CHAPTER 8

DIFFERENTIAL EQUATIONS

8.1 Oscillations

PREREQUISITES

1. Recall how to use the chain rule for differentiation (Section 2.2).
2. Recall how to differentiate trigonometric functions (Section 5.2).
3. Recall how to convert between cartesian and polar coordinates (Section 5.1).
4. Recall how to graph trigonometric functions (Section 5.5).

PREREQUISITE QUIZ

1. Differentiate the following expressions:

   (a) $\sin 3x^2$

   (b) $\cos(x^3 + 2)$

2. Convert the polar coordinates $(2,\pi/3)$ to cartesian coordinates.
3. Convert the cartesian coordinates $(-5/2,0)$ to polar coordinates.
4. Sketch the graph of $y = 2\cos(x/2)$.

GOALS

1. Be able to solve differential equations of the form $x'' = -\omega^2 x$.
2. Be able to convert from $A \sin \omega t + B \cos \omega t$ to $\alpha \cos(\omega t - \theta)$ and sketch the graph.

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STUDY HINTS

1. Notation and definitions. In the force law $F = -kx$, k is called the spring constant. The frequency of oscillations, measured in radians per second, is $\omega = \sqrt{k/m}$.

2. Spring equation. $d^2x/dt^2 = -\omega^2x$ describes simple harmonic motion, and it is known as the spring equation. If $x = x_0$ and $dx/dt = v_0$ at $t = 0$, the solution is $x = x_0\cos \omega t + (v_0/\omega)\sin \omega t$. You should memorize this solution.

3. Uniqueness. Just be aware that the spring equation has a unique solution if the initial values for position and velocity are given. The proof of uniqueness is usually not needed for solving problems.

4. Graphing $x = \alpha \cos(\omega t - \theta)$. In order to graph $x = A \cos \omega t + B \sin \omega t$, convert it to $x = \alpha \cos(\omega t - \theta)$ by using the relationships $\alpha = \sqrt{A^2 + B^2}$ and $\theta = \tan^{-1}(B/A)$. Knowing this, we also know that the maximum occurs and equals α when $\omega t - \theta = 0$, i.e., $t = \theta/\omega$, which is the phase shift. The graph repeats itself when $\omega t - \theta = 2\pi$; therefore, the period is $2\pi/\omega$. Finally, the amplitude is α .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We want to show that $f(t + 2\pi/3) = f(t)$. $f(t + 2\pi/3) = \cos 3(t + 2\pi/3) = \cos(3t + 2\pi)$. Since the cosine function has period 2π , $\cos(3t + 2\pi) = \cos(3t) = f(t)$.

5. The solution of $d^2x/dt^2 + \omega^2x = 0$ is $x = x_0\cos \omega t + (v_0/\omega)\sin \omega t$ where $x_0 = x(0)$ and $v_0 = dx/dt$ at $t = 0$. Here, $\omega = 3$, so the solution is $x = 1 \cos 3t + (-2/3)\sin 3t = \cos 3t - 2 \sin 3t/3$.

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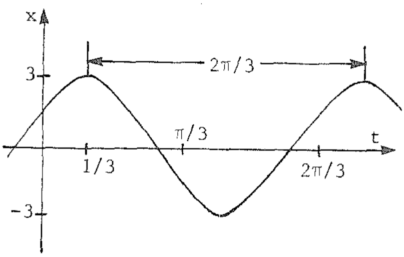
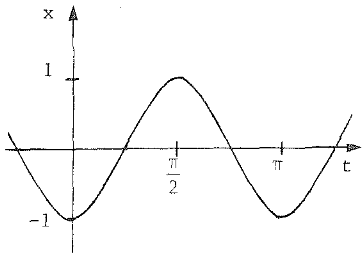
9. 

A function of the form $x = \alpha\cos(\omega t - \theta)$ has period $2\pi/\omega$, amplitude α, and phase shift θ/ω. In this case, the period is $2\pi/3$; the amplitude is 3; the phase shift is $1/3$.

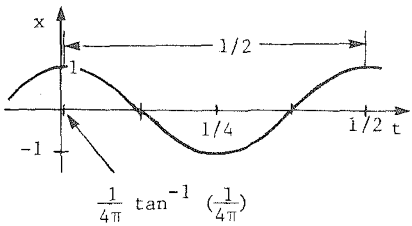
13. 

The solution of $d^2x/dt^2 + \omega^2 x = 0$ is $x = x_0\cos\omega t + (v_0/\omega)\sin\omega t$ where $x_0 = x(0)$ and $v_0 = dx/dt$ at $t = 0$. Here, $\omega = 2$, so the solution is $x = -\cos 2t$.

17. Here, $\omega = 2$, $y_0 = 1$, and $v_0 = 3$, so the solution is $y = y_0\cos\omega t + (v_0/\omega)\sin\omega t = \cos 2t + (3/2)\sin 2t$.

21. The frequency $\omega/2\pi = 2$ is given, so $\omega = 4\pi$.

(a) By definition, $\omega = \sqrt{k/m}$, so $4\pi = \sqrt{k/1}$; therefore, $k = 16\pi^2$ is the spring constant.

(b) 

Here $x_0 = v_0 = 1$, so the solution of the spring equation becomes $x = \cos 4\pi t + (1/4\pi)\sin 4\pi t$. To graph it, rewrite $(1, 1/4\pi)$ in polar coordinates: $\left(\sqrt{1 + 1/16\pi^2}, \tan^{-1}(1/4\pi)\right)$, or approximately $(1, 0.08)$, so $x \approx \cos(4\pi t - 0.08)$. This is used to sketch the graph.

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25. (a) The equation of motion is $m(d^2x/dt^2) = f(x)$, so this specific equation is $27(d^2x/dt^2) = -3x + 2x^3$.

(b) The linearized equation of motion is $m(d^2x/dt^2) = f'(x_0)(x - x_0)$ where $f(x_0) = 0$. We are told that $f(0) = 0$. $f'(x) = -3 + 6x^2$ implies $f'(0) = -3$, so the linearized equation is $27(d^2x/dt^2) = -3x$.

(c) The period of linearized oscillations is $2\pi/\sqrt{-f'(x_0)/m}$, which is $2\pi/\sqrt{3/27} = 6\pi$.

29. Since $f(t)$ satisfies the spring equation, $f(t)$ has the form $A \cos \omega t + B \sin \omega t$. Therefore, $f \circ g = f(g(t)) = A \cos \omega(at + b) + B \sin \omega(at + b)$. Now if $f \circ g$ satisfies the spring equation, $d^2(f \circ g)/dt^2 + \omega^2(f \circ g) = 0$. Now, $d(f \circ g)/dt = -a\omega A \sin \omega(at + b) + a\omega B \cos \omega(at + b)$, so $d^2(f \circ g)/dt^2 = -a^2\omega^2 A \cos \omega(at + b) - a^2\omega^2 B \times \sin \omega(at + b)$. Hence $d^2(f \circ g)/dt^2 + \omega^2(f \circ g) = 0 = -a^2\omega^2[A \cos \omega(at + b) + B \sin \omega(at + b)] + \omega^2[A \cos \omega(at + b) + B \sin \omega(at + b)] = \omega^2[A \cos \omega(at + b) + B \sin \omega(at + b)](1 - a^2)$. Since this is zero, one of the above factors must be zero. Since $\omega \neq 0$ and $(f \circ g)$ can be nonzero, $1 - a^2$ must be 0 . Therefore $a = \pm 1$.

Notice that the choice of b does not affect the differentiation of $f \circ g$. Thus, there is no restriction on b .

33. (a) By the definition of antiderivatives $V'(x_0) = -f(x_0) = 0$ and $V''(x_0) = -f'(x_0) > 0$. By the second derivative test, x_0 is a local minimum of V .

(b) By the chain rule, we have $dE/dt = (1/2)m(2)(dx/dt)(d^2x/dt^2) + V'(x)(dx/dt) = m(dx/dt)(dx^2/dt^2) + (-f(x))(dx/dt) = (dx/dt)[m(dx^2/dt^2) - f(x)] = (dx/dt)(0) = 0$.

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33. (c) By part (b), we know that E is constant. If dx/dt and $x - x_0$ are sufficiently small at $t = 0$, then $E = (1/2)m(dx/dt)^2 + V(x)$ is also small. Since E must remain a small constant, the two terms which comprise E must both be small also. Therefore, dx/dt and $x - x_0$ will both remain small.

SECTION QUIZ

1. Solve the differential equation $d^2y/dx^2 + \pi^2 y = 0$, assuming that $dy/dx = 3$ when $x = -2$ and $y = 2$ when $x = 1$.

2. Let $x = 8\sin(t/4) + 3\cos(t/4)$.

   (a) What differential equation of the form $x''(t) = -kx(t)$ does x solve? Remember to specify the initial position and velocity.

   (b) Convert the given equation into the form $x = \alpha\cos(\omega t - \theta)$.

   (c) Sketch the graph of x.

   (d) What is the spring constant if the mass is 2 ?

