9. General Relativity as a Hamiltonian System*

In this lecture we discuss the Einstein field equations of general relativity from the point of view of Hamiltonian systems. In order to motivate the discussion, we digress to include some background and motivational material.

Background.

The basic tenet of special relativity, that the speed of light is constant independent of the movement of source or observer, is reflected in a simple mathematical structure on $R^4 = R^3 \times R$ (space $\times$ time), viz the Minkowski metric:

$$v \cdot w = v^1 w^1 + v^2 w^2 + v^3 w^3 - v^4 w^4$$

or, as a matrix:

$$\begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ 0 & & +1 & \\ & & & -1 \end{pmatrix}$$

(use units such that $c = 1$).

The physically meaningful concepts in special relativity are those invariant under the Lorentz group; i.e. the group of linear isometries of the Minkowski metric.

As Einstein showed in 1905, the above picture - forced by concrete experiments (namely the Michaelson-Morley experiment) - has consequences of a non intuitive nature such as length contractions, time dilatation etc. All this is described in most elementary texts, such as Taylor-Wheeler [1].

* This and the next lecture are based on Fisher-Marsden [1,2,5,6].

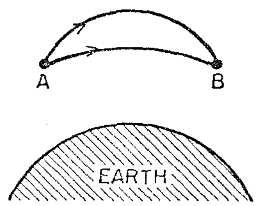
Later Einstein had the following brilliant insight: it is physically impossible to distinguish gravitational forces from acceleration forces. Indeed, by Galileo's fameous experiment we know that gravitational mass is the same as inertial mass ("principle of equivalence"). But acceleration is a purely geometrical (or kinematical) phenomena. Therefore it should be possible to geometrize space time in such a way that the gravitational fields are part of the geometry itself.

This is what Einstein did in his papers of 1915-17. (See Lanczos [2] for more historical facts).

It is fairly obvious how to generalize Minkowski space. We just use a Lorentz manifold V ; i.e. a 4-manifold with a symmetric bilinear form $< , >_\alpha$ on each tangent space $T_x V$, which has signature $(+++-)$.

We want the following to hold: particles in free fall (in the gravitational field) should follow geodesics on V .

Thus we are asserting that a body moving under the force of gravitation alone (e.g. a satellite circling the earth) should travel along a geodesic in an appropriate differentiable manifold. Such a manifold is certainly not flat 3-space, since the motion of a satellite would not then be geodesic. It is also easy to see that the manifold cannot be a curved three-dimensional Riemannian space: consider the case of two projectiles P_1, P_2 launched at the same time from A with trajectories as indicated in the Figure, both passing through B (this is easily arranged). It is clear that not both P_1 and P_2 can be geodesic with respect to any 3-space metric; since B can be moved arbitrarily close to A, there are no normal neighborhoods of A (in which there are unique minimizing geodesics).



On the other hand, we do get unique trajectories if we require the projectile to pass through B at a given time. So we are compelled to consider a manifold of dimension at least four. Finally, it is almost obvious that this 4-manifold cannot be Riemannian (metric tensor positive definite): Riemannian manifolds are isotropic in the sense that there are no intrinsically defined, distinguished directions. But space time is not isotropic; for example the geodesic connecting (you,now) with (Sirius, 1 second later) could not be traversed by a material particle required to travel at a speed below that of light. One has to distinguish between possible particle trajectories (timelike curves), impossible particle trajectories (spacelike curves), and possible photon trajectories (null curves).

All in all, one is led to consider a four-dimensional Lorentz manifold whose metric tensor g has signature (+++-). This is quite natural since it tells us that locally (in the tangent space, or in a normal neighborhood) the universe looks like Minkowski-space. As stated above, in this manifold, the "world line" or space time trajectory of a freely falling particle is a geodesic. Furthermore, it is assumed that this geodesic does not depend on the mass of the particle (an orange and a grapefruit behave the same way in the same gravitational field). This is another way of stating the principle of equivalence.

Less obvious than the above is the following. In Newtonian gravitational theory, the gravitational potential φ must satisfy $\nabla^2\varphi = 0$ exterior to matter. Since the metric is supposed to geometrize these potentials, what conditions should we impose on the metric?

Using this analogy and a good deal of intuition and guesswork, Einstein was led to the (empty space) field equations*

$$R_{\alpha\beta} = 0 \quad .$$

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* The curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ is defined on vector fields by

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \quad \text{where} \quad (\nabla_X Y)^\alpha = X^\beta \frac{\partial Y^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}X^\beta Y^\gamma$$

(summation on repeated indices) and

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\rho\beta}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right)$$

Here $g_{\alpha\beta}$ is the components of the metric in a chart, and $g^{\alpha\rho}$ is the inverse matrix. We write $x^\alpha = (x^j,t)$ for local coordinates. Also raising indices corresponds to identifying T_xV and T^*_xV via g, as usual (see lecture 2) e.g.: $X_\beta = g_{\beta\gamma}X^\gamma$ etc.

The Ricci tensor $Ric = R_{\alpha\beta}$ is $R_{\alpha\beta} = R_{\alpha\delta\beta}{}^{\delta}$ (a contraction), and the scalar curvature is $R = R_\alpha{}^\alpha$. One writes $A_{\alpha\beta|\gamma}$ for the covariant derivative of a tensor.

