CHAPTER 7

BASIC METHODS OF INTEGRATION

7.1 Calculating Integrals

PREREQUISITES

1. Recall how to integrate polynomials (Sections 2.5, 4.4, and 4.5).
2. Recall integration formulas involving exponentials and logarithms (Section 6.3).
3. Recall integration formulas involving trigonometric functions and their inverses (Section 5.2 and 5.4).
4. Recall the relationship between the integral and area (Section 4.6).

PREREQUISITE QUIZ

1. Perform the following integrations:

   (a) $\int_0^4 (x^3 + x)dx$

   (b) $\int (e^x - 1/\sqrt{1 - x^2})dx$

   (c) $\int (\cos x - 1/x + 2/x^2)dx$

2. Suppose $g(x) \geq f(x)$ on $[0,2]$ and $f(x) \geq g(x)$ on $[2,3]$. Write an expression for the area between the graphs of $g(x)$ and $f(x)$ on $[0,3]$.

GOALS

1. Be able to evaluate integrals involving sums of polynomials, trigonometric functions, exponentials, and inverse trigonometric functions.
2. Be able to use integration for solving area and total change problems.

STUDY HINTS

1. Definite integrals. In the box preceding Example 2, restrictions are placed on a and b. Can you explain why? If $n = -2, -3, -4, \ldots$ a and b must have the same sign to avoid the discontinuity in the integrand at $x = 0$. If n is not an integer, one must impose condition that avoid roots of negative numbers. Finally, recall that $\ln x$ is undefined for $x \leqslant 0$.
2. Checking answers. Remember that integration is the inverse of differentiation. Thus, you should always check your answer by differentiating it to get the integrand. Differentiation can often detect a wrong sign or a wrong factor.
3. Word problems. Be sure all quantities are expressed in compatible units. (See Example 9.)
4. Review of integration methods. The material in this section is review. If any of the examples didn't make sense, go back to the appropriate sections in Chapters 1-6 and review until you understand the examples.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By combining the sum rule, the constant multiple rule, and the power rule for antidifferentiation, we get $\int(3x^2 + 2x + x^{-3})dx = 3x^3/3 + 2x^2/2 + x^{-2}/(-2) + C = x^3 + x^2 - 1/2x^2 + C$.

5. We guess that the antiderivative of $\sin 2x$ is $a \cos 2x$. Differentiation gives $-2a \sin 2x$, so a should be $-1/2$. Thus, $\int(\sin 2x + 3x)dx = -\cos 2x/2 + 3x^2/2 + C$. *

9. Using the sum rule, the constant multiple rule, and the power rule for antidifferentiation, we get $F(x) = \int(x^8 + 2x^2 - 1)dx = x^9/9 + 2x^3/3 - x + C$. Note that $F(-a) = -F(a)$, so $F(a) - F(-a) = 2F(a)$. Therefore, $\int_{-2}^{2}(x^8 + 2x^2 - 1)dx = 2F(2) = 2(512/9 + 16/3 - 2) = 1084/9$.

13. Using the power rule for antidifferentiation, $\int_{16}^{81} \sqrt[4]{s}\, ds = \int_{16}^{81} s^{1/4} ds = [s^{5/4}/(5/4)]\Big|_{16}^{81} = (4/5)[(81)(3) - (16)(2)] = 844/5$.

17. $\int_{-\pi}^{\pi} \cos x\, dx = \sin x \Big|_{-\pi}^{\pi} = 0$.

21. By the constant multiple rule, $\int_0^1 [3/(x^2 + 1)]\, dx = 3\int_0^1 [1/(x^2 + 1)]\, dx = 3 \tan^{-1} x\Big|_0^1 = 3\pi/4$.

25. From the basic trigonometric antidifferentiation formulas, $\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = 1$.

29. By the logarithm differentiation formula, $\int_1^5 (dt/t) = \ln t\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$.

33. By the sum rule, the integral is $\int_{-200}^{200}(90x^{21} - 80x^{33} + 5580x^{97})dx + \int_{-200}^{200} 1\, dx$. Since the first integrand is an odd function, the antiderivative will be an even function, $F(x)$; therefore $F(200) = F(-200)$ and the integral is 0. Thus, we are left with $\int_{-200}^{200} 1\, dx = x\Big|_{-200}^{200} = 400$.

37. (a) According to the fundamental theorem of calculus, the derivative of the integral is the integrand. Using the chain rule, we have $(d/dx)\left[e^{(x^2)}/2 + C\right] = 2xe^{(x^2)}/2 = xe^{(x^2)}$. Therefore, the formula is correct.

*Throughout the student guide, we take $\cos 2x/2$ to mean $(1/2)\cos 2x$, not $\cos (2x/2)$.

37. (b) According to Example 7, $\int \ln x\, dx = x \ln x - x + C$. Thus, $\int_1^e \left(2xe^{(x^2)} + 3 \ln x\right)dx = \left(e^{(x^2)} + 3x \ln x - 3x\right)\Big|_1^e = e^{(e^2)} + 3e -$ $3e - e - 0 + 3 = e^{(e^2)} - e + 3$.

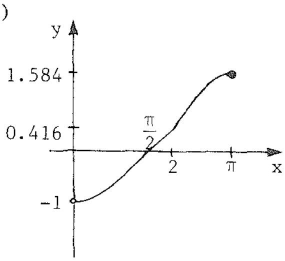
41. We may write $g(t) = \int_{t^2}^{4} \sqrt{e^x + \sin 5x^2}\, dx = -\int_4^{t^2} \sqrt{e^x + \sin 5x^2}\, dx$ as $f(t^2)$, where $f(u) = \int_4^u -\sqrt{e^x + \sin 5x^2}\, dx$. By the fundamental theorem of calculus (alternative version), $f'(u) = -\sqrt{e^u + \sin 5u^2}$. By the chain rule, $g'(t) = f'(t^2)\cdot(d/dt)t^2 = -2t\sqrt{e^{(t^2)} + \sin 5t^4}$.

45. Property 4 of the definite integral, the endpoint additivity rule, is used in this exercise.

(a) $\int_{-\pi/2}^{\pi/2} f(x)g(x)dx = \int_{-\pi/2}^{\pi/2} \sin x = -\cos x\Big|_{-\pi/2}^{\pi/2} = 0$.

(b) $\int_1^3 g(x)h(x)dx = \int_1^2 dx/x^2 + \int_2^3 2\, dx/x^2 = (-1/x)\Big|_1^2 + (-2/x)\Big|_2^3 =$ $-1/2 + 1 - 2/3 + 1 = 5/6$.

(c)



If $0 < x \leqslant 2$, then $\int_{\pi/2}^x f(t)g(t)dt = \int_{\pi/2}^x \sin t\, dt = -\cos t\Big|_{\pi/2}^x = -\cos x$. If $2 \leqslant x \leqslant \pi$, then $\int_{\pi/2}^x f(t)g(t)dt = \int_{\pi/2}^2 \sin t\, dt + \int_2^x (2 \sin t)dt = -\cos t\Big|_{\pi/2}^2 + (-2\cos t)\Big|_2^x = -\cos(2) - 2\cos x + 2\cos(2) = \cos(2) - 2\cos x$. Therefore,

$$\int_{\pi/2}^x f(t)g(t)dt = \begin{cases} -\cos x & \text{if } 0 < x \leqslant 2 \\ \cos(2) - 2\cos x & \text{if } 2 \leqslant x \leqslant \pi \end{cases} .$$

49. Since the function is positive on $[1,4]$, the area is $\int_a^b f(x)dx$. $\int_1^4 [(x^2 + 2)/\sqrt{x}]dx = \int_1^4 (x^{3/2} + 2x^{-1/2})dx = [x^{5/2}/(5/2) + 2x^{1/2}/(1/2)]\Big|_1^4 =$ 16.4 .

53. The y-intercept of $y = 1 - x^2/4$ is 1 , so the white area in the center is a circle of radius 1 . We will find the area in the first quadrant and then multiply by 4 to find the total area. The x-intercept

53. (continued)

of $y = 1 - x^2/4$ is 2 . The area of the white quarter circle is $\pi/4$, so the area of the entire shaded region is $4\left[\int_0^2 (1 - x^2/4)dx - \pi/4\right] =$ $4[(x - x^3/12)|_0^2 - \pi/4] = 4(4/3 - \pi/4) = 16/3 - \pi$.

57. Integrating the identity $\sin^2 x + \cos^2 x = 1$ from 0 to $\pi/2$ and using the sum rule gives $\int_0^{\pi/2} \sin^2 x dx + \int_0^{\pi/2} \cos^2 dx = \int_0^{\pi/2} 1\, dx$. The two integrals on the left-hand side are equal by assumption and the right-hand side is $\pi/2$, so $\int_0^{\pi/2} \sin^2 x\, dx = \pi/4$. (Note that we did this problem without ever finding $\int \sin^2 x\, dx$.)