3. As a money-saving concept, the latest lines of economical cars are not equipped with shock absorbers. After going over a pothole, it has been determined that these 800,000 gram cars have a spring constant of 400,000. Initially, after going over a pothole, the car is 10 cm. from equilibrium and bouncing with a velocity of 5 cm/sec.

   (a) Write an equation of the form $x = A\cos\omega t + B\sin\omega t$ describing the car's vertical motion.

   (b) Sketch the graph of the solution.

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ANSWERS TO PREREQUISITE QUIZ

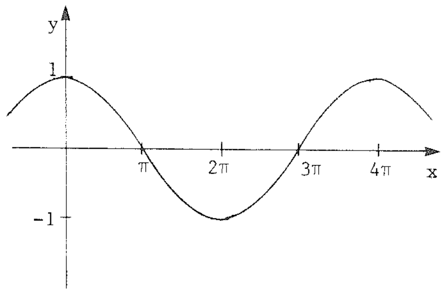
1. (a) $6x \cos 3x^2$

   (b) $-3x^2 \sin(x^3 + 2)$

2. $(1,\sqrt{3})$

3. $(-5/2,0)$ or $(5/2,\pi)$

4.



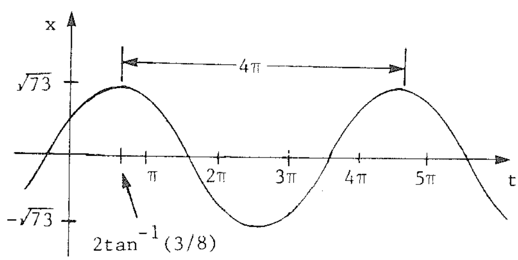
ANSWERS TO SECTION QUIZ

1. $y = -2 \cos \pi x + (3/\pi)\sin \pi x$

2. (a) $x'' = -x/16$; $x_0 = 3$, $v_0 = 2$

   (b) $x = \sqrt{73} \cos(t/2 - \tan^{-1}(8/3))$

   (c)



   (d) 1/2

3. (a) $x = 10 \cos(\sqrt{2}t/2) + 5\sqrt{2} \sin(\sqrt{2}t/2)$

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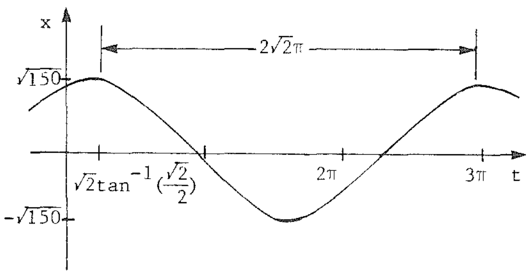
3. (b)



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8.2 Growth and Decay

PREREQUISITES

1. Recall how to differentiate exponential functions (Section 6.3).

PREREQUISITE QUIZ

1. Differentiate the following:

   (a) $\exp(3t)$

   (b) $\exp(x^2)$

   (c) $\exp(-2t + 4)$

GOALS

1. Be able to solve differential equations of the form $f'(t) = \gamma f(t)$.
2. Be able to understand the concept of half-life and compute it.

STUDY HINTS

1. Decay and growth. $f'(t) = \gamma f(t)$ is known as the decay or growth equation depending on whether γ is negative or positive. Regardless of sign, the solution should be memorized; it is $f(t) = f(0)\exp(\gamma t)$.
2. Half-life. Rather than memorizing the formula for half-life, it is easiest to apply the definition. By definition, $f(t_{1/2}) = (1/2)f(0)$; from the solution, we also have $f(t_{1/2}) = f(0)\exp(-\kappa t_{1/2})$. Therefore $1/2 = \exp(-\kappa t_{1/2})$. See how this is used in Example 5, Method 2.
3. Doubling time. This is similar to the half-life concept except that the rate constant is positive rather than negative.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

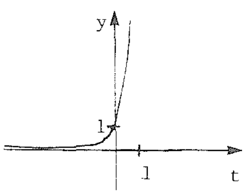
1. This exercise is similar to Example 1. We have $\gamma = -0.11$, so a differential equation for the iron's temperature is $dT/dt = -0.11(T - 20)$.

5. The solution of $f'(t) = \gamma f(t)$ is $f(t) = f(0)\exp(\gamma t)$. Here, $f(0) = 2$ and $\gamma = -3$, so $f(t) = 2\exp(-3t)$.

9. The solution of $f'(t) = \gamma f(t)$ is $f(t) = f(0)\exp(\gamma t)$. Here $\gamma = 8$, so $y(t) = y(0)\exp(8t)$. Substituting $y = 2$ and $t = 1$ gives $2 = y(0)e^8$, so $y(0) = 2e^{-8}$. Therefore, $y(t) = 2e^{-8}(e^{8t}) = 2\exp(8t - 8)$.

13. As in Example 3, we let $f(t) = T - 20$, so $f'(t) = -(0.11)f(t)$ and $f(0) = 210 - 20 = 190$. Therefore, $f(t) = 190\exp(-0.11t)$. We want to find out when $T = 100$ or $f(t) = 80$. Thus $80 = 190\exp(-0.11t)$ or $-0.11t = \ln(8/19)$, i.e., $t = [\ln(8/19)]/(-0.11) \approx 7.86$ minutes.

17.



Rearrangement yields $f' = 3f$, so $\gamma = 3$ and $f(0) = 1$; therefore, the solution is $f(t) = e^{3t}$.

21. Since $\gamma = 3$ is positive, x must be an increasing function. See Fig. 8.2.1.

25. If the decay law is $f'(t) = -\kappa f(t)$, then the half-life equation is $t_{1/2} = (1/\kappa)\ln 2$. Here, $\kappa = 0.000021$, so $t_{1/2} = (1/0.000021)\ln 2 \approx$ 33,000 years.

29. Rearrangement yields $\kappa = \ln 2/t_{1/2} = \ln 2/450000000$. After 90% decays, 0.10 gram is left, so $0.10 = e^{-(\ln 2/450000000)t}$, i.e., $\ln(0.10) = -(\ln 2/450000000)t$, i.e., $t = -\ln(0.10)450000000/\ln 2 \approx 1.5 \times 10^9$ years .

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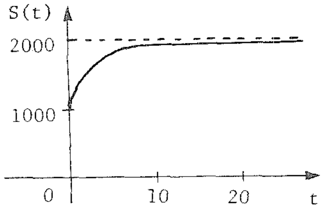
33. $f(t)/f(0) = 2 = e^{\gamma(10)}$ implies $\gamma = \ln 2/10$. Then, $3000 = 100e^{(\ln 2/10)t}$ implies $\ln 30 = (\ln 2/10)t$, i.e., $t = 10 \ln 30/\ln 2 \approx 49$ minutes.

37. As in Example 8, if $P(t) = 4P_0$, then $4 = e^{0.075t}$, so $\ln 4 = 0.075t$, i.e., $t = \ln 4/(0.075) \approx 18.5$ years.

41. (a) Differentiation gives $S'(t) = 300e^{-0.3t}$.

(b) $\lim_{t\to\infty} S(t) = 2000 - 1000\cdot 0 = 2000$, so 2000 books will be sold eventually. This is the difference between a constant and a natural decay.

(c)



45. (a) Upon differentiation of the solution, we get $da/dt = \int_1^t [h(s)/s^2]ds + th(t)/t^2 + C$; therefore, $t(da/dt) = t\int_1^t [h(s)/s^2]ds + tC + t^2h(t)/t^2 = a + h$.

(b) Here, $h(s) = e^{-1/s}$, so the solution is $t\int_1^t (e^{-1/s}/s^2)ds + tC$. Substituting $u = -1/s$ yields $du = ds/s^2$, so $\int_1^t (e^{-1/s}/s^2)ds = \int_{-1}^{-1/t} e^u du = e^u\Big|_{-1}^{-1/t} = \exp(-1/t) - 1/e$. Also, $a(1) = 1\int_1^1 [h(s)/s^2]ds + 1(C) = 0 + C = 1$, so $C = 1$. Thus, the solution is $a(t) = t/e^{-1/t} - t/e + t$.

SECTION QUIZ

1. Element Z decays exponentially. 80% remains after one month. What is the half-life of element Z ?

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2. A population obeys exponential growth. In 25 years, the population increases from 500 to 750. How long would it take for the same population to increase from 3 million to 4 million?

3. Solve the differential equation $y' = y/2$, assuming $y(3) = 1$.

4. Solve and sketch the solution of $5y' = -y$ if $y(0) = 3$.

5. Suppose an object shrinks exponentially. Initially, it weighs 15 grams. Exactly one hour later, it weighs 14 grams. When will it weigh 4 grams?