Following this, Einstein incorporated matter, or other external sources via

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} \doteq T_{\alpha\beta}$$

where $T_{\alpha\beta}$ is a given energy momentum tensor of the sources ($T_{\alpha\beta}$ is divergence free and $R_{\alpha\beta}$ is not, so Einstein modified $R_{\alpha\beta}$ to $G_{\alpha\beta}$ - fortunately $R_{\alpha\beta} = 0$ is equivalent to $G_{\alpha\beta} = 0$) .

Sometimes a "cosmological constant" λ is also included:

$$G_{\alpha\beta} - \lambda g_{\alpha\beta} = T_{\alpha\beta} \quad .$$

Later these equations were "justified" on theoretical grounds by Cartan-Weyl. They proved that any symmetric divergence free 2-tensor depending on g, Dg, D^2g had to have the form* $\mu G_{\alpha\beta} - \lambda g_{\alpha\beta}$.

There is another piece of motivation that lends insight into the nature of the field equations which is due to Pirani [1]. This proceeds as follows:

Let u be the tangent to a timelike geodesic $x(t)$ (timelike means $\langle u,u \rangle < 0$), so $\nabla_u u = 0$. Consider the Jacobi field (or deviation vector) η along $x(t)$; it satisfies Jacobi's equation:

$$\nabla_u \nabla_u \eta + R(\eta, u) u = 0$$

where R is the curvature tensor. Regarded as a map $\tilde{R}_u$ in η , $Ric(u,u)$ is its trace.

We are supposing $Ric = 0$. Let e_i, $i=1,2,3$ be vectors

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* It is usually assumed that the tensor depends linearly on D^2g, but see Rund-Lovelock [1].

orthogonal to u at a point p where $t = 0$. Then extend e_i to be Jacobi fields with initial condition $\nabla_u e_i = 0$ at p. Then $\nabla_u(u\cdot e) = \nabla_u u\cdot e_i + u\cdot\nabla_u e_i = u\cdot\nabla_u e_i$ so $\nabla_u\nabla_u(u\cdot e_i) = u\cdot\nabla_u\nabla_u e_i = -u\cdot R(e_i u)u = 0$ (we have $< R(v,u)U,u > = 0$ always by skew symmetry of $< R(u,v)w,z >$ in w,z). Hence $u\cdot e_i = 0$ for all time. Choose e_i to be eigenvectors of $\tilde{R}_u$ on the space orthogonal to u. We denote this restriction by $\tilde{R}_u^{\perp}$. Thus $\tilde{R}_u^{\perp} e_i = \lambda_i e_i$, and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ because $\tilde{R}_u$ is zero.

Now these vectors e_i span a three volume. Multiply e_i by ϵ (so that we can be sure exp maps the field ϵe_i onto geodesics close to the geodesic through p). A satisfactory approximation of the volume of the cube spanned by these vectors is $(\text{vol}) = \epsilon^3 e_1 \wedge e_2 \wedge e_3$.

We compute:

$$^4g_{\alpha\beta}\,dx^\alpha dx^\beta = -dt^2 + g_{ij}\,dx^i dx^j \quad .$$

$$\frac{d^2}{ds^2}(\text{vol})\Big|_p = \nabla_u\nabla_u(\text{vol})\Big|_p$$

$$= \epsilon^3\{(\nabla_u\nabla_u e_1) \wedge e_2 \wedge e_3 + e_1 \wedge (\nabla_u\nabla_u e_2) \wedge e_3 + e_1 \wedge e_2 \wedge \nabla_u\nabla_u e_3$$

$$+ \text{First derivative terms}\}\Big|_p \; .$$

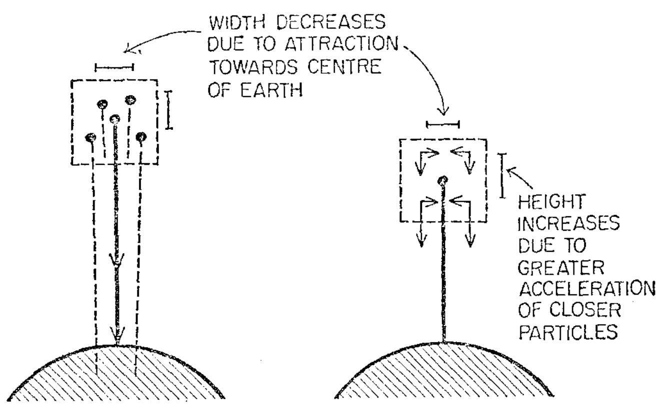
Since the e_i's are Jocobi fields and eigenvectors (at p) of $\tilde{R}_u^{\perp}$, and since $\nabla_u e_i\big|_p = 0$, we have

$$(1) \quad \frac{d^2}{dt^2}(\text{vol})\Big|_p = -(\lambda_1 + \lambda_2 + \lambda_3)(\text{vol})\Big|_p = 0 \quad \text{if} \quad \frac{d(\text{vol})}{dt}\Big|_p = 0$$

as the condition equivalent to Ric = 0 .

