CHAPTER 9

APPLICATIONS OF INTEGRATION

9.1 Volumes by the Slice Method

PREREQUISITES

1. Recall how to derive the integration formula for area by using the infinitesimal argument (Section 4.6).
2. Recall the various methods of integration (Chapter 7).

PREREQUISITE QUIZ

1. The area under the graph of a positive function $f(x)$ is $\int_a^b f(x)\,dx$. Sketch a typical graph of $f(x)$ and use it to explain the geometric meaning of $f(x)\,dx$.
2. Evaluate $\int (x+1)^2\,dx$.
3. Evaluate $\int e^{4y}\,dy$.

GOALS

1. Be able to compute volumes by using the slice method.
2. Be able to compute volumes by using the disk method.

STUDY HINTS

1. Basic formula. All of the volume formulas are based upon $V = \int_a^b A(x)\,dx$, where $A(x)$ is a solid's cross-sectional area. Remembering this and deriving the formulas in this section will prove to be more beneficial than just memorizing.

2. Slice method. Study Examples 1, 2, and 3 to see how elementary geometry is used to compute the cross-sectional area. Note that $A(x)$ in the volume formula corresponds to $\ell(x)$ in the area formula.

3. Radius dependent upon height. Example 2 shows how similar triangles are commonly used as an aid in ccmputing a cross-sectional area.

4. Disk method. By this method, each cross-sectional area is simply a circle whose radius is $f(x)$; therefore, $A(x) = \pi[f(x)]^2$. Learn this derivation rather than memorizing the formula. Note that the formula works even if $f(x) < 0$.

5. Washer method. The area is the difference between that of two circular regions; therefore, if $g(x) \geqslant f(x)$, then $A(x) = \pi[g(x)]^2 - \pi[f(x)]^2$. Note that this reduces to the disk method if $f(x) = 0$. Again, it is best to learn the derivation. WARNING: The integrand is not $[f(x) - g(x)]^2$.

6. Step function argument (p. 425). This is just for the theoretically inclined. Except in honors courses, most instructors will not expect their students to reproduce the argument.

7. Cavalieri's principle. Basically, it states that two volumes with equal cross-sectional areas have equal volumes. Thus, the "tilted" solids in Exercises 1-4 have the same volume as those which stand "straight up." Note that $\Sigma_{i=1}^{n} \pi r_i^2 \Delta x_i$ in the discussion is simply the disk method formul

SOLUTIONS TO EVERY OTHER ODD EXERCISE

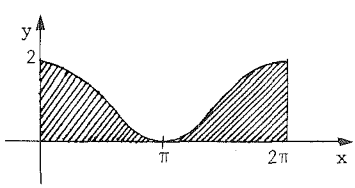
1. Let the x-axis be vertical as in Example 2. Then apply the slice method. Each infinitesimal slice has a circular base with area $\pi r^2 = \pi(1)^2$ and a thickness dx. Thus, $V = \int_0^3 \pi dx = 3\pi$.

5. Since the cross-section is a square, its area is $\{[(x - 6)^2 - 1]/6\}^2$. Therefore, the volume is $V = (1/36)\int_0^5 [(x - 6)^4 - 2(x - 6)^2 + 1]\, dx = (1/36)[(x - 6)^5/5 - 2(x - 6)^3/3 + x]\Big|_0^5 = 2125/54$.

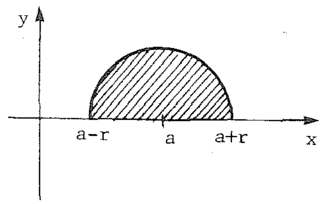
9. As shown in Example 2, the volume from height x_1 to height x_2 is $(\pi r^2/h^2)(h^2x - hx^2 + x^3/3)\Big|_{x_1}^{x_2}$. The volume of a short cone with height from x to h is $\pi r^2(h - x)^3/3h^2$. When we bisect the cone, we want $\pi r^2(h - x)^3/3h^2 = V/2$, where $V = \pi r^2 h/3$, the volume of the entire cone. Therefore, $(h - x)^3 = h^3/2$, i.e., $x = (1 - 1/\sqrt[3]{2})h$. Similarly, we equate $\pi r^2(h - x)^3/3h^2$ to $V/4$ and $(3/4)V$. Thus, the cuts are at $x_1 = (1 - \sqrt[3]{1/4})h$, $x_2 = (1 - \sqrt[3]{1/2})h$, and $x_3 = (1 - \sqrt[3]{3/4})h$, respectively.

13. The volume of the entire cylinder is $\pi(5)^2(20) = 500\pi\ cm^3$ since the radius is 5 cm. The volume of the wedge is determined by $r = 5$ and $\tan\theta = 5/5 = 1$, so the removed volume is $2(5)^3/3 = 250/3\ cm^3$. Therefore, the entire solid's volume is $(500\pi - 250/3) \approx 1487.5\ cm^3$.

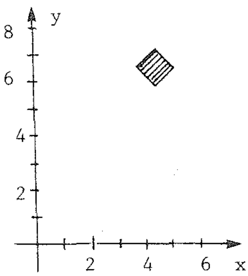
17.



Here, $f(x) = \cos x + 1$, so the disk method gives the volume as $V = \pi\int_0^{2\pi} (\cos x + 1)^2 dx = \pi\int_0^{2\pi} (\cos^2 x + 2\cos x + 1)dx = \pi\int_0^{2\pi} ((1 + \cos 2x)/2 + 2\cos x + 1)dx = \pi(3x/2 + \sin 2x/4 + 2\sin x)\Big|_0^{2\pi} = 3\pi^2$.

21. 

The equation of the entire circle is $(x - a)^2 + y^2 = r^2$, so the equation of the semicircle is $y = \sqrt{r^2 - (x - a)^2}\, dx$. By the disk method, $V = \pi\int_{a-r}^{a+r}\left(\sqrt{r^2 - (x - a)^2}\right)^2 dx = \pi\int_{a-r}^{a+r}[r^2 - (x - a)^2]\, dx = \pi[xr^2 - (x - a)^3/3]\big|_{a-r}^{a+r} = \pi(2r^3 - 2r^3/3) = 4\pi r^3/3$

25. 

The square in Exercise 23 is centered at $(9/2,13/2)$. Since each side of the square has length 1 , each vertex is $\sqrt{2}/2$ from the center; therefore, the vertices are $(9/2,13/2\pm\sqrt{2}/2)$ and $(9/2\pm\sqrt{2}/2,13/2)$. The square can be divided in two regions. One is the region between $y = x + (4 + \sqrt{2})/2$ and $y = -x + (22 - \sqrt{2})/2$ on $[(9 - \sqrt{2}/2,9/2]$ and the other lies between $y = -x + (22 + \sqrt{2})/2$ and $y = x + (4 - \sqrt{2})/2$ on $[9/2,(9 + \sqrt{2})/2]$. For revolution around the x-axis, the disk method gives the volume as $V = \pi\int_{(9-\sqrt{2})/2}^{9/2}\{[x + (4 + \sqrt{2})/2]^2 - [-x + (22 - \sqrt{2})/2]^2\}dx + \pi\int_{9/2}^{(9+\sqrt{2})/2}\{[-x + (22 + \sqrt{2})/2]^2 - [x + (4 - \sqrt{2})/2]^2\}dx = \pi\int_{(9-\sqrt{2})/2}^{9/2}(26x - 117 + 13\sqrt{2})\, dx + \pi\int_{9/2}^{(9+\sqrt{2})/2}(-26x + 117 + 13\sqrt{2})dx = \pi[(13x^2 - 117x + 13\sqrt{2}x)\big|_{(9-\sqrt{2})/2}^{9/2} + (-13x^2 + 117x + 13\sqrt{2}x)\big|_{9/2}^{(9+\sqrt{2})/2}] = \pi[13(-1 + 9\sqrt{2})/2 - 117(\sqrt{2}/2) + 13\sqrt{2}(\sqrt{2}/2) - 13(1 + 9\sqrt{2})/2 + 117(\sqrt{2}/2) + 13\sqrt{2}(\sqrt{2}/2) = 13\pi$.