61. The area is given by the integral $\int_a^b [dx/(1 + x^2)] = \tan^{-1} x|_a^b = \tan^{-1} b -$ $\tan^{-1} a$. By the definition of the inverse tangent function, the value of $\tan^{-1} x$ always lies in the interval $(-\pi/2, \pi/2)$, so the difference between two values must be less than π , regardless of the length of the interval $[a,b]$.

65. (a) Guess that the antiderivative has the form ae^{-kt} . We differentiate to get $-ake^{-kt}$, so $a = -R/k$. $A = \int_0^T Re^{-kt}dt = (-R/k)e^{-kt}|_0^T =$ $(-R/k)(e^{-kT} - 1) = (R/k)(1 - e^{-kt})$.

(b) For this problem, $k = 0.0825$; $R = 4(12)(230) = 11040$; $T = 5$. Therefore, $A = (11040/0.0825)(1 - e^{-(0.0825)(5)}) = \$45,231.46$.

69. Using the identity, $\int_1^e [dt/t(t + 1)] = \int_1^e (dt/t) - \int_1^e [dt/(t + 1)] =$ $\ln t\,|_1^e - \ln(t + 1)|_1^e = 1 - 0 - \ln(e + 1) + \ln(2) \approx 0.380$. (The second integral was found by guessing that $\int [dt/(t + 1)] = a \ln|t + 1| + C$. Then differentiation showed that $a = 1$.)

SECTION QUIZ

1. Evaluate the following integrals:

   (a) $\int[(x^4 + 3x^3 - 1)/x]\,dx$

   (b) $\int_{-93}^{93}(x^5 + 8x^3 - 27x + 1)\,dx$

   (c) $\int(x^{3/2} + 1/\sqrt{x} + e^x)\,dx$

   (d) $\int_{\pi}^{\pi/2}(2\sin\theta + \cos\theta)\,d\theta$

   (e) $\int_{1/3}^{1/3}\left[dt/\sqrt{t + t^2}\right]$

   (f) $\int(3/u\sqrt{u^2 - 1})\,du$, $u > 0$

   (g) $\int[(t^4 + 2t^2 + 2)/(t^2 + 1) - e^t + \csc t \cot t]\,dt$

2. $\int[dt/(1 + t^2)] = \tan^{-1}t + C$ and $\int[dt/(1 + t^2)] = -\cot^{-1}t + C$. Since the integrals are equal, we have $\tan^{-1}t + C = -\cot^{-1}t + C$. The C's cancel, leaving $\tan^{-1}t = -\cot^{-1}t$. Obviously, $\tan^{-1}t \neq -\cot^{-1}t$. What is the fallacy in this argument?

3. Show that $\int e^x \sin x\,dx \neq (1/2)e^x(\sin x + \cos x) + C$.

4. Once again, Guilty Gary had borrowed his neighbor's tools without asking. This time, he was spray painting his house when a friendly policeman stopped to compliment Guilty Gary on the fine job he was doing. Startled, Guilty Gary forgot to turn off the spray before turning around. When he realized he was spraying the policeman, he accidentally increased the spray rate. For 15 seconds, paint was coming out at $(0.25 + t)$ liters/min., where t is in minutes. How much paint did Guilty Gary spray on the policeman?*

*Dear Reader: I realize that many of you hate math but are forced to complete this course for graduation. Thus, I have attempted to maintain interest with "entertaining" word problems. They are not meant to be insulting to your intelligence. Obviously, most of the situations will never happen; however, calculus has several practical uses and such examples are found throughout Marsden and Weinstein's text. I would appreciate your comments on whether my "unusual" word problems should be kept for the next edition.

ANSWERS TO PREREQUISITE QUIZ

1. (a) 72

   (b) $e^x - \sin^{-1}x + C$

   (c) $\sin x - \ln|x| - 2/x + C$

2. $\int_0^2 [g(x) - f(x)]\,dx + \int_2^3 [f(x) - g(x)]\,dx$

ANSWERS TO SECTION QUIZ

1. (a) $x^4/4 + x^3 - \ln|x| + C$

   (b) 186

   (c) $2x^{5/2}/5 + 2\sqrt{x} + e^x + C$

   (d) -1

   (e) 0

   (f) $3\sec^{-1}u + C$

   (g) $t^3/3 + t + \tan^{-1}t - e^t - \csc t + C$

2. The arbitrary constants have different values.

3. Differentiate the right-hand side.

4. 3/32 liters

7.2 Integration by Substitution

PREREQUISITES

1. Recall how to differentiate rational, trigonometric, logarithmic, and exponential functions (Sections 1.5, 5.2, and 6.3).
2. Recall how to differentiate by the chain rule (Section 2.2).
3. Recall how to integrate basic functions (Section 7.1).

PREREQUISITE QUIZ

1. Differentiate the following expressions:

   (a) $\cos x + \sec 2x + 1/2x$

   (b) $e^{\sin x} + 3^{x/2} + x^{-3}$

   (c) $x^6 + 3x^2 - \ln(x^3 + 4x)$

2. Evaluate $\int (x^6 + \cos x + 1/x)dx$.
3. Evaluate $\int [1/(1 + x^2) + e^x - 3] dx$.

GOALS

1. Be able to recognize the types of integrals which may be evaluated by substitution.
2. Be able to evaluate integrals using the technique of substitution.

STUDY HINTS

1. Integration by substitution. Be sure you understand this technique well. It is one of the most important techniques of integration.
2. Choosing a substitution. Practice and study the examples. Note that in the first three examples, u is an expression which is raised to a power.

3. <u>More common substitutions</u>. If x^n appears, see if x^{n-1} also appears in the integrand, where n again does not have to be an integer. If so, try substituting $u = x^n$. Two other good choices for u are used to derive the shifting and scaling rules.

4. <u>Shifting rule</u>. Do not memorize the rule. It is simply a substitution of $u = x + a$.

5. <u>Scaling rule</u>. Do not memorize the rule. It is simply a substitution of $u = bx$.

6. <u>Differential notation</u>. As an alternative, you may differentiate both sides of $u = g(x)$ to get $du = g'(x)dx$, and then solving to get $du/g'(x) = dx$. All of the methods presented are the same. Use the one which you feel most comfortable with.

7. <u>Giving an answer</u>. Remember to express your answer in terms of the original variable. And don't forget the arbitrary constant.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x^2 + 4$, so $du = 2x\,dx$. Then $\int 2x(x^2 + 4)^{3/2}dx = \int u^{3/2}du = u^{5/2}/(5/2) + C = 2u^{5/2}/5 + C = 2(x^2 + 4)^{5/2}/5 + C$.

   Differentiating, we get $(d/dx)[2(x^2 + 4)^{5/2}/5 + C] = (2/5)(5/2)(x^2 + 4)^{3/2}(2x) = 2x(x^2 + 4)^{3/2}$. Differentiation yields the integrand, so the answer is verified.

5. Let $u = \tan\theta$, so $du = \sec^2\theta\, d\theta$. Then $\int(\sec^2\theta/\tan^3\theta)d\theta = \int(du/u^3) = u^{-2}/(-2) + C = -1/2 \tan^2\theta + C$.

   Upon differentiating, we get $(d/d\theta)[-1/2 \tan^2\theta + C] = 2(2)\tan\theta\sec^2\theta/(2\tan^2\theta)^2 = \sec^2\theta/\tan^3\theta$. Differentiation yields the integrand, so the answer is verified.

9. Let $u = x^4 + 2$, so $du = 4x^3dx$. Then $\int(x^3/\sqrt{x^4 + 2})dx = \int(x^3/u^{1/2})(du/4x^3) = (1/4)\int u^{-1/2}du = (1/4)[u^{1/2}/(1/2)] + C = \sqrt{x^4 + 2}/2 + C$.

Differentiating the answer gives $(d/dx)[(x^4 + 2)^{1/2}/2 + C]$ $(1/2)(1/2)(x^4 + 2)^{-1/2}(4x^3) = x^3/\sqrt{x^4 + 2}$, which is the integrand. Thus, the answer is verified.

13. Let $u = \cos(r^2)$, so $du = -\sin(r^2)\cdot 2r\,dr$. Then $\int 2r\sin(r^2)\times \cos^3(r^2)dr = \int -u^3du = -u^4/4 + C = -\cos^4(r^2)/4 + C$.

Differentiating the answer gives $(d/dr)[-\cos^4(r^2)/4 + C] = (-1/4)(4)(\cos^3(r^2))(-\sin(r^2))(2r) = 2r\sin(r^2)\cos^3(r^2)$, which is the integrand. Thus, the answer is verified.

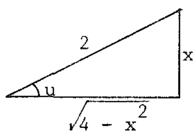
17. Let $u = \theta + 4$, so $du = d\theta$. Then $\int\sin(\theta + 4)d\theta = \int\sin u\,du = -\cos u + C = -\cos(\theta + 4) + C$. (The shifting rule may have been applied to this integral.)

Differentiating the answer yields $(d/d\theta)[-\cos(\theta + 4) + C] = \sin(\theta + 4)$, so the answer is verified.