We can interpret this more physically as follows. Imagine ourselves in a freely falling elevator and watch a collection of freely falling particles. The particles are initially at rest with respect to each other, but due to motion towards the earth's center, they will pick up a relative motion (see the following Figure).

The condition (1) says that the 3-volume (up to second order) is remaining constant during the motion. This geometric property is directly verifiable in the case of the Newtonian gravitational field, so is a reasonable condidate for generalization. Thus we shall adopt Ric = 0 as the Einstein field equations in our Lorentz four manifold.*



The general program

Let V be a spacetime with M a three-dimensional spacelike section without boundary (a spacelike section is a submanifold such that for $0 \neq v \in T_xM, \langle v,v \rangle > 0$) . Assume for the moment that M is compact, so that there exists a neighborhood U of M in which the timelike geodesics (that is geodesics whose tangent vectors v have

* See also J. Wheeler [2].

$< v,v > < 0)$ orthogonal to M have no focal points. If we let t measure proper time on these geodesics, with $t = 0$ on $M = M_0$, then the function t is well-defined in U. The surfaces M_t given by $t =$ constant form a one parameter family of space sections, all diffeomorphic to M. Let g_t be the induced Riemannian metric on M_t. Via the aformentioned diffeomorphism, we can regard g_t as a curve in the space of positive definite metrics on M. The fact that V is Ricci flat implies that g_t satisfies certain differential equations. We want to work these out.

We also want to go the other way: given M, a positive definite metric g_0, and a symmetric tensor $k_0 = \dot{g}_0$ (the second fundamental form of M in V) we want to find the curve g_t describing the time evolution of the geometry of M, and then to past together the resulting 3-manifolds M_t to obtain a piece of spacetime.

The Space of Riemannian Metrics.

Fix a 3-manifold M which we shall take to be compact for simplicity. This is supposed to represent a model for the spatial universe. Let $S_2(M)$ be the set of all C^∞ symmetric two tensors on M and let $\mathfrak{M} \subset S_2(M)$ be the cone of positive definite ones; i.e. riemannian metrics.

The "time evolution" of the universe will be represented by a curve of metrics $g(t) \in \mathfrak{M}$. Of course there is no global time scale and physically this evolution takes place relative to a given system of "clocks" and a global "frame". This point will be briefly discussed below. We want $\mathfrak{M}$ to be the configuration space for a dynamical system. The first job therefore will be to construct a metric on $\mathfrak{M}$.

As was the case with hydrodynamics, one should properly work with metrics in the Sobolev class H^S . This space is denoted $\mathfrak{M}^S$. For simplicity most of the development will be done in $\mathfrak{M}$. Below we shall discuss briefly the existence questions, which make use of H^S . This also comes into play in lecture 10. Since $\mathfrak{M}$ is an open cone in $S_2(M)$ (using the H^S or C^∞ topology), for $g \in \mathfrak{M}$, we have

$$T_g\mathfrak{M} = S_2(M)$$

so

$$T\mathfrak{M} = \mathfrak{M} \times S_2(M)$$

Define a metric $\mathcal{G}$ on $\mathfrak{M}$ as follows:

$$\mathcal{G}_g(h,k) = \int_M \{h\cdot k - (\mathrm{tr}h)(\mathrm{tr}k)\}d\mu_g \tag{1}$$

where $g \in \mathfrak{M}$, $h,k \in T_g\mathfrak{M}$, $h\cdot k$ is the induced inner product, $h\cdot k = h^{ij}k_{ij}$, $\mathrm{tr}h = h^i{}_i$ is the trace and μ_g is the volume determined by g . Observe that $h\cdot k$ and $\mathrm{tr}h$ both depend upon g . Thus $\mathcal{G}$ is a non constant metric. $\mathcal{G}$ is called the deWitt metric. Although $\mathcal{G}$ is not positive definite, we can easily demonstrate that $\mathcal{G}$ is weakly non degenerate: suppose $\mathcal{G}_g(h,k) = 0$ for all $h \in S_2$.

Then $\mathcal{G}_g(k, k - \frac{1}{2}(\mathrm{tr}k)g) = 0$. But this equals $\int_M k\cdot k$ so $k = 0$.

The first thing we will want to do is work out the spray of $\mathcal{G}$.

Proposition. The spray S of $\mathcal{G}$ is given as follows: Its principal part is

$$S: \ T\mathfrak{M} \to S_2 \times S_2$$

$$S(g,k) = (k,\ k \times k - \tfrac{1}{2}(\mathrm{tr}k)k - \tfrac{1}{8}\{k\cdot k - (\mathrm{tr}k)^2\}g) \tag{2}$$

Note. The factor $\frac{1}{8}$ here depends on $\dim M = 3$.

Proof. We use the formulas for the spray in lecture 2. We first must compute the derivative of $\mathcal{G}_g$ with respect to g . We do this in three steps:

Lemma. The derivative of $g \mapsto \mu_g$ in direction $h \in S_2$ is given by $\frac{1}{2}(\mathrm{tr}h)\mu_g$.

Proof. Let $g(t) = g + th$. The derivative in question is $\frac{d}{dt}\mu_{g(t)}\Big|_{t=0}$. Using the local formula $\mu_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n$ we get the result from the formula $\frac{d}{dt}\det(g_{ij} + th_{ij})\Big|_{t=0} = \mathrm{tr}(h_{ij})\det(g_{ij})$.

