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1. Infinite Dimensional Manifolds.

Basic Calculus.

We shall let E, F, G, ... denote Banach spaces. Let $U \subset E$ be open and let $f : U \to F$ be a given mapping. We say f is Fréchet differentiable at $x_0 \in U$ if there is a continuous (= bounded) linear map $Df(x_0) : E \to F$ such that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\|h\| < \delta$ implies

$$\|f(x_0 + h) - f(x_0) - Df(x_0)\cdot h\| \leq \varepsilon\|h\| .$$

The map $Df(x_0)$ is necessarily unique.

Let $L(E, F)$ denote the space of all continuous linear maps from E to F together with the operator norm

$$\|T\| = \sup_{\|x\|\leq 1} \|T\cdot x\|$$

so that $L(E, F)$ is a Banach space. Let $L_s(E, F)$ denote the same space with the strong operator topology; i.e. the topology of pointwise convergence.

If f is Fréchet differentiable at each $x \in U$ and if $x \mapsto Df(x) \in L(E, F)$ (resp. $L_s(E, F)$) is continuous, we say f is of class C^1 (resp. T^1) .

By induction it is not hard to formulate what it means for f to be of class C^r or T^r . For our purposes we shall be mostly dealing with C^r although T^r does arize in certain problems (see Abraham [6] and Chernoff-Marsden [2]).

The usual rules of calculus hold. Foremost amongst these is the chain rule:

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x) .$$

To obtain substantial results, one often employs the following:

Inverse Function Theorem. Let $f : U \subset E \to F$ be C^r , $r \geq 1$. Assume $Df(x_0)$ is an isomorphism for some $x_0 \in U$. Then there exists open neighborhoods U_0 of x_0 and V_0 of $f(x_0)$ such that $f : U_0 \to V_0$ is bijective and has a C^r inverse $f^{-1} : V_0 \to U_0$. (We say f is a local diffeomorphism.)

The proof of this is essentially the same as one learns in advanced calculus where E , F are taken to be R^n . For details, see Lang [1] or Dieudonné [1].

Implicit Function Theorem. Let $U \subset E$, $V \subset F$ be open and $f : U \times V \to G$ be C^r , $r \geq 1$. For $x_0 \in U$, $y_0 \in V$, assume $D_2f(x_0, y_0)$ (the derivative with respect to y) is an isomorphism of F onto G. Then there is a unique C^r map $g : U_0 \times W_0 \to V$ where U_0, W_0 are sufficiently small neighborhoods of x_0 and $f(x_0, y_0)$ respectively, such that

$$f(x, g(x, w)) = w$$

for all $(x, w) \in U_0 \times W_0$.

Indeed, this follows from the inverse function theorem applied to the map $\Phi : U \times V \to E \times G$

$$\Phi(x, y) = (x, f(x, y))$$

which is a local diffeomorphism.

Results like these are central to the study of submanifolds which we deal with later. They, in turn, are crucial to several of the applications.

In applications, the spaces E, F, ... are often spaces of functions and $f : E \to F$ may be some sort of non-linear differential operator. Then $Df(x_0)$ will be what is called the linearization of f about x_0.

Manifolds.

Let M be a (Hausdorff) topological space. We say M is a C^∞ manifold modelled on the Banach space E when it has the following additional structure: there is an open covering $\{U_\alpha\}$ of M together with homeomorphisms

$$\varphi_\alpha : U_\alpha \to V_\alpha \subset E$$

where V_α is open in E such that for all α, β, the overlap map (or coordinate change)

$$\varphi_\alpha \circ \varphi_\beta^{-1}$$

(defined on $\varphi_\beta(U_\alpha \cap U_\beta)$) is a C^∞ map.

By a <u>chart</u> (or <u>coordinate patch</u>) we mean a homoemorphism $\varphi : U \subset M \to V \subset E$ of open sets such that for all α, the map

$$\varphi \circ \varphi_\alpha^{-1}$$

(defined on $\varphi_\alpha(U_\alpha \cap U)$) is C^∞.

