{5/2}\text{)},$$
$$9 \cdot 7a_3 - 9a_3 + a_2 = 0 \qquad \text{(coefficient of } x^{7/2}\text{)};$$

so

$$a_1 = -\frac{1}{10}a_0, \quad a_2 = -\frac{a_1}{28} = \frac{1}{280}a_0,$$

and

$$a_3 = -\frac{a_2}{54} = -\frac{1}{280 \cdot 54}a_0.$$

Thus

$$y = a_0\left(x^{3/2} - \frac{1}{10}x^{5/2} + \frac{1}{280}x^{7/2} - \frac{1}{280 \cdot 54}x^{9/2} + \cdots\right).$$

The general solution is a linear combination of the two we have found:

$$y = c_1\left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \frac{1}{144}x^3 - \cdots\right)$$
$$+ c_2\left(x^{3/2} - \frac{1}{10}x^{5/2} + \frac{1}{280}x^{7/2} - \cdots\right). \blacktriangle$$

The equation that determines r, obtained by setting the coefficient of the lowest power of x in the equation to zero, is called the *indicial equation*.

The Frobenius method requires modification in two cases. First of all, if the indicial equation has a repeated root r_1, then there is one solution of the form $y_1(x) = a_0x^{r_1} + a_1x^{r_1+1} + \cdots$ and there is a second of the form $y_2(x) = y_1(x)\ln x + b_0x^{r_1} + b_1x^{r_1+1} + \cdots$. This second solution can also be found by the method of reduction of order. (See Exercise 48, Section 12.7.) Second, if the roots of the indicial equation differ by an integer, the method may again lead to problems: one may or may not be able to find a genuinely new solution. If $r_2 = r_1 + N$, then the second series $b_0x^{r_2} + b_1x^{r_2+1} + \cdots$ is of the same form, $a_0x^{r_1} + a_1x^{r_1+1} + \cdots$, with the first N coefficients set equal to zero. Thus it would require very special circumstances to obtain a second solution this way. (If the method fails, one can use reduction of order, but this may lead to a complicated computation).

We conclude with an example where the roots of the indicial equation differ by an integer.

Example 6 Find the general solution of Bessel's equation[12] $x^2y'' + xy' + (x^2 - k^2)y = 0$ with $k = \frac{1}{2}$.

Solution We try $y = x^r\sum_{i=0}^{\infty} a_i x^i$. Then

$$y = \sum_{i=0}^{\infty} a_i x^{i+r},$$

$$x^2y = \sum_{i=0}^{\infty} a_i x^{i+r+2} = \sum_{i=2}^{\infty} a_{i-2}x^{i+r},$$

$$y' = \sum_{i=0}^{\infty} (i+r)a_i x^{i+r-1} = \sum_{i=-1}^{\infty} (i+r+1)a_{i+1}x^{i+r},$$

$$xy' = \sum_{i=0}^{\infty} (i+r)a_i x^{i+r},$$

$$y'' = \sum_{i=-2}^{\infty} (i+r+1)(i+r+2)a_{i+2}x^{i+r},$$

and

$$x^2y'' = \sum_{i=0}^{\infty} (i+r-1)(i+r)a_i x^{i+r}.$$

Setting the coefficient of x^r in $x^2y'' + xy' + (x^2 - k^2)y = 0$ equal to zero, we get

$$0 = (r-1)ra_0 + ra_0 - \tfrac{1}{4}a_0,$$

so the indicial equation is $0 = (r-1)r + r - \frac{1}{4} = r^2 - \frac{1}{4}$, the roots of which are $r_1 = -\frac{1}{2}$ and $r_2 = \frac{1}{2}$, which differ by the integer 1.