The latter formula may be proven from the fact that the derivative det at the identity is trace, so $\frac{d}{dt}\det(g_{ij} + t\cdot h_{ij}) = \det(g_{ij})\frac{d}{dt}\det(1 + tg^{-1}h)$ $= \det(g_{ij})\ \mathrm{tr}(h_{ij})$. Note $g\colon T_xM \to T_x^*M$ so $g^{-1}h$ is a linear map from $T_x M$ to itself and $\mathrm{tr}h$ is the trace of this map. In coordinate language $g^{-1}h$ raises one index on h. □

Lemma. The derivative of $g \mapsto h \cdot k$ in direction h_1 is given by $-2\, h_1 \cdot (h \times k)$ where $h \times k = hg^{-1}k$ or $(h \times k)_{ij} = h_{i\ell} k^{\ell}{}_j$, in coordinates.

Proof. Now $h \cdot k = \mathrm{tr}(g^{-1}hg^{-1}k)$ and as usual, $\frac{d}{dt}\, g(t)^{-1} = -\, g^{-1}h_1 g^{-1}$, where $g(t) = g + th_1$. Thus we get for the derivative

$$-\,\mathrm{tr}(g^{-1}h_1\, g^{-1}hg^{-1}k) - \mathrm{tr}(g^{-1}hg^{-1}h_1 g^{-1}k)$$

and this gives the result. □

In a similar way, one proves

Lemma. The derivative of $g \mapsto \mathrm{tr}(h)$ in direction h_1 is given by $-h_1 \cdot h$.

Continuing with the proof the proposition, we have from lecture 2 that if we write $S(g,k) = (k, S_g(k))$, S_g should satisfy:

$$G_g(S_g(k),h) = \tfrac{1}{2}\, D_g\{G_g(k,k)\} \cdot h - D_g G_g(k,h) \cdot k \qquad (3)$$

From the lemmas we get

$$D_g G_g(k,h) \cdot h_1 = \int_M (-\,2h_1 \cdot (k \times h) + (\mathrm{tr}h)h_1 \cdot k + (\mathrm{tr}k)h_1 \cdot h)\, d\mu_g$$

$$+ \int_M [h \cdot k - (\mathrm{tr}h)(\mathrm{tr}k)] \cdot \tfrac{1}{2}\, \mathrm{tr}h_1\, d\mu_g \quad .$$

Thus the right hand side of (3) becomes

$$\int_M h\cdot(k \times k) - \tfrac{1}{2}h\cdot k(\mathrm{tr}k) - (\mathrm{tr}h)\; k\cdot k + \tfrac{1}{2}(\mathrm{tr}k)^2\mathrm{tr}h + \frac{1}{4}[k\cdot k - (\mathrm{tr}k)^2]\mathrm{tr}h\} d\mu_g \tag{4}$$

while the left side is $\int_M \{S_g(k)\cdot h - \mathrm{tr}(S_g k)\;\mathrm{tr}(h)\}\; d\mu_g$ which becomes, on substituting the stated expression for S_g, using $g\cdot h = \mathrm{tr}h$, $\mathrm{tr}g = 3$, and $\mathrm{tr}(k \times k) = k\cdot k$,

$$\int_M h\cdot(k \times k) - \tfrac{1}{2}(h\cdot k)\mathrm{tr}k - \frac{1}{8}\{k\cdot k\;\mathrm{tr}h - (\mathrm{tr}k)^2\;\mathrm{tr}h\}\; d\mu_g$$

$$- \int_M k\cdot k\;\mathrm{tr}h - \tfrac{1}{2}(\mathrm{tr}k)^2\;\mathrm{tr}h - \frac{1}{8}\{k\cdot k\cdot 3\;\mathrm{tr}h - (\mathrm{tr}k)^2 3\mathrm{tr}h\}\; d\mu_g$$

which equals (4) above. □

<u>The Gravitational Potential.</u>

We have established our metric $\mathcal{G}$ on $\mathfrak{M}$ and have determined its spray. We now proceed to consider a potential and will compute its gradient. The spray S of $\mathcal{G}$ is simply algebraic, whereas the gradient of the potential will involve non linear differential operators.

Define $V: \mathfrak{M} \to R$ by $V(g) = 2\int_M R(g)\; d\mu_g$ where $R(g)$ is the scalar curvature (remember g is a three dimensional metric) and as usual μ_g is the volume associated with g.

<u>Proposition.</u> <u>The gradient of</u> V <u>with respect to the metric</u> $\mathcal{G}$ <u>on</u> $\mathfrak{M}$ <u>is</u>

$$\mathrm{grad}\; V(g) = -2\;\mathrm{Ric}(g) + \tfrac{1}{2}\; R(g)g \in S_2(M) \quad .$$

Proof. Let $g(t) = g + th$. Then

$$dV(g)\cdot h = 2\frac{d}{dt}\int_M R(g(t))\, d\mu_{g(t)}\Big|_{t=0}$$

The derivative may be done in two parts. The $\mu_{g(t)}$ part is taken care of by the lemmas. For the scalar curvature we use:

Lemma. $\frac{d}{dt}R(g(t))\Big|_{t=0} = \Delta(\mathrm{tr}h) + \delta\delta h - \mathrm{Ric}(g)\cdot h$

where $\delta\delta h = h^{ij}{}_{|i|j}$ is the double covariant divergence.

