8. Completeness Theorems and Nonlinear Wave Equations.

As we have mentioned in earlier lectures, the question of the completeness of a flow is a fundamental one; i.e. can solutions be indefinitely extended in time?

In order to deal with this question, one usually proceeds as follows. One first establishes a local existence theorem and then one uses some kind of estimates (so called a priori estimates) to show that this solution does not move to ∞ in a finite time, and hence can be extended to exist for all time. (See the results at the end of lecture 6.)

Below we shall illustrate this general procedure with a couple of examples. We begin by describing a general technique based on energy estimates.

Liapunov Methods.

The concept of Liapunov stability (see lecture 6) can be used effectively as a completeness theorem. Below we shall apply this theorem to nonlinear wave equations.

Theorem. *Let E be a Banach space and F_t a local flow on E with fixed point at 0. Suppose that for any bounded set $B \subset E$ there is an $\epsilon > 0$ such that integral curves beginning in B exist for a time interval $\geq \epsilon$.*

Let $H : E \to R$ be a smooth function invariant under the flow.

(a) *If $H(u) \geq \text{const.}\|u\|^2$, then the flow is complete.*

(b) _If_ $H(0) = 0$, $DH(0) = 0$ _and_ $D^2H(0)$ _is positive or negative definite, then there is a neighborhood_ U _of_ 0 _such that any integral curve starting in_ 0 _is defined for all_ t ; _moreover_, 0 _is stable_.

Proof. (a) Let $u \in E$. Since H is conserved we have the a priori estimate $\|u\|^2 \leq$ constant, so u remains in a Bounded set B . But because of the assumption on the flow, the integral curve beginning at u can be indefinitely extended.

(b) From the assumptions, there are constants α , β such that

$$\alpha\|u\|^2 \leq |D^2H(0)\cdot(u, u)| \leq \beta\|u\|^2 .$$

Hence, by Taylor's theorem, in a small neighborhood U_0 of 0 , we have

$$\gamma\|u\|^2 \leq |H(u)| \leq \delta\|u\|^2 .$$

Because H is conserved, this shows that there are neighborhoods U , V of 0 such that if $u \in U$, it remains in V as long as it is defined. Hence we have completeness as in (a). Since V can be arbitrarily small, we also have stability. □

Nonlinear Wave Equations.

The following equation has been of considerable interest in quantum field theory:

$$\frac{\partial^2\varphi}{\partial t^2} = \nabla^2\varphi - m^2\varphi - \alpha\varphi^p \tag{1}$$

on R^n , where φ is a scalar function, $m > 0$, $\alpha \in R$ and $p \geq 2$ is an integer. The constant α is called the coupling constant and the non-linear term $\alpha\varphi^p$ represents some sort of self interaction of the field φ.

This equation in the same sense as the linear wave equation (see lecture 2) is Hamiltonian. The energy function is

$$H(\varphi, \dot{\varphi}) = \tfrac{1}{2}\langle\dot{\varphi}, \dot{\varphi}\rangle + \tfrac{1}{2}\langle\nabla\varphi, \nabla\varphi\rangle + \frac{m^2}{2}\langle\varphi, \varphi\rangle + \frac{\alpha}{p+1}\int \varphi^{p+1}dx .$$

We chose the phase space to be $H^1 \times L_2$ as for the linear wave equation.

We want to apply the previous theorem to discuss global solutions. In order to do this we need a local existence theory and we need to know H is smooth. For the latter, the key thing is whether or not φ^{p+1} is integrable. To answer this one uses a generalization of the Sobolev inequalities. We shall discuss these points in turn, but let us first state the results corresponding to cases (a), (b) of the previous theorem.

Theorem. (a) Suppose $n = 2$, $\alpha > 0$ and p is odd, or else $n = 3$ and $p = 3$. Then the flow of (1) is complete.

(b) Suppose $n = 2$ with p, α arbitrary or $n = 3$, $p = 2, 3, 4$, α arbitrary. Then there is an $\varepsilon > 0$ such that if $\varphi, \dot{\varphi}$ is in the $H^1 \times L_2$ ε-ball about 0 then the corresponding solutions exist for all $t \in R$ (actually if the initial data is C^∞, so is the solution). Furthermore the 0 solution is Liapunov stable in the $H^1 \times L_2$ topology.