6. A stranger is trying to decide what to eat at a Mexican restaurant. He asks the waiter, "What's this - Jalepeño peppers?" The waiter tells him, "Try it. You'll like it." After one bite, the stranger's tongue feels like it's at $60^\circ C$. If ice water requires 90 seconds to bring his tongue temperature back down to $38^\circ C$ (normal tongue temperature is $37^\circ C$), what is the decay constant? Assume the tongue obeys Newton's law of cooling.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $3\exp(3t)$

   (b) $2x \exp(x^2)$

   (c) $-2\exp(-2t + 4)$

ANSWERS TO SECTION QUIZ

1. 3.11 months

2. 17.74 years

3. $y = e^{(x-3)/2}$

4. $y = 3e^{-x/5}$

5. 19.16 hours

6. $-0.0348 \text{ sec}^{-1} = -2.090 \text{ min}^{-1}$

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8.3 The Hyperbolic Functions

PREREQUISITES

1. Recall how to differentiate exponential functions (Section 6.3).

2. Recall how to apply the chain rule for differentiating (Section 2.2).

PREREQUISITE QUIZ

1. Differentiate the following with respect to t :

   (a) $e^t + e^{-t}$

   (b) $(e^t + e^{-t})/(e^t - e^{-t})$

   (c) $\exp(t^2 + t)$

GOALS

1. Be able to define the hyperbolic trigonometric functions as a function of exponentials.

2. Be able to differentiate and integrate expressions involving hyperbolic functions.

3. Be able to solve differential equations of the form $x'' = \omega^2 x$.

STUDY HINTS

1. Definitions. You should memorize $\sinh t = (e^t - e^{-t})/2$ and $\cosh t = (e^t + e^{-t})/2$. They are the same except that $\sinh t$ has a minus sign. Remembering that $\sinh 0 = 0$ and $\cosh 0 = 1$ may help. As with sin and cos , sinh is odd and cosh is even. One usually pronounces sinh as "cinch", cosh as it is written, and tanh as "tanch."

2. Other hyperbolic functions. Notice the similarities of formulas (4) with their trigonometric counterparts.

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3. Derivatives of hyperbolic functions. Except for the sign of the derivative and the fact that they are hyperbolic functions, the formulas are the same as their trigonometric counterparts. Note that the "commonly" used functions sinh , cosh , and tanh have a positive sign in front of the derivative, whereas the others have a negative sign.

4. Half-angle formulas. Formulas (8) are useful for integration. They are analogous to the trigonometric half-angle formulas; note that the negative sign is associated with $\sin^2 x$ and $\sinh^2 x$.

5. Antiderivatives of hyperbolic functions. As usual, the simplest antiderivatives are determined by reversing the differentiation formulas.

6. The equation $d^2x/dt^2 = \omega^2 x$. Memorize the fact that the solution is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$, where $x = x_0$ and $dx/dt = v_0$ at $t = 0$. Alternatively, one can memorize $x = A \cosh \omega t + B \sinh \omega t$ and derive the solution by determining A and B .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrangement of formula (3) yields $\cosh^2 x = 1 + \sinh^2 x$. By definition, $\tanh^2 x + \operatorname{sech}^2 x = \sinh^2 x/\cosh^2 x + 1/\cosh^2 x = (\sinh^2 x + 1)/\cosh^2 x = \cosh^2 x/\cosh^2 x = 1$.

5. Prove the identity by the method of Example 3. $(d/dx)\cosh x = (d/dx)(e^x + e^{-x})/2 = (e^x - e^{-x})/2 = \sinh x$.

9. Using the fact that $(d/dx)\sinh x = \cosh x$ and the chain rule, we have $(d/dx)\sinh(x^3 + x^2 + 2) = (3x^2 + 2x)\cosh(x^3 + x^2 + 2)$.

13. Using the fact that $(d/dx)\sinh x = \cosh x$ and the chain rule, we have $(d/dx)\sinh(\cos(8x)) = \cosh(\cos(8x))\cdot(d/dx)\cos(8x) = -8 \sin 8x \cosh(\cos 8x)$.

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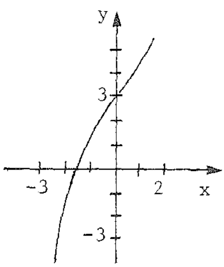
17. Since $(d/dx)\coth x = \text{csch}^2 x$, the chain rule gives $(d/dx)\coth 3x = -3\ \text{csch}^2 3x$.

21. Since $(d/dx)\cosh x = \sinh x$ and $(d/dx)\tanh x = \text{sech}^2 x$, the quotient rule gives $(d/dx)[\cosh x/(1 + \tanh x)] = [\sinh x(1 + \tanh x) - \cosh x(\text{sech}^2 x)]/(1 + \tanh x)^2 = [\sinh x(1 + \tanh x) - \text{sech}\ x]/(1 + \tanh x)^2$.

25. The solution of $d^2x/dt^2 - \omega^2 x = 0$ is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$ where $x = x_0$ and $dx/dt = v_0$ when $t = 0$. Here, $\omega = 3$, so the solution is $y = 0 \cosh 3t + (1/3)\sinh 3t = \sinh 3t/3$.

29. The solution of $d^2x/dt^2 - \omega^2 x = 0$ is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$ where $x = x_0$ and $dx/dt = v_0$ when $t = 0$. Here, $\omega = 3$, so the solution is $x = \cosh 3t + \sinh 3t/3$.

33.



Begin with the graph of $y = \sinh x$ as shown in Fig. 8.3.3. Shift it up 3 units to obtain the graph of $y = 3 + \sinh x$.

37. Substitute $u = 3x$ to get $\int \cosh 3x\, dx = \sinh 3x/3 + C$.

41. Use the identity $\sinh^2 x = (\cosh 2x - 1)/2$ to get $\int \sinh^2 x\, dx = \int [(\cosh 2x - 1)/2]\, dx = \sinh 2x/4 - x/2 + C$.

45. Let $u = \cosh x$, so $du = \sinh x\, dx$, and $\int \cosh^2 x \sinh x\, dx = \int u^2 du = u^3/3 + C = \cosh^3 x/3 + C$.

49. Using the technique of implicit differentiation, we have $3(y + x\, dy/dx)\text{sech}^2(3xy) + (\cosh y)dy/dx = 0$. Thus, $dy/dx = -3y\ \text{sech}^2 3xy/(\cosh y + 3x\ \text{sech}^2 3xy)$.

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53. By the definition of $\cosh x$ and $\sinh x$, $(\cosh x + \sinh x)^n = [(e^x + e^{-x}) + (e^x - e^{-x})]^n/2^n = (2e^x)^n/2^n = e^{nx}$. Also, $\cosh nx + \sinh nx = (e^{nx} + e^{-nx})/2 + (e^{nx} - e^{-nx})/2 = 2e^{nx}/2 = e^{nx}$. Therefore, $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx = e^{nx}$.

SECTION QUIZ

1. Differentiate the following functions of x:

   (a) $\sinh 3x$

   (b) $\cosh x \sinh 2x$

   (c) $\tan x \sinh 2x$

   (d) $\coth x/\operatorname{csch} 2x$

   (e) $\tanh(x/2) - \operatorname{sech} x$

2. Write $\cosh(x/2)$ in terms of exponentials.

3. Solve the following differential equations:

   (a) $f''(x) = 16f(x)$; $f'(0) = 5$; $f(0) = 2$.

   (b) $f''(x) = -25f(x)$; $f'(0) = 3$; $f(0) = 1$.

   (c) $d^2y/dx^2 = 9y$; $(dy/dx)|_0 = 6$; $y(0) = 0$.

4. Perform the following integrations:

   (a) $\int x \cosh 2x\, dx$

   (b) $\int e^x \cosh x\, dx$

   (c) $\int \sinh^5 x \cosh x\, dx$

5. One day, two teen-agers decided to equip their grandfather's electric wheelchair with rocket jets. When the elderly man went for his afternoon ride down the street, the faulty rockets did not work immediately. When Grandpa had ridden 100 m. down the street, the rocket jets began firing. At that time, he was at an equilibrium position and v_0 was 1 . As he accelerated down the street, the ride became bumpier and bumpier,

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5. (continued)

and his height off the seat can be described by $d^2x/dt^2 = x$, where x is the position of the chair.

(a) Solve the differential equation.

(b) Sketch the graph of the solution.

(c) How fast was Grandpa moving, i.e., find dx/dt.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $e^t - e^{-t}$

(b) $4/(e^t - e^{-t})^2$

(c) $(2t + 1)\exp(t^2 + t)$

ANSWERS TO SECTION QUIZ

1. (a) $3 \cosh 3x$

(b) $\sinh x \sinh 2x + 2 \cosh x \cosh 2x$

(c) $\sec^2 x \sinh 2x + 2 \tan x \cosh 2x$

(d) $(-\operatorname{csch}^2 x \operatorname{csch} 2x + 2 \coth x \operatorname{csch} 2x \coth 2x)/\operatorname{csch}^2 2x$

(e) $(1/2)\operatorname{sech}^2(x/2) - \operatorname{sech} x \tanh x$

2. $(e^{x/2} + e^{-x/2})/2$

3. (a) $f(x) = 2 \cosh 4x + (5/4)\sinh 4x$

(b) $f(x) = \cos 5x + (3/5)\sin 5x$

(c) $y = 2 \sinh 3x$

4. (a) $x \sinh 2x/2 - \cosh 2x/4 + C$

(b) $e^{2x}/4 + x/2 + C$

(c) $\sinh^6 x/6 + C$

5. (a) $x = \sinh t$

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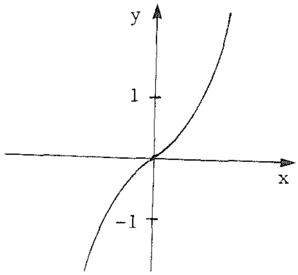
5. (b)



(c) dx/dt = cosh t

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8.4 The Inverse Hyperbolic Functions

PREREQUISITES

1. Recall the definition of an inverse function and how to differentiate them (Section 5.3).
2. Recall how to differentiate the hyperbolic trigonometric functions (Section 8.3).