29. A doughnut can be made by revolving a circle around the x-axis. To form the desired doughnut, revolve the circle centered at $(0,(R + r)/2)$ with radius $(R - r)/2$. The equation of the circle is $x^2 + (y - (R + r)/2)^2 = (R - r)^2/4$. Solving for y , we get $\pm\sqrt{(R - r)^2/4 - x^2} + (R + r)/2$. Therefore, the volume is $\pi\int_{-(R-r)/2}^{(R-r)/2}\Big[\sqrt{(R - r)^2/4 - x^2} +$

29. (continued)

$$(R + r)/2\Big]^2 dx - \pi\int_{-(R-r)/2}^{(R-r)/2}\Big[-\sqrt{(R - r)^2/4 - x^2} + (R + r)/2\Big]^2 dx =$$

$$\pi\int_{-(R-r)/2}^{(R-r)/2} 2(R + r)\sqrt{(R - r)^2/4 - x^2}\,dx = 2\pi(R + r)\int_{-(R-r)/2}^{(R-r)/2}\sqrt{(R - r)^2/4 - x^2}\,dx .$$

Note that the integral is the area of a semicircle centered at the origin with radius $(R - r)/2$, which is $\pi(R - r)^2/8$. Thus, the volume of the doughnut is $\pi^2(R + r)(R - r)^2/4$.

SECTION QUIZ

1. From elementary geometry, we know that the volume of a cone is $(1/3)\pi r^2 h$, where r is the radius of the base and h is the height of the cone. Suppose we revolve $y = x/2$ on $[0,2]$ around the x-axis, then the base radius is 1 and $h = 2$. Thus, the volume is $2\pi/3$. On the other hand, $V = \pi\int_a^b [f(x)]^2 dx = \pi[f(x)]^3/3\big|_a^b$. For the cone, $f(x) = x/2$, $a = 0$, and $b = 2$, so the volume is $\pi(x/2)^3/3\big|_0^2 = (\pi x^3/24)\big|_0^2 = \pi/3$. What's wrong?

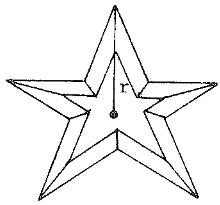
2. The line $y = x + 1$ is revolved about the x-axis to form a solid of revolution.

   (a) A vertical cut is made at $x = 9$. What is the volume of the resulting solid between $x = 0$ and $x = 9$?

   (b) Where should another cut be made parallel to the y-axis to get two equal volumes from the solid in (a)?

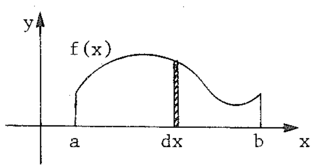
3. The curve $y = x^5$ on $[0,1]$ is revolved around the y-axis. Use the disk method to find the volume of the resulting solid.

4. The cross-sectional area of a solid at height h is given by $h|\cos h|$. The solid extends from $h = 0$ to $h = 3\pi/2$. Is enough information given to compute the volume? If not, what is missing? If yes, compute it.

5. 

Bruce, the boozing butcher has just returned from his afternoon vodka break. As he begins to trim the fat off a piece of rib roast, the alcohol begins to take effect and all he can see before his eyes are stars. Consequently, the beef is cut into the shape shown. Define the "radius" r of the cross-sectional stars to be the segment from the center to one of the outer vertices. When our boozing friend was sober, he determined the area of the star to be $5r^2/4$. If the bottom base has area 125 cm^2 , the top base has area 20 cm^2 , and the height is 6 cm , what is the volume of the meat that Boozer Bruce cut?

ANSWERS TO PREREQUISITE QUIZ

1. 

The shaded region is a very thin "rectangle". Its width is dx and since it is so thin, its height, $f(x)$, is "constant". Therefore, $f(x)dx$ is the area of the region.

2. $(x + 1)^3/3 + C$

3. $e^{4y}/4 + C$

ANSWERS TO SECTION QUIZ

1. In performing the integration, a factor of 2 was forgotten when the substitution $u = x/2$ was made.

2. (a) 333π

   (b) $\sqrt[3]{1001/2} - 1 \approx 6.94$

3. $5\pi/7$

4. Yes; $5\pi/2 - 1$

5. 390 cm^3

9.2 Volumes by the Shell Method

PREREQUISITES

1. Recall how to compute volumes by the disk method (Section 9.1).

PREREQUISITE QUIZ

1. Find the volume of the solid obtained by revolving the graph of $y = x^2 - 1$ on $[2,4]$ around the x-axis.
2. Repeat Question 1 for $y = \sec x$ on $[0,\pi/4]$.

GOALS

1. Be able to compute volumes by using the shell method.

STUDY HINTS

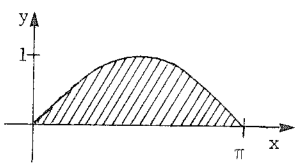
1. Shell method. Learn the derivation. It is simply a summation of infinitesimal cylindrical volumes, dV. Each shell has radius x, so its circumference is $2\pi x$. Multiplying by the thickness dx gives the area of the base as $2\pi x\,dx$. Finally, for a region between the curves $f(x)$ and $g(x)$, the height is $f(x) - g(x)$. Thus, we sum $dV = 2\pi x\,dx(f(x) - g(x))$ to get $V = 2\pi\int_a^b x[f(x) - g(x)]dx$. If $g(x) = 0$, this reduces to $V = 2\pi\int_a^b xf(x)dx$. Note that if $f(x) < 0$, we need to use $|f(x)|$.
2. Useful trick. At this point in your studies, you do not know how to integrate $\sqrt{1 - x^2}$; however, many times it is possible to make a substitution so that even though you cannot compute an integral directly, you can determine it by computing the area under the curve using elementary geometry. See how this method is used in Example 6.

3. Step function argument. Again, as with the disk method, you will probably not need to regurgitate the step function argument unless you are enrolled in an honors course. Ask your instructor.

4. Choosing a method. In most cases, if $y = f(x)$ is revolved around the y-axis to generate a solid, the volume is best found by using the shell method. Similarly, revolution around the x-axis implies the use of the disk method. If you have a thorough understanding of these two methods, it is possible to do the same problem using either method. However, the simplest method should be used to promote efficiency.

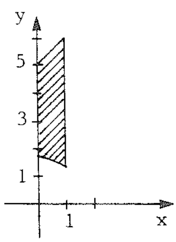
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



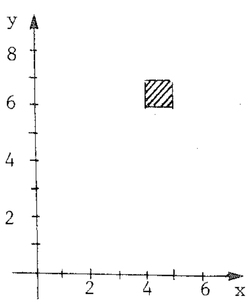
By the shell method, the volume is $2\pi\int_0^{\pi} x \sin x\, dx$. Integration by parts with $u = x$ and $v = -\cos x$ yields $2\pi(-x \cos x + \sin x)\big|_0^{\pi} = 2\pi(\pi) = 2\pi^2$.

5.

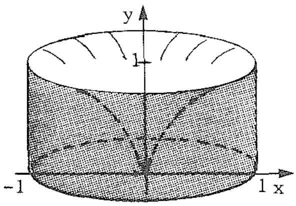


For revolution around the y-axis, the shell method gives $V = 2\pi\int_0^1 x(5 + x)dx - 2\pi\int_0^1 x\sqrt{3 - x^2}\, dx$. Let $u = 3 - x^2$, so $du/2 = -x\, dx$; therefore, $V = 2\pi(5x^2/2 + x^3/3)\big|_0^1 + 2\pi\int_3^2\sqrt{u}\, du/2 = 17\pi/3 + 2\pi u^{3/2}/3\big|_3^2 = \pi(17 + 4\sqrt{2} - 6\sqrt{3})/3$.

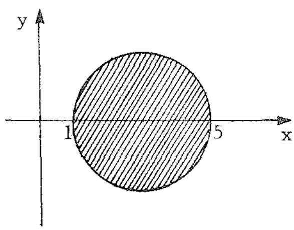
9.



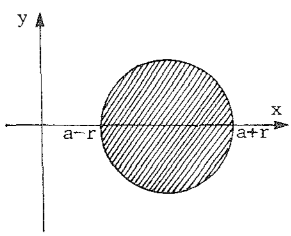
The description of the square is the region between $y = 6$ and $y = 7$ on $[4,5]$. For revolution around the y-axis, the shell method gives $V = 2\pi\int_4^5 7x\, dx - 2\pi\int_4^5 6x\, dx = 2\pi(7x^2/2 - 6x^2/2)\big|_4^5 = 2\pi(1/2)x^2\big|_4^5 = \pi(25 - 16) = 9\pi$.

13. 