Notice that the conditions p odd, $\alpha > 0$ is precisely what makes the last term of $H \geq 0$, so $H(\varphi, \dot{\varphi}) \geq \text{const}(\|\varphi\|^2_{H'} + \|\dot{\varphi}\|^2_{L_2})$ which is (a) of the previous theorem.

The other restrictions on n, p come from the Sobolev theorem in the following form. (See Nirenberg [1], Cantor [1]).

<u>Sobolev-Nirenberg-Gagliardo inequality</u>: <u>Suppose</u>

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a)\frac{1}{q}$$

<u>where</u> $\frac{j}{m} \leq a \leq 1$ (<u>if</u> $m - j - \frac{n}{r}$ <u>is an integer</u> ≥ 1, <u>only</u> $a < 1$ <u>is allowed). Then for</u> $f : R^n \to R^k$,

$$\|D^j f\|_{L^p} \leq (\text{const})\|D^m f\|^a_{L^r} \cdot \|f\|^{1-a}_{L^q}$$

<u>for a constant independent of</u> f.

For example suppose $n = 3$ and $f \in H^1$. Then taking $m = 1$, $r = 2$, $q = 2$, $j = 0$, $a = 1$ we find that $f \in L^6$ and

$$\left(\int f^6 dx\right)^{1/6} \leq (\text{const})\left(\int (\nabla f)^2\right)^{1/2}.$$

Such results can be used to prove smoothness of H above and smoothness results in the following:

<u>Local Existence Theory</u>.

<u>Theorem</u>. <u>Let</u> E <u>be a Banach space</u>, $A : D \subset E \to E$ <u>linear, the generator of a semi-group</u> U_t <u>and let</u> $J : E \to E$ <u>be smooth with</u> DJ

bounded on bounded sets. Then

$$\frac{du}{dt} = Au + J(u)$$

defines a unique local flow whose local time of existence is uniformly > 0 on bounded sets. (The evolution operator, F_t is C^∞ for fixed t.)

This result is due to Segal [1] who, based on earlier work of Jorgens, pointed out how it can be used to prove the results (a) on the wave equation (the result (b) is due to, amongst others, Chadam [1], Marsden [10]).

The proof of this result is remarkably simple. Namely, we convert the differential equation to the following integral equation:

$$u(t) = U_t u_0 + \int_0^t U_{t-s} J(u(s))ds \ . \tag{2}$$

The key thing is that the unbounded operator A now disappears and only the bounded operator U_t and the smooth operator J are involved. One can now use the usual Picard method to solve (2). Also one verifies that the solution lies in D if u_0 does and that the solution satisfies the equation (for the latter, J should be C^1 and not merely Lipschitz).

The point is that using the Sobolev-Nirenberg-Gagliardo inequalities one can verify that J has the requisite smoothness: take

$$u = \binom{\varphi}{\dot\varphi} \in H^1 \times L_2$$

$$Au = \begin{pmatrix} -m^2 I & I \\ \Delta & 0 \end{pmatrix}$$

$$Ju = \begin{pmatrix} 0 \\ -\alpha\varphi^p \end{pmatrix}$$

(so one has to check $\varphi \mapsto \varphi^p$ of H^1 to L_2 is smooth). Then the global existence claims follow by the Liapunov method.

We hasten to add that the method depends crucially on the positivity of the linearized energy norm. For other systems of interest, such as the coupled Maxwell-Dirac equations these ideas can give local solutions but they do not help determine if one has global solutions. That particular problem remains largely open. (See Gross [1].)

Quantum Mechanical Completeness Theorems.

Recall that Stone's theorem asserts that every self adjoint operator H on a Hilbert space $\mathcal{H}$ determines a one parameter unitary group (or flow) $U_t = e^{itH}$, defined for all $t \in R$. "Completeness" therefore amounts to the question of verifying self adjointness. Actually this is not such a simple question and is an active area of current research. (See, e.g. Simon [1].)

Let us recall a couple of definitions. Let $\mathcal{H}$ be a Hilbert space and $H : D \subset \mathcal{H} \to \mathcal{H}$ a linear operator, with D dense.