The collection of all charts yields what is called a <u>maximal atlas</u>.

Let M and N be manifolds and $f : M \to N$ a continuous map. We say f is of <u>class</u> C^r if for every chart $\varphi : U \subset M \to V \subset E$ of M and $\psi : U_1 \subset N \to V_1 \subset F$ of N, the map

$$\psi \circ f \circ \varphi^{-1}$$

of the open set $\varphi(f^{-1}(U_1) \cap U)$ to F is C^r. By the chain rule one sees that this holds for all charts if it holds for <u>some</u> covering of N and M by charts.

Submanifolds.

Let M be a manifold and let $S \subset M$. In applications S is often defined by some restrictive condition; e.g. by constraints of the form $f(x) = c$. It is important to know whether or not S is smooth, e.g. has no sharp corners. Below we give a useful condition for this, but first let us formulate the definition.

We say S is a <u>submanifold</u> of M (where M is modelled on E) if we can write $E = F \oplus G$ (topological sum) and for every $x \in S$, there is a chart $\varphi : U \subset M \to V \subset E$ of M where $x \in U$ such that

$$\varphi(U \cap S) = V \cap (F \times \{w\})$$

where $w \in G$.

In other words, the chart φ "flattens out" S making it lie in the subspace F.

One sees that the above charts define a manifold structure for S; S will be modelled on F. The conditions ensure that the manifold structure on S is compatible with that on M.

Vector Bundles.

By a <u>vector bundle</u> we mean a manifold E together with a submanifold $M \subset E$ and a projection $\pi : E \to M$ (i.e. $\pi \circ \pi = \pi$) such that for each $x \in M$, the fibre $E_x = \pi^{-1}(x)$ is a linear space with x as the zero element; there should also be a covering by charts (called vector bundle charts) of the form

$$\varphi : \pi^{-1}(U) \subset E \to V \times F$$

where U is open in M, $V \subset E$, the model space for M and F is some fixed Banach space such that the overlap maps are linear isomorphisms when restricted to each fiber.

Intuitively, one thinks of a vector bundle over M as a collection of linear spaces E_x, one attatched to each $x \in M$. For $v \in E_x$, $\pi(v) = x$ is the base point to which v is attatched. As we shall see, the quantities v can be vectors, tensors, differential forms, spinors, etc.

The Tangent Bundle.

The most basic vector bundle attatched to a manifold M is its tangent bundle TM. It was an important observation in the historical development of manifold theory, that the tangent space to a manifold can be defined completely intrinsically. For example there is no need to have a space in which the manifold is embedded; one might think such an embedding is necessary by thinking of surfaces in R^3.

There are two useful and equivalent ways to define T_xM, the fibre of TM above $x \in M$.

First, we can use curves. Indeed, intuitively a tangent vector $v \in T_xM$ ought to be $c'(0)$ for some curve $c(t)$ in M with $c(0) = x$. So consider all curves $c : R \to M$ with $c(0) = x$ and say $c_1 \sim c_2$ if $c_1'(0) = c_2'(0)$ in some (and hence every) chart about x , where $c' = dc/dt$ in that chart. Then T_xM is defined to be the set of equivalence classes, and TM is the disjoint union of the T_xM 's.

The above definition is useful because it is closely connected with our intuition. There is a second definition which brings out the vector bundle structure of TM more clearly. This goes as follows.

Fix $x \in M$ again and look at charts φ defined on neighborhoods of x . Consider pairs (φ, e) where $e \in E$, the model space of M . Say

$$(\varphi_1, e_1) \sim (\varphi_2, e_2)$$

if

$$D(\varphi_2 \circ \varphi_1^{-1})(\varphi_1(x)) \cdot e_1 = e_2$$

Then T_xM is the set of equivalence classes of such pairs. Clearly T_xM is a linear space. Moreover a chart φ induces naturally a vector bundle chart on TM by using the definition, and these charts make M manifestly a submanifold.