PREREQUISITE QUIZ

1. (a) On what intervals is $y = -x^2 + 4$ invertible?

   (b) Find a decreasing function which is an inverse of $y = -x^2 + 4$.
2. Differentiate the following with respect to x :

   (a) $\cosh(x^2 + 1)$

   (b) $\sinh x$
3. Let $f(x) = x^5 + x^3 + x$. Find $(f^{-1})'(2)$.

GOALS

1. Be able to differentiate and integrate expressions involving the inverse trigonometric hyperbolic functions.

STUDY HINTS

1. Inverse hyperbolic derivatives. Study the method of deriving the derivative of $\sinh^{-1}x$ and note its similarity to that for $\sin^{-1}x$ (Chapter 5). All of the others are derived analogously. The only difference between $(d/dx)\sinh^{-1}x$ and $(d/dx)\cosh^{-1}x$ *is that the first has a plus sign in the* denominator and the second has a minus sign. A similar statement may be said for the denominator of the derivatives of $\text{sech}^{-1}x$ and $\text{csch}^{-1}x$. The derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ look the same, but $\tanh^{-1}x$ is defined for $|x| < 1$; $\coth^{-1}$ for $|x| > 1$. Think about the graph to determine if the sign is correct.

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2. Inverse hyperbolic logarithmic expressions. Again, study how to derive the formula $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$. The others are derived by a similar method. The formulas are normally not worth memorizing. Learn to derive them (for exams), or in many cases (for homework), one can simply look them up. Consult your instructor to see what is expected on exams.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since $(d/dx)\cosh^{-1}x = 1/\sqrt{x^2 - 1}$, the chain rule gives $(d/dx)\cosh^{-1}(x^2 + 2) = [1/\sqrt{(x^2 + 2)^2 - 1}]2x = 2x/\sqrt{x^4 + 4x^2 + 3}$.

5. Since $(d/dx)\tan^{-1}x = 1/(1 - x^2)$, the chain rule and product rule give $(d/dx)x \tan^{-1}(x^2 - 1) = \tan^{-1}(x^2 - 1) + 2x^2/[1 - (x^2 - 1)^2] =$ $\tan^{-1}(x^2 - 1) - 2x^2/(x^4 - 2x^2) = \tan^{-1}(x^2 - 1) + 2/(2 - x^2)$.

9. Since $(d/dx)\sinh^{-1}x = 1/\sqrt{x^2 + 1}$, the chain rule gives $(d/dx)\exp(1 + \sinh^{-1}x) = [\exp(1 + \sinh^{-1}x)]/\sqrt{x^2 + 1}$.

13. $\tanh^{-1}x = (1/2)\ln[(1 + x)/(1 - x)]$, so $\tanh^{-1}(0.5) = (1/2)\ln(1.5/0.5) =$ $\ln 3/2 \approx 0.55$.

17. Let $y = \cosh x = (e^x + e^{-x})/2$. Multiply through by $2e^x$ and rearrange to get $(e^x)^2 - 2ye^x + 1 = 0$. By the quadratic formula, $e^x = (2y \pm \sqrt{4y^2 - 4})/2$. Since $e^x > 0$, we take the positive square root to get $e^x = y + \sqrt{y^2 - 1}$. Thus, $x = \cosh^{-1}y = \ln(y + \sqrt{y^2 - 1})$. Change the variables to get the desired result.

21. Differentiate $\tanh^{-1}x = (1/2)\ln[(1 + x)/(1 - x)]$. By the chain rule, we get $(1/2)[(1 - x)/(1 + x)]\cdot[(1 - x) + (1 + x)]/(1 - x)^2 =$ $1/(1 + x)(1 - x) = 1/(1 - x^2)$.

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25. By the chain rule, we get $\left[1/(x + \sqrt{x^2 - 1})\right] \cdot [1 + (1/2)(x^2 - 1)^{-1/2} \times (2x)] = \left[1/(x + \sqrt{x^2 - 1})\right]\left[1 + x/\sqrt{x^2 - 1}\right] = \left[1/(x + \sqrt{x^2 - 1})\right]\left[(\sqrt{x^2 - 1} + x)/\sqrt{x^2 - 1}\right] = 1/\sqrt{x^2 - 1}$. Differentiation yields the integrand, so the formula is verified.

29. Substitute $u = 2x$, so $du/2 = dx$. Therefore, $\int [dx/(1 - 4x^2)] = (1/2)\int [du/(1 - u^2)] = (1/4)\ln|(1 + u)/(1 - u)| + C = (1/4)\ln|(1 + 2x)/(1 - 2x)| + C$.

33. Substitute $u = \sin x$, so $du = \cos x\, dx$; therefore, $\int (\cos x/\sqrt{\sin^2 x + 1})dx = \int (du/\sqrt{u^2 + 1}) = \sinh^{-1} u + C = \sinh^{-1}(\sin x) + C = \ln(\sin x + \sqrt{\sin^2 x + 1}) + C$.

37. By definition, $\cosh^{-1}(\sqrt{x^2 + 1}) = \ln(\sqrt{x^2 + 1} + \sqrt{x^2}) = \ln(\sqrt{x^2 + 1} + x)$ if $x \geqslant 0$ or $\ln(\sqrt{x^2 + 1} - x)$ if $x < 0$. Now, if $x > 0$, differentiation yields $f'(x) = 1/\sqrt{x^2 + 1}$. If $x < 0$, we get $f'(x) = -1/\sqrt{x^2 + 1}$. $f'(x)$ is not continuous at $x = 0$ because $\lim_{x\to 0-} f'(x) = -1$ and $\lim_{x\to 0+} f'(x) = 1$. Thus, $\cosh^{-1}(\sqrt{x^2 + 1})$ is not even once differentiable for all x .

SECTION QUIZ

1. Differentiate the following functions:

   (a) $\sinh^{-1}(x/2)$

   (b) $\cosh^{-1} x \operatorname{sech}^{-1}(2x)$, $x < -1$

   (c) $\coth^{-1} x \tan^{-1}(x^2)$, $|x| > 1$

2. Perform the following integrations:

   (a) $\int (x/\sqrt{x^4 + 1})dx$

   (b) $\int [(1 + x)/(1 - x^2)]\, dx$

   (c) $\int [dx/(x^2 - 1)]$

   (d) $\int \sinh^{-1} x\, dx$

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3. A highway patrolwoman had just stopped a driver with alcoholic breath. The driver explained that he was an astrophysicist and had spotted a flying pink elephant travelling along the path $y = \cosh^{-1}x$.

   (a) Use logarithms to determine the elephant's position at $x = 2$.

   (b) Sketch the elephant's flight path.

   (c) If the y-axis points north and the x-axis points east, estimate the elephant's flight direction at $x = 2$. (Choose from N , S , E , W , NW , SW , NE , and SE).

ANSWERS TO PREREQUISITE QUIZ

1. (a) $(-\infty,0)$ and $(0,\infty)$

   (b) $f^{-1}(x) = -\sqrt{-x + 4}$

2. (a) $2x \sinh(x^2 + 1)$

   (b) $\cosh x$

3. 1/93

ANSWERS TO SECTION QUIZ

1. (a) $1/\sqrt{x^2 + 4}$

   (b) $\operatorname{sech}^{-1}2x/\sqrt{x^2 - 1} + \cosh^{-1}x/x\sqrt{1 + x^2}$

   (c) $\tan^{-1}(x^2)/(1 - x^2) + 2x \coth^{-1}x/(1 + x^2)$

2. (a) $\sinh^{-1}(x^2)/2 + C = \ln(x^2 + \sqrt{x^4 + 1})/2 + C$

   (b) $-\ln|1 - x| + C$ [Factor the denominator and simplify the integrand first].

   (c) $-\tanh^{-1}x + C$ if $|x| < 1$; $-\coth^{-1}x + C$ if $|x| > 1$; or $(1/2)\ln|(1 - x)/(1 + x)| + C$ if $x \neq 1$.

   (d) $x \sinh^{-1}x - \sqrt{x^2 + 1} + C = x \ln(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + C$

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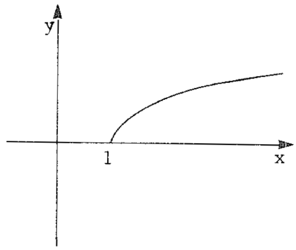
3. (a) $\ln(2 + \sqrt{3}) \approx 1.32$

(b)



(c) NE (The actual angle is $\pi/6$.)

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8.5 Separable Differential Equations

PREREQUISITES

1. Recall the geometric interpretation of the linear approximation (Section 1.6).
2. Recall the rules of integration (Chapter 7).