The adjoint $H^* : D^* \subset \mathcal{H} \to \mathcal{H}$ is defined as follows:

$$D^* = \{x \in \mathcal{H} \mid \exists z \in \mathcal{H} \text{ such that } \langle z,y\rangle = \langle x,Hy\rangle \text{ for all } y \in D\}$$

and $H^*x = z$.

An operator is <u>symmetric</u> if $\langle Hx, y\rangle = \langle x, Hy\rangle$ for all $x, y \in D$. Equivalently, $H^* \supset H$; i.e. $D^* \supset D$ and $H^* = H$ on D.

An operator is <u>self adjoint</u> if $H^* = H$.

Often self adjointness is not so easy to check because it depends crucially on the correct choice of D. For example Δ is self adjoint on $H^2(R^n)$, but not on C_0^∞ ($\mathcal{H} = L_2(R^n)$.)*

One is led to introduce another notion. Recall that the <u>closure</u> $\bar{H}$ of an operator H is that operator whose graph is the closure of the graph of H. (This operator $\bar{H}$ always is well defined for symmetric operators.)

A symmetric operator H is called <u>essentially self adjoint</u> if its closure $\bar{H}$ is self adjoint.

It can be shown that this is equivalent to saying that H has <u>at most one self adjoint extension</u>.

For example, Δ with domain C_0^∞ is essentially self adjoint and its closure is Δ with domain H^2.

Since there is a unique way of recovering a self adjoint operator from an essentially self adjoint one, there is no loss in trying to verify the condition of essential self adjointness. This is what is done in practice.

If an operator is not essentially self adjoint this means some

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*$C_0^\infty = C^\infty$ functions with compact support.

additional information e.g. boundary conditions -- must be specified in order to uniquely determine the dynamics.

Consider the Hamiltonian operator for the Hydrogen atom, $H = -\Delta + \frac{1}{r}$ on R^3. Despite the fact that the solution of the Schrodinger equation has been known explicitly for a half century, only in 1950 was this operator shown to be essentially self adjoint on C_0^∞ (the domain of the closure turns out to be H^2). This was done by T. Kato (see Kato [6] for details and references).

More generally, on R^3, $-\Delta + V$ is essentially self adjoint if $V \in L_2 + L_\infty$. The L_∞ part is trivial, being a bounded operator. To handle the L_2 part (the part of $1/r$ near the origin) one uses a Sobolev estimate in the form: let $V \in L_2$. Then for all $\varepsilon > 0$ there is an M_ε such that for all $f \in H^2$,

$$\|Vf\|_{L_2} < M_\varepsilon \|f\|_{L_2} + \varepsilon \|\Delta f\|_{L_2} .$$

One can then use:

<u>Kato's Criterion</u>. <u>Let</u> A <u>be (essentially) self adjoint on</u> $\mathcal{H}$ <u>with domain</u> D_A. <u>Let</u> B <u>be symmetric</u>, $D_B \supset D_A$ <u>and assume for some</u> $0 < \lambda < 1$

$$\|Bx\| \leq C\|x\| + \lambda\|Ax\|$$

<u>for all</u> $x \in D_A$. <u>Then</u> $A + B$ <u>is (essentially) self adjoint on</u> D_A.

This result is a rather elementary result in operator theory.

We won't go into the details here.

The above method is the basic one by which one handles local singularities such as occur in the Hydrogen atom. On the other hand there can be problems at ∞ such as occur when an atom is placed in an external field. This situation is covered by a theorem of Ikebe-Kato [1]:

Theorem. Let $V : R^3 \to R$ be such that V is smooth and $V(x) \geq V_0(\|x\|)$ where $V_0(r)$ is monotone decreasing and for $H > V_0$,

$$\int^{\infty} \frac{dr}{\sqrt{H - V(r)}} = + \infty .$$

Then $-\Delta + V$ is essentially self adjoint on C_0^∞ (the C^∞ functions with compact support).

Note. If $V_0(r) = -r^\alpha$, $\alpha \leq 2$ then we have the validity of the assumptions.

The result is too intricate to go into here (an exposition of the proof, generalized to manifolds will appear in Chernoff-Marsden [1]).

Ikebe-Kato then go on to combine this result with the previous type of result. The final result covers most (non-relativistic) cases of interest.