We leave it to the reader to check the equivalence of the two definitions. We use $\pi : TM \to M$ for the projection.

One often uses definitions involving derivations for T_xM in the case of finite dimensional manifolds. For infinite dimensional manifolds this is possible but is rather cumbersome(see Schwartz [1], p. 105 for a discussion).

Let $f : M \to N$ be a C^r map, $r \geq 1$. Then there is a bundle map $Tf : TM \to TN$ naturally induced; i.e. Tf maps fibres to fibres and the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[f]{} & N \end{array} .$$

Commutativity of this diagram means nothing more than for $x \in M$ 6, and $v \in T_xM$, $Tf(v) \in T_{f(x)}N$.

Using the first definition of TM, we define $Tf(v) = \frac{d}{dt} f(c(t))\big|_{t=0}$ where $v = c'(0)$. (Remember $c'(0)$ stands for the equivalence class of curves and in a chart for TM, it really is the derivative.)

Actually this definition is very useful for doing computations, as we shall see later.

Using the second definition, if the local representative for f is $f_{\varphi,\psi} : V \subset E \to V_1 \subset F$ relative to charts φ on M and

ψ on N , then the local representative for Tf is, in the corresponding charts for TM and TN ,

$$(Tf)_{\varphi,\psi} : V \times E \to V_1 \times F$$

$$(x, e) \mapsto (f_{\varphi\psi}(x), Df_{\varphi,\psi}(x)\cdot e)$$

One checks that this is consistent with the equivalence relation and so yields a well defined map Tf .

In the language of tangents, the chain rule can be neatly expressed by saying that

$$T(f\circ g) = Tf\circ Tg \ .$$

Submersions.

Let E be a topological vector space, and $F \subset E$ a closed subspace. We say F splits if there is another closed subspace G such that

$$E = F \oplus G \quad \text{(topological sum)} \ .$$

For example if E is a Hilbert space this is always the case, for we can choose $G = F^{\perp}$. However in a general Banach space a closed subspace need not have a closed complement.

We say topological vector space rather than Banach space here because we want to use the case where E is T_xM . The latter does not carry canonically the structure of a Banach space, but it does have the structure of a topological vector space (if a norm is assigned to each tangent space T_xM, one speaks of a Finsler structure).

Now let M and N be Banach manifolds and $f : M \to N$ a C^∞ map. We want to know when $S = f^{-1}(w)$ is a submanifold of M, where $w \in N$ is fixed. We say f is a submersion on S if for all $x \in S$, $T_xf : T_xM \to T_{f(x)}N$ is surjective and kernel T_xf splits.

Theorem. Let f be a submersion on S as just described. Then S is a smooth submanifold of M.

Proof. Work in a chart $U \subset E$ for M. Write $E = E_0 \oplus E_1$ where $E_0 = \ker Df(x)$, for x fixed. Consider the map Φ defined near x to $E_0 \times F$ by

$$\Phi(x_0, x_1) = (x_0, f(x_0, x_1)) .$$

Since E_0 is kernel $Df(x)$ we see that D_2f is an isomorphism from E_1 to F and so $D\Phi$ at x is an isomorphism. The map Φ is therefore a local diffeomorphism by the inverse function theorem. Clearly Φ yields a chart showing S is a submanifold. □

Since S is modelled on E_0, this argument also shows:

Corollary. $T_xS = \text{kernel } T_xf$.

To make effective use of this result one must be judicious in the choice of N. The space N must be large enough so f maps into N, but only just large enough to ensure that $T_x f$ will be surjective.

There is a similar result for <u>immersions</u>. Here $f : M \to N$ should be injective and have injective tangent at each $x \in M$ and the image should split. Also, the map f should be closed. Then $f(M)$ will be a submanifold of N.

The reader can work this case out for himself. We have stressed the submersion case because it is more useful for the sort of applications that we have in mind.

<u>Differential Forms</u>.