This is a straightforward but somewhat lengthy computation which we shall omit. (See Lichnerowicz [2]).

Since we are taking M compact with no boundary, the two terms $\Delta(\mathrm{tr}h)$, $\delta\delta h$ drop out by Stokes theorem. Hence we get

$$dV(g)\cdot h = -2\int_M \mathrm{Ric}(g)\cdot h\, d\mu_g + \int_M R(g)\mathrm{tr}(h)\, d\mu_g \quad ,$$

It is now easy to verify that the formula in the proposition satisfies

$$\mathcal{G}_g(\mathrm{grad}\, V(g), h) = dV(g)\cdot h$$

if we remember that $\mathrm{tr}(\mathrm{Ric}(g)) = R(g)$ and $\mathrm{tr}g = 3$. □

The Energy Condition; Coordinate Invariance and Conservation Laws

If we consider the Lagrangian $L(g,k) = \frac{1}{2}\mathcal{G}_g(k,k) - V(g)$ on $\mathfrak{M}$, then we have computed above the corresponding spray to be $S_g(k) - \mathrm{grad}\, V(g)$. Thus an integral curve $g(t)$ satisfies

$$\frac{d^2g}{dt^2} = S_g\left(\frac{dg}{dt}\right) - \mathrm{grad}V(g) \quad .$$

These equations have an important property which is not shared by the usual non relativistic field theories. This is that not only is the total energy conserved, but it is pointwise conserved. Actually this law is intimately connected with another conservation law which we shall develop first.

Theorem. Let $\pi = ((\mathrm{tr}k)g - k) \otimes \mu_g$, the conjugate momentum. Then along an integral curve of L above, $\frac{\partial}{\partial t}\delta\pi = 0$. Here, $\delta\pi$ is defined by $\delta((\mathrm{tr}k)g - k) \otimes \mu_g$; $\delta h = h_{i\,j}^{\;\;li}$. In particular if $\delta\pi = 0$ at $t = 0$, then this condition is maintained. Furthermore, this law is the conservation law associated with the invariance of L under the (left) action of the group of diffeomorphisms $\mathcal{D}$ on $\mathfrak{M}$ by $\Phi_\eta(g) = \eta_* g$.

In otherwords, we get a free conservation law just because our theory is invariant under coordinate transformations. The actual form of V is irrelevant.

Proof. We are considering the action of $\mathcal{D}$ on $\mathfrak{M}$ as stated. See lecture 4 for the relevant properties of $\mathcal{D}$ which are used here. Consider X a vector field on M , so X is in the Lie algebra of $\mathcal{D}$. The one parameter subgroup corresponding to X is its flow $F_t \in \mathcal{D}$. Since

$$\frac{d}{dt} F_{t*}g\Big|_{t=0} = - L_X g \quad ,$$

we see that the corresponding infinitesimal generator on $\mathfrak{M}$ is

$g \mapsto -L_X g \in S_2(M)$. Hence by our conservation laws (lecture 6) ,

$$(g,k) \mapsto \mathcal{G}_g\,(k, -L_X g)$$

is a conserved quantity. At this point, we need the following:

Lemma. $\int_M L_X g \cdot k \, d\mu_g = -2\int_M X \cdot \delta k \, d\mu_g$.

Proof. It is easy to derive the following formula $L_X g = X^i{}_{|j} + X^j{}_{|i}$. From this it follows that $\delta(k \cdot X) = (\delta k) \cdot X + k \cdot \nabla X = (\delta k) \cdot X + \frac{1}{2} k \cdot L_X g$. Since, by Stokes theorem $\int_M \delta(k \cdot X)\, d\mu = 0$, we get the lemma . □

$$\begin{aligned} \text{Now} \quad \mathcal{G}_g(k, L_X g) &= \int_M \{k \cdot L_X g - (\mathrm{tr}k)(\mathrm{tr}L_X g)\} d\mu \\ &= \int_M (L_X g) \cdot (k - (\mathrm{tr}k)g)\, d\mu \\ &= \int_M L_X g \cdot \pi \\ &= -2 \int_M X \cdot \delta\pi \end{aligned}$$

Thus for any vector field X , $\int_M X \cdot \delta\pi$ is conserved. Hence $\delta\pi$ itself is conserved. □

This result could also be obtained from Noethers theorem. Notice that the bundle in question is $S^2(M)$, and since $\mathcal{L}$ depends on second derivatives of the fields g , since $R(g)$ does, one would have to use the second jet bundle. That approach seems more complicated.

The energy conservation law is as follows.

Theorem. For the equations for L above, we have

$$\frac{\partial}{\partial t}\{\mathcal{H}(g,k)\,\mu_g\} + 2\,\delta\delta\pi = 0$$

where

$$\mathcal{H}(g,k) = \tfrac{1}{2}\,\hat{\mathcal{G}}_g(k,k) + 2\,R(g)$$

is the energy density. In particular if $\delta\pi = 0$, $\mathcal{H} = 0$ at $t = 0$ then these conditions are maintained in time.