21. Let $u = t^2 + 2t + 3$, so $du = (2t + 2)dt = 2(t + 1)dt$. Then $\int[(t + 1)/\sqrt{t^2 + 2t + 3}]dt = \int u^{-1/2}(du/2) = (1/2)(u^{1/2}/(1/2)) + C = u^{1/2} + C = \sqrt{t^2 + 2t + 3} + C$.

Differentiating the answer gives $(d/dt)(\sqrt{t^2 + 2t + 3} + C) = (1/2)(t^2 + 2t + 3)^{-1/2}(2t + 2) = (t + 1)/\sqrt{t^2 + 2t + 3}$, which is the integrand. Thus, the answer is verified.

25. Use the hint to get $\int\cos^3\theta d\theta = \int(\cos\theta)(\cos^2\theta)d\theta = \int(\cos\theta)(1 - \sin^2\theta)d\theta$ Now let $u = \sin\theta$, so $du = \cos\theta\,d\theta$. Then $\int\cos^3\theta\,d\theta = \int(1 - u^2)du = u - u^3/3 + C = \sin\theta - \sin^3\theta/3 + C$.

29. 

Let $x = 2 \sin u$, so $dx = 2 \cos u \, du$; therefore, $\int \sqrt{4 - x^2} \, dx = \int \sqrt{4 - 4 \sin^2 u} \times (2 \cos u \, du) = 4\int \cos^2 u \, du = 2\int (1 + \cos 2u) du$ (using the half-angle formula) $= 2(u + \sin 2u/2) + C = 2u + 2 \sin u \times \cos u + C$. Since $u = \sin^{-1}(x/2)$ (See the figure), the integral is $2 \sin^{-1}(x/2) + x\sqrt{4 - x^2}/2 + C$.

33. Let $u = \ln t$, so $du = dt/t$; therefore, $\int [\sin(\ln t)/t] dt = \int \sin u \, du = -\cos u + C = -\cos(\ln t) + C$.

37. (a) Let $u = \sin x$, so $\cos x \, dx = du$. Then $\int \sin x \cos x \, dx = \int u \, du = u^2/2 + C = \sin^2 x/2 + C$.

(b) Let $u = \cos x$, so $\sin x \, dx = -du$. Then $\int \sin x \cos x \, dx = \int u(-du) = -u^2/2 + C = -\cos^2 x/2 + C$.

(c) From $\sin 2x = 2 \sin x \cos x$, we get $\sin x \cos x = \sin 2x/2$. Hence $\int \sin x \cos x \, dx = (1/2)\int \sin 2x \, dx = (1/2)[-\cos 2x/2] + C = -\cos 2x/4 + C$ by the scaling rule.

To show that the three answers we got are really the same, we need to show that they differ from one another by constants. $\sin^2 x/2 - (-\cos^2 x/2) = 1/2$, which is a constant. Also, $-\cos^2 x/2 - (-\cos 2x/4) = -\cos^2 x/2 + (2 \cos^2 x - 1)/4 = -1/4$, which is a constant. Thus, we have shown that all three answers are equivalent.

SECTION QUIZ

1. Evaluate the following integrals:

   (a) $\int (t + 4)(t + 5)^{3/2} dt$ [Hint: Let $u = t + 5$.]

   (b) $\int \left[2e^{3x}/\left(1 + e^{6x}\right)\right] dx$

   (c) $\int \left(5/\sqrt{4u - u^2}\right) du$ [*Hint: Complete the square.*]

   (d) $\int \sec^5 t \tan t \, dt$

   (e) $\int x\left(e^{4x}\right)^x dx$

2. What is wrong with the following statements?

   (a) $\int (3 - x)^{3/2} dx = 2(3 - x)^{5/2}/5 + C$.

   (b) $\int (x - 3)^{-2} dx = (x - 3)^{-3}/(-3) + C$.

   (c) $\int \cos 5x \, dx = 5 \sin 5x + C$.

   (d) $\int [(4x)^2 + (4x) + 1] dx = (1/4)[(4x)^3/3 + (4x)^2/2 + (4x)]$.

3. The rate at which Schizophrenic Sam spends listening to voices t months after his psychiatric visit is given by $t^2/(t + 4)$, $0 \leqslant t \leqslant 2.5$. Find a formula describing the amount of time spent in auditory hallucination since his last psychiatric appointment. (Hint: Let $u = t + 4$

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-\sin x + 2 \sec 2x \tan 2x - 1/2x^2$

   (b) $(\cos x)e^{\sin x} + (1/2)(\ln 3)3^{x/2} - 3x^{-4}$

   (c) $6x^5 + 6x - (3x^2 + 4)/(x^3 + 4x)$

2. $x^7/7 + \sin x + \ln |x| + C$

3. $\tan^{-1} x + e^x - 3x + C$

ANSWERS TO SECTION QUIZ

1. (a) $2(t + 5)^{7/2}/7 - 2(t + 5)^{5/2}/5 + C$; let $u = t + 5$.

(b) $(2/3)\tan^{-1}\left(e^{3x}\right) + C$; let $u = e^{3x}$.

(c) $5\sin^{-1}[(u - 2)/2] + C$; let $v = u - 2$.

(d) $\sec^5 t/5 + C$; let $u = \sec x$.

(e) $\exp(4x^2)/8 + C$; let $u = 4x^2$.

2. (a) Minus sign is missing.

(b) Exponent and denominator should be -1 rather than -3 .

(c) Answer should be $\sin 5x/5 + C$.

(d) Arbitrary constant is missing.

3. $(t + 4)^2/2 - 8(t + 4) + 16 \ln|t + 4|$.

7.3 Changing Variables in the Definite Integral

PREREQUISITES

1. Recall how to integrate by the method of substitution (Section 7.2).

2. Recall the fundamental theorem of calculus (Section 4.4).

PREREQUISITE QUIZ

1. Express the integral $\int_a^b F'(x)dx$ in terms of F .

2. Perform the following integrations:

   (a) $\int \cos x \sin x \, dx$

   (b) $\int [3x/(1 + x^4)] \, dx$

   (c) $\int (2x - 3)^4 dx$

GOALS

1. Be able to change the limits of integration while integrating by substitution.

STUDY HINTS

1. Changing limits. Once you have learned integration by substitution for indefinite integrals, definite integrals can easily be computed by changing the limits every time you change variables. Simply choose $u = g(x)$ and then change the limits from $x = a$ and b to $u = g(a)$ and $g(b)$.

2. Substitution may not work. If the integral becomes harder, try a different substitution or maybe integration by substitution is not the correct method to be used for solving the integral. When you are proficient, going through a mental checklist shouldn't take long.

3. Use of tables. Don't rely on tables. By the time you finish this course, you should be able to derive the integration formulas found in the tables. They are provided for your convenience . Check with your instructor regarding access to a table during an exam.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x + 2$, so $du = dx$; therefore, $\int_{-1}^{1}\sqrt{x+2}\,dx = \int_{1}^{3}\sqrt{u}\,du = [(u^{3/2})/(3/2)]\,|_1^3 = 2(\sqrt{27} - 1)/3 = 2\sqrt{3} - 2/3$.

5. Let $u = x^2 + 2x + 1$, so $du = (2x + 2)dx = 2(x + 1)dx$; therefore, $\int_2^4 (x + 1)(x^2 + 2x + 1)^{5/4}dx = (1/2)\int_9^{25} u^{5/4}du = [(1/2)u^{9/4}/(9/4)]\,|_9^{25} = 2[(25)^{9/4} - (9)^{9/4}]/9$.

9. Let $u = x^2$, so $du = 2xdx$; therefore, $\int_0^1 x\exp(x^2)dx = (1/2)\int_0^1 e^u du = (e^u/2)\,|_0^1 = (e - 1)/2$.

13. Let $u = \cos x$, so $du = -\sin x\,dx$; therefore, $\int_{-\pi/2}^{\pi/2} 5\cos^2 x \sin x\,dx = -\int_0^0 5u^2 du = 0$.

17. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$; therefore, $\int_{\pi/8}^{\pi/4}\tan\theta\,d\theta = \int_{\pi/8}^{\pi/4}(\sin\theta/\cos\theta)d\theta = -\int_{\cos(\pi/8)}^{\sqrt{2}/2}(du/u) = -\ln|u|\Big|_{\cos(\pi/8)}^{\sqrt{2}/2} = \ln(\cos(\pi/8)) - \ln(\sqrt{2}/2) = \ln(\sqrt{2}\cos(\pi/8))$. We used the identity $\tan\theta = \sin\theta/\cos\theta$ to solve this problem.

21. Division yields $\int_1^3[(x^3 + x - 1)/(x^2 + 1)]dx = \int_1^3[x - 1/(x^2 + 1)]dx = (x^2/2 - \tan^{-1}x)\,|_1^3 = 4 - \tan^{-1}(3) + \pi/4$.