Given a linear topological space E, *we let* E^* *denote the* dual space; i.e. the space of all continuous linear maps $\ell : E \to R$.

Let M be a manifold and TM its tangent bundle. We can form a new bundle T^*M over M whose fibre over $x \in M$ is the dual space T_x^*M. It is not hard to see that this is a vector bundle. It is called the <u>cotangent bundle</u>.

In general if $\pi : E \to M$ is a vector bundle, a <u>section</u> s of E is a map

$$s : M \to E$$

such that

$$\pi \circ s = \text{identity} .$$

In other words, $s(x) \in E_x$ for each $x \in M$.

A section X of the tangent bundle is called a vector field while a section α of the cotangent bundle is called a one-form or a covector field.

Let $f : M \to R$. Then since $TR = R \times R$, $T_x f : T_x M \to R$, or in other words $T_x f \in T_x^* M$. Thus the tangent of f naturally induces a one form on M. So regarded, it is denoted df and is called the differential of f.

We can generalize $T^* M$ as follows. Let $\wedge^k M$ be the vector bundle over M whose fiber at $x \in M$ is the k-multilinear alternating continuous maps $T_x M \times \ldots \times T_x M \to R$. A section of the bundle $\wedge^k M$ is called a k-form. We regard real-valued functions as 0-forms.

Let α be a k-form and β an ℓ-form. Then the wedge product $\alpha \wedge \beta$ is defined as

$$(\alpha \wedge \beta)_x(v_1,\ldots,v_{k+\ell}) = \sum_\sigma (\text{sgn }\sigma)\alpha_x(v_{\sigma(1)}\ldots,v_{\sigma(k)})\beta_x(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)})$$

where the sum is over all permutations σ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+\ell)$ and $\text{sgn }\sigma = \pm 1$ is the sign of σ.

Note: We use the conventions of Bourbaki [1]. Compare with Abraham [2].

In the case of R^3 we can identify one forms and two forms with vectors. When we do so, $\alpha \wedge \beta$ is seen to be just the cross product.

If $f : M \to N$ is a C^r mapping and α is a C^r k-form on N we get a C^{r-1} k-form $f^*\alpha$ on M defined by

$$(f^*\alpha)_x(v_1, \ldots, v_k) = \alpha_{f(x)}(Tf \cdot v_1, \ldots, Tf \cdot v_k) .$$

We call $f^*\alpha$ the <u>pull back</u>* of α by f .

If $X_1, \ldots, X_k$ are vector fields on M and α is a k-form we get a real valued function $\alpha(X_1, \ldots, X_k)$ defined by

$$\alpha(X_1, \ldots, X_k)(x) = \alpha_x(X_1(x), \ldots, X_k(x)) .$$

Notice that the differential mapped a 0-form to a 1-form. This can be generalized as follows. If α is a k-form, define the $k + 1$ form $d\alpha$ by

$$d\alpha_x(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i (D\alpha_x \cdot v_i)(v_0, \ldots, \hat{v}_i, \ldots, v_k)$$

where $\hat{v}_i$ denotes that v_i is missing and $D\alpha_x$ is the derivative of α in charts; note $\alpha : U \subset E \to \Lambda^k(E)$ so $D\alpha_x : E \to \Lambda^k(E)$. One can check that d is chart independent. The operator d plays a fundamental role in calculus on manifolds. It is called the <u>exterior derivative</u>.

It is not hard, but a little tedious, to verify:

*Note that the positioning of the stars agrees with Bourbaki [1], Lang [1] but is the opposite of Abraham [2].

(i) d is real linear

(ii) $d \circ d = 0$

and (iii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.

Condition (ii) is a generalization of the familiar identity $\nabla \times (\nabla f) = 0$ from vector analysis.

If α is a k-form and X a vector field, define the <u>interior product</u> $i_X\alpha = X \lrcorner\, \alpha$ by

$$(i_X\alpha)_x(v_2,\ldots,v_k) = \alpha_x(X(x),\ v_2,\ldots,v_k)$$

so $i_X\alpha$ is a $k-1$ form.