The volume is $2\pi\int_0^1 x\sqrt{x}\,dx = 2\pi(2x^{5/2}/5)\Big|_0^1 = 4\pi/5$. If we put the resulting volume of Example 5 of the previous section on top of the solid generated here, it will produce a cylinder of radius 1 and height 1.

17. 

Use the method of Example 6. The volume of the top half is $2\pi\int_1^5 x[4-(x-3)^2]^{1/2}dx = 2\pi\int_{-2}^2 (u+3)\sqrt{4-u^2}\,du = 2\pi\int_{-2}^2 u\sqrt{4-u}\,du + 6\pi\int_{-2}^2\sqrt{4-u^2}\,du = 0 + 6\pi(2\pi) = 12\pi^2$. Thus, the total volume is $24\pi^2$.

21. (a) 

By symmetry, we only need to revolve the upper semicircle. Thus, $f(x) = [a^2-(x-b)^2]^{1/2}$, and the volume is $V = 2\pi\int_{b-a}^{b+a} x[a^2-(x-b)^2]^{1/2}dx$. Using the method of Example 6, let $u = x - b$ to get $2\pi\int_{-a}^a (u+b) \times \sqrt{a^2-u^2}\,du = 2\pi\int_{-a}^a u\sqrt{a^2-u^2}\,du + 2\pi\int_{-a}^a b\sqrt{a^2-u^2}\,du = 0 + 2\pi b(\pi a^2)$. Thus, the total volume is $2\pi^2a^2b$.

(b) The volume of $T_{a,b}$ is $2\pi^2a^2b$ by part (a). Substituting $a+h$ for a, we compute the volume of $T_{a+h,b}$ to be $2\pi^2(a+h)^2b = 2\pi^2b(a^2+2ah+h^2)$. Therefore, the difference in the two volumes is $2\pi^2b(2ah+h^2)$.

(c) As in Exercise 19, we expect the surface area to be the derivative $(d/dh)[\text{volume}(T_{a+h,b}) - \text{volume}(T_{a,b})]$, evaluated at $h = 0$. Thus, the surface area is $2\pi^2b(2a+2h)\Big|_{h=0} = 4\pi^2ab$.

SECTION QUIZ

1. The curve $y = x^4$ on $[0,1]$ is revolved around the x-axis to generate a solid of revolution. Use the shell method to compute the volume. Compute the same volume using the disk method.

2. The region between the curves $y = x$ and $y = x^2$ on $[0,2]$ is revolved around the y-axis. What is the volume of the resulting solid?

3. (a) The line $y = x + 5$ on $[2,3]$ is revolved around the line $x = 1$. What is the volume of the solid generated by revolving the region between $y = x + 5$ and the x-axis?

   (b) Find a general formula for the volume generated by revolving the region under $y = f(x)$ on $[a,b]$ about the line $x = A$. Assume $A < a < b$, and $f(x) \geqslant 0$ for x in $[a,b]$.

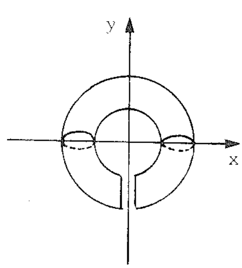
4. Suppose $f(x) = \cos x$ on $[0,\pi]$. The region between $f(x)$ and the x-axis is revolved around the y-axis. Which of the following is true?

   (a) The volume can not be computed because $f(x) < 0$ for some x in $[0,\pi]$.

   (b) The volume can be computed; it is negative.

   (c) The volume can be computed; it is positive.

   (d) The volume is $2\pi^2 - 4\pi$ because the volume on $[0,\pi]$ is $\pi^2 - 2\pi$ and symmetry can be applied.

5. 

As a practical joke, you give your hungry little cousin a stale doughnut. She breaks her front teeth trying to bite into it. Angrily, she throws the doughnut at you. Seeing her front teeth still stuck in it, you realize its value as a conversation piece. In order to stand the doughnut up, you drill a 1-centimeter diameter hole along the y-axis as shown.

5. (a) The inner radius of the doughnut is 2 centimeters. Its outer radius is 4 centimeters. What was the original volume of the doughnut? (*Hint:* See Example 6.)

(b) Suppose you revolve the line $x = 1$ between $y = -2$ and $y = -4$ around the y-axis and you subtract the resulting volume from the doughnut's original volume. Neglecting the volume of the teeth, is your answer an approximation or is it exact for the volume of the doughnut with the drilled hole? Explain. (*Hint:* Take a coffee break and buy yourself a doughnut.)

(c) What answer did you get for the volume by using the method described in part (b)?

ANSWERS TO PREREQUISITE QUIZ

1. $(163.07)\pi$

2. π

ANSWERS TO SECTION QUIZ

1. $2\pi\int_0^1 y(1 - y^{1/4})dy = \pi/9$; $\pi\int_0^1 x^8 dx = \pi/9$

2. 3π

3. (a) $68\pi/3$

   (b) $2\pi\int_a^b (x - A)f(x)dx$

4. c

5. (a) $3\pi^2$

   (b) Approximate; the "curvature" at $y = -2$ is greater than that at $y = -4$.

   (c) $3\pi^2 - 2\pi$

9.3 Average Values and the Mean Value Theorem for Integrals

PREREQUISITES

1. Recall how to use the summation notation (Section 4.1).
2. Recall the intermediate value theorem (Section 3.1).
3. Recall the meaning of the mean value theorem for differentiation (Section 3.6).

PREREQUISITE QUIZ

1. State the mean value theorem for derivatives.
2. What conditions are necessary to apply the mean value theorem?
3. Explain the intermediate value theorem and state any necessary conditions.
4. Suppose $a_1 = 1$, $a_2 = 4$, $a_3 = -2$ and $b_j = j$. Compute the following:
   (a) $\Sigma_{i=1}^{3} a_i b_i$
   (b) $\Sigma_{i=2}^{5} b_i$

GOALS

1. Be able to compute the average of a function on a given interval.
2. Be able to state the mean value theorem for integrals and understand its meaning.

STUDY HINTS

1. Notation. Remember that the bar over the function indicates that the average value is desired. The interval is indicated as a subscript in $\overline{f(x)}_{[a,b]}$.

2. Average value. If you remember that the average is weighted, you should have no problem deriving the formula. It is simply the area under the curve divided by the length of the interval, i.e., the integral divided by the length. Fig. 9.3.1 may help you remember this.

3. Clarification. The numbers m and M as used on p. 435 are often chosen to be the minimum and the maximum values of $f(x)$ on $[a,b]$.

4. Integral mean value theorem. As with several other previous theorems, this is an existence theorem. It states that the average is attained somewhere in the interval, possibly many times; however, it doesn't specify exactly where it occurs, nor does the theorem help you find it. Note that continuity on a closed interval is required.

5. Example 5. Notice that the solution uses the fact that $\int_a^a f(x)dx = 0$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The average value is $\overline{f(x)}_{[a,b]} = \int_a^b f(x)dx/(b - a)$. In this case, it is $[1/(1 - 0)]\int_0^1 x^3 dx = (x^4/4)|_0^1 = 1/4$.

5. This is just like Exercise 1 except that the limits of integration range from 0 to 2 . Thus, the average is $[1/(2 - 0)]\int_0^2 x^3 dx = (1/2)(x^4/4)|_0^2 = 2$.

9. The average is $[1/(1 - 0)]\int_0^1 \sin^{-1}x\ dx$. Using the formula for integrating inverse functions from Section 7.4, we get $\int_0^1 \sin^{-1}x\ dx = x\sin^{-1}x|_0^1 - \int_0^{\pi/2}\sin y\ dy = x\sin^{-1}x|_0^1 + \cos y|_0^{\pi/2} = \pi/2 - 1$.

13. The average is $[1/(3 - 1)]\int_1^3(x^3 + \sqrt{1/x})dx = (1/2)\int_1^3(x^3 + x^{-1/2})dx = (1/2)(x^4/4 + 2\sqrt{x})|_1^3 = (1/2)(20 + 2\sqrt{3} - 2) = 9 + \sqrt{3}$.