PREREQUISITE QUIZ

1. Evaluate the following:
   (a) $\int [dx/(1 - x)]$
   (b) $\int (e^y + y^4 - \sin y)dy$
2. (a) Write a formula which approximates $f(x_0 + \Delta x)$.
   (b) How is the linear approximation related to the tangent line of a graph?
   (c) Use the linear approximation to estimate $(0.97)^2$.

GOALS

1. Be able to recognize separable differential equations and solve them.
2. Be able to use direction fields or the Euler method for estimating solutions of differential equations.

STUDY HINTS

1. Separable equations. These must be first-order differential equations. They are separable in the sense that everything involving y can be placed on one side of the equals sign, and everything involving x can be placed on the other side.

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2. Method of solution. Begin by thinking of dx and dy as separate entities. Multiply and divide to put all of the terms involving y and all of the terms involving x on separate sides of the equation. Then integrate both sides. See Examples 1, 2, and 3.

3. Electric circuits. The result of Example 4 should not be memorized; however, the result is useful for doing the exercises.

4. Transforming an equation. Example 6 demonstrates an interesting method of solution. By letting a new variable represent a derivative, the original equation became a separable first-order equation.

5. Direction fields. These fields are represented by tiny lines which are the tangent lines to the solution curves. If $y(x)$ is a solution of $dy/dx = f(x,y)$, then the slope of the graph at the point (x,y) is $dy/dx = f(x,y)$. Line segments through (x,y) with slope $f(x,y)$ are used to depict the direction field. By drawing a curve which follows the direction of these lines, one can sketch a solution without knowing an explicit formula. See Figures 8.5.7 and 8.5.8.

6. Euler method. This method is based upon the concept of the linear approximation (Section 1.6). By starting at a point and moving along the tangent line for a short distance, one should remain near the curve. You may find it more efficient to understand the concept and derive the formula, rather than memorizing it.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrange $dy/dx = \cos x$ to get $dy = \cos x\, dx$. Integrating both sides yields $\int dy = \int \cos x\, dx$, so $y = \sin x + C$. The initial condition $y(0) = 1$ implies $C = 1$, so $y = \sin x + 1$.

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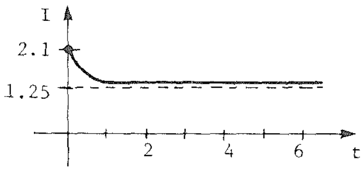
5. Rearrange the equation to get $dy/y = dx/x$. Integration yields $\ln|y| = \ln|x| + C$. Exponentiation gives $y = e^C x$. Substituting $y(1) = 2$ gives $e^C = -2$, so $y = -2x$ is the solution.

9. Multiply through by $dx/(1 + y)$ to get $dy/(1 + y) = dx/(1 + x)$. Integrating both sides yields $\ln(1 + y) = \ln(1 + x) + C$. Substituting $y = 1$ and $x = 0$ gives $\ln 2 = \ln 1 + C = C$. Thus, $\ln(1 + y) = \ln(1 + x) + \ln 2$, or upon exponentiating, $1 + y = 2(1 + x)$, i.e., $y = 2x + 1$.

13. 

This is Example 4 with $L = 3$, $R = 8$, $E = 10$, and $I_0 = 2.1$. Thus, the solution is $I = 10/8 + (2.1 - 10/8)\exp(-8t/3) = 1.25 + 0.85\exp(-8t/3)$. At $t = 0$, $I = 2.1$; then the graph drops exponentially toward $E/R = 1.25$.

17. $dy/dt = by - rxy = b(s/c) - r(b/r)(s/c) = bs/c - bs/c = 0$ and $dx/dt = -sx + cxy = -s(b/r) + c(b/r)(s/c) = -sb/r + sb/r = 0$. Since both x and y are constant, this is the equilibrium point which also solves the predator-prey equations.

21. Substituting into $y = (T_0/mg)[\cosh(mgx/T_0) - 1] + h$ yields $y = (T_0/9.8)[\cosh (9.8/T_0)x - 1]$. Now, $y'(x) = (T_0/9.8)(9.8/T_0) \times [\sinh (9.8/T_0)x] = \sinh(9.8/T_0)x$. As T_0 gets large, $9.8/T_0$ approaches 0 and $\sinh x = 0$. Thus, the solution is not only straight, but also constant as T_0 gets large.

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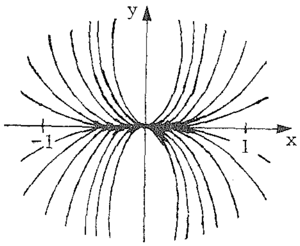
25. (a)



(b) Differentiation gives the differential equation, $dy/dx = 3cx^2$.

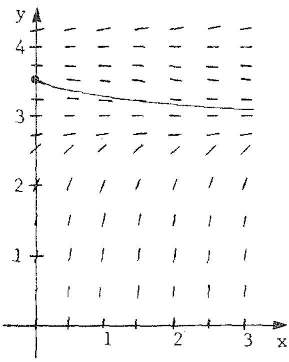
(c) The slopes must be negative reciprocals for the curves to be orthogonal, so $dy/dx = -1/3cx^2$. Separating variables gives $dy = -dx/3cx^2$ and integration yields $y = 1/3cx + C$.

29. Starting with $(x_0, y_0) = (0,1)$, we want to find $y_{10} = y(1)$. We use the recursive formula $y_n = (y_{n-1} - x_{n-1}^2)(0.1) + y_{n-1}$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1	1.1	1.209	1.3259	1.44949	1.578439	1.711283	1.846411	1.982052	2.116258	2.246883

The ten-step Euler procedure gives us $y(1) \approx 2.2469$.

33.



This exercise is analogous to Example 10. We have $dy/dx = (y - 4)(y - 3)$. As shown by the direction field, $\lim_{x\to\infty} y(x)$ for $y(0) = 3.5$ is 3 .

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37. Differentiate to get $f''(x) = 2y(dy/dx)e^x + e^x y^2 + 4y^5 + 4x\cdot 5y^4(dy/dx)$. Differentiate again, so $f'''(x) = e^x[2y(dy/dx) + y^2 + 2(dy/dx)^2 + 2y(d^2y/dx^2) + 2y(dy/dx)] + 20y^4(dy/dx) + 20y^4(dy/dx) + 20x[4y^3(dy/dx) + y^4(d^2y/dx^2)]$. Substitute $y(0) = 1$ into the given equation giving $dy/dx = 1$. Substitute these into the equation for $f''(x)$ to get $f''(0) = 2 + 1 + 4 = 7$. Substitute these into the equation for $f'''(x)$ to get $f'''(0) = 2 + 1 + 2 + 14 + 2 + 20 + 20 = 61$.

SECTION QUIZ

1. (a) Solve the differential equation $dy/dx = (x + 4)(y - 2)/(x + 7)$, given $y(0) = 1$.
   (b) Find the interval in x for which the solution is valid.
2. (a) Solve $dy/dx = y/2$, assuming $y(0) = 1$.
   (b) Plot the solution to part (a) for $x = x_i/10$, where $x_i = 0$, ... , 10 . Then, draw a smooth curve through the plotted points.
   (c) Use Euler's method to approximate $y(1)$ for $dy/dx = y/2$, $y(0) = 1$. Use ten steps.
   (d) Plot the eleven points obtained from part (c) onto the same graph as in (b) and connect the points with straight line segments.
3. Solve the differential equation $y'' = 1 + (y')^2$, assuming $(dy/dx)|_0 = 0$ and $y_0 = 2$.
4. Which of the following equations are separable?
   (a) $dy/dx = x^3y - y \ln x + y$
   (b) $dy/dx = x^2 - xy^2$
   (c) $(y/(x - 1))dy/dx = x(y^2 - y)$
   (d) $(x + y)dy/dx = xy$

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5. At the annual witches' convention, new advances in brewology (the science of brewing magic potions) were discussed. One of the newest potions prevents broom thefts. Depending on the location (x,y), broom thieves will be swept off their feet and taken for a ride. The ride is designed to take the path given by $dy/dx = (x^2 + x + 1)/y$. Determine the path $y(x)$ if $y(1) = 1$.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-\ln|1 - x| + C$

   (b) $e^y + y^5/5 + \cos y + C$

2. (a) $f(x_0) + f'(x_0)\Delta x$

   (b) The linear approximation and the equation of a tangent line have the same formula.

   (c) 0.94

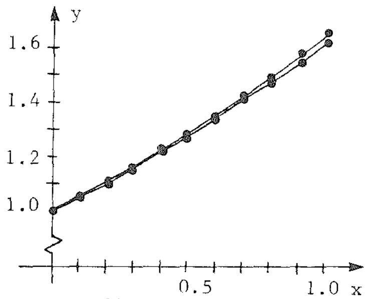
ANSWERS TO SECTION QUIZ

1. (a) $y = -343e^x/(x + 7)^3 + 2$

   (b) $x \geqslant -7$

2. (a) $y = e^{x/2}$

   (b),(d)



   (c) $y(1) \approx 1.6289$

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3. $y = -\ln|\cos x| + 2$

4. a and c

5. $y^2/2 = x^3/3 + x^2/2 + x - 4/3$

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8.6 Linear First-Order Equations

PREREQUISITES

1. Recall basic rules of differentiation, especially the product and chain rules (Chapters 1 and 2).
2. Recall basic methods of integration, especially exponential functions and substitution (Chapter 7).