A Classical Analogue of the Ikebe-Kato Theorem.

There is a theorem in classical mechanics which yields completeness of a Hamiltonian system under the same conditions as in the Ikebe-Kato theorem. (See Weinstein-Marsden [1].) The argument works well on manifolds just as easily as on R^n.

Let us begin by considering the one dimensional case.

Let R^+ be the nonnegative reals and $V_0 : R^+ \to R$ a nonincreasing C^1 function. Consider the Hamiltonian system with the usual kinetic energy and potential V_0 ; i.e. if $c(t)$ is a solution curve we have

$$c''(t) = -\frac{dV_0}{dx}(c(t)) .$$

By monotonicity of V_0 , if $c'(0) \geq 0$ then $c'(t) \geq 0$ for all $t \geq 0$. Thus if $H = [c'(t)/2]^2 + V_0(c(t))$ is the constant total energy,

$$c'(t) = 2(H - V_0(c(t)))^{\frac{1}{2}} .$$

<u>Definition</u>. The potential V_0 is <u>positively complete</u> iff

$$\int_{x_1}^{x} \frac{dx}{(2(H - V_0(x)))^{1/2}} \to \infty \quad \text{as} \quad x \to \infty$$

for all $x_1 \geq 0$ and H such that $V_0(x_1) < H$.

It is easy to see that if this holds for some x_1 , H , such that $V_0(x_1) < H$ then it holds for all such x_1 , H (use the fact that improper integrals with asymptotic integrands are simultaneously convergent or divergent).

Since the above integral is just the time required for $c(t)$ to move from x_1 to x we see that V_0 is positively complete iff all integral curves $c(t)$ with $c(0) \geq 0$, $c'(0) \geq 0$ are defined for all $t \geq 0$. (The case when $c'(0) = 0$ is easily disposed of.)

Below we will use the notation $\tilde{c}(x_0, H)(t)$ for the integral curve with $\tilde{c}(x_0, H)(0) = x_0$ and energy H.

Example. The function $-x^{\alpha}$ for $\alpha \geq 0$ is positively complete iff $\alpha \leq 2$. The same is true for

$$-x[\log(x + 1)]^{\alpha}, -x \log(x + 1)[\log(\log(x + 1) + 1]^{\alpha} \quad \text{etc.}$$

Consider now the general case.

Theorem.* Let M be a complete Riemannian manifold (actually M may be infinite-dimensional) and let V be a C^1 function on M. Suppose there is a point $p \in M$ and a positively complete V_0 on R^+ such that for all $m \in M_1$ [with $d(m, p)$ sufficiently large], $V(m) \geq V_0(d(m, p))$ where d is the Riemannian distance on M. Then the flow on TM of the Hamiltonian vector field with (the usual kinetic energy $K(v) = \langle v, v\rangle/2$ and) potential V is a complete flow (i.e. integral curves are defined for all $t \in R$).

Examples. If $V(m) \geq -(\text{Constant})d(m, p)^2$ for sufficiently large $d(m, p)$ the conditions hold. This is satisfied if $\|\text{grad } V(m)\| \leq (\text{Constant})d(m, p)$ (for sufficiently large $d(m, p)$).

Proof of Theorem. Let $c : [0, b[\to TM$ be an integral curve, $0 < b < \infty$.

*See W. Gordon [1], D. Ebin [2], and A. Weinstein and J. Marsden [1].

As usual, it will suffice to show that the curve $c_0(t)$, the projection of $c(t)$ on M, remains in a bounded set for all $t \in [0, b[$ (a similar argument holds for $t \in]-b, 0]$). (In infinite dimensions one uses an argument of Ebin [2].)