25. If we make the substitution $u = x^3 + 3x^2 + 1$, we have $du = (3x^2 + 6x)dx$, so $\int_0^1\left[(x^2 + 3x)/\sqrt[3]{x^3 + 3x^2 + 1}\right]dx = \int_{u=1}^{5}[(x^2 + 3x)/\sqrt[3]{u}(3x^2 + 6x)]\,du = \int_{u=1}^{5}[(x + 3)/\sqrt[3]{u}(3x + 6)]\,du$. There is no simple way to express the quantity $(x + 3)/(3x + 6)$ in terms of u . (We

25. (continued)

would have to solve the equation $u = x^3 + 3x^2 + 1$ for x in terms of u.) We conclude that the substitution was not effective in this case.

29. We use the first half of formula 65 because $a = 3 > 0$ and $b = 2$, $c = 1$. Therefore, $\int_0^1 \left(dx/\sqrt{3x^2 + 2x + 1}\right) = (1/\sqrt{3}) \ln \left|2(3)x + 2 + 2\sqrt{3}\sqrt{3x^2 + 2x + 1}\right| \Big|_0^1 = (1/\sqrt{3})[\ln|8 + 6\sqrt{2}| - \ln|2 + 2\sqrt{3}|] = (1/\sqrt{3}) \times \ln[(4 + 3\sqrt{2})/(1 + \sqrt{3})]$.

33. Let $u = x^2 + 2x + 2$, so $du = (2x + 2)dx = 2(x + 1)dx$. Thus, the area is $\int_0^1 [(x + 1)/(x^2 + 2x + 2)^{3/2}]dx = \int_2^5 (du/2u^{3/2}) = [(1/2)u^{1/2}/(-1/2)]\big|_2^5 = (-1/\sqrt{u})\big|_2^5 = -1/\sqrt{5} + 1/\sqrt{2} = -\sqrt{5}/5 + \sqrt{2}/2 = (5\sqrt{2} - 2\sqrt{5})/10$.

37. (a) By substituting $u = \cos x$, we have $du = -\sin x\, dx$, $u(\pi/2) = 0$, and $u(0) = 1$. Thus, $\int_0^{\pi/2} \cos^2 x \sin x\, dx = -\int_1^0 u^2 du = (-u^3/3)\big|_1^0 = 1/3$.

(b) Let F be the antiderivative of f. Then, $\int_a^b f(g(x))g'(x)dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a))$. This is the same as $\int_{g(a)}^{g(b)} f(u)du = F(u)\big|_{g(a)}^{g(b)}$. Thus, the formula is valid independent of the relationship between $g(a)$ and $g(b)$.

SECTION QUIZ

1. Evaluate the following integrals:

(a) $\int_{-1}^1 (x + 1)^5 x\, dx$ [Hint: Let $u = x + 1$.]

(b) $\int_{-3}^3 x \exp(-x^2)dx$

(c) $\int_0^1 [t^5/(1 + t^3)^3]dt$ [Hint: Let $u = 1 + t^3$.]

(d) $\displaystyle\int_0^{-\sqrt{1/2}} \left[2y/(1 - y^4)^{1/2}\right] dy$ [Hint: Let $u = y^2$.]

2. Consider the integral $\int_{-\pi}^{\pi} 3x^2 \sin(x^3)[\exp(\cos(x^3))]^2 dx$.

   (a) Rewrite the integral by substituting $u = x^3$.

   (b) Rewrite the integral in (a) by substituting $v = \cos u$.

   (c) Rewrite the integral in (b) by substituting $w = 2v$.

   (d) Evaluate the integral.

   (e) What substitution can be used to evaluate the integral in one step?

3. Lost in a magic cave, you read, scribbled on the wall, "RIGHT $\int_0^{\pi/4} 72 \sin 4x\, dx$, LEFT $\int_{-2}^{0} 6(t+2)^3 dt$, RIGHT $\int_{-1/2}^{1/2} (70/\pi) \times [2dx/(1+4x^2)]$" . From that sign, an arrow points to a combination lock marked "EXIT." What combination will probably bring you back to the outside world?

ANSWERS TO PREREQUISITE QUIZ

1. $F(b) - F(a)$

2. (a) $\sin^2 x/2 + C$ or $-\cos^2 x/2 + C$

   (b) $(3/2) \tan^{-1}(x^2) + C$

   (c) $(2x - 3)^5/10 + C$

ANSWERS TO SECTION QUIZ

1. (a) $128/7 - 32/3$

   (b) 0

   (c) 1/24

   (d) $\pi/6$

2. (a) $\int_{-\pi^3}^{\pi^3} \sin u\, [\exp(\cos u)]^2 du$

(b) $-\int_{-\cos(\pi^3)}^{\cos(\pi^3)} [\exp v]^2\, dv$

(c) $-\int_{-2(\cos(\pi^3))}^{2(\cos(\pi^3))} (e^w\, dw/2)$

(d) $\{\exp[-2\cos(\pi^3)] - \exp[2\cos(\pi^3)]\}/2$

(e) Let $u = 2\cos(x^3)$.

3. 36 RIGHT, 24 LEFT, 35 RIGHT (Apologies to the female readers!)

7.4 Integration by Parts

PREREQUISITES

1. Recall the product rule for differentiation (Section 1.5).
2. Recall the definition of an inverse function (Section 5.3).

PREREQUISITE QUIZ

1. Differentiate the following expressions:
   (a) $x \ln x$
   (b) $(5x - 3)(x^2 - 4x + 1)$
   (c) $x^2 \tan x$
2. If $f(x) = 2x + 3$, find a formula for $f^{-1}(x)$.

GOALS

1. Be able to use the technique of integration by parts.
2. Be able to integrate inverse functions.

STUDY HINTS

1. Integration by parts. Memorize the formula $\int u\,dv = uv - \int v\,du$ and don't forget the minus sign. The formula will be used quite often during your studies of mathematics. Learn the formula well enough so that it becomes second nature to you.
2. Choice of u. When integrating by parts, the factor x^n , where n is a positive integer, is often chosen to be u . After n repetitions of integration by parts, this factor will be eliminated. Another common situation is when one of the factors is $\ln x$, in which case we let $u = \ln x$. Note the choice of u in Examples 1 , 2 , and 3 .

3. I method. Example 4 demonstrates a technique which is useful for integrals involving exponentials and trigonometric functions. The method works because repeated differentiation of the factors yields the original factors.

4. "Other" I methods. Try Example 4 again by letting $u = e^x$ both times. Now, try again with $u = \sin x$ in the first step and then letting $u = e^x$ in the second step. The last method demonstrates that you must let u be the exponential or the trigonometric functions throughout the integration. Don't change in the middle.

5. Integrating inverse functions. Once you learn the formula $\int u\, dv = uv - \int v\, du$, simply rename the variables to get $\int y\, dx = xy - \int x\, dy$. All we did was replace u with y and v with x. Finally, remember to express your answer to $\int y\, dx$ in terms of x, not y.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x + 1$ and $dv = \cos x\, dx$, so $du = dx$ and $v = \sin x$; therefore, $\int (x + 1)\cos x\, dx = (x + 1)\sin x - \int \sin x\, dx = (x + 1)\sin x + \cos x + C$.

5. Let $u = x^2$ and $dv = \cos x\, dx$, so $du = 2x\, dx$ and $v = \sin x$; therefore, $\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx$. For the integral $\int 2x \sin x\, dx$, apply integration by parts again with $u = 2x$, so $dv = \sin x\, dx$, $du = 2\, dx$ and $v = -\cos x$; therefore $\int 2x \sin x\, dx = -2x \cos x - \int -\cos x\, (2\, dx) = -2x \cos x + 2 \sin x + C$. Hence $\int x^2 \cos x\, dx = x^2 \sin x - [-2x \cos x + 2 \sin x] + C = (x^2 - 2)\sin x + 2x \cos x + C$.

9. Let $u = \ln(10x)$, so $dv = dx$, $du = (10/10x)dx = dx/x$, and $v = x$; therefore, $\int \ln(10x)dx = x\ln(10x) - \int x(dx/x) = x\ln(10x) - x + C$.

13. Let $u = s^2$, so $dv = e^{3s}ds$, $du = 2s\,ds$, and $v = e^{3s}/3$; therefore, $\int s^2e^{3s}ds = s^2e^{3s}/3 - \int 2se^{3s}ds/3$. Apply parts again: Let $u = 2s$, so $dv = e^{3s}ds/3$, $du = 2\,ds$, and $v = e^{3s}/9$; therefore, $\int s^2e^{3s}ds = s^2e^{3s}/3 - 2se^{3s}/9 + \int 2e^{3s}/9 = s^2e^{3s}/3 - 2se^{3s}/9 + 2e^{3s}/27 + C = e^{3s}(9s^2 - 6s + 2)/27 + C$.

17. Let $u = t^2$, so $dv = 2t\cos(t^2)dt$, $du = 2t\,dt$, and $v = \sin t^2$ (v is found by substituting $w = t^2$). Thus, $\int 2t^3\cos t^2\,dt = t^2\sin t^2 - \int 2t\sin t^2dt$. Again, we substitute $w = t^2$ to get the answer $t^2\sin t^2 + \cos t^2 + C$.