Setting the coefficient of x^{r+1} equal to zero gives

$$0 = r(r+1)a_1 + (r+1)a_1 - \tfrac{1}{4}a_1 = \left[(r+1)^2 - \tfrac{1}{4}\right]a_1,$$

and the general recursion relation arising from the coefficient of x^{i+r}, $i \geqslant 2$, is

$$0 = r(r+i)a_i + (r+i)a_i - \tfrac{1}{4}a_i + a_{i-2} = \left((r+i)^2 - \tfrac{1}{4}\right)a_i + a_{i-2}.$$

Let us work first with the root $r_1 = -\frac{1}{2}$. Since $-\frac{1}{2}$ and $+\frac{1}{2}$ are both roots of the indicial equation, the coefficients a_0 and a_1 are arbitrary. The recursion relation is

$$a_i = -\frac{a_{i-2}}{(-1/2+i)^2 - 1/4} = -\frac{a_{i-2}}{i^2 - i} = -\frac{a_{i-2}}{i(i-1)}$$

for $i \geqslant 2$. Thus

$$a_2 = -\frac{a_0}{2\cdot 1}, \qquad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1},$$

$$a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \ldots, \qquad a_{2k} = \frac{a_0(-1)^k}{(2k)!}.$$

[12] This equation was extensively studied by F. W. Bessel (1784–1846), who inaugurated modern practical astronomy at Königsberg Observatory.

Similarly

$$a_3 = -\frac{a_1}{3\cdot 2}, \qquad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}, \ldots, \qquad a_{2k+1} = \frac{a_1(-1)^k}{(2k+1)!}.$$

Our general solution is then

$$x^{-1/2}\left[a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\right]$$

which we recognize to be $a_0(\cos x)/\sqrt{x} + a_1(\sin x)/\sqrt{x}$.

Notice that in this case we have found the general solution from just one root of the indicial equation. ▲

Exercises for Section 12.8

In Exercises 1–4, find solutions of the given equation in the form of power series: $y = \sum_{i=0}^{\infty} a_i x^i$.

1. $y'' - xy' - y = 0$.
2. $y'' - 2xy' - 2y = 0$.
3. $y'' + 2xy' = 0$.
4. $y'' + xy' = 0$.

In Exercises 5–8, find the first three nonzero terms in the power series solution satisfying the given equation and initial conditions.

5. $y'' + 2xy' = 0$, $y(0) = 0$, $y'(0) = 1$.
6. $y'' + 2x^2y = 0$, $y(0) = 1$, $y'(0) = 0$.
7. $y'' + 2xy' + y = 0$, $y(0) = 0$, $y'(0) = 2$.
8. $y'' - 2xy' + y = 0$, $y(0) = 0$, $y'(0) = 1$.
9. *Airy's equation* is $y'' = xy$. Find the first few terms and the recursion relation for a power series solution.
10. *Tchebycheff's equation* is $(1 - x^2)y'' - xy' + \alpha^2 y = 0$. Find the first few terms and the recursion relation for a power series solution. What happens if $\alpha = n$ is an integer?

In Exercises 11–14, use the Frobenius method to find the first few terms in the general solution of the given equation.

11. $3xy'' - y' + y = 0$.
12. $2xy'' + (2 - x)y' - y = 0$.
13. $3x^2y'' + 2xy' + y = 0$.
14. $2x^2y'' - 2xy' + y = 0$.
15. Consider Bessel's equation of order k, namely, $x^2y'' + xy' + (x^2 - k^2)y = 0$.
    (a) Find the first few terms of a solution of the form $J_k(x) = a_0x^k + a_1x^{k+1} + \cdots$.
    (b) Find a second solution if k is not an integer.
16. *Laguerre functions* are solutions of the equation $xy'' + (1 - x)y' + \lambda y = 0$.
    (a) Find a power series solution by the Frobenius method.
    (b) Show that there is a polynomial solution if λ is an integer.
17. Verify that the power series solutions of $y'' + \omega^2 y = 0$ are just $y = A\cos\omega x + B\sin\omega x$.