17. Let t be the number of hours after midnight, then

$$f(t) = \begin{cases} 50 & 0 \leqslant t \leqslant 3 \\ 145/3 + 5t/9 & 3 \leqslant t \leqslant 12 \\ 15 + 10t/3 & 12 \leqslant t \leqslant 15 \\ 90 - 5t/3 & 15 \leqslant t \leqslant 24 \end{cases}.$$

The formula for average values gives us $[1/(24 - 0)][\int_0^3 50\,dt + \int_3^{12}(145/3 + 5t/9)dt + \int_{12}^{15}(15 + 10t/3)dt + \int_{15}^{24}(90 - 5t/3)dt] = (1/24)[(50t)|_0^3 + (145t/3 + 5t^2/18)|_3^{12} + (15t + 5t^2/3)|_{12}^{15} + (90t - 5t^2/6)|_{15}^{24}] = (1/24)[150 + (435 + 37.5) + (45 + 135) + (810 - 292.5)] = (1/24)(1320) = 55°F$.

21. We apply the fundamental theorem of calculus: $\overline{f'(x)}_{[a,b]} = [1/(b - a)] \times \int_a^b f'(x)dx = f(x)|_a^b/(b - a) = [f(b) - f(a)]/(b - a)$. This can only be 0 if $f(b) = f(a)$.

25. According to the mean value theorem for integrals, there is some t_0 in $[a,b]$ such that $\overline{f'(t)}_{[a,b]} = [1/(b - a)]\int_a^b f'(t)dt$. By the fundamental theorem of calculus, the right-hand side is $[f(b) - f(a)]/(b - a)$. We have shown that the average derivative is attained at some t_0 in $[a,b]$, so $f'(t_0) = [f(b) - f(a)]/(b - a)$, which is the mean value theorem for derivatives.

29. We need to show that $\lim_{x\to x_0} F(x) = F(x_0)$. By the definition used in Section 11.1 , we must show that $|F(x) - F(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. By definition, $F(x) - F(x_0) = \int_a^x f(s)ds - \int_a^{x_0} f(s)ds = \int_{x_0}^x f(s)ds$. Now, by the extreme value theorem, $|f(s)|$ has a maximum $|M|$ on $[a,b]$, so $|f(s)| \leqslant |M|$ on $[a,b]$ and $|\int_{x_0}^x f(s)ds| \leqslant |M||x - x_0|$. Thus, if we let $\delta = |x - x_0| = \varepsilon/|M|$, we have $|F(x) - F(x_0)| \leqslant |M|\varepsilon/|M| = \varepsilon$.

SECTION QUIZ

1. Find the average of the following functions on the given intervals:

   (a) $g(t) = t\sqrt{t - 1}$ on $[2,3]$

   (b) $f(x) = 2^x$ on $[1,3]$

   (c) $y = \sin(x/2)$ for $0 \leqslant x \leqslant \pi$

2. Suppose $\overline{f(x)}_{[0,3]} = 8$, $\overline{f(x)}_{[3,5]} = 2$, and $\overline{f(x)}_{[5,10]} = 5$.

   (a) What is $\overline{f(x)}_{[0,10]}$?

   (b) Find $\overline{f(x)}_{[0,5]}$, if possible.

   (c) Find $\overline{f(x)}_{[3,8]}$, if possible .

3. Consider the function in Question 2.

   (a) Can we conclude that $f(x_0) = 2$ for some x_0 in $[3,5]$? Explain.

   (b) If f is not differentiable somewhere in $[0,10]$, but it is continuous, can we conclude that $f(x_0) = 5$ for some x_0 in $[0,10]$? Explain.

   (c) If f is continuous, $f(7.5) = 5$, and $f(10) = 7$, then can we conclude that $f(5) = 3$? Explain.

4. True or false: A continuously differentiable function can not attain its average at a critical point.

5. On a typical day, the Smith family, consisting of three talkative teenagers and their parents, uses the telephone for $3 + t^2$ hours, where t is the number of days before or after Wednesday, 12 noon. Thus, $t = 1$ on Tuesday or Thursday, etc.

   (a) What is the average number of hours the Smith family talks on the phone daily? (Hint: Integrate on the interval $-3.5 \leqslant t \leqslant 3.5$.)

   (b) Show that the mean value theorem for integrals is valid for this situation.

ANSWERS TO PREREQUISITE QUIZ

1. If f is continuous on $[a,b]$ and differentiable in (a,b), then $f'(x_0) = [f(b) - f(a)]/(b - a)$ for some x_0 in (a,b).
2. Continuity and differentiability.
3. Continuity is a necessity. If f is continuous, and $f(x_1)$, $f(x_2)$ lie on opposite sides of the line $y = C$, then one must cross the line somewhere in (x_1,x_2).
4. (a) 3

   (b) 14

ANSWERS TO SECTION QUIZ

1. (a) $(44\sqrt{2} - 16)/15$

   (b) $3/\ln 2$

   (c) $2/\pi$
2. (a) 5.3

   (b) 5.6

   (c) Not possible
3. (a) No, f may not be continuous.

   (b) Yes, particularly in $(5,10)$ by the mean value theorem for integrals

   (c) No, there are numerous functions which satisfy $f(7.5) = 5$, $f(10) =$ and $\int_5^{10} f(x)dx = 25$. The function doesn't have to be symmetric.
4. False; consider $f(x) = x^3$ on $[-1,1]$.
5. (a) 28/3 hours/day

   (b) $x_0 = (-3\pm\sqrt{139/3})/2 \approx -4.90$ and 1.90; 1.90 is in the interval $(-3.5,3.5)$.

9.4 Center of Mass

PREREQUISITES

1. Recall how integration formulas for areas and volumes were derived by using the infinitesimal argument (Sections 4.6, 9.1, and 9.2).
2. Recall how to use the summation notation (Section 4.1).

PREREQUISITE QUIZ

1. Use an infinitesimal argument to derive $V = 2\pi\int_a^b xf(x)dx$.
2. Use an infinitesimal argument to derive $A = \int_a^b [f(x) - g(x)]dx$, the area between two graphs such that $f(x) \geqslant g(x)$ on $[a,b]$.
3. (a) In general, does $\Sigma_{i=1}^n m_i x_i = (\Sigma_{i=1}^n m_i)(\Sigma_{i=1}^n x_i)$?

   (b) In general, does $\Sigma_{i=1}^n m_i x_i / \Sigma_{i=1}^n m_i = \Sigma_{i=1}^n x_i$?

GOALS

1. Be able to state and understand the consolidation principle.
2. Be able to find the center of mass on a line.
3. Be able to find the center of mass for a plane region.

STUDY HINTS

1. Consolidation principle. This allows you to concentrate the entire mass of an object at its center of mass. This principle is important in the derivation of formulas. "Negative" masses may also be used if a mass needs to be subtracted. The usefulness of this principle is illustrated quite well in Example 7.
2. Center of mass on the line. By remembering that the center of mass is simply a weighted average of position, you should be able to recall the formula $\bar{x} = \Sigma_{i=1}^n m_i x_i / \Sigma_{i=1}^n m_i$.

3. Warning. If you think the m_i's cancel in the center of mass formula (3), you need to review Section 4.1.

4. Center of mass in the plane. The only difference between $\bar{x}$ and $\bar{y}$ is that one formula uses x_i and the other uses y_i .

5. Symmetry principle. Properly used, this saves a lot of needless work. Remember that uniform density is a requirement. Two axes of symmetry is all that is needed to determine the center of mass of a region in the plane.

6. Center of mass of a region. It is recommended that you learn to derive the formula with the aid of Fig. 9.4.10. Again, this is a weighted average where the mass is density ρ times the area of the region. The center of mass of the "infinitesimal rectangle" is simply $(x,f(x)/2)$. Now, use the consolidation principle and the fact that integration is a continuous summation. The same formula holds even if $f(x)$ is negative. Exercise 28 gives a more general formula.

7. Density cancels. In all of the formulas, mass is used. For region in the plane, mass is proportional to density $\times$ area. Density cancels when it is uniform. Be careful when density is not uniform.

8. Step functions. As with the other step function arguments presented in this chapter, you will probably not be expected to recall the material in the supplement.

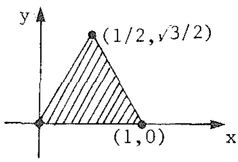
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. In this case, $M_1 = m_1$, $M_2 = m_2 + m_3$, $X_1 = x_1$, and $X_2 = (m_2x_2 + m_3x_3)/(m_2 + m_3)$. The center of mass is $(M_1X_1 + M_2X_2)/(M_1 + M_2) = [m_1x_1 + (m_2 + m_3)(m_2x_2 + m_3x_3)/(m_2 + m_3)]/[m_1 + (m_2 + m_3)] = (m_1x_1 + m_2x_2 + m_3x_3)/(m_1 + m_2 + m_3)$, which is the same formula derived in Example 1.