Proof. Let $\mathcal{K}_g(k) = \tfrac{1}{2}\{k\cdot k - (\mathrm{tr}k)^2\}$ the kinetic energy density.

Then

$$\frac{\partial}{\partial t}(\mathcal{K}\mu_g) = \hat{\mathcal{G}}_g\!\left(k, \frac{dk}{dt}\right)\mu_g + D_g\mathcal{K}_g(k)\cdot k\,\mu_g$$

$$+ \frac{\mathcal{K}}{2}\,\mathrm{tr}(k)\,\mu_g$$

where $\hat{\mathcal{G}}$ is the pointwise Dewitt metric. Using the lemmas on p. 213-214,

$$D_g\mathcal{K}_g(k)\cdot k = -\,k\cdot(k\times k) + k\cdot k(\mathrm{tr}k)$$

and

$$\hat{\mathcal{G}}_g\!\left(k, \frac{dk}{dt}\right) = k\cdot\, k\times k = \tfrac{1}{2}(\mathrm{tr}k)k\cdot k - \frac{1}{4}\,\mathcal{K}(\mathrm{tr}k)$$

$$-\,\{\mathrm{tr}k\,(k\cdot k) - \tfrac{1}{2}(\mathrm{tr}k)^3 - \frac{3}{4}\,\mathcal{K}(\mathrm{tr}k)\}$$

$$+\,2\,\mathrm{Ric}(g)\cdot k - 2\,R(g)\,\mathrm{tr}(k)$$

$$-\,\tfrac{1}{2}\,r(g)\,\mathrm{tr}(k) + \frac{3}{2}\,R(g)\,\mathrm{tr}(k)$$

Adding we get

$$\frac{\partial}{\partial t}(\mathcal{K}\mu_g) = 2\,\mathrm{Ric}(g)\cdot k - R(g)\,\mathrm{tr}(k) \quad .$$

On the other hand,

$$\frac{\partial}{\partial t}\,(2\,R(g)\,\mu_g) = R(g)tr(k)\,\mu_g + (2\,\Delta(trk)$$

$$+ 2\delta\delta k - 2\,Ric(g)k)\,\mu_g \quad .$$

Hence adding,

$$\frac{\partial}{\partial t}\,(\mathcal{H}\,\mu_g) = 2(\Delta(trk) - \delta\delta k)\mu_g$$

$$= -\,2\delta\delta\pi \quad . \quad \square$$

There is good a priori evidence, including a theorem, that any genuinely relativistic theory, in the absense of external fields, must have $\mathcal{H}$ pointwise constant (see Fischer-Marsden [1]).

Therefore one selects out the subset $\mathcal{C}$ of $T\mathfrak{M}$ defined by:

$$\mathcal{C} = \{(g,k) \mid \tfrac{1}{2}\{k\cdot k - (trk)^2\} + 2R(g) = 0 \text{ and}$$

$$\delta(k - (trk)g) = 0\} \tag{5}$$

The previous results prove that our Hamiltonian flow on $T\mathfrak{M}$ leaves $\mathcal{C}$ invariant and we thus select out $\mathcal{C}$ as the physically meaningful subset. It is rather analogous to what one does in electromagnetism. In general, $\mathcal{C}$ is not a manifold. This point is discussed in lecture 10.

Thus for $(g,k) \in \mathcal{C}$, the evolution equations become,

$$\left.\begin{aligned} \frac{\partial g}{\partial t} &= k \\ \frac{\partial k}{\partial t} &= S_g(k) - \text{grad } V(g) \\ &= k \times k - \tfrac{1}{2}(trk)k + 2\,Ric(g) \end{aligned}\right\} \tag{6}$$

The extra terms have dropped out in view of $\mathcal{H} = 0$.

Relationahip with the Four Geometry.

We now form $L = M \times R$ and construct a Lorentz metric on L as follows.

$$\tilde{g}_{(x,t)}((v,r),(w,s)) = g_x(t)(v,w) - rs$$

where (v,r), $w,s) \in T_{(x,t)}(M \times R) \approx T_xM \times R$ and g is the time dependent metric on M . In coordinates, the formula reads:

$$\tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = g_{ij}\ dx^i dx^j - dt^2$$

where $x^\beta = (x^i, t)$; $i=1,2,3$, $\beta=0,1,2,3$.

Theorem. The Lorentz metric $\tilde{g}$ is Ricci flat if and only if g satisfies the evolution equations (6) above, together with the initial constraints (5).

This result therefore establishes the equivalence between solving the intiial value problem for the three metric g and Ricci-flatness of the four metric $\tilde{g}$ i.e. the Einstein field equations. Note that we have taken a special form for $\tilde{g}$, namely we have assumed $\tilde{g}_{0i} = 0$, $\tilde{g}_{00} = -1$. This point is discussed below.