★18. Find the first few terms of the general solution for Bessel's equation of order $\frac{3}{2}$.

★19. (a) Verify that the solution of Legendre's equation does not converge for all x unless $\lambda = n(n+1)$ for some nonnegative integer n.
(b) Compute $P_1(x)$, $P_2(x)$, and $P_3(x)$.

★20. Use Wronskians and Exercise 49 of Section 12.7 to show that the solution found in Example 1 is the general solution.

★21. Use Wronskians and Exercise 49 of Section 12.7 to show that the solution found in Example 3 is the general solution.

★22. Prove that the Legendre polynomials are given by *Rodrigues' formula*:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

★23. (a) Solve $x^2y' + (x - 1)y - 1 = 0$, $y(0) = 1$ as a power series to obtain $y = \sum n!\,x^n$, which converges only at $x = 0$. (b) Show that the solution is

$$y = \frac{e^{-1/x}}{x}\int \frac{e^{1/x}}{x}\,dx.$$

Hint: Show that the right-hand side is a polynomial whose coefficients satisfy the same recursion relation as P_n.

Review Exercises for Chapter 12

In Exercises 1–8, test the given series for convergence. If it can be summed using a geometric series, do so.

1. $\sum_{i=1}^{\infty} \frac{1}{(12)^i}$
2. $\sum_{i=1}^{\infty} \frac{1}{100(i+1)}$
3. $\sum_{i=1}^{\infty} \frac{3^{i+1}}{5^{i-1}}$
4. $\sum_{i=1}^{\infty} \frac{8}{9^i}$
5. $1 + 2 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$
6. $100 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots$
7. $\sum_{i=1}^{\infty} \frac{9}{10 + 11i}$
8. $\sum_{i=1}^{\infty} \frac{6}{7 + 8i}$

In Exercises 9–24, test the given series for convergence.

9. $\sum_{n=1}^{\infty} 5^{-n}$
10. $\sum_{n=1}^{\infty} \frac{4^n}{(2n+1)!}$
11. $\sum_{k=1}^{\infty} \frac{k}{3^k}$
12. $\sum_{n=1}^{\infty} \frac{2n}{n+3}$
13. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$
14. $\sum_{n=1}^{\infty} \frac{2n}{n^2+3}$
15. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$
16. $\sum_{j=0}^{\infty} \frac{(-1)^j j}{j^2+8}$
17. $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$
18. $\sum_{i=1}^{\infty} \frac{i}{i^3+8}$
19. $\sum_{n=1}^{\infty} ne^{-n^2}$
20. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 - \sin^2 99n}$
21. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$
22. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}}\right)$
23. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$
24. $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$

Sum the series in Exercises 25–32 to within 0.05.

25. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \cdots$
26. $\frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \frac{5}{32} - \cdots$
27. $\sum_{i=1}^{\infty} \frac{(-1)^i}{2^i + 3}$
28. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{6n^2 - 1}$
29. $\sum_{n=1}^{\infty} \frac{1-n}{3^n}$
30. $\sum_{n=1}^{\infty} \frac{n^2}{n!}$
31. $\sum_{n=1}^{\infty} \frac{\cos n}{n^4 + 1}$
32. $\sum_{n=1}^{\infty} \frac{\cos n}{4^n + 1}$

Tell whether each of the statements in Exercise 33–46 is true or false. Justify your answer.