5. Using the formula $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$, we get $\bar{x} = (1 \cdot 7 + 3 \cdot 3 + 5 \cdot 5 + 7 \cdot 1)/(1 + 3 + 5 + 7) = 48/16 = 3$.

9. Using the formulas $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$ and $\bar{y} = \Sigma_{i=1}^{n} m_i y_i / \Sigma_{i=1}^{n} m_i$, we get $\Sigma_{i=1}^{2} m_i x_i = 10(1) + 20(1) = 30$; $\Sigma_{i=1}^{2} m_i y_i = 10(0) + 20(2) = 40$; $\Sigma_{i=1}^{2} m_i = 10 + 20 = 30$. Thus, $\bar{x} = 30/30 = 1$ and $\bar{y} = 40/30 = 4/3$.

13. (a)

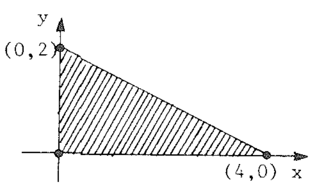


All of the sides of the triangle must have unit length. Thus, the third vertex may be determined by solving $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{(x - 1)^2 + (y - 0)^2} = 1$ or by using plane geometry. Thus, the thrid vertex is $(1/2, \sqrt{3}/2)$. $\bar{x} = m(0 + 1 + 1/2)/3m = 1/2$, where m is the mass of the object. $\bar{y} = m(\sqrt{3}/2 + 0 + 0)/3m = \sqrt{3}/6$. Thus, *the center of mass is* $(1/2, \sqrt{3}/6)$.

(b) If the mass at $(0,0)$ is doubled, then $\bar{x} = [2m(0) + m(1/2) + m(1)]/(2m + m + m) = 3/8$ and $\bar{y} = [2m(0) + m(\sqrt{3}/2) + m(0)]/(2m + m + m) = \sqrt{3}/8$. Thus, the center of mass is $(3/8, \sqrt{3}/8)$.

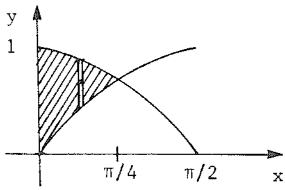
17. Using the formulas $\bar{x} = \int_a^b xf(x)dx / \int_a^b f(x)dx$ and $\bar{y} = (1/2)\int_a^b [f(x)]^2 dx / \int_a^b f(x)dx$, we get $\bar{x} = \int_1^3 x(4/x^2)dx / \int_1^3 (4/x^2)dx = \int_1^3 (4/x)dx / (-4/x)|_1^3 = 4 \ln x|_1^3/(8/3) = 4 \ln 3/(8/3) = 3 \ln 3/2$. And $\bar{y} = (1/2)\int_1^3 (4/x^2)^2 dx / \int_1^3 (4/x^2)dx = (1/2)\int_1^3 16x^{-4}dx/(8/3) = 3(x^{-3}/-3)|_1^3 = 26/27$. Thus, the center of mass is $((3/2)\ln 3, 26/27) \approx (1.65, 0.96)$.

21.



As shown in the figure, we want to find the center of mass of the region under $f(x) = -x/2 + 2$. Therefore, $\bar{x} = \int_0^4 x(-x/2 + 2)dx / \int_0^4 (-x/2 + 2)dx = (-x^3/6 + x^2)|_0^4 / (-x^2/4 + 2x)|_0^4 = 4/3$. $\bar{y} = (1/2)\int_0^4 (-x/2 + 2)^2 dx / \int_0^4 (-x/2 + 2)dx = -(-x/2 + 2)^3/3|_0^4/4 = (8/3)/4 = 2/3$. Therefore, the center of mass is $(4/3, 2/3)$.

25. First, recall that $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$. Since the masses are independent of time, we differentiate to get the velocity at the center of mass, $v = d\bar{x}/dt = \Sigma_{i=1}^{n} m_i (dx_i/dt) / \Sigma_{i=1}^{n} m_i$. By definition, $P = \Sigma_{i=1}^{n} m_i v_i = \Sigma_{i=1}^{n} m_i (dx_i/dt)$, and $M = \Sigma_{i=1}^{n} m_i$. Rearrange $P = Mv$ to get $v = P/M = \Sigma_{i=1}^{n} m_i (dx_i/dt)/\Sigma_{i=1}^{n} m_i = d\bar{x}/dt = v$.

29. 

Note that $\cos x \geqslant \sin x$ on $[0,\pi/4]$. On an infinitesimal strip, the center of mass is $(x,(\cos x + \sin x)/2)$ (see figure). We take a weighted average by weighing the center of mass against its mass, i.e., area. The area of the rectangular infinitesimal strip is $(\cos x - \sin x)dx$. Now the continuous sum of $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$ becomes $\int_0^{\pi/4} x(\cos x - \sin x)dx / \int_0^{\pi/4} (\cos x - \sin x)dx$. Integration by parts with $u = x$ and $v = \sin x + \cos x$ yields $[x(\sin x + \cos x)|_0^{\pi/4} - \int_0^{\pi/4}(\sin x + \cos x)dx] / \sin x + \cos x)|_0^{\pi/4} = [(\pi/4)\sqrt{2} - (-\cos x + \sin x)|_0^{\pi/4}]/(\sqrt{2} - 1) = (\sqrt{2}\pi/4 - 1)/(\sqrt{2} - 1) \approx 0.27 = \bar{x}$. Similarly, the continuous sum of $\bar{y} = \Sigma_{i=1}^{n} m_i y_i / \Sigma_{i=1}^{n} m_i$ is $\int_0^{\pi/4} (1/2)(\cos x + \sin x)(\cos x - \sin x)dx / \int_0^{\pi/4}(\cos x - \sin x)dx = (1/2)\int_0^{\pi/4}(\cos^2 x - \sin^2 x)dx/(\sqrt{2} - 1)$. By a trigonometric substitution, we get $(1/2)\int_0^{\pi/4} \cos 2x \, dx/(\sqrt{2} - 1) = (1/2)(\sin 2x/2)|_0^{\pi/4}/(\sqrt{2} - 1) = 1/4(\sqrt{2} - 1)$. Therefore, the center of mass is $((\sqrt{2}\pi/4 - 1)/(\sqrt{2} - 1), 1/4(\sqrt{2} - 1)) \approx (0.27, 0.60)$.

SECTION QUIZ

1. (a) A square region with vertices at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$ has density x^2 at (x,y) . Explain why the symmetry principle applies for a vertical axis.

1. (b) A square region with vertices at $(3,-1)$, $(3,1)$, $(5,1)$, and $(5,-1)$ has density x^2 at (x,y) . Explain why the symmetry principle does not apply for a vertical axis.

2. Find the centers of mass of each individual square region described in Question 1.

3. Find the center of mass of the combined regions described in Question 1.

4. The earth has a radius of 6.4×10^3 km and a mass of 6.0×10^{24} kg. The moon has a radius of 1.6×10^3 km and a mass of 7.4×10^{22} kg. The moon is 3.9×10^5 km from earth.

   (a) Assuming the earth and moon are perfect spheres, where is the center of mass of the earth-moon system?

   (b) What information was not necessary and why?

5. A famous artist likes to focus on the center of mass. For her latest painting, the lovely artist has selected three objects. All of the individual centers of mass are located in the same plane. One object is a circular plate with center at $(-3,4)$ and radius $\sqrt{5/\pi}$. A square plate has vertices at $(2,2)$, $(4,2)$, $(2,4)$, and $(4,4)$. Each plate has density 1 kg/m^2 and each unit on the xy-plane is 1 m. The third object is a frog whose mass is 2 kg and whose center of mass is at the origin.

   (a) Where is the center of mass for the three objects?

   (b) At the end of the day, the artist kisses the frog and he turns into a handsome prince whose mass increases to 70 kg and whose center of mass remains at $(0,0)$. Where is the new center of mass for the three objects?

ANSWERS TO PREREQUISITE QUIZ

1. This is the shell method. Revolving a thin rectangle at x around the y-axis gives us a cylinder with base circumference or "length" $2\pi x$. Its height is $f(x)$ and its width is dx, so its volume is $2\pi x f(x) dx$. Integrate to get the entire volume.

2. Consider a thin rectangle at x. Its height is $f(x) - g(x)$ and its width is dx. Therefore, its area is $[f(x) - g(x)]dx$. Integrate to get the entire area.