21. Let $u = \ln(\cos x)$, so $du = (-\sin x/\cos x)dx = -\tan x\,dx$; therefore, $\int \tan x\ln(\cos x)\,dx = -\int u\,du = -u^2/2 + C = -[\ln(\cos x)]^2/2 + C$.

25. We will use formula (8) with the role of x and y reversed. Let $x = \sqrt{1/y - 1}$, $y = 1/(x^2 + 1)$; therefore, $\int\sqrt{1/y - 1}\,dy = \int x\,dy = xy - \int y\,dx = \sqrt{1/y - 1}/(x^2 + 1) - \int[dx/(x^2 + 1)] = y\sqrt{1/y - 1} - \tan^{-1}x + C = y\sqrt{1/y - 1} - \tan^{-1}\sqrt{1/y - 1} + C$.

29. If we choose $f(x) = x$ and $G(x) = \sin x$, then $F(x) = x^2/2$ and $g(x) = \cos x$. We obtain $\int x\sin x\,dx = (x^2/2)\sin x - (1/2)\int x^2\cos x\,dx$. The new integral on the right is more complicated than the one we started with, so this choice of f and G is not suitable.

33. Note that $\ln x^3 = 3\ln x$, so let $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$. We get $\int_1^3 \ln x^3\,dx = 3\int_1^3 \ln x\,dx = 3[x\ln x|_1^3 - \int_1^3 1\,dx] = 3(3\ln 3 - 2)$.

37. Use the formula for integrating inverse functions, so $y = \cos^{-1}(4x)$ and $dy = (-4/\sqrt{1 - 16x^2})dx$. Therefore, $\int_{1/8}^{1/4} \cos^{-1}(4x)dx =$ $x \cos^{-1}(4x)\Big|_{1/8}^{1/4} + 4\int_{1/8}^{1/4} (x/\sqrt{1 - 16x^2})dx$. Let $u = 1 - 16x^2$, so $du = -32x\, dx$ or $-du/32 = x\, dx$. Hence $\int_{1/8}^{1/4} \cos^{-1}(4x)dx = (1/4)(0) -$ $(1/8)(\pi/3) - 4\int_{3/4}^{0} u^{-1/2}du/32 = -\pi/24 - \sqrt{u}/4\Big|_{3/4}^{0} = \sqrt{3}/8 - \pi/24 \approx 0.09$.

41. First, substitute $\theta = 2x$, so $\int_{-\pi}^{\pi} \exp(2x) \sin (2x)\, dx$ becomes $(1/2)\int_{-2\pi}^{2\pi} e^{\theta} \sin \theta\, d\theta$. From Example 4, $\int e^x \sin x\, dx = e^x(\sin x - \cos x)/$ $2 + C$. Thus, we get $(1/2)[(1/2)e^{\theta}(\sin \theta - \cos \theta)]\Big|_{-2\pi}^{2\pi} = [-e^{2\pi} + e^{-2\pi}]/4$.

45. Use the formula for inverse functions with $y = \sin^{-1}2x$ and $dy =$ $\left(2/\sqrt{1 - 4x^2}\right)dx$. Thus, $\int_0^{1/2\sqrt{2}} \sin^{-1}2x\, dx = x \sin^{-1}2x\Big|_0^{1/2\sqrt{2}} - \int_0^{1/2\sqrt{2}}\left(2x/\sqrt{1 - 4x^2}\right)dx$. Substitute $u = 1 - 4x^2$ to get $(1/2\sqrt{2})(\pi/4) + (1/4) \times$ $\int_1^{1/2}(du/\sqrt{u}) = \pi/8\sqrt{2} + (1/4)2u^{1/2}\Big|_1^{1/2} = \pi/8\sqrt{2} + 1/2\sqrt{2} - 1/2 = (\pi - 4)/8\sqrt{2} -$ $1/2$.

49. Let $u = x$, so $dv = \sin ax\, dx$, $du = dx$, and $v = -\cos ax/a$; therefore, $\int_0^{2\pi} x \sin ax\, dx = -x \cos ax/a\Big|_0^{2\pi} + \int \cos ax\, dx/a = - x \cos ax/$ $a\Big|_0^{2\pi} + \sin ax/a^2\Big|_0^{2\pi} = (-2\pi \cos 2\pi a)/a + (\sin 2\pi a)/a^2$. Since the terms in the numerator are finite as a tends to infinity, $\int_0^{2\pi} x \sin ax\, dx$ approaches 0 as a approaches ∞ .

Geometrically, $\sin ax$ oscillates more as a approaches ∞ . Note that the area of each hump gets smaller as a gets larger. Finally, the area of one hump above the x-axis almost cancels the area of the next hump below the x-axis; therefore, the total area approaches 0 .

53. (a) $\int \cos^n x\, dx = \int(\cos^{n-1}x)(\cos x)dx$. We apply integration by parts with $u = \cos^{n-1}x$, and $dv = \cos x\, dx$, so $du = (n - 1)\cos^{n-2}x \times$ $(-\sin x)dx$ and $v = \sin x$. Therefore, $\int\cos^n x\, dx = \cos^{n-1}x \sin x -$ $[-(n - 1)\int \cos^{n-2}x(\sin x\, dx)(\sin x)] = \cos^{n-1}x \sin x + (n-1)\int\cos^{n-2}x \times$ $(1 \cos^2 x)dx = \cos^{n-1} x \sin x + (n - 1)\int\cos^{n-2}x - (n - 1)\int\cos^n x\, dx$.

53. (a) (continued)

Rearrangement yields $n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \times \int \cos^{n-2} x\,dx$. Division by n yields the desired result.

(b) Letting $n = 2$, we have $\int \cos^2 x\,dx = \cos^{2-1} x \sin x/2 + [(2-1)/2]\int \cos^{2-2} x\,dx = \cos x \sin x/2 + (1/2)\int dx = (1/2) \times (\cos x \sin x + x) + C$. Letting $n = 4$, we have $\int \cos^4 x\,dx = (\cos^3 x \sin x)/4 + (3/4)\int \cos^2 x\,dx = (\cos^3 x \sin x)/4 + (3/4) \times (\cos x \sin x/2 + x/2) + C = (1/4)[\cos^3 x \sin x + (3/2)\cos x \sin x + 3x/2] + C$.

57. (a) By the fundamental theorem of calculus, $Q = \int (dQ/dt)dt = \int i\,dt = \int EC(\alpha^2/\omega + \omega)e^{-\alpha t}\sin(\omega t)dt$.

(b) Let $A = EC(\alpha^2/\omega + \omega)$ and let $u = e^{-\alpha t}$, then $dv = \sin(\omega t)\,dt$, $du = -\alpha e^{-\alpha t}dt$, and $v = -\cos(\omega t)/\omega$; therefore, $Q = A\int e^{-\alpha t} \times \sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \int \alpha e^{-\alpha t}\cos(\omega t)\,dt/\omega]$. Now, let $u = \alpha e^{-\alpha t}/\omega$, so $dv = \cos(\omega t)\,dt$, $du = -\alpha^2 e^{-\alpha t}/\omega$, and $v = \sin(\omega t)/\omega$; therefore, $Q = A\int e^{-\alpha t}\sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2 - \int \alpha^2 e^{-\alpha t}\sin(\omega t)\,dt/\omega^2]$. Rearrangement yields $A(1 + \alpha^2/\omega^2)\int e^{-\alpha t}\sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2] + C$. Division by $1 + \alpha^2/\omega^2 = (\omega^2 + \alpha^2)/\omega^2$ gives us $Q = EC[(\alpha^2 + \omega^2)/\omega][\omega^2/(\omega^2 + \alpha^2)][-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2] + C = -ECe^{-\alpha t}[\cos(\omega t) + \alpha\sin(\omega t)/\omega] + C$. $Q(0) = -EC + C = 0$, so $C = EC$. Therefore, $Q(t) = EC\{1 - e^{-\alpha t}[\cos(\omega t) + \alpha\sin(\omega t)/\omega]\}$.

61. In each case, we must consider the case $n = 0$ separately because an n will appear in the denominator. It is always the case that $b_0 = (1/\pi)\int_0^{2\pi} f(x)\cdot 0\,dx = 0$.

(a) $a_0 = (1/\pi)\int_0^{2\pi} \cos 0\,dx = (1/\pi)\int_0^{2\pi} dx = (1/\pi)x\Big|_0^{2\pi} = 2$; $a_n = (1/\pi) \times \int_0^{2\pi} \cos nx\,dx = (1/\pi)(\sin nx)/n\Big|_0^{2\pi} = 0$; $b_n = (1/\pi)\int_0^{2\pi} \sin nx\,dx =$

61. (a) (continued)

$(1/\pi)(-\cos nx)/n\,\Big|_0^{2\pi} = 0$. Thus, $a_0 = 2$ and all other Fourier coefficients are 0 .