33. If $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
34. Every geometric series $\sum_{i=1}^{\infty} r^i$ converges.
35. Convergence or divergence of any series may be determined by the ratio test.
36. $\sum_{i=1}^{\infty} 1/2^i = 1$.
37. $e^{2x} = 1 + 2x + x^2 + x^3/3 + \cdots$.
38. If a series converges, it must also converge absolutely.
39. The error made in approximating a convergent series by a partial sum is no greater than the first term omitted.
40. $\cos x = \sum_{k=0}^{\infty} (-1)^k x^{2k}/(2k)!$.
41. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=0}^{\infty} b_k$ are both convergent, then $\sum_{j=1}^{\infty} a_j + \sum_{k=0}^{\infty} b_k = b_0 + \sum_{i=1}^{\infty}(a_i + b_i)$.
42. $\sum_{i=1}^{\infty} (-1)^i [3/(i+2)]$ converges conditionally.
43. The convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty}(a_n + a_{n+1})$.
44. The convergence of $\sum_{n=1}^{\infty}(a_n + a_{n+1})$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.
45. The convergence of $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ implies the convergence of $\sum_{n=1}^{\infty} |a_n|$.
46. The convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n^2$.

47. If $0 \leqslant a_n \leqslant ar^n$, $r < 1$, show that the error in approximating $\sum_{i=1}^{\infty} a_i$ by $\sum_{i=1}^{n} a_i$ is less than or equal to $ar^{n+1}/(1-r)$.
48. Determine how many terms are needed to compute the sum of $1 + r + r^2 + \cdots$ with error less than 0.01 when (a) $r = 0.5$ and (b) $r = 0.09$.

Find the sums of the series in Exercises 49–52.

49. $\sum_{n=1}^{\infty} 1/9^n$
50. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ [*Hint:* Use partial fractions.]
51. $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ [*Hint:* Write the numerator as $n + 1 - 1$.]
52. $\sum_{n=1}^{\infty} \frac{1+n}{2^n}$ [*Hint:* Differentiate a certain power series.]

Find the radius of convergence of the series in Exercises 53–58.

53. $1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{4!} + \cdots$
54. $1 + 3x + 5x^2 + 7x^3 + \cdots$
55. $\sum_{n=0}^{\infty} \frac{x^n}{(3n)!}$
56. $\sum_{n=0}^{\infty} \frac{(x - 1/2)^n}{(n+1)!}$
57. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n$
58. $\sum_{n=0}^{\infty} \frac{n^n}{n!} x^n$

Find the Maclaurin series for the functions in Exercises 59–66.

59. $f(x) = \cos 3x + e^{2x}$
60. $g(x) = \frac{1}{1 - x^3}$
61. $f(x) = \ln(1 + x^4)$
62. $g(x) = \frac{1}{\sqrt{1 - x^4}}$
63. $f(x) = \frac{d}{dx}(\sin x - x)$
64. $g(k) = \frac{d^2}{dk^2}(\cos k^2)$
65. $f(x) = \int_0^x \frac{(e^t - 1)}{t}\, dt$
66. $g(y) = \int_0^y \sin t^2\, dt$

Find the Taylor expansion of each function in Exercises 67–70 about the indicated point, and find the radius of convergence.

67. e^x about $x = 2$
68. $1/x$ about $x = 1$
69. $x^{3/2}$ about $x = 1$
70. $\cos(\pi x)$ about $x = 1$

Find the limits in Exercises 71–74 using series methods.

71. $\lim_{x\to 0} \frac{1-\cos \pi x}{x^2}$.
72. $\lim_{x\to 0} [6\frac{\sin \pi x}{x^5} - \frac{6\pi - \pi^3 x^2}{x^4}]$.
73. $\lim_{x\to 0} \frac{(1+x)^{3/2} - (1-x)^{3/2}}{x^2}$.
74. $\lim_{x\to \pi} [1 + \cos x - \frac{(x-\pi)^2}{2} + \frac{(x-\pi)^4}{24}]$.

In Exercises 75–78 find the real part, the imaginary part, the complex conjugate, and the absolute value of the given complex number.

75. $3 + 7i$
76. $2 - 10i$
77. $\sqrt{2-i}$
78. $(2+i)/(2-i)$

In Exercises 79–82, plot the given complex numbers, indicating r and θ on your diagram, and write them in polar form $z = re^{i\theta}$.