The proof turns on the Gauss-Codazzi equations which relate the curvatures on L,M with the second fundamental form and the unit normal. This result which we assume here, is the following, for the case at hand:

<u>Lemma</u>. <u>Let</u> S_{ij} <u>be the second fundamental form on</u> M <u>and</u> Z^{μ} <u>the unit normal to</u> M , <u>so</u> $Z^{\mu} = (0,1)$. <u>Let</u> ${}^{(4)}R_{\alpha\beta\gamma\delta}$ <u>be the curvature tensor on</u> L , ${}^{(3)}R_{ijk\ell}$ <u>that on</u> M . <u>Then</u>

$$\text{(i)} \quad {}^{(4)}R_{0i0j} = \frac{\partial S_{ij}}{\partial t} - (S \times S)_{ij}$$

$$\text{(ii)} \quad {}^{(4)}R_{ijk\ell} = {}^{(3)}R_{ijk\ell} + S_{ik}S_{j\ell} - S_{i\ell}S_{jk}$$

$$\text{(iii)} \quad {}^{(4)}R_{0ijk} = S_{ik|j} - S_{ij|k}$$

Now if $g, \tilde{g}$ are related as before, we assert that

$$S_{ij} = -\tfrac{1}{2} k_{ij}$$

where $k_{ij} = \frac{\partial g}{\partial t}$. Indeed, we have $S_{ij} = -Z_{i|j} = -T^{\alpha}_{ij} Z_{\alpha} = -\Gamma^{0}_{ij}$. But from the formula for the Christoffel symbols, we compute that $\Gamma^{0}_{ij} = \frac{1}{2} g_{ij,0}$, and so our claim holds.

Now suppose $\tilde{g}$ is Ricci flat. Then in particular,

$$0 = {}^{(4)}R_{ij} = \tilde{g}^{\alpha\beta}\, {}^{(4)}R_{\alpha i j\beta} = \tilde{g}^{00}\, {}^{(4)}R_{0ij0} + \tilde{g}^{k\ell}\, {}^{(4)}R_{kij\ell} \, .$$

Applying (i), (ii) of the lemma with $S_{ij} = -\frac{1}{2} k_{ij}$ gives

$$0 = -\tfrac{1}{2}\frac{\partial k_{ij}}{\partial t} + \frac{1}{4}(k \times k)_{ij} + g^{k\ell}\{{}^{(3)}R_{kij\ell} + \tfrac{1}{4}(k_{kj}\, k_{i\ell} - k_{k\ell}\, k_{ij})\}$$

or $$\frac{\partial k_{ij}}{\partial t} = \tfrac{1}{2}(k \times k)_{ij} + 2\,{}^{(3)}R_{ij} + \tfrac{1}{2}(k \times k)_{ij} - \tfrac{1}{2}(\mathrm{tr}k)k$$

$$= (k \times k)_{ij} - \tfrac{1}{2}\, k_{ij}(\mathrm{tr}k) + 2\,{}^{(3)}R_{ij}$$

which is the correct equation of motion for k, according to (6). Similarly from

$$0 = {}^{(4)}R_{oi} = \tilde{g}^{\alpha\beta}\, {}^{(4)}R_{\alpha oi\beta} = -{}^{(4)}R_{ooio} + \tilde{g}^{k\ell}\, {}^{(4)}R_{koi\ell}$$

$$= \tilde{g}^{k\ell}\, {}^{(4)}R_{koi\ell}$$

we obtain from (iii)

$$0 = \tfrac{1}{2} g^{k\ell}\, \{k_{k\ell|i} - k_{ki|\ell}\}$$

or $$\delta(\mathrm{tr}k\; g - k) = 0$$

Similarly from ${}^{4}R_{oo} = 0$ we obtain the energy statement. The converse is proved in exactly the same way. □

<u>The Lapse and Shift</u>.

Although any Lorentz metric $\tilde{g}$ can be put in the form $\tilde{g} = g_{ij}\, dx^i dx^j - dt^2$ by a suitable coordinate change (namely in gaussian, or normal, coordinates), the above description is incomplete since it singles out this coordinate system as special. The situation can be remedied however, by introducing what are called the lapse and shift functions. The shift function is a time dependent vector field X prescribed in advance, corresponding to a choice of coordinate system. Now we set

$$\tilde{g}_{\alpha\beta} = -(1 - \langle X,X \rangle)dt^2 - 2X_i dt\, dx^i + g_{ij}dx^i dx^j$$

and this corresponds to the evolution equations

$$\frac{\partial g}{\partial t} = k - L_X g$$

$$\frac{\partial k}{\partial t} = S_g(k) - 2\,\mathrm{Ric}(g) - L_X k$$

This can all be seen very simply as a change from what we might call "space" to "body" coordinates. Namely, if $\bar{g}$ is is a solution for no shift and η_t is the flow of X, then $g = (\eta_t^{-1})^* \bar{g}$ solves the above. $\tilde{g}$ above is just the metric in the induced coordinate change on $M \times R$.