PREREQUISITE QUIZ

1. Differentiate the following:
   (a) $\exp(-\sin x)$
   (b) $t \exp(t^2)$
2. Evaluate the following integrals:
   (a) $\int t \exp(t^2) dt$
   (b) $\int (x + 2)\exp(2x^2 + 8x) dx$
   (c) $\int te^t dt$

GOALS

1. Be able to solve linear first-order differential equations.

STUDY HINTS

1. Linearity defined. Differential equations are linear in the sense that, in the usual usage, dy/dx and y appear only to the first power.
2. Method of solution. Equation (6) gives the solution of $dy/dx = P(x)y + Q(x)$. You should not memorize the solution. Instead, learn the method of solution, which is summarized in the box in the middle of p. 409. As always, practice is the best way to learn.

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3. Choice of method. Always look for the simplest solution. For instance, Example 1 is separable. You get another (simpler) solution by noting that the right-hand side is $x(y + 1)$. (However, note that the equations in Example 2 are not separable.)

4. Integrating factor defined. These are multiplicative factors which transform an expression into one which can be integrated. $\exp(-\int P(x)dx)$ is the integrating factor which is discussed in this section.

5. Applications. You should understand how the results in Examples 3-6 were derived. There is no need to memorize the results; however, the results may be useful for solving the exercises.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Here, $P(x) = 1/(1 - x)$ and $Q(x) = 2/(1 - x) + 3$. $-\int P(x)dx = -\ln|1 - x|$, dropping the integration constant. Thus we have $[1/(1 - x)][dy/dx - y/(1 - x)] = [1/(1 - x)][2/(1 - x) + 3] = (d/dx)[y/(1 - x)] = 2(1 - x)^{-2} + 3/(1 - x)$. Integration gives $y/(1 - x) = 2/(1 - x) - 3\ln|1 - x| + C$ or $y = 2 + (-3\ln|1 - x| + C)(1 - x)$.

5. $P(x) = \cos x$ and $Q(x) = 2\cos x$. $-\int P(x)dx = -\sin x$, so $\exp(-\sin x)(dy/dx - y\cos x) = 2\exp(-\sin x)\cos x$. Letting $u = -\sin x$, we get $y\exp(-\sin x) = -2\exp(-\sin x) + C$ or $y = -2 + C\exp(\sin x)$. When $y = 0$, $x = 0$, so $0 = -2 + C$ or $C = 2$. Thus, the solution is $y = -2 + 2\exp(\sin x)$.

9. The equation becomes $dI/dt = -RI/L + (E_0\sin\omega t + E_1)/L$. $P(t)$ is still the same, so $\exp(tR/L)(dI/dt + RI/L) = \exp(tR/L)(E_0\sin\omega t + E_1)/L = (d/dt)\exp(tR/L)\cdot I$. Using formula 82 of the integration table on the inside back cover of the text, we integrate to get $I\exp(tR/L) = (E_0/L)[\exp(tR/L)/((R/L)^2 + \omega^2)](R\sin\omega t/L - \omega\cos\omega t) + (E_1/L)(L/R)\exp(tR/L) +$

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9. (continued)

C . Multiplying by $\exp(-tR/L)$ gives $I = (E_0/L)(R \sin \omega t/L - \omega\cos \omega t)/((R/L)^2 + \omega^2) + E_1/R + C \exp(-tR/L)$. (See Example 3 for more details.)

13. From the solution to Example 4, $y = (1 - \exp(-2.67 \times 10^{-7}t))(2.51 \times 10^6)$. We want to find t so that $y(t) = 0.9(2.51 \times 10^6)$, i.e., $0.9 = 1 - \exp(-2.67 \times 10^{-7}t)$ or $0.1 = \exp(-2.67 \times 10^{-7}t)$. Therefore, $-2.67 \times 10^{-7}t = \ln(0.1)$ and $t \approx 8.63 \times 10^6$ seconds $\approx$ 100 days.

17. Use the result of Example 5: $v = (mg/\gamma)[1 - e^{-\gamma t/m}]$. The distance travelled is $\int v\,dt = mgt/\gamma + ge^{-\gamma t/m} + C$. Since no distance has been travelled at $t = 0$, we have $c = -g$. As $t \to \infty$, v approaches $mg/\gamma = 64$, so $m/\gamma = 64/g$. Now, we want to know when $v = 0.9(64) = 57.6 = 64(1 - e^{-gt/64})$, or $\ln(1 - 57.6/64) = -gt/64$, i.e., $t = (-64/g)\ln(1 - 57.6/64) \approx 15.0$ seconds. At this time, the person has fallen 951 meters.

21. The acceleration is $dv/dt = [F(M_0 - rt) - (-r)Ft]/(M_0 - rt)^2 - [(gM_0 - grt)(M_0 - rt) - (-r)(gM_0t - grt^2/2)]/(M_0 - rt)^2 = FM_0/(M_0 - rt)^2 - [g(M_0 - rt)^2 + grt(M_0 - rt/2)]/(M_0 - rt)^2$. Substituting $M_1 = M_0 - rt$, we get $FM_0/M_1^2 - [gM_1^2 + grt(2M_0 - rt)/2]/M_1^2 = FM_0/M_1^2 - [2gM_1^2 + g(M_0 - M_1)(M_0 + M_0 - rt)]/2M_1^2 = FM_0/M_1^2 - [2gM_1^2 + g(M_0^2 - M_1^2)]/2M_1^2 = FM_0/M_1^2 - g(M_0^2 - M_1^2)/2M_1^2$.

25. Any solution y of $y' = P(x)y + Q(x)$ must be of the form $y = \exp(\int P(x)dx)\{\int [Q(x)\exp(-\int P(x)dx)dx] + C\}$. Since $y(0) = y_0$, we can show that C is unique: $y_0 = \exp(\int P(x)dx)|_{x=0}\{\int [Q(x) \times \exp(-\int P(x)dx)dx]|_{x=0} + C\}$. Therefore, $C = y_0\exp(-\int P(x)dx)|_{x=0} - \int [Q(x)\exp(-\int P(x)dx)dx]|_{x=0}$. Now, the right side of this equation is

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25. (continued)

a constant, determined by operations on y_0, $P(x)$, and $Q(x)$, and evaluated for $x = 0$. Hence C is a constant, not a function, and there is exactly one y. [Alternatively, if y_1 and y_2 are two solutions, look at the equation for $w = y_1 - y_2$.]

29. (a) The equation is $(d/dt)(Mv) = F - Mg - \gamma v$, or $(d/dt)[(M_0 - rt)v] = F - (M_0 - rt)g - \gamma v = (M_0 - rt)dv/dt - rv$. Rearrangement yields $dv/dt = rv/(M_0 - rt) - \gamma v/(M_0 - rt) + F/(M_0 - rt) - g$. $P(x) = (r - \gamma)/(M_0 - rt)$, so $-\int P(x)dx = \ln(M_0 - rt) - (\gamma/r)\ln(M_0 - rt)$ and $\exp(-\int P(x)dx) = (M_0 - rt)^{1-\gamma/r}$. Therefore, $(M_0 - rt)^{1-\gamma/r}[dv/dt - (r - \gamma)v/(M_0 - rt)] = F(M_0 - rt)^{-\gamma/r} - g(M_0 - rt)^{1-\gamma/r} = (d/dt)[v(M_0 - rt)^{1-\gamma/r}]$. Integrate to get $v(M_0 - rt)^{1-\gamma/r} = -F(M_0 - rt)^{1-\gamma/r}/(r - \gamma) + g(M_0 - rt)^{2-\gamma/r}/(2r - \gamma) + C$ or $v = F/(\gamma - r) - g(M_0 - rt)/(\gamma - 2r) + C/(M_0 - rt)^{1-\gamma/r}$. $v = 0$ when $t = 0$, so $0 = F/(\gamma - r) - gM_0/(\gamma - 2r) + C/M_0^{1-\gamma/r}$ or $C = [gM_0/(\gamma - 2r) - F/(\gamma - r)]M_0^{1-\gamma/r}$. Thus, the solution is $v = F/(\gamma - r) - g(M_0 - rt)/(\gamma - 2r) + \{[gM_0/(\gamma - 2r) - F/(\gamma - r)]M_0^{1-\gamma/r}\}(M_0 - rt)^{\gamma/r-1}$.

(b) At burnout, $M_0 - rt = M_1$, so $v = F/(\gamma - r) - gM_1/(\gamma - 2r) + [gM_0/(\gamma - 2r) - F/(\gamma - r)](M_0/M_1)^{1-\gamma/r}$.

SECTION QUIZ

1. Solve $x(dy/dx) + 2y = x^2 - x + 1$ if $y(1) = 1$.

2. Solve $y'(x) - y = e^{3x}$ if $y(0) = 1$.

3. Find $x(y)$ if $dy/dx = 1/(x + y)$. [Hint: What is the relationship between dy/dx and dx/dy?].

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4. A stuntwoman is going down Niagara Falls inside a barrel. Due to resistance from the water, her velocity can be described by $m(dv/dt) = mg - 0.7v$, where m is her mass (50 kg) and g is $9.8 m/s^2$, the acceleration due to gravity.