3. (a) No

   (b) No

ANSWERS TO SECTION QUIZ

1. (a) The mass at $(-x,y)$ equals the mass at (x,y).

   (b) The mass at $(4 - x_0,y)$ does not equal the mass at $(4 + x_0,y)$.

2. $(0,0)$ and $(204/49,0) \approx (4.16,0)$

3. $\approx (4.14,0)$

4. (a) 4.8×10^3 km from the earth on the line between the earth and the moon.

   (b) The radii are not needed because the consolidation principle can be used.

5. (a) $(-3/11,32/11)$

   (b) $(-3/79,32/79)$

9.5 Energy, Power, and Work

PREREQUISITES

1. Recall the physical interpretation of integrating rates of change (Section 4.6).

PREREQUISITE QUIZ

1. Given the following quantities for $f(x)$, what is $\int f(x)dx$?
   (a) Velocity (meters per second)
   (b) Water flow rate (gallons per minute)
   (c) Melting rate of a candle (grams per minute)

GOALS

1. Be able to state the relationship between power and energy and apply it for problem solving.
2. Be able to state the relationship between work and force and apply it for problem solving.

STUDY HINTS

1. Psychology. Many students fear this section because it is "Physics." Remember that you are enrolled in a math course and being a physics major is not a requirement. You will find the text self contained in what you need to know.
2. Units. The unit of work is the joule which is $1\ \mathrm{kg\cdot m^2/s^2}$. This can be remembered by using $dW = Fdx = ma\ dx$ and using the units of m, a, and dx. Another unit to know is the watt which is 1 joule/second.

3. Energy vs. power. Energy is the total sum of power, which is rate. They are related just as distance (integration) is to velocity (differentiation).

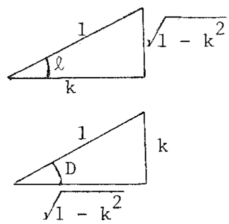
4. Important formulas. Memorize the fact that work is the integral of force with respect to position, not time. It is not essential to know that $dK/dx = F$. However, note that work is a form of kinetic energy; this is a special form of the equation $dW/dx = F$.

5. Good example. Example 4 should be studied thoroughly. It uses the important relationships $dW = Fdx$, $F = mg$, and $m = \rho V$. At this point, we still need to determine the volume term, which is $\pi r^2 dx$, by using similar triangles. Remember that gravity acts in a downward direction.

6. Sunshine formula. This is an interesting application of the calculus you have learned so far. Consult your instructor to see if you need to study it. None of the equations should be memorized. The energy equation (3) is the important one for doing the exercises.

SOLUTIONS TO EVERY OTHER ODD EXERCISE (SUPPLEMENT)

1. We use the formula, $E = \sqrt{\cos^2 \ell - \sin^2 D} + \sin \ell \sin D \cos^{-1}(-\tan \ell \tan D)$ where $\sin D = \sin \alpha \cos(2\pi T/365)$. On June 21, we have $T = 0$, so $\sin D = \sin \alpha$, or $D = \alpha$. Also, we have $\cos \ell = \sin \alpha$ and $\sin \ell = \cos \alpha$, $\tan \ell = \cot \alpha$, so $E = \sqrt{\sin^2 \alpha - \sin^2 \alpha} + \cos \alpha \sin \alpha \times \cos^{-1}(-\tan \ell \cot \ell) = \cos \alpha \sin \alpha \cos^{-1}(-1) = (\pi/2)\sin(2\alpha)$. Evaluating at $\alpha = 23.5^\circ$, we find $E \approx 1.15$, which is about 1.25 times the energy received at the equator on June 21. This excess is due, of course, to the long day at the Arctic Circle.

5\. 

To simplify the calculations, let $k = \sin\alpha\cos(2\pi T/365)$. On a day on which the sun does not set, $S = 12 = (24/2\pi)\cos^{-1}\left[-\tan\ell\left\{k/\sqrt{1-k^2}\right\}\right]$. The equation requires $\tan\ell\left\{k/\sqrt{1-k^2}\right\} = 1$ or $\tan^2\ell(k^2) = 1 - k^2$. Equivalently, $k^2(\tan^2\ell + 1) = 1$, which implies $k^2\sec^2\ell = 1$. Thus, $k = \cos\ell$ or $\ell = \cos^{-1}k$. Using the notation of the text, $\sin D = k$. Now substituting into the energy equation, $E = \sqrt{\cos^2\ell - \sin^2 D} + \sin\ell\sin D\cos^{-1}(-\tan\ell\tan D) = \sqrt{k^2 - k^2} + \sqrt{1-k^2}(k)\cos^{-1} \times \left[-\left\{\sqrt{1-k^2}/k\right\}\left\{k/\sqrt{1-k^2}\right\}\right] = \sqrt{1-k^2}(k)\cos^{-1}(-1) = \pi\sin\ell\sin\alpha\cos(2\pi T/365) = \pi\sin\ell\sin D$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1\. We use $E = \int_a^b P dt$. During one hour, $0 \leqslant t \leqslant 3600$ seconds, so the energy output is $E = \int_0^{3600} 1050\sin^2(120\pi t)dt$. By the half-angle formula, $E = (1/2)(1050)\int_0^{3600}(1 - \cos 240\pi t)dt = 525(t - \sin 240\pi t/240\pi)\Big|_0^{3600} = 525(3600) = 1{,}890{,}000$ joules.

5\. Work is the integral of force; $W = \int_a^b F\,dx$. In this case, $W = \int_0^1 3x\,dx = (3x^2/2)\Big|_0^1 = 3/2$.

9\. Power is dE/dt , so the units are joules/second. As in Example 2, the work done in one second $= (1\text{ kg})(9.8\text{ m/sec}^2)(10\text{ m}) = 98$ joules. Therefore, the required power is 98 joules/1 second = 98 watts.

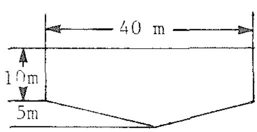
13\. Recall that $\Delta K = \int_a^b F\,dx$. In this case, $\Delta K = \int_0^1(-3x)dx = (-3x^2/2)\Big|_0^1 = -3/2$, so kinetic energy decreases by 1.5 joules.

17. If the depth of the water is between 0 and 2 , then the volume, with a thickness dx , is (15)(30) dx . If the water is below the 2m level we use similar triangles to find that the volume is (15)[15 + 5(5 - x)] dx Since a cubic meter of water has mass 10^3 kilograms, we multiply the volume by 1000 to find the total mass. Each layer of water with thickness dx is lifted x meters against gravity with $g = 9.8$ m/sec^2 , so $W = 1000g[\int_0^2 450\, x\, dx + \int_2^5 (600 - 75x)x\, dx] = 9800[225x^2|_0^2 + (300x^2 - 25x^3)|_2^5] = 9800(900 + 6300 - 2925) = 9800(4275) = 41{,}895{,}000$ joules.

21. Since -3 newtons is needed to stretch the spring 5 cm, +3 newtons is needed to compress the spring 5 cm. Work is the integral of force, so $W = F(\Delta x) = (3)(0.05) = 0.15$ joule since the force is constant.

SECTION QUIZ

1. While doing pull-ups at your local athletic club, you realize that you must lift your 70 kg body off the ground by 0.5 meter. How much work is done by your arm muscles?

2. 

Algae needs to be removed from a drydock which is 300 m long. The cross-sectional area is shown at the left. How much energy is needed to pump the entire dock dry?

3. Late last night, a vampire was seen breaking into the blood bank. He had sucked the blood out of 50 cylindrical test tubes. Each test tube was 10 cm long and had a diameter of 1 cm. If blood has a density of 1.03 g/cm^3 , how much work was done by the vampire while sucking up his dinner?

ANSWERS TO PREREQUISITE QUIZ

1. (a) Net distance travelled

   (b) Volume of water flowing through over time

   (c) Amount of candle which melted

ANSWERS TO SECTION QUIZ

1. 343 joules

2. 9.3×10^9 joules

3. 0.2 joules

9.R Review Exercises for Chapter 9

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) Use the disk method and the half-angle formula, so $V = \pi\int_0^\pi \sin^2 x\,dx = \pi\int_0^\pi [(1 - \cos 2x)/2]\,dx = \pi(x/2 - \sin 2x/4)\Big|_0^\pi = \pi(\pi/2) = \pi^2/2$.