Let $n = c_0(0)$ and H the energy of $c(t)$. Let

$$f_1(t) = d(c_0(t), p) \quad \text{and} \quad f_2(t) = \tilde{c}(d(n, p), H)(t)$$

(notation as above). Now

$$\begin{aligned} f_1(t) &\leq d(p, n) + \int_0^t \|c(s)\| ds \\ &= d(p, n) + \int_0^t (2[H - V(c_0(s))])^{1/2} ds \\ &\leq d(p, n) + \int_0^t (2[H - V_0(f_1(s))])^{1/2} ds \, . \end{aligned}$$

Also we have

$$f_2(t) = d(p, n) + \int_0^t (2[H - V_0(f_2(s))])^{1/2} ds \, .$$

It follows from these and monotonicity of V_0 that

$$f_1(t) \leq f_2(t) \leq \tilde{c}(d(n, p), H)(b)$$

for all $t \in [0, b[$. This is an elementary comparison argument. (See the lemma below.) We conclude that $f_1(t) = d(c_0(t), p)$ remains bounded for $t \in [0, b[$ and so the result follows. $\square$

Remarks. (1) The completeness for $t \geq 0$ is preserved if a dissipative vector field Y is added to the Hamiltonian vector field (i.e. Y is vertical $[T\pi(Y) = 0]$ and $Y \cdot K \leq 0$ where K is the kinetic energy). This is easy to see.

(2) This proof also gives an estimate for the growth of $d(c_0(t), p)$ in terms of V_0 ; for example if $V_0 = -x^2$ then $d(c_0(t), p) - d(n, p)$ grows like e^t .

When is the Sum of two Complete Vector Fields Complete?

Unfortunately, not always. For example consider

$$X : R^2 \to R^2 , \quad Y : R^2 \to R^2 ,$$

$$X(x, y) = (y^2, 0) , \quad Y(x, y) = (0, x^2) .$$

Each of X , Y has a complete flow, but $X + Y$ does not.

Using the sort of argument in the previous theorem however, one can get a result.

Theorem. *Let H be a Hilbert space and let X and Y be locally Lipschitz vector fields which satisfy the following:*

(a) *X and Y are bounded and Lipschitz on bounded sets,*

(b) *there is a constant $\beta \geq 0$ such that $\langle Y(x), x\rangle \leq \beta\|x\|^2$ for all $x \in H$,*

(c) *there is a locally Lipschitz monotone increasing function $c(t) > 0$, $t \geq 0$ such that $\int^{\infty} \frac{dx}{c(x)} = +\infty$ and $\langle X(x_0), x_0\rangle \leq \|x_0\| c(\|x_0\|)$*

or, stronger, if $x(t)$ *is an integral curve of* X ,

$$\frac{d}{dt}\|x(t)\| \leq c(\|x(t)\|) .$$

Then X , Y *and* $X+Y$ *are positively complete (i.e. complete for* $t \geq 0$.)

Note. One may assume $\|X(x_0)\| \leq c(\|x_0\|)$ in (c) instead of $\frac{d}{dt}\|x(t)\| \leq c(\|x(t)\|)$.

Proof. We begin with an elementary comparison lemma:

Lemma. *Suppose* $r'(t) = c(r(t))$ *and* $r_0 \geq 0$. *Then* $r(t) \geq 0$ *is defined for all* $t \geq 0$. *Suppose* $f(t) \geq 0$, *is continuous and*

$$f(t) \leq r_0 + \int_0^t c(f(s))ds , \quad t \in [0, T[.$$

Then

$$f(t) \leq r(t) \quad \text{for} \quad t \in [0, T[.$$

This lemma is not hard to prove. See Hartman [1] for such results.

Proof of Theorem. Let $u(t)$ be an integral curve of $X+Y$. By assumption (a), it suffices to show $u(t)$ is bounded on finite t-intervals, say $t \in [0, T[$. Now using (b),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= \langle u(t), X(u(t)) + Y(u(t))\rangle \\ &\leq \beta\|u(t)\|^2 + \langle u(t), X(u(t))\rangle . \end{aligned}$$

By assumption (c) we have for an integral curve $x(t)$ of X, $\langle x(t), X(x(t))\rangle = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \|x(t)\|c(\|x(t)\|)$. Therefore $\langle x_0, X(x_0)\rangle \leq \|x_0\|c(\|x_0\|)$ for any $x_0 \in H$. Thus we get

$$\frac{d}{dt}\|u(t)\| \leq \beta\|u(t)\| + c(\|u(t)\|)$$

and hence

$$\frac{d}{dt}(e^{-\beta t}\|u(t)\|) \leq c(\|u(t)\|) .$$

By the lemma, $e^{-\beta t}\|u(t)\|$ is bounded, so $\|u(t)\|$ is bounded. $\square$