Define the <u>Lie derivative</u> $L_X\alpha$ by

$$L_X\alpha = di_X\alpha + i_Xd\alpha$$

so $L_X\alpha$ is a k-form if α is. This object is an extremely useful tool in Hamiltonian mechanics as we shall soon see.

Following are some of the basic operations on vector fields. If X is a vector field on M and $f : M \to R$, we set

$$X(f) : M \to R\ ,\ x \mapsto df_x \cdot X(x)\ .$$

By convention $i_X f = 0$, so we see that $X(f) = L_X f$. Notice that $X(f)$ is nothing but the derivative of f in the direction of X .

Clearly $X(f)$ is a derivation in f : $X(fg) = fX(g) + gX(f)$. As we have mentioned, this property is sometimes used (in finite dimensions) to characterize vector fields.

Let X and Y be vector fields. Then their bracket $[X, Y]$ is a vector field on M such that in local coordinates

$$[X, Y] = DY \cdot X - DX \cdot Y \ .$$

As a derivation, we have

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \ .$$

Let f be a C^r diffeomorphism of M onto N , $r \geq 1$. That is, f is C^r , a bijection with C^r inverse. Given a vector field X on M , set

$$f_*X = Tf \circ X \circ f^{-1}$$

a vector field on N . Similarly if Y is a vector field on N , set

$$f^*Y = Tf^{-1} \circ Y \circ f \ .$$

These are characterised by

$$(f_*X)(h) = (X(h \circ f)) \circ f^{-1}$$

which follows from the chain rule.

In the following table we summarize some of the useful identities connecting the various operations which we have introduced. The proofs are straightforward algebraic manipulations. These identities are quite convenient in various applications as we shall see in the next lecture.

1. Vector fields on M with the bracket $[X, Y]$ form a Lie algebra; i.e., $[X, Y]$ is real bilinear, skew symmetric and Jacobi's identity holds: $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

2. For a diffeomorphism f , $f_*[X, Y] = [f_*X, f_*Y]$ and $(f\circ g)_*X = f_*g_*X$.

3. The forms on a manifold are a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$ for k and ℓ forms α and β respectively.

4. If f is a map, $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, $(f\circ g)^*\alpha = g^*f^*\alpha$.

5. d is a real linear map on forms and: $dd\alpha = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\alpha \wedge d\beta$ for α a k-form.

6. For α a k-form and $X_0, \ldots, X_k$ vector fields:

$$d\alpha(X_0,\ldots,X_k) = \sum_{i=0}^{k} (-1)^i X_i(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k) + \\ + \sum_{i<j} (-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k) .$$

7. For a map f , $f^* d\alpha = df^* \alpha$.

8. (Poincaré lemma). If $d\alpha = 0$ then α is locally exact; i.e. there is a neighborhood U about each point on which $\alpha = d\beta$.

9. $i_X \alpha$ is real bilinear in X , α and for $h : M \to R$, $i_{hX}\alpha = h i_X \alpha = i_X h\alpha$. Also $i_X i_X \alpha = 0$, $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k \alpha \wedge i_X \beta$.

10. For a diffeomorphism f , $f^* i_X \alpha = i_{f^* X} f^* \alpha$.

11. $L_X \alpha = d i_X \alpha + i_X d\alpha$.

12. $L_X \alpha$ is real bilinear in X , α and $L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta$.

13. For a diffeomorphism f , $f^* L_X \alpha = L_{f^* X} f^* \alpha$.

14. $(L_X\alpha)(X_1,\ldots,X_k) = X(\alpha(X_1,\ldots,X_k)) - \sum_{i=1}^{k} \alpha(X_1,\ldots,[X,X_i],\ldots,X_k)$.