(b) $a_0 = (1/\pi)\int_0^{2\pi} x\,dx = (1/\pi)(x^2/2)\Big|_0^{2\pi} = 2\pi$. a_n is determined by letting $u = x$, $dv = \cos nx\,dx$, $du = dx$, and $v = \sin nx/n$, so $a_n = (1/\pi)\int_0^{2\pi} x\cos nx\,dx = (1/\pi)(x\sin nx/n\Big|_0^{2\pi} - \int_0^{2\pi}\sin nx\,dx/n) = (1/\pi)(0 + \cos nx/n^2\Big|_0^{2\pi}) = 0$. b_n is determined by letting $u = x$, $dv = \sin nx\,dx$, $du = dx$, and $v = -\cos nx/n$, so $b_n = (1/\pi)\int_0^{2\pi} x\sin nx\,dx = (1/\pi)(-x\cos nx/n\Big|_0^{2\pi} - \int_0^{2\pi}\cos nx\,dx/n) = (1/\pi)(-2\pi/n - \sin nx/n^2\Big|_0^{2}) = (1/\pi)(-2\pi/n - 0) = -2/n$. Thus, $a_0 = 2\pi$, $b_n = -2/n$ if $n \neq 0$, and all other Fourier coefficients are 0 .

(c) $a_0 = (1/\pi)\int_0^{2\pi} x^2\,dx = (1/\pi)(x^3/3)\Big|_0^{2\pi} = 8\pi^2/3$. a_n is determined by letting $u = x^2$, $dv = \cos nx\,dx$, $du = 2x\,dx$, and $v = \sin nx/n$, so $a_n = (1/\pi)\int_0^{2\pi} x^2\cos nx\,dx = (1/\pi)(x^2\sin nx/n\Big|_0^{2\pi} - \int_0^{2\pi} 2x\sin nx\,dx/n)$. Using the result from part (b) , $a_n = (1/\pi)\times(0) - (1/\pi)(2/n)\int_0^{2\pi} x\sin nx\,dx = -(2/n)(-2/n) = 4/n^2$. b_n is determined by letting $u = x^2$, $dv = \sin nx\,dx$, $du = 2x\,dx$, and $v = -\cos nx/n$, so $b_n = (1/\pi)\int_0^{2\pi} x^2\sin nx\,dx = (1/\pi)\times(-x^2\cos nx/n\Big|_0^{2\pi} + \int_0^{2\pi} 2x\cos nx\,dx/n)$. Using the result from part (b), $b_n = (1/\pi)(-4\pi^2/n) + (1/\pi)(2/n)\int_0^{2\pi} x\cos nx\,dx = -4\pi/n$. Thus, $a_0 = 8\pi^2/3$, $a_n = 4/n^2$ if $n \neq 0$, $b_0 = 0$, and $b_n = -4\pi/n$ if $n \neq 0$.

(d) This problem requires using $\int_0^{2\pi}\sin mx\sin nx\,dx$, $\int_0^{2\pi}\cos mx\times\cos nx\,dx$, and $\int_0^{2\pi}\sin mx\cos nx\,dx$. By Exercise 50(a), $\int_0^{2\pi}\sin mx\cos nx\,dx = (n\sin mx\sin nx + m\cos mx\cos nx)/(n^2 - m^2)\Big|_0^{2\pi} = 0$ if $n^2 \neq m^2$. If $n^2 = m^2$, $\int_0^{2\pi}\sin mx\cos nx\,dx =$

61. (d) (continued)

$(-\cos 2mx)/4m|_0^{2\pi} = 0$. Using the product formula, $\int_0^{2\pi} \sin mx \times \sin nx\, dx = (1/2)\int_0^{2\pi} [\cos(mx - nx) - \cos(mx + nx)]\, dx = (1/2) \times [\sin(mx - nx)/(m - n) - \sin(mx + nx)/(m + n)]|_0^{2\pi} = 0$ provided $m \neq n$. If $m = n$, then we apply the half-angle formula: $\int_0^{2\pi} \sin^2 mx\, dx = \int_0^{2\pi} [(1 - \cos 2mx)/2]\, dx = (x - \sin 2mx/2m)/2|_0^{2\pi} = \pi$. Applying the product formula, we get $\int_0^{2\pi} [\cos(mx + nx) + \cos(mx - nx)]\, dx = [\sin(mx + nx)/(m + n) + \sin(mx - nx)/(m - n)]|_0^{2\pi} = 0$ provided $m \neq n$. If $m = n$, then the half-angle formula implies $\int_0^{2\pi} \cos^2 mx\, dx = \int_0^{2\pi} [(1 + \cos 2mx)/2]\, dx = [(x + \sin 2mx/2m)/2]|_0^{2\pi} = \pi$.

$a_0 = (1/\pi)\int_0^{2\pi} (\sin 2x + \sin 3x + \cos 4x)dx = (1/\pi)(-\cos 2x/2 - \cos 3x/3 + \sin 4x/4)|_0^{2\pi} = 0$. $a_n = (1/\pi)[\int_0^{2\pi}(\sin 2x + \sin 3x + \cos 4x)(\cos nx\, dx)] = (1/\pi)(\pi) = 1$ if $n = 4$, and $a_n = 0$ if $n \neq 4$. $b_n = (1/\pi)[\int_0^{2\pi}(\sin 2x + \sin 3x + \cos 4x)(\sin nx\, dx)] = 1$ if $n = 2$ or 3 , and $b_n = 0$ otherwise. Therefore, $a_4 = b_2 = b_3 = 1$ and all other Fourier coefficients are 0 .

SECTION QUIZ

1. Which statement is the formula for integration by parts?

   (a) $\int uv\, dx = uv - v\int u\, dx$

   (b) $\int y\, dx = xy - \int x\, dy$

   (c) $\int u\, dv = uv - v\int du$

2. Evaluate the following integrals:

   (a) $\int x^5 \ln x \, dx$

   (b) $\int x^5 \cos(x^2) dx$

   (c) $\int x^4 e^x dx$

   (d) $\int e^{4x} \sin 4x \, dx$

   (e) $\int \sin^{-1}(3x) dx$

3. Comment on the following: Differentiation of both sides shows that $\int \ln x \, dx = x \ln x - x + C$. Thus, $\int x \ln x \, dx = x(x \ln x - x) - \int (x \ln x - x) dx$.

   (a) Is integration by parts used correctly? Explain.

   (b) Evaluate $\int x \ln x \, dx$.

4. Jumping Janet's new inheritance is a porcupine farm with an odd-shaped plot of land. A new fence is needed, so Janet decides to make the holes for the posts. In preparation to hop away from flying quills, Janet jumps along the boundary on a pogo stick. The land is bounded by $y = x^2 \cos x$ and $y = -(x - 3\pi/4)^2 + 9\pi^2/16$. If one unit equals 10 meters, how much land area is allotted to each of Jumping Janet's 25 porcupines? (Hint: Draw a graph to determine the limits of integration.)

ANSWERS TO PREREQUISITE QUIZ

1. (a) $1 + \ln x$

   (b) $15x^2 - 46x + 17$

   (c) $2x \tan x + x^2 \sec^2 x$

2. $(x - 3)/2$

ANSWERS TO SECTION QUIZ

1. b

2. (a) $x^6 \ln x/6 - x^6/36 + C$

(b) $(x^4/2)\sin(x^2) + x^2\cos(x^2) - \sin(x^2) + C$

(c) $x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$

(d) $(e^{4x}\sin 4x - e^{4x}\cos 4x)/8 + C$

(e) $x \sin^{-1}(3x) - \sqrt{1 - 9x^2}/3 + C$

3. (a) The integration is being performed correctly; however, it is more useful to let $u = \ln x$ and $dv = x\,dx$.

(b) $x^2 \ln x/2 - x^2/4 + C$

4. $(9\pi^3/4 + 9\pi^2 - 8)\ m^2$/porcupine

7.R Review Exercises for Chapter 7

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the sum rule, power rule, and the basic trigonometric rules for antidifferentiation, we have $\int(x + \sin x)dx = x^2/2 - \cos x + C$.

5. Using integration formulas for sums, exponentials, logarithms, rational powers, and trigonometric functions, we get $\int(e^x - x^2 - 1/x + \cos x)dx = e^x - x^3/3 - \ln|x| + \sin x + C$.

9. Integrate by substitution. Let $u = x^3$, then $du = 3x^2dx$ or $du/3 = x^2dx$; therefore, $\int x^2 \sin x^3\, dx = \int \sin u\, du/3 = -\cos u/3 + C = -\cos x^3/3 + C$.

13. Integrate by substitution. Let $u = x + 2$, so $du = dx$; therefore, $\int(x + 2)^5dx = \int u^5du = u^6/6 + C = (x + 2)^6/6 + C$.

17. Substitute $u = \cos 2x$, so $du = -2 \sin 2x\, dx$; therefore, $\int 2 \cos^2 2x \sin 2x\, dx = -\int u^2du = -u^3/3 + C = -\cos^3 2x + C$.

21. Factor out $1/\sqrt{4} = 1/2$ and substitute $u = t/2$, so $du = dt/2$. Thus, $\int(1/\sqrt{4 - t^2} + t^2)dt = (1/2)\int\left[dt/\sqrt{1 - (t/2)^2}\right] + \int t^2dt = (1/2)\int\left[2\, du/\sqrt{1 - u^2}\right] + t^3/3 = \sin^{-1}u + t^3/3 + C = \sin^{-1}(t/2) + t^3/3 + C$.