This therefore takes care of coordinate changes on $M \times R$ corresponding to changes in M. For changes along R one introduces the lapse and things now become more involved. We now introduce $N: M \times R \to R$ and

$$\tilde{g}_{\alpha\beta} = - N dt^2 + g_{ij}\, dx^i dx^j$$

with

$$\begin{cases} \dfrac{\partial g}{\partial t} = Nk \\[2ex] \dfrac{\partial k}{\partial t} = NS_g(k) - 2N\,\mathrm{Ric}(g) + 2\,\mathrm{Hess}\,N \end{cases}$$

where $\mathrm{Hess}\,N = N_{|i|j}$ is the Hessian of N. We shall not go into details here except to remark that this can be handled by the following device. Set $\mathcal{T}$ = the C^∞ maps ξ ; $M \times R \to R$. If one knows how to treat relativistic particles by extension of the Lagrangian to a homegeneous degenerate one, (see Lancos [1]) then we are motivated to extend our Lagrangian from $\mathfrak{M}$ to $\mathfrak{M} \times \mathcal{T}$ with $\mathcal{T}$ generalizing a single time parameter R . This procedure leads to the above equations of motion, with the degeneracy reflected in the arbitrariness of N . One can use the symmetry groups $\mathcal{D}$ and $\mathcal{T}$ to construct a reduced phase space using the methods of lecture 6 to recover a result of Fadeev [1]. See Marsden-Fischer [1]. However we shall not pursue the matter further here.

Remarks on existence of solutions

The original theorem concerning existence of solutions for the Einstein system is due to Fourès-Bruhat [1]. The result was improved on by Lichnerowicz [1] using Leray systems. See also Choquet-Bruhat [1] and Dionne [1]. The method involves the theory of second order partial differential equations which are quasi-linear and "strictly hyperbolic". Actually, there is a simpler theory of quasi-linear first order systems which is applicable here (cf. Fischer-Marsden [2,3]).

The way this goes is a bit complicated and will not be presented in detail here. We will illustrate with the wave equation how one reduces a second order system to a first order one. The method for relativity is more complicated, but the basic idea is the same.

First of all let us consider the linear problem in R^n: Let u be a vector-valued function $u:R^n \to R^m$. The system

$$(1) \qquad \frac{\partial u}{\partial t} = \sum_{i=1}^{n} A^i(x) \frac{\partial u}{\partial x^i} + B(x)\cdot u$$

is said to be symmetric hyperbolic if the $m \times m$ matrices A^i are symmetric for all $1 \leq i \leq n$. The system is first order and linear in u. Under fairly mild restrictions (A^i, B should be of class H^s, $s > (n/2) + 1$), there exists a unique solution u_t in H^s (all time) for any initial condition u_0 in H^s. This result is due basically to Petrovsky [1], Friedrichs [1], and others. A proof may be found in Courant-Hilbert [1] Vol. II; see also Kato [3,4] and Dunford-Schwartz [1]. Using standard techniques of reducing second order systems to first order, this theorem may be used to solve the wave equation in R^n:

EXAMPLE. The wave equation.

The equation is

$$\frac{\partial^2 f}{\partial t^2} = \nabla^2 f; f = f(x^1, \ldots, x^n, t) .$$

Put, formally,

$$\begin{bmatrix} f \\ \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^n} \\ \frac{\partial f}{\partial t} \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ u_{n+1} \end{bmatrix}$$

Then the wave equation for f is the same as the following symmetric hyperbolic system for u :

$$\left\{\begin{aligned} \frac{\partial u_0}{\partial t} &= u_{n+1} \\ \frac{\partial u_1}{\partial t} &= \frac{\partial u_{n+1}}{\partial x^1} \\ &\vdots \\ \frac{\partial u_n}{\partial t} &= \frac{\partial u_{n+1}}{\partial x^n} \\ \frac{\partial u_{n+1}}{\partial t} &= \frac{\partial u_1}{\partial x^1} + \dots + \frac{\partial u_n}{\partial x^n} \end{aligned}\right. .$$

In this case

$$A^1 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 1 & \dots & 0 \end{bmatrix} \qquad \text{etc.}$$

are symmetric $(n+2) \times (n+2)$ matrices.

Thus, using the linear theory for general first order symmetric hyoerbolic systems, we get an existence theorem for the wave equation, namely that if $(f_0, (\partial f_0/\partial t)) \in H^{s+1} \times H^s$ there is a unique solution $f_t \in H^{s+1}$, $-\infty < t < \infty$, satisfying the given initial conditions.

The hyperbolicity of $(\partial^2 f/\partial t^2) = \Delta^2 f$ is reflected in the symmetry of the A^i. If we had used $(\partial^2 f/\partial t^2) = -\Delta^2 f$, the A^i would not have come out symmetric - the Cauchy problem in this case is not well posed.

Now consider the nonlinear problem in R^n. In this case we have a system of the form

$$\frac{\partial u}{\partial t} = \sum_i A^i(x,t,u) \frac{\partial u}{\partial x^i} + B(x,t,u) \quad ,$$

where the A^i and B are matrices which are polynomial in u (or more generally, satisfy Sobolev's "condition T"; cf. Sobolev [1]). The system is quasi-linear and the matrices A^i are symmetric. The nonlinear theorem is obtained from the linear theory by adapting the Picard method. In this case also, unique solutions exist in H^s, but only for short time, in contrast to the linear theory.

The Einstein system above is rather like the wave equation and one can show that in the appropriate variables, obtained in a way not unlike that for the wave equation, it is symmetric hyperbolic.

The verification that it is symmetric hyperbolic uses "harmonic coordinates"; cf. Lichnerowicz [1].

Thus we get existence and uniqueness of smooth solutions for short time (which can be extended to maximal solutions as well). These solutions depend continuously on the initial data.

For details of all of this, see Fischer-Marsden [2].