   (a) If $v = 0$ when $t = 0$, find the velocity function.

   (b) When does her speed become 25 m/s (freeway driving speed)?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-(\cos x)\exp(-\sin x)$

   (b) $(1 + 2t^2)\exp(t^2)$

2. (a) $\exp(t^2)/2 + C$

   (b) $\exp(2x^2 + 8x)/4 + C$

   (c) $te^t - e^t + C$

ANSWERS TO SECTION QUIZ

1. $y = x^2/4 - x/3 + 1/2 + 7/12x^2$

2. $y = (e^{3x} + e^x)/2$

3. $x = -y - 1 + Ce^y$

4. (a) $v = (mg/0.7)[1 - \exp(-0.7t/m)]$

   (b) 2.60 seconds

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8.R <u>Review Exercises for Chapter 8</u>

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrangement gives us $dy/y = 3\ dt$. Then, integrating gives $\ln y = 3t + C$, so for $k = e^C$, $y = ke^{3t}$. Substitute $y(0) = 1$, giving $1 = ke^0 = k$. So the final solution is $y = e^{3t}$.

5. With $P(t) = 3$ and $Q(t) = 1$, note that $\int P(t)dt = 3t$. Subtract $3t$ and multiply by e^{-3t} , giving $e^{-3t}(dy/dt - 3y) = e^{-3t} = (d/dt)(ye^{-3t})$. Integrate, so $ye^{-3t} = (-1/3)e^{-3t} + C$, and $y = (-1/3) + Ce^{3t}$. Substitute $y(0) = 1$ to get $1 = (-1/3) + C$, so $C = 4/3$ and the solution is $y = (4e^{3t} - 1)/3$.

9. This is a case of natural growth with $\gamma = 4$, so $f(x) = Ce^{4x}$. Substitute $f(0) = 1$ to get $C = 1$. Therefore, $f(x) = e^{4x}$.

13. This is a case of simple harmonic motion with $\omega = 1$, $x_0 = 1$, and v_0 is unknown. The solution is $x(t) = \cos t + v_0 \sin t$. Substituting $x(\pi/4) = 0$ gives $0 = \sqrt{2}/2 + v_0\sqrt{2}/2$, so $v_0 = -1$. Therefore, the solution is $x(t) = \cos t - \sin t$.

17. $dy/dx = e^{x+y} = e^x e^y$, so rearrangement gives $dy/e^y = e^x dx$. Integration yields $-e^{-y} = e^x + C$. Substituting $y(0) = 1$ gives $-1/e = 1 + C$, so $C = 1 - 1/e$. Thus, $e^{-y} = 1/e - 1 - e^x$ or $y = -\ln|1/e - 1 - e^x|$.

21. Separate variables to get $dy/(y + 1) = dt/(1 - t)$ and integrate to get $\ln(y + 1) = \ln(1 - t) + C$. $y(0) = 0$ implies $C = 0$, so $y + 1 = 1 - t$ or $y = -t$.

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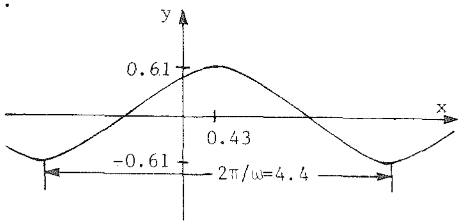
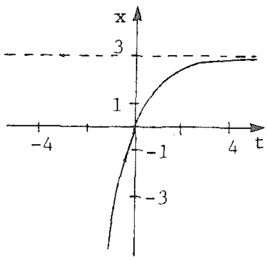
25\.



This is simple harmonic motion with $\omega = \sqrt{2}$, $y_0 = 1/2$, and $v_0 = 1/2$, so the solution is $y(x) = (1/2)\cos(\sqrt{2}x) + (1/2\sqrt{2}) \times \sin(\sqrt{2}x)$. Change $(1/2, 1/2\sqrt{2})$ to polar coordinates: $(\sqrt{3/8}, \tan^{-1}(1/\sqrt{2}))$ so $y(x) = \sqrt{3/8}\cos(\sqrt{2}x - \tan^{-1}(1/\sqrt{2}))$. To plot the graph, $y(x) \approx 0.612\cos(1.41x - 0.196\pi)$.

29\.



This is a first-order equation with $P(t) = -1$ and $Q(t) = 3$. The integrating factor is e^t . Thus, $e^t(dx/dt + x) = 3e^t = (d/dt)xe^t$. Integration gives $xe^t = 3e^t + C$. $x(0) = 0$ implies $0 = 3 + C$ or $C = -3$, so $x = 3 - 3e^{-t}$. Since $\lim_{t\to\infty} e^{-t} = 0$, we have $\lim_{t\to\infty} x(t) = 3$.

33\. Using the hint, $dw/dx + w = x$. With $P(x) = -1$ and $Q(x) = x$, note that $\int P(x)dx = -x$. Multiply by e^x , giving $e^x(dw/dx + w) = xe^x = (d/dx)(we^x)$. Integrate to get $we^x = xe^x - e^x + C$. Divide by e^x , yielding $w = x - 1 + Ce^{-x}$. Substitute $y'(0) = 1 = w(0)$ to get $1 = -1 + C$, so $C = 2$. Therefore $w = x - 1 + 2e^{-x} = dy/dx$. Integrate again to get $y = x^2/2 - x - 2e^{-x} + D$. Substitute $y(0) = 0$ to get $0 = -2 + D$, meaning $D = 2$. Then the solution is $y = x^2/2 - x - 2e^{-x} + 2$.

37\. Since $(d/dx)\sinh x = \cosh x$, the chain rule gives $(d/dx)\sinh(3x^2) = 6x\cosh(3x^2)$.

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41. Using the product rule with $(d/dx)\sinh^{-1}x = 1/\sqrt{x^2 + 1}$ and $(d/dx)\cosh x = \sinh x$, we get $(d/dx)[(\sinh^{-1}x)(\cosh 3x)] = \cosh 3x/\sqrt{x^2 + 1} + 3 \sinh 3x \sinh^{-1}x$.

45. Substitute $u = \sinh x$, so $du = \cosh x\, dx$; therefore, the integral becomes $\int [du/(1 + u^2)] = \tan^{-1}u + C = \tan^{-1}(\sinh x) + C$.

49. Integrate by parts with $u = x$ and $dv = \sinh x\, dx$, so $du = dx$ and $v = \cosh x$. Therefore, $\int x \sinh x\, dx = x \cosh x - \int \cosh x\, dx = x \cosh x - \sinh x + C$.

53. (a) Here, $m = 10$ and $\sqrt{k/m} = 8$, so $k/10 = 64$, i.e., the spring constant is $k = 640$.

(b) The force is $m(d^2x/dt^2)$. $dx/dt = 80 \cos(8t)$ and $d^2x/dt^2 = -640 \sin(8t)$, so the force is $-6400 \sin(8t)$. At $t = \pi/16$, the force is $-6400 \sin(\pi/2) = -6400$ newtons.

57. This is natural growth and it obeys $f(t) = f(0)e^{kt}$. In this case, $f(0) = 100{,}000$ and $f(10) = 200{,}000$, so $2 = e^{k(10)}$ or $k = \ln 2/10$. We want to determine t for $f(t) = 10$ million $= 100{,}000e^{(\ln 2/10)t}$, i.e., $\ln 100 = (\ln 2/10)t$ or $t = 10 \ln 100/\ln 2 \approx 66.4$ years.

61. Let $x(t)$ be the temperature above $18\,^\circ C$ in $^\circ C$. Then $dx/dt = kx$ for some constant k . Separate variables to get $dx/x = k\, dt$. Integration yields $\ln x = kt + C$. Exponentiate to get $x = e^{kt}e^{C}$. Let $D = e^C$, so $x = De^{kt}$. Now $x(0) = 82$ and $x(8) = 62$, so substitute each of these, giving $82 = D$ and $62 = 82\, e^{8k}$. Therefore $31/41 = e^{8k}$, so $\ln(31/41) = 8k$, and thus $(1/8) \ln(31/41) = k$. Therefore, $x = 82 \exp[(1/8) \ln(31/41)t]$. At $50\,^\circ C$, $x = 32$, so $32/82 = \exp[(1/8) \ln(31/41)t]$. Take logs to get $\ln(16/41) = (1/8) \ln(31/41)t$, or $t = 8 \ln(16/41)/\ln(31/41) \approx 27$ minutes.

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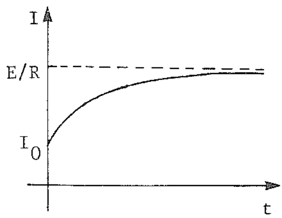
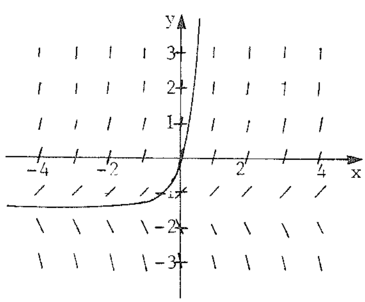
65. 