   (b) Use the shell method and integrate by parts, so $V = 2\pi\int_0^\pi x \sin x\,dx = 2\pi(-x \cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx) = 2\pi(-x \cos x + \sin x)\Big|_0^\pi = 2\pi(\pi) = 2\pi^2$.

5. Use the result of Example 5, Section 9.2 with $R = 1$ and $r = 1/3$. The volume of the original ball is $(4/3)\pi$. The volume of the removed material is $(4/3)\pi[1 - (1 - 1/9)^{3/2}]$. Thus, the volume of the resulting solid is $(4/3)\pi(8/9)^{3/2} = 64\sqrt{2}\pi/81$.

9. The average value is $\overline{f(t)}_{[a,b]} = [1/(b - a)]\int_a^b f(t)dt$. In this case, it is $[1/(1 - 0)]\int_0^1 (1 + t^3)dt = 1(t + t^4/4)\Big|_0^1 = 5/4$.

13. The average value of $g(x)$ is $\int_0^2 g(x)dx/(2 - 0) = \int_0^2 3f(x)dx/2 = (3/2)\int_0^2 f(x)dx = (3/2)(4) = 6$.

17. The average of f on $[a,b]$ is $\mu = [1/(b - a)]\int_a^b f(t)dt = [1/(1 - 0)]\int_0^1 x^2 dx = (x^3/3)\Big|_0^1 = 1/3$. The variance is $[1/(b - a)] \times \int_a^b [f(t) - \mu]dt = [1/(1 - 0)]\int_0^1 [x^2 - 1/3]^2 dx = \int_0^1 (x^4 - 2x^2/3 + 1/9)dx = (x^5/5 - 2x^3/9 + x/9)\Big|_0^1 = 1/5 - 2/9 + 1/9 = 4/45$. The standard deviation is the square root of the variance, which is $\sqrt{4/45} = 2/3\sqrt{5} = 2\sqrt{5}/15$.

21. The average of $f(x)$ on $[a,b]$ is $\mu = [1/(b - a)]\int_a^b f(x)dx$, so $\mu = [1/(2 - 0)]\int_0^2 f(x)dx = (1/2)[\int_0^1 dx + \int_1^2 2\,dx] = (1/2)[x\Big|_0^1 + 2x\Big|_1^2] = 3/2$. The variance is $[1/(b - a)]\int_a^b [f(x) - \mu]^2 dx$, so variance $= (1/2)\int_0^2 [f(x) - \mu]^2 dx = (1/2)[\int_0^1 (1 - 3/2)^2 dx + \int_1^2 (2 - 3/2)^2 dx] = (1/2)((x/4)\Big|_0^1 + (x/4)\Big|_1^2) = 1/4$. The square root of the variance is the standard deviation, which is $1/2$.

25. Use the formulas $\bar{x} = \int_a^b xf(x)dx/\int_a^b f(x)dx$ and $\bar{y} = (1/2)\int_a^b [f(x)]^2 dx/\int_a^b f(x)dx$. Thus, the center of mass is $\bar{x} = \int_0^2 x(x^4)dx/\int_0^2 x^4 dx = \left(x^6/6\big|_0^2\right)/\left(x^5/5\big|_0^2\right) = (32/3)/(32/5) = 5/3$ and $\bar{y} = \int_0^2 (x^4)^2 dx/2\int_0^2 x^4 dx = \left(x^9/9\big|_0^2\right)/2(32/5) = (512/9)/2(32/5) = 40/9$. Therefore, the center of mass is $(5/3, 40/9)$.

29. Use the formulas from Exercise 28, Section 9.4 with $g(x) = x^3$ and $f(x) = -x^2$. Therefore, $\bar{x} = \int_0^1 x(x^3 + x^2)dx/\int_0^1 (x^3 + x^2)dx = (x^5/5 + x^4/4)\big|_0^1/(x^4/4 + x^3/3)\big|_0^1 = (9/20)/(7/12) = 27/35$; $\bar{y} = (1/2)\int_0^1 (x^3 - x^2) \times (x^3 + x^2)dx/\int_0^1 (x^3 + x^2)dx = (1/2)\int_0^1 (x^6 - x^4)dx/(7/12) = (x^7/7 - x^5/5)\big|_0^1/(7/6) = (-2/35)/(7/6) = -12/245$. Thus, the center of mass is $(27/35, -12/245)$.

33. Use the equation $\Delta K = \int_a^b F\,dx = \int_2^4 30\sin(\pi x/4)dx = 30(4/\pi) \times [-\cos(\pi x/4)]\big|_2^4 = (-120/\pi)(-1 - 0) = (120/\pi)$ joules.

37. (a) Consider a rectangular slab with width dx. Then, at a depth h, the force on the slab is $\rho gh\,dh\,dx$ because the pressure acts on an area $dh\cdot dx$. Integrate from 0 to $f(x)$ to find the total force on the rectangular slab. Then integrate the pressure of each slab from a to b to find the total force on the dam. $F = \int_a^b [\int_0^{f(x)} \rho gh\,dh]dx = \int_a^b [\rho g(h^2/2)\big|_0^{f(x)}]dx = (1/2)\int_a^b \rho g[f(x)]^2 dx$.

(b) If we rotate $f(x)$ around the x-axis, then the volume V obtained is $\pi\int_a^b [f(x)]^2 dx$; therefore, $F = V\rho g/2\pi$.

(c) We use the formula in (b). Rotation of the dam face results in a volume consisting of two cones and a circular cylinder. The cones have radii of 100 m and heights of 125 m. The cylinder has a radius of 100 m and a height of 50 m. Therefore, $V = 2(\pi r^2 h/3) + \pi r^2 h = 2\pi(100)^2(125)/3 + \pi(100)^2(50) = 4{,}000{,}000\pi/3$, which means $F = (4{,}000{,}000\pi/3)(1000)(9.8)/2\pi = (19.6/3)(10^9) \approx (6.53 \times 10^9)$ newtons $= (2/3)(\rho g \times 10^6)$ newtons.

41. Suppose that $f(a) < 0$ and $f(b) > 0$. Let $I = \int_a^b f(x)dx$. If $I = 0$, the mean value theorem for integrals tells us that there is an x_0 in $[a,b]$ with $(b - a)f(x_0) = I = 0$, so that $f(x_0) = 0$, as required. If $I > 0$, then let $a_1 = a - I/f(a)$, and extend f to the interval $[a_1,b]$ by setting $f(x) = f(a)$ for x in $[a_1,b]$. This is still continuous, and now one may compute that $\int_{a_1}^b f(x)dx = 0$. By the mean value theorem for integrals, there is an x_0 in $[a_1,b]$ with $f(x_0) = 0$. Since $f(x) = f(a) < 0$ on $[a_1,a]$, this x_0 must be in $[a,b]$. The case $I < 0$ is handled in a similar way; extend the function past b instead. (See the hint on p.A.53 for another method).

TEST FOR CHAPTER 9

1. True or false:

   (a) The volume of the solid of revolution obtained by revolving the graph of $f(x) \geqslant 0$ around the x-axis is $V = 2\pi\int_a^b xf(x)dx$.

   (b) If $f(x) < 0$, it can be revolved around the x-axis to form a solid of revolution.

   (c) The center of mass of a plane region with uniform density always lies on the region.

   (d) The average of $f(x)$ on $[a,b]$ is attained at least once by f in (a,b).

   (e) The average of $f(x)$ on $[a,b]$ never occurs at a or b.

2. (a) Find the volume which results from revolving $y = x^2 - 1$, $0 \leqslant x \leqslant 2$, around the x-axis.

   (b) Does it matter that $f(x) < 0$ on part of the interval? Why?

3. A force $F(x) = 1/x^2$ is applied to a particle on the interval $1 \leqslant x \leqslant 10$. Find the work done by the force in moving the particle from $x = 1$ to $x = 10$.

4. Let $f(x) = \sqrt{x}$ on $[0,1]$. Find the volume of the solid of revolution generated by revolving the graph of $f(x)$ around the following lines:

   (a) $x = 3$

   (b) $y = -1$

5. Find the center of mass of the region between $g(x) = \sqrt{x + 4}$ and the x-axis on the interval $0 \leqslant x \leqslant 4$.

6. Compute the average of $x \cos(x^2 + \pi)$ on $[0,\sqrt{\pi}]$.

7. What is the center of mass of the region bounded by $y = x^3$ and the x-axis on $[-1,2]$?

8. (a) Show that the mean value theorem for integrals applies to $f(x) = x^2 - 1$ on $[-1,0]$.