15. Locally, $(L_X\alpha)_x \cdot (v_1,\ldots,v_k) = D\alpha_x \cdot X(x) \cdot (v_1,\ldots,v_k)$

$$+ \sum_{i=1}^{k} \alpha_x \cdot (v_1,\ldots,DX_x \cdot v_i,\ldots,v_k) .$$

16. The following identities hold:

$$L_{fX}\alpha = f L_X \alpha + df \wedge i_X \alpha$$

$$L_{[X,Y]}\alpha = L_X L_Y \alpha - L_Y L_X \alpha$$

$$i_{[X,Y]}\alpha = L_X i_Y \alpha - i_Y L_X \alpha$$

$$L_X d\alpha = d L_X \alpha$$

$$L_X i_X \alpha = i_X L_X \alpha .$$

TABLE 1

Flows of Vector Fields.

By a flow (or a one parameter group of diffeomorphisms) we mean a collection of (smooth) maps $F_t : M \to M$, $t \in R$ such that

$$\begin{cases} F_{t+s} = F_t \circ F_s \\ F_0 = \text{identity} \end{cases} .$$

The term dynamical system is also used. For each $x \in M$, $t \mapsto F_t(x)$ is the trajectory of x . The condition $F_{t+s} = F_t \circ F_s$ expresses nothing more than "causality".

If X is a vector field, we say it has flow F_t if

$$\frac{d}{dt} F_t(x) = X(F_t(x)) .$$

In other words, $F_t(x)$ solves the system of differential equations determined by X . In finite dimensions these are ordinary differential equations. In infinite dimensions certain types of partial differential equations can be handled; we shall discuss this point below.

If the individual solution curves (or integral curves) of X are unique; i.e. if

$$\frac{d}{dt} c(t) = X(c(t))$$

$$c(0) = x_0$$

has a unique solution, then one can prove the above flow property

$F_{t+s} = F_t \circ F_s$ rather easily.

An important point is that, in general, the flow of a vector field need not be defined for all $t \in R$ for each $x \in M$. For example a trajectory can leave the manifold in a finite time. To ensure this doesn't happen, one requires certain estimates to establish that a trajectory remains in a bounded region for bounded t-intervals. If $F_t(x)$ is defined for all $t \in R$ we say X has a complete flow.

Local Existence and Uniqueness Theorem. *If* X *is a* C^r *vector field*, $r \geq 1$, *then* X *has a locally defined, unique* C^r *flow* F_t.

This result is proved by the Picard iteration method, as one learns in elementary courses on differential equations. Thus it is similar to the proof of the inverse function theorem. Actually it can be deduced from the latter directly; see Robbin [1]. (In fact the other cornerstone of differential analysis, the Frobenius theorem can also be so deduced; see Penot [4]).

One usually proceeds by using the above theorem to deduce the existence of the local flow and then use special properties of X to prove completeness. (For example we shall do this in certain cases for Hamiltonian vector fields.)

The Heat Equation.

In these lectures we are concerned to a great extent with partial differential equations. For these, the above theorem is rather limited. To see why, consider the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

where u is a function of $x \in R$ and $t \in R$ and u is given at time $t = 0$. Here $\Delta = \partial^2/\partial x^2$ is the Laplacian.

Of course this equation can be solved explicitly:

$$u(x, t) = \frac{1}{\sqrt{4t\pi}} \int_{-\infty}^{\infty} e^{-|x-y|^2/4t} u_0(y)\,dy \ , \quad t > 0 \ .$$

The solution is actually good only for $t \geq 0$. Nevertheless it yields a well defined <u>continuous</u> semi flow F_t on $L_2(R)$. In general it won't be t-differentiable for all $u_0 \in L_2(R)$.

Indeed Δ is not a bounded operator on $L_2(R)$ so we cannot use the existence theorem. Rather Δ is defined only on a domain $D \subset L_2(R)$, a dense linear subspace consisting of those $f \in L_2$ whose derivatives (in the distribution sense) of order ≤ 2 also lie in L_2.