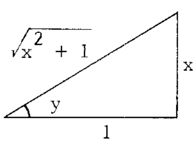
25. Integrate by parts. Let $u = x^2$, then $dv = \cos x\, dx$, $du = 2x\, dx$, and $v = \sin x$; therefore, $\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx$. Now, repeat the integration by parts with $u = 2x$, $dv = \sin x\, dx$, $du = 2dx$, and $v = -\cos x$; therefore, $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - \int 2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$.

29. Integrate by parts. Let $u = \ln 3x$, then $dv = x^2dx$, $du = 3dx/3x = dx/x$, and $v = x^3/3$; therefore, $\int x^2 \ln 3x\, dx = x^3 \ln 3x/3 - \int x^2dx/3 = x^3 \ln 3x/3 - x^3/9 + C$.

33. Integrate by parts with $u = x$, $dv = \cos 3x\,dx$, $du = dx$, and $v = \sin 3x/3$; therefore, $\int x \cos 3x\,dx = x \sin 3x/3 - \int \sin 3x\,dx/3 = x \sin 3x/3 + \cos 3x/9 + C$.

37. Substitute $w = x^2$, then $dw = 2x\,dx$; therefore, $\int x^3 e^{(x^2)}dx = \int x^2\left[e^{(x^2)}\right](x\,dx) = \int we^w dw/2$. Now, integrate by parts with $u = w$, so $dv = e^w dw/2$, $du = dw$, and $v = e^w/2$; therefore, $\int x^3 e^{(x^2)}dx = we^w/2 - \int e^w dw/2 = we^w/2 - e^w/2 + C = x^2 e^{(x^2)}/2 - e^{(x^2)}/2 + C$.

41. Substitute $w = \sqrt{x}$, then $dw = (1/2\sqrt{x})dx = dx/2w$; therefore, $\int e^{\sqrt{x}}dx = \int e^w(2w\,dw)$. Now, integrate by parts with $u = 2w$, so $dv = e^w dw$, $du = 2\,dw$, and $v = e^w$. Thus, $\int e^{\sqrt{x}}dx = 2we^w - \int 2e^w dw = 2we^w - 2e^w + C = e^w(2w - 2) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$.

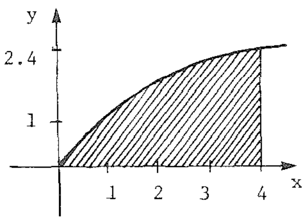
45. 

Using the formula for integrating inverse functions, we have $y = \tan^{-1}x$, and $x = \tan y$. Thus, $\int \tan^{-1}x\,dx = \tan y(\tan^{-1}x) - \int \tan y\,dy = \tan(\tan^{-1}x)(\tan^{-1}x) - \int(\sin y/\cos y)dy$. Let $u = \cos y$, so $du = -\sin y\,dy$; therefore, $\int \tan^{-1}x\,dx = x\tan^{-1}x + \int(du/u) = x\tan^{-1}x + \ln|u| + C = x\tan^{-1}x + \ln|\cos(\tan^{-1}x)| + C = x\tan^{-1}x + \ln\left(1/\sqrt{x^2 + 1}\right) + C = x\tan^{-1}x - \ln(1 + x^2)/2 + C$.

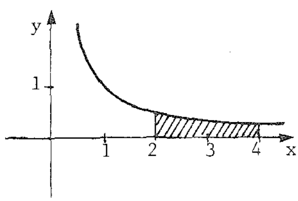
49. Let $u = x$, so $dv = \sin 5x\,dx$, $du = dx$, and $v = -\cos 5x/5$; therefore, $\int_0^{\pi/5} x \sin 5x\,dx = -x\cos 5x/5\Big|_0^{\pi/5} + \int_0^{\pi/5}\cos 5x\,dx/5 = -(\pi/5)(-1/5) + \sin 5x/25\Big|_0^{\pi/5} = \pi/25$.

53. Let $u = \tan^{-1}x$, then $dv = x\,dx$, $du = 1/(1 + x^2)$, and $v = x^2/2$. Integrating by parts and using long division, we have $\int_0^{\pi/4} x\tan^{-1}x\,dx = x^2\tan^{-1}x/2\Big|_0^{\pi/4} - \int_0^{\pi/4}[x^2dx/2(1 + x^2)] = (\pi^2/32)\tan^{-1}(\pi/4) - \int_0^{\pi/4}(dx/2) + \int_0^{\pi/4}[dx/2(1 + x^2)] = (\pi^2/32)\tan^{-1}(\pi/4) + (-x/2 + \tan^{-1}x/2)\Big|_0^{\pi/4} = ((\pi^2 + 16)/32)\tan^{-1}(\pi/4) - \pi/8$.

57. Integrate by parts with $u = x$, $dv = \sqrt{2x + 3}$, $du = dx$, and $v = (2x + 3)^{3/2}/3$ (using integration by substitution); therefore, $\int_0^1 x\sqrt{2x + 3}\, dx = x(2x + 3)^{3/2}/3\big|_0^1 - \int_0^1 (2x + 3)^{3/2}dx/3 = 5^{3/2}/3 - (2x + 3)^{5/2}/15\big|_0^1 = 5\sqrt{5}/3 - 25\sqrt{5}/15 + 9\sqrt{3}/15 = 3\sqrt{3}/5$.

61. 

The area under the curve of $f(x)$ on $[a,b]$ is $\int_a^b f(x)dx$. Let $u = x^2 + 9$, so $du = 2x\, dx$; therefore, $\int_0^4 \left\{3x\, dx/\sqrt{x^2 + 9}\right\} = \int_9^{25} (3\, du/2\sqrt{u}) = (3/2)\sqrt{u}(2)\big|_9^{25} = 3(5 - 3) = 6$.

65. 

The area under the curve of $f(x)$ on $[a,b]$ is $\int_a^b f(x)dx$. Thus, the area is $\int_2^4 (dx/x) = \ln x\big|_2^4 = \ln 4 - \ln 2 = \ln(4/2) = \ln 2$.

69. In $[0,\pi/2]$, $e^x + \cos x > -x^3 - 2x - 6$, so the area is $\int_0^{\pi/2} [(e^x + \cos x) - (-x^3 - 2x - 6)]\, dx = (e^x + \sin x + x^4/4 + x^2 + 6x)\big|_0^{\pi/2} = e^{\pi/2} - 1 + 1 + \pi^4/64 + \pi^2/4 + 3\pi = e^{\pi/2} + (\pi^4 + 16\pi^2 + 192\pi)/64 \approx 18.225$.

73. *In the first method*, let $u = \sin(\pi x/2)$, then $dv = \cos(\pi x)dx$, $du = (\pi/2)\cos(\pi x/2)dx$, and $v = \sin(\pi x)/\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = \sin(\pi x/2)\sin(\pi x)/\pi - \int \sin(\pi x)\cos(\pi x/2)dx/2$. Now, let $u = \cos(\pi x/2)$, so $dv = \sin(\pi x)dx/2$, $du = -(\pi/2)\sin(\pi x/2)dx$, $v = -\cos(\pi x)/2\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = \sin(\pi x/2)\sin(\pi x)/\pi + \cos(\pi x/2)\cos(\pi x)/2\pi + (1/4)\int \sin(\pi x/2)\cos(\pi x)dx$. Rearrangement yields $\int \sin(\pi x/2)\cos(\pi x)dx = 4[\sin(\pi x/2)\sin(\pi x)/\pi + \cos(\pi x/2)\cos(\pi x)/2\pi]/3 + C$.

By the second method, let $u = \cos(\pi x)$, then $dv = \sin(\pi x/2)dx$, $du = -\pi \sin(\pi x)dx$, and $v = -2\cos(\pi x/2)/\pi$; therefore, $\int \sin(\pi x/2) \times \cos(\pi x)dx = 2\cos(\pi x)\cos(\pi x/2)/\pi - \int 2\cos(\pi x/2)\sin(\pi x)dx$. Now, let

73. (continued)

$u = \sin(\pi x)$, then $dv = 2\cos(\pi x/2)dx$, $du = \pi\cos(\pi x)dx$, and $v = 4\sin(\pi x/2)/\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = -2\cos(\pi x)\cos(\pi x/2)/\pi - 4\sin(\pi x)\sin(\pi x/2)/\pi + 4\int \sin(\pi x/2)\cos(\pi x)dx$. Rearrangement yields $\int \sin(\pi x/2)\cos(\pi x)dx = [4\sin(\pi x)\sin(\pi x/2)/\pi + 2\cos(\pi x)\cos(\pi x/2)/\pi]/3 + C$.

77. (a) Let $u = \ln x$, then $du = dx/x$; therefore, $\int (\ln x/x)dx = \int u\,du = u^2/2 + C = (\ln x)^2/2 + C$.