79. $1 - i$
80. $\frac{1+i}{1-i}$
81. $ie^{\pi i/2}$
82. $(1+i)e^{i\pi/4}$
83. Solve for z: $z^2 - 2z + \pi i = 0$.
84. Solve for z: $z^8 = \sqrt{5} + 3i$.

Find the general solution of the differential equations in Exercises 85–96.

85. $y'' + 4y = 0$
86. $y'' - 4y = 0$
87. $y'' + 6y' + 5y = 0$
88. $y'' - 6y' - 2y = 0$
89. $y'' + 3y' - 10y = e^x + \cos x$
90. $y'' - 2y' - 3y = x^2 + \sin x$
91. $y'' - 6y' + 9y = \cos\left(\frac{x}{2}\right)$
92. $y'' - 10y + 25 = \cos(2x)$
93. $y'' + 4y = \frac{x}{\sqrt{x^2+1}}$. (Express your answer in terms of integrals.)
94. $y'' - 3y' - 3y = \frac{\sin x}{\sqrt{\cos^2 x + 1}}$. (Express your answer in terms of integrals.)
95. $y''' + 2y'' + 2y' = 0$
96. $y''' - 3y'' + 3y' - y = e^x$

In Exercises 97–100, identify the equation as a spring equation and describe the limiting behavior as $t \to \infty$.

97. $x'' + 9x + x' = \cos 2t$.
98. $x'' + 9x + 0.001x' = \sin(50t)$.
99. $x'' + 25x + 6x' = \cos(\pi t)$.
100. $x'' + 25x + 0.001x' = \cos(60\pi t)$.

In Exercises 101–104, find the first few terms of the general solution as a power series in x.

101. $y'' + 2xy = 0$
102. $y'' - (4\sin x)y = 0$
103. $y'' - 2x^2y' + 2y = 0$
104. $y'' + y' + xy = 0$

Find the first few terms in an appropriate series for at least one solution of the equations in Exercises 105–108.

105. $5x^2y'' + y' + y = 0$.
106. $xy'' + y' - 4y = 0$.
107. Bessel's equation with $k = 1$.
108. Legendre's equation with $\lambda = 3$.
109. The current I in the electric circuit shown in Figure 12.R.1 satisfies

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dE}{dt},$$

where E is the applied voltage and L, R, C are constants.

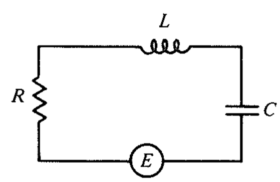


Figure 12.R.1. An electric circuit.

(a) Find the values of m, k, γ that make this equation a damped spring equation.
(b) Find $I(t)$ if $I(0) = 0$, $I'(0) = 0$ and $L = 5$, $C = 0.1$, $R = 100$, and $E = 2\cos(60\pi t)$.

110. Verify formula (16) in Section 12.7.
111. Verify that $\sum_{n=0}^{\infty} x^2(1+x^2)^{-n}$ is a convergent geometric series for $x \neq 0$ with sum $1 + x^2$. It also converges to 0 when $x = 0$. (This shows that the sum of an infinite series of continuous terms need not be continuous.)
112. A beam of length L feet supported at its ends carries a concentrated load of P lbs at its center. The maximum deflection D of the beam from equilibrium is

$$D = \frac{2L^3P}{EI\pi^4}\sum_{n=1}^{\infty}\frac{|\sin(n\pi/2)|}{n^4}.$$

(a) Use the formula $\sum_{n=1}^{\infty}(1/n^4) = \pi^4/90$ to show that

$$\sum_{k=1}^{\infty}\frac{1}{(2k)^4} = \left(\frac{1}{2^4}\right)\left(\frac{\pi^4}{90}\right).$$

[*Hint:* Factor out 2^{-4}.]

(b) Show that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \left(\frac{15}{16}\right)\left(\frac{\pi^4}{90}\right);$$

hence $D = (1/48)(L^3P/EI)$. [*Hint:* A series is the sum of its even and odd terms.]