   (b) What is the average of $f(x)$ on $[-1,0]$?

9. Find the volume of the solid of revolution generated by revolving the region between the graphs of $2x$ and x^3 on $[0,1]$ around each axis.

10. One hot day in India, an elephant was down at the river cooling itself. The elephant filled its trunk with water to squirt on its back. Unfortunately, the last trunkful of water had a fly in it, which caused a sneeze attack. Suppose the elephant's cylindrical trunk has a diameter of 8 cm and a length of 1.5 m. How much work is minimally required to force the water out of its trunk if it is pointing straight up into the sky?

ANSWERS TO CHAPTER TEST

1. (a) False; $V = \pi\int_a^b [f(x)]^2 dx$.

   (b) True

   (c) False; the center of mass of a "ring" lies at the center, in the hole.

   (d) False; f should be continuous.

   (e) False; consider $f(x) = 1$ on $[-2,2]$.

2. (a) $(46/15)\pi$

   (b) No, because $[-f(x)]^2 = [f(x)]^2$. Thus, the formula is the same for any f .

3. $-9/10$

4. (a) $16\pi/5$

   (b) $17\pi/6$

5. $((40 + 24\sqrt{2})/35, (9 + 18\sqrt{2})/28)$

6. 0

7. $(511/85, 2047/238)$

8. (a) f is integrable and the average is attained at $x_0 = -\sqrt{1/3}$, which is in $[-1,0]$.

   (b) $-2/3$

9. $25\pi/21$ around x-axis; $14\pi/15$ around y-axis

10. 55 joules

COMPREHENSIVE TEST FOR CHAPTERS 7-9 (Time limit: 3 hours)

1. True or false. If false, explain why.

(a) The differential equation $dy/dx = y/(x + 1)(y + xy)$ is separable.

(b) $\int [dt/(1 + t^2)] = \tan^{-1} t + C = -\cot^{-1} t + C$ is valid for all t.

(c) $\int [dt/(1 - t^2)] = \tanh^{-1} t + C = \coth^{-1} t + C$ is valid for all $t \neq 1$.

(d) Revolving $y = x^2$ around the x-axis yields the same volume as revolving $y = x^2 + 2x + 1$ around the line $y = 1$ on any interval $[0,a]$, $a > 0$.

(e) Force is the derivative of work with respect to time.

(f) One solution of $dy/dx = -3x$ is $y = -3x^2/2$.

(g) The mean value theorem for integrals requires differentiability before it can apply to a function.

(h) $\int_0^5 f(t)g(t)dt = (\int_0^5 f(t)dt)g(0) + f(0)\int_0^5 g(t)dt$.

(i) The derivative of $\cosh x$ exists for all real x.

(j) $\int \tan^3\theta\, d\theta = \tan^4\theta/4 + C$

2. Multiple choice.

(a) Which is the solution of $y'' + 4y = 0$, where $y'(0) = y(0) = 1$?

(i) $\cosh 2t + (1/2)\sinh 2t$

(ii) $\cos 2t + (1/4)\sin 2t$

(iii) $\sin 2t + (1/2)\cos 2t$

(iv) $(1/2)\sin 2t + \cos 2t$

(b) The average value of $x \exp(x^2)$ on $[0,1]$ is:

(i) $e/2$

(ii) $(e - 1)/2$

(iii) $(x^2 + \ln x)\big|_0^1$

(iv) $(e + 1)/2$

2. (c) What is the most efficient way to evaluate $\int \sec^3\theta \tan\theta \, d\theta$?

(i) Integrate by parts with $u = \sec^3\theta$, $dv = \tan\theta \, d\theta$.

(ii) Integrate by parts with $u = \tan\theta$, $dv = \sec^3\theta \, d\theta$.

(iii) Substitute $u = \sec\theta$.

(iv) Substitute $u = \tan\theta$.

(d) Suppose f is symmetric with respect to the origin and $b > a$. Furthermore, suppose $f \geqslant 0$ on $[-b,-a]$, then $\bar{f}_{[-a,b]}$ is:

(i) Positive

(ii) Negative

(iii) Zero

(iv) There is not enough information.

(e) A rabbit population grows exponentially. It takes five years for the population to increase from 1000 to 2000 . How long does it take for the population to grow from 20,000 to 30,000 ?

(i) About 50 years

(ii) 2-4 years

(iii) 4-6 years

(iv) There is not enough information.

3. Perform the following integrations:

(a) $\int \cosh^{-1}x \, dx$

(b) $\int y^2 (\ln y)^2 dy$

(c) $\int_0^1 \left[3t^3/\sqrt{t^2 + 4}\right] dt$

(d) $\int [(x + 1)/(x^2 + 2x + 5)] dx$

(e) $\int_3^4 (\ln t/t) dt$

4. Short answers.

(a) Solve the differential equation $y' = -y/4$, assuming $y(0) = 3$.

(b) Differentiate $\tanh(x^2 + 3)$.

(c) Express $\sinh t$ as a sum or difference of exponentials.

(d) Sketch the graph of $y = \cosh x$.

(e) If $F' = f$, what is $\int f(3t)dt$?

5. Find the center of mass of the region between $y = x^2$ and $y = x^3$ on $[0,1]$.

6. The temperature of a swimming pool changes according to the formula $dT/dt + T = 80^\circ$. Initially, $T(0) = 50^\circ$. How long does it take for the temperature to change from 50° to 75° ?

7. Solve the following differential equations with the given conditions

(a) $dy/dx = xy$, $y(0) = 1$

(b) $dy/dx = \sin x + y$, $y(0) = 1$

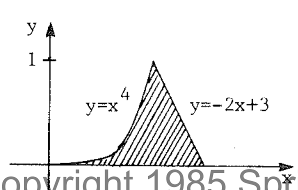
(c) $d^2y/dx^2 + 7x = 0$, $y(0) = 1$, $y'(0) = 1$

8. Let R be the region between x and x^3 on $[-1,2]$. What is the volume of the solid of revolution obtained by revolving R around each of the following?

(a) The x-axis.

(b) The y-axis.

9. Suppose the price of land in Manhattan increases at an instantaneous rate of 15% per year. How long does it take for the value of the land to triple?

10. 

A volcanic mountain top has a shape which can be described as the solid of revolution formed by revolving the shaded region around the y-axis. Each unit represents 1 kilometer and the mountain top has a uniform

10. density of 5000 kg/m^3. A violent eruption disintegrates the mountain top sending the particles to form a cloud cover at $y = 3$. How much energy was expended by the volcano in forming the cloud cover?

ANSWERS TO COMPREHENSIVE TEST

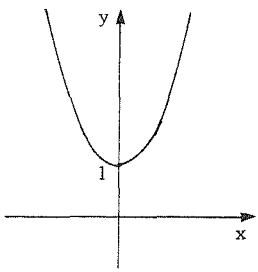
1. (a) True
   (b) True
   (c) False; it is $\tanh^{-1} + C$ if $|t| > 1$, $\coth^{-1} t + C$ if $|t| < 1$.
   (d) True
   (e) False; it is the derivative with respect to position.
   (f) True
   (g) False; it only requires integrability.
   (h) False; the right-hand side should be obtained from integration by parts.
   (i) True
   (j) False; differentiating the right-hand side yields $\tan^3\theta \sec^2\theta$.
2. (a) iv
   (b) ii
   (c) iii
   (d) i
   (e) ii
3. (a) $x \cosh^{-1}x - \sqrt{x^2 - 1} + C$
   (b) $(y^3/3)[(\ln y)^2 - 2(\ln y)/3 + 2/9] + C$
   (c) $16 - 7\sqrt{5}$
   (d) $(1/2)\ln|x^2 + 2x + 5| + C$
   (e) $[(\ln 4)^2 - (\ln 3)^2]/2$

4. (a) $y = 3 \exp(-y/4)$

(b) $2x \operatorname{sech}^2(x^2 + 3)$

(c) $(e^t - e^{-t})/2$

(d)



(e) $[F(3t)]/3 + C$

5. $(3/5, 12/35)$

6. ln 6

7. (a) $y = \exp(x^2/2)$

(b) $y = (3e^x - \sin x - \cos x)/2$

(c) $\cos\sqrt{7}x + (1/\sqrt{7})\sin\sqrt{7}x$

8. (a) $340\pi/21$

(b) $124\pi/15$

9. ln 3/1.15 year

10. 3.3×10^{14} joules