(b) Letting $x = 3\tan u$, we have $dx = 3\sec^2 u\,du$ and $u = \tan^{-1}(x/3)$; therefore, $\int_{\sqrt{3}}^{3\sqrt{3}} \left[dx/x^2\sqrt{x^2+9}\right] = \int_{\pi/6}^{\pi/3}\left[(3\sec^2 u\,du)/9\tan^2 u\sqrt{9\tan^2 u + 9}\right] = \int_{\pi/6}^{\pi/3}[3\sec^2 u\,du/(9\tan^2 u)(3\sec u)] = (1/9)\int_{\pi/6}^{\pi/3}(\sec u\,du/\tan^2 u) = (1/9)\int_{\pi/6}^{\pi/3}(\cos u\,du/\sin^2 u)$. Let $v = \sin u$, so $dv = \cos u\,du$; therefore, $\int_{\sqrt{3}}^{3\sqrt{3}}\left(dx/x^2\sqrt{x^2+9}\right) = (1/9)\int_{1/2}^{\sqrt{3}/2}(dv/v^2) = (1/9)(-1/v)\Big|_{1/2}^{\sqrt{3}/2} = (1/9)(-2/\sqrt{3} + 2) = (2/9)(-\sqrt{3}/3 + 1)$.

81. (a) Let $u = e^{25x}$, so $dv = \cos 5x\,dx$, $du = 25e^{25x}$, and $v = \sin 5x/5$; therefore, $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/5 - 5\int(\sin 5x)e^{25x}dx$. Now, let $u = e^{25x}$, so $dv = 5\sin 5x$, $du = 25e^{25x}dx$, and $v = -\cos 5x$. Thus, $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/5 + e^{25x}\cos 5x - 25\int e^{25x}\cos 5x\,dx$. Rearrangement gives us $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/130 + e^{25x}\cos 5x/26$. Evaluating at the limits, 0 and t , gives us $e^{25t}\sin 5t/130 + e^{25t}\cos 5t/26 - 1/26$. Multiplying this by $100e^{-25t}$ gives us $(100/26)(\sin 5t/5 + \cos 5t - e^{-25t})$.

(b) $Q(1.01) = (100/26)[(\sin 5.05)/5 + \cos 5.05 - e^{-25.25}] \approx (3.8462) \times (-0.18871 + 0.33123 - 1.0816 \times 10^{-11}) \approx (3.8462)(0.14252) \approx 0.548$ coulomb.

85. Let $u = x$, then $dv = e^{ax}\cos(bx)\,dx$, $du = dx$, and $v = \int e^{ax}\cos(bx)\,dx$. Now, let $u = e^{ax}$, so $dv = \cos(bx)\,dx$, $du = ae^{ax}dx$, and $v = \sin(bx)/b$; therefore, $\int e^{ax}\cos(bx)\,dx = e^{ax}\sin(bx)/b - \int ae^{ax}\sin(bx)dx/b$. Repeat integration by parts with $u = ae^{ax}$, $dv = \sin(bx)\,dx/b$, $du = a^2e^{ax}dx$, and $v = -\cos(bx)/b^2$; therefore, $\int e^{ax}\cos(bx)dx = e^{ax}\sin(bx)/b + ae^{ax}\cos(bx)/b^2 - \int a^2e^{ax} \times \cos(bx)\,dx/b^2$. Rearrangement yields $\int e^{ax}\cos(bx)dx = [e^{ax}\sin(bx)/b + ae^{ax}\cos(bx)/b^2][b^2/(a^2 + b^2)] = [be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2)$.

Thus, $\int xe^{ax}\cos(bx)\,dx = x[be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) - \int[be^{ax}\sin(bx) + ae^{ax}\cos(bx)]dx/(a^2 + b^2)$. For $\int be^{ax}\sin(bx)dx/(a^2 + b^2)$, let $u = be^{ax}/(a^2 + b^2)$, so $dv = \sin(bx)\,dx$, $du = abe^{ax}dx/(a^2 + b^2)$, and $v = -\cos(bx)/b$. Thus, $\int be^{ax}\sin(bx)dx/(a^2 + b^2) = -e^{ax}\cos(bx)/(a^2 + b^2) + \int ae^{ax}\cos(bx)dx/(a^2 + b^2)$.

Using the result for $\int e^{ax}\cos(bx)\,dx$ in the first part of this problem, we have $\int xe^{ax}\cos(bx)\,dx = x[be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) + [e^{ax}\cos(bx)/(a^2 + b^2)]\int ae^{ax}\cos(bx)\,dx/(a^2 + b^2) - \int ae^{ax}\cos(bx)\,dx/(a^2 + b^2) = [bxe^{ax}\sin(bx) + axe^{ax}\cos(bx)]/(a^2 + b^2) + e^{ax}\cos(bx)/(a^2 + b^2) - [2a/(a^2 + b^2)][be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) + C = xe^{ax}[b\sin(bx) + a\cos(bx)]/(a^2 + b^2) + e^{ax}[(b^2 - a^2) \times \cos(bx) - 2ab\sin(bx)]/(a^2 + b^2)^2 + C$.

TEST FOR CHAPTER 7

1. True or false:

(a) Substituting $u = -x^2$ into $\int_0^1 \exp(-x^2)dx$ yields $\int_0^{-1} e^u du$.

(b) If f and g are integrable functions and a , b , c are real numbers such that $a < b < c$, then $\int_a^b [f(x) + g(x)]dx = \int_a^c f(x)dx + \int_c^b g(x)dx + \int_c^b f(x)dx + \int_a^c g(x)dx$.

(c) For any constant $a > 0$, $\int_0^1 a^x dx = a - 1$.

(d) $\int_0^1 [dx/(1 + x^2)] = \tan^{-1}(1) - \tan^{-1}(0) = 45 - 0 = 45$.

(e) The area of the region bounded by $x = 2 - y^2$ and the y-axis is $\int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)dy$.

2. Show that $\int [dx/(x^3 - x^2)] = 1/x - \ln|x| + \ln|x - 1| + C$.

3. Substitute $x = \sin\theta$ into the integral $\int_{-1}^1 \sqrt{1 - x^2}\, dx$ and write the integral as a definite integral in terms of θ .

4. Evaluate the following by making a substitution:

(a) $\int [(x + 2)/(x^2 + 4x - 3)]dx$

(b) $\int [e^t/(1 + e^{2t})]dt$

(c) $\int [\sin x/(1 + \cos^2 x)]dx$

5. Evaluate the following integrals using integration by parts:

(a) $\int t^2 \sin t\, dt$

(b) $\int \ln(x^4)dx$

(c) $\int x^2 e^{-x} dx$

6. Let $u = 1/x$ and $dv = dx$, so integration by parts gives $\int(dx/x) = (1/x)x - \int(-1/x^2)x\, dx = 1 + \int(dx/x)$. Subtracting $\int(dx/x)$ from both sides yields $0 = 1$. Explain what went wrong.

7. Find the area under the graph of $f(x)$ on $[0,1]$ if $f(x)$ is:

(a) $x\exp(-x^2)$

(b) $x^3\exp(-x^2)$

8. Evaluate $\int\tan^{-1}(3x)dx$.

9. A particle's acceleration at time t is $\sqrt{(1-t)/(t+1)}$. If its velocity at $t = 0$ is 0 , what is the velocity as a function of t? (Hint: $\sqrt{(1-x)/(1-x)} = 1$.)

10. An oddly dressed gentleman, nicknamed Odd Ollie, came into the Odd Furniture Store. He bought a table top whose edges, according to the salesman, are given by $f(x) = x/(x+1)$, and the lines $x = 0$, $x = 1$, and $y = 0$. Each unit represents 1 meter. Odd Ollie also has a collection of "DO NOT REMOVE UNDER PENALTY OF LAW" tags, which he wants to make into a tablecloth. The average tag measures 3 cm. by 5 cm. Odd Ollie is willing to cut the tags to match the shape of his table. How many tags does Odd Ollie minimally need for his tablecloth?

ANSWERS TO CHAPTER TEST

1. (a) False; substitution yields $\int_0^{-1}(-e^u/2\sqrt{-u})du$.

(b) True

(c) False; it is $(\ln a)(a-1)$.

(d) False; $\tan^{-1}(1) - \tan^{-1}(0) = \pi/4$.

(e) True

2. Differentiate the right-hand side and simplify.

3. $\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$.

4. (a) $\ln\sqrt{x^2 + 4x - 3} + C$

(b) $\tan^{-1}(e^t) + C$

(c) $-\tan^{-1}(\cos x) + C$

5. (a) $-t^2\cos t + 2t \sin t + 2 \cos t + C$

(b) $4x \ln x - 4x + C$

(c) $-(x^2 + 2x + 2)e^{-x} + C$

6. The integral $\int(dx/x)$ may have different additive constants on each side.

7. (a) $1/2 - 1/2e$

(b) $1/2 - 1/e$

8. $x \tan^{-1}(3x) + (1/6)\ln(1 + 9x^2) + C$

9. $\sin^{-1}t + \sqrt{1 - t^2} - 1$

10. 205 tags