(c) Use the first two nonzero terms in the series for D to obtain a simpler formula for D. Show that this result differs at most by 0.23% from the theoretical value.

113. The deflection $y(x,t)$ of a string from its straight profile at time t, measured vertically at location x along the string, $0 \leqslant x \leqslant L$, is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

where A_n, L and c are constants.

(a) Explain what this equation means in terms of limits of partial sums for x, t fixed.

(b) Initially (at $t = 0$), the deflection of the string is

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

Find the deflection value as an infinite series at the midpoint $x = L/2$.

114. In the study of saturation of a two-phase motor servo, an engineer starts with a transfer function equation $V(s)/E(s) = K/(1 + s\tau)$, then goes to the first-order approximation $V(s)/E(s) = K(1 - s\tau)$, from which he obtains an approximate equation for the saturation dividing line.

(a) Show that $1/(1 + s\tau) = \sum_{n=0}^{\infty}(-s\tau)^n$, by appeal to the theory of geometric series. Which values of $s\tau$ are allowed?

(b) Discuss the replacement of $1/(1 + s\tau)$ by $1 - s\tau$; include an error estimate in terms of the value of $s\tau$.

115. Find the area bounded by the curves $xy = \sin x$, $x = 1$, $x = 2$, $y = 0$. Make use of the Taylor expansion of $\sin x$.

116. A wire of length L inches and weight w lbs/inch, clamped at its lower end at a small angle $\tan^{-1}P_0$ to the vertical, deflects $y(x)$ inches due to bending. The displacement $y(L)$ at the upper end is given by

$$y(L) = \frac{2P_0}{3L^{1/3}} \frac{\int_0^{L^{3/2}} u(az)\,dz}{u(aL^{3/2})},$$

where $a = \frac{2}{3}\sqrt{W/EI}$, and

$$u(az) = \frac{az^{-1/3}}{2} \sum_{k=0}^{\infty} (-1)^k \frac{a^{2k}z^{2k}}{2^{2k}k!\,\Gamma(k + 2/3)}.$$

The values of the *gamma function* Γ may be found in a mathematical table or on some calculators as $\Gamma(x) = (x - 1)!$ $[\Gamma(\frac{2}{3}) = 1.3541$, $\Gamma(\frac{5}{3}) = 0.9027$, $\Gamma(\frac{8}{3}) = 1.5046$, $\Gamma(\frac{11}{3}) = 4.0122]$. The function u is the *Bessel function* of order $-\frac{1}{3}$.

(a) Find the smallest positive root of $u(az) = 0$ by using the first four terms of the series.

(b) Evaluate $y(L)$ approximately by using the first four terms of the series.

117. (a) Use a power series for $\sqrt{1 + x}$ to calculate $\sqrt{5/4}$ correct to 0.01. (b) Use the result of part (a) to calculate $\sqrt{5}$. How accurate is your answer?

118. In each of the following, evaluate the indicated derivative:

(a) $f^{(12)}(0)$, where $f(x) = x/(1 + x^2)$;

(b) $f^{(10)}$, where $f(x) = x^6 e^{x+1}$.

119. Let

$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots.$$

Determine the numbers B_1, B_2, and B_3. (The B_i are known as the *Bernoulli numbers*.)

120. Show in the following two ways that $\sum_{n=1}^{\infty} na^n = a/(1 - a)^2$ for $|a| < 1$.

(a) Consider

$$S_n = a + 2a^2 + 3a^3 + \cdots + na^n,$$

$$aS_n = a^2 + 2a^3 + \cdots + (n-1)a^n + na^{n+1}$$

and subtract.

(b) Differentiate $\sum_{n=0}^{\infty} a^n = 1/(1 - a)$ with respect to a, and then subtract your answer from $\sum_{n=0}^{\infty} a^n = 1/(1 - a)$.