By Example 4 of Section 8.5, the solution is $I = E/R + (I_0 - E/R)e^{-Rt/L}$. If $I_0 < E/R$, then the graph begins below E/R and increases, rather than decreases, toward E/R .

69. Let $x(t)$ be the number of gallons of antifreeze in the radiator at t minutes. Note that $x(0) = 4/3$. Now, $dx/dt = -(\text{flow out}) = -(1/2)x/4 = -x/8$. This is exponential decay, so $x(t) = (4/3)e^{-t/8}$. When the mixture is 95% fresh water, $x(t) = (0.05)(4) = 1/5$, so solve $1/5 = (4/3)e^{-t/8}$. Multiply by $3/4$ and take logs, so $\ln(3/20) = -t/8$, or $t = 8\ln(20/3) \approx 15.2$ minutes. If you drain the radiator and then add fresh water, you will spend $[4/(1/2)]2 = 16$ minutes. Therefore, draining the radiator first is no faster.

73. 

$P(x) = 3$, so the integrating factor is e^{-3x} ; therefore, $e^{-3x}(y' - 3y) = 4e^{-3x} = (d/dx)ye^{-3x}$. Integration gives $ye^{-3x} = -4e^{-3x}/3 + C$ or $y = -4/3 + Ce^{3x}$.

77. Starting with $(x_0,y_0) = (0,1)$, we want $y_{10} = y(1)$ and $y_{20} = y(1)$. For y_{10} , we use the formula $y_n = y_{n-1}(0.1) + y_{n-1} = (1.1)y_{n-1}$. $y_0 = 1$; $y_1 = 1(1.1)$; $y_2 = 1(1.1)(1.1)$; $y_3 = 1(1.1)(1.1)(1.1)$; ... ; $y_{10} = 1(1.1)^{10} = 2.5937$. For the twenty-step method, $y_n = y_{n-1}(0.05) + y_{n-1} = (1.05)y_{n-1}$, which implies $y_{20} = 1(1.05)^{20} = 2.6533$. For the exact solution, $dy/dx = y$ implies $\int(dy/y) = \int dx$, so $\ln|y| = x + C$. $y(0) = 1$ implies $C = 0$; therefore $y = e^x$, and $y(1) = e \approx 2.718282$

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77. (continued)

The ten-step method gives us an error of 4.58% , while the twenty-step method has a 2.39% error.

81. (a) Differentiate w to get $w' = y'$. Multiply w by a to get $aw = ay + b = y'$. Therefore $w' = aw$, a case of natural growth, so $w(t) = Ce^{at} = y(t) + b/a$. Thus, $y(t) = Ce^{at} - b/a$.

(b) From part (a), $dw/dx = aw$. Separate variables to get $dw/w = a\,dt$, and integrate to get $\ln w = at + k$. Exponentiation gives $w = e^k e^{at} = Ce^{at}$ for $C = e^k$. Substitute $w = y + b/a$ and subtract b/a to get $y = Ce^{at} - b/a$.

(c) With $P(t) = a$ and $Q(t) = b$, note that $\int P(t)\,dt = at$. Subtract ay and multiply by e^{-at} to get $e^{-at}(y' - ay) = e^{-at} = (d/dt)(ye^{-at})$. Integrate, giving $ye^{-at} = (-b/a)e^{-at} + C$. Multiply by e^{at} to get $y = -b/a + Ce^{-at}$. All three answers are the same.

85. (a) In other words, find $y = f(x)$ such that $f(x) = \sqrt{1 + [f'(x)]^2}$, i.e., $y^2 = 1 + (y')^2$. Now, if y is constant, y' becomes zero and we are left with $y^2 = 1$ or $y = \pm 1$. The solution $y = -1$ is not valid since the integrand on the right is positive. Thus, $y = 1$ is one solution.

If y is not constant, then rearrangement yields $(y')^2 = y^2 - 1$ or the differential equation $y' = \sqrt{y^2 - 1} = dy/dx$. Separating variables yields $dy/\sqrt{y^2 - 1} = dx$. Integration gives us $\cosh^{-1} y = x + C$ or $y = \cosh(x + C)$.

(b) We recognize $\int_a^b f(x)dx$ as the area under $f(x)$ on $[a,b]$. In Chapter 10, we will derive $\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ for the length of $f(x)$ on $[a,b]$. Thus the formula equates area and arc length.

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TEST FOR CHAPTER 8

1. True or false:

   (a) If y is a function of x, then $y' + y = 0$ can be solved by integrating to get $y + y^2/2 = C$.

   (b) The domain of $\cosh^{-1}(x^2 + 1)$ is all real x.

   (c) The most general solution to the differential equation $y' = -ay$, where a is constant and $y(0) = 2$, is $y = 2e^{-ax}$.

   (d) As long as $y(x_0)$ is specified for some constant x_0, $y'' - y = 0$ has a unique solution for $y(x)$.

   (e) For all x, $\cosh^2 x = 1 + \sinh^2 x$.

2. Solve the following differential equations with the given conditions:

   (a) $d^2x/dt^2 + 9x = 0$, $x(0) = 1$, $x'(0) = 1$

   (b) $d^2x/dt^2 - 9x = 0$, $x(0) = 1$, $x'(0) = 1$

   (c) $d^2x/dt^2 + 9t = 0$, $x(0) = 2$, $x'(1) = 2$

3. (a) Find a solution of the form $y = A\cos(\omega t - \theta)$ for $4y'' = -y$, assuming $y(0) = 1$ and $y'(0) = 4$.

   (b) Sketch the graph of y.

4. Find the solution of $dx/dt + x = \sin t + 2e^{-t}$ if $x(0) = 1$.

5. Solve the differential equation $dy/dx + \sin^2 y = 1$, assuming $y(0) = 0$.

6. Solve the following differential equations with the given initial conditions:

   (a) $dy/dx = -2x$; $y(0) = 1$

   (b) $f'(x) + x^2 f(x) = 0$; $f(0) = A$, a constant

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7. Evaluate the following:

(a) $(d/dt)\sqrt{\cosh 5t}$

(b) $\int \tanh(x/3)\,dx$

(c) $\int \left[(2x + 5)/\sqrt{1 + x^2}\right]dx$

(d) $(d/dy)\tanh^{-1}(e^{-y+2})$

8. (a) Find an approximate solution for $y(2)$ if $y' = x^2 + 2y$ and $y(0) = 0$ by using a 10-step Euler method.

(b) Compare the answer in (a) with the exact solution.

9. An electric circuit is governed by the equation $C(dV/dt) + V/R = I_0\cos(\omega t)$ where C, R, I_0, and ω are constants. Find $V(t)$ satisfying $V(0) = 0$.

10. Scientific investigators have recently concluded that stupid question asking obeys the law of exponential decay. At the age of five, the average person's stupid questioning peaks, and then decays exponentially.

(a) Suppose a young boy asked an average of 1 stupid question daily when he was five. He is now thirteen and asks an average of 0.65 stupid questions daily. Write a formula for $q(t)$, the average number of stupid questions asked daily, in terms of t, the person's age.

(b) How many years does it take for this person's stupid question asking to decrease by 50% ?

ANSWERS TO CHAPTER TEST

1. (a) False; integration needs to be done with respect to x, not y.

(b) True

(c) True; it is the only solution.

(d) False; $y'(x_1)$ must also be specified for some x_1.

(e) True

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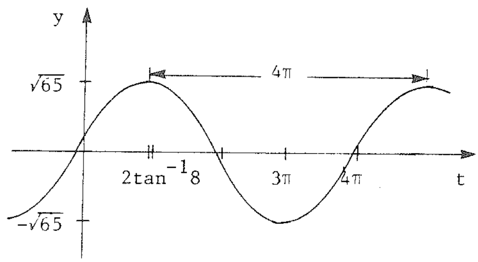
2\. (a) $x = \cos 3t + (1/3) \sin 3t$

(b) $x = \cosh 3t + (1/3) \sinh 3t$

(c) $x = -3t^3/2 + 13t/2 + 2$

3\. (a) $y = \sqrt{65}\cos(t/2 - \tan^{-1} 8)$

(b)



4\. $x = e^{-t}/2 + 2te^{-t} + (\sin t - \cos t)/2$

5\. $y = \tan^{-1} x$

6\. (a) $y = e^{-2x}$

(b) $f(x) = A \exp(-x^3/3)$

7\. (a) $5 \sinh 5t/2\sqrt{\cosh 5t}$

(b) $3 \ln [\cosh(x/3)] + C$

(c) $2\sqrt{1 + x^2} + 5 \sinh^{-1} x + C$

(d) $e^{-y+2}/(e^{-2y+4} - 1)$

8\. (a) $y(2) \approx 5.378$

(b) $y(x) = -x^2/2 - x/2 - 1/4 + e^{2x}/4$, so $y(2) = 10.3995$

9\. $V = [I_0R / (R^2C^2\omega^2 + 1)] [\cos \omega t + (RC\omega)\sin \omega t - \exp(-t/RC)]$

10\. (a) $q(t) = \exp[-0.0538(t - 5)]$

(b) 12.9 years

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