One cannot remedy the situation by passing to the C^∞ functions. Indeed this is not a Banach but a Fréchet space and it is not hard to show that for these spaces the local existence theorem is false.

However there is a general theorem which can cover the situation, called the Hille Yosida theorem. We don't want to go into this now, so we just state two useful special cases: (See Yosida [1] for details. The "Schrodinger case" - Stone's theorem - will be considered later).

Parabolic Case. Let H be a Hilbert space and $A : D \subset H \to H$ a (linear) self adjoint operator, $A \leq 0$. Then the equation $\frac{du}{dt} = Au$ defines a unique linear semi-flow F_t, $t \geq 0$ on H. The equation $\frac{du}{dt} = Au$ is satisfied for $u_0 \in D$ and $u_t = F_t(u_0)$.

For example, this covers the case of the heat equation. For the wave equation, $\frac{\partial^2 u}{\partial t^2} = \Delta u$, we use:

Hyperbolic Case. Let H, A be as above. Then the equation

$$\begin{cases} \frac{du}{dt} = v \\ \\ \frac{dv}{dt} = Au \end{cases}$$

defines a unique flow F_t, $t \in R$ on $D \times H$.

Densely Defined Vector Fields.

In view of the above examples, it is useful to extend our notions about vector fields so as to include more interesting examples.

By a manifold domain $D \subset M$, we mean a dense subset D in a manifold M such that D is also a manifold and the inclusion $i : D \to M$ is smooth and Ti has dense range.

By a densely defined vector field, we mean a map $X : D \to TM$ such that for $x \in D$, $X(x) \in T_x M$. A flow (or semi-flow) for X will be a collection of maps $F_t : D \to D$, $t \in R$ (or $t \geq 0$) [perhaps locally defined] such that

$$F_{t+s} = F_t \circ F_s$$

$$F_0 = \text{identity}$$

and for $x \in D$,

$$\frac{d}{dt} F_t(x) = X(F_t(x))$$

where $\frac{d}{dt}$ is taken regarding $F_t(x)$ as a curve in M .

Such a generalization allows the more interesting examples like the heat and wave equation and non-linear generalizations of them to be included.

What about a local existence theorem? Since we already have a good theorem (the Hille-Yosida theorem) available for the linear case, it is natural to linearize. There is a theorem, the Nash-Moser theorem which is suitable for these purposes. The exact hypotheses are too complicated to give in full here, but basically the spaces must be "decent" (technically, they must admit "smoothing operators") and if X is the vector field: $X : D \subset E \to E$, the linear operators $DX(x) : E \to E$ must have flows (or semi-flows) which vary smoothly with x . Then X will have a local flow.

For further information, see J. T. Schwartz [1], J. Marsden [1] and M. L. Gromov [1]. There are also a number of special techniques available, some of which are discussed later.

Flows and Lie Derivatives.

There is a very fundamental link between the flow of a vector field and the Lie derivative.

Theorem. Let X be a C^r vector field on M, $r \geq 1$ and α a k form on M. Let F_t be the flow of X. Then

$$\frac{d}{dt} F_t^* \alpha = F_t^* (L_X \alpha) .$$

Actually the proof is very simple, once we have the identities in table 1. Indeed if we differentiate in a chart

$$\frac{d}{dt} (F_t^* \alpha)_x (v_1, \ldots, v_k)$$

$$= \frac{d}{dt} \alpha_{F_t(x)} (D \cdot F_t \cdot v_1, \ldots, DF_t \cdot v_k)$$

we get exactly the expression for $L_X \alpha$ in formula 15 of table 1.

For example if $L_X \alpha = 0$ then $F_t^* \alpha = \alpha$; i.e. α is preserved by the flow.

The above theorem extends also to densely defined vector fields. We just need that each $F_t : D \to D$ be C^1 and that α be smooth on M (rather than on D) .

One of the points we wish to make here is that these geometrical ideas, culminating for example in the above theorem, can be applied to partial differential as well as to ordinary differential equations.