121. In highway engineering, a *transitional spiral* is defined to be a curve whose curvature varies directly as the arc length. Assume this curve starts at $(0, 0)$ as the continuation of a road coincident with the negative x axis. Then the parametric equations of the spiral are

$$x = k\int_0^{\phi} \frac{\cos\theta}{\sqrt{\theta}}\,d\theta, \qquad y = k\int_0^{\phi} \frac{\sin\theta}{\sqrt{\theta}}\,d\theta.$$

(a) By means of infinite series methods, find the ratio x/y for $\phi = \pi/4$.

(b) Try to graph the transitional spiral for $k = 1$, using accurate graphs of $(\cos\theta)/\sqrt{\theta}$, $(\sin\theta)/\sqrt{\theta}$ and the area interpretation of the integral.

★122. The free vibrations of an elastic circular membrane can be described by infinite series, the terms of which involve trigonometric functions and *Bessel functions*. The series

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(n+i)!}(x/2)^{n+2i}$$

is called the *Bessel Function* $J_n(x)$; n is an integer $\geqslant 0$.

(a) Establish convergence by the ratio test.

(b) The frequencies of oscillation of the circular membrane are essentially solutions of the equation $J_n(x) = 0$, $x > 0$. Examine the equation $J_0(x) = 0$, and see if you can explain why $J_0(2.404) = 0$ is possible.

(c) Check that J_n satisfies Bessel's equation (Example 6, Section 12.8).

★123. Show that g defined by $g(x) = e^{-1/x^2}$ if $x \neq 0$ and $g(0) = 0$ is infinitely differentiable and $g^{(i)}(0) = 0$ for all i. [*Hint:* Use the definition of the derivative and the following lemma provable by l'Hôpital's rule: if $P(x)$ is any polynomial, then $\lim_{x\to 0} P(x)g(x) = 0$.]

★124. Let $f(x) = (1 + x)^{\alpha}$, where α is a real number. Show by an induction argument that $f^{(i)}(x) = \alpha(\alpha - 1) \ldots (\alpha - i + 1)(1 + x)^{\alpha - i}$, and hence show that $(1 + x)^{\alpha}$ is analytic for $|x| < 1$.

★125. True or false: The convergence of $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ implies absolute convergence of $\sum_{n=1}^{\infty} a_n b_n$.

★126. (a) Show that if the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is R, then the radius of convergence of $\sum_{n=1}^{\infty} a_n x^{2n}$ is $\sqrt{R}$.
(b) Find the radius of convergence of the series $\sum_{n=0}^{\infty} (\pi/4)^n x^{2n}$.

★127. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = f(x)/(1 - x)$.
(a) By multiplying the power series for $f(x)$ and $1/(1 - x)$, show that $g(x) = \sum_{i=0}^{\infty} b_i x^i$, where $b_i = a_0 + \cdots + a_i$ is the ith partial sum of the series $\sum_{i=0}^{\infty} a_i$.
(b) Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{i \to \infty} b_i$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?
(c) Let $e^x/(1 - x) = \sum_{i=0}^{\infty} b_i x^i$. What is $\lim_{i \to \infty} b_i$?

★128. (a) Find the second-order approximation at $T = 0$ to the day-length function S (see the supplement to Chapter 5) for latitude 38° and your own latitude.
(b) How many minutes earlier (compared with $T = 0$) does the sun set when $T = 1, 2, 10, 30$?
(c) Compare the results in part (b) with those obtained from the exact formula and with listings in your local newspaper.
(d) For how many days before and after June 21 is the second-order approximation correct to within 1 minute? Within 5 minutes?

★129. Prove that e is irrational, as follows: if $e = a/b$ for some integers a and b, let $k > b$ and let $\alpha = k!(e - 2 - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{k!})$. Show that α is an integer and that $\alpha < 1/k$ to derive a contradiction.