3. Elliptic Operators and Function Spaces.

In this lecture we shall discuss some of the basic spaces of functions which are used in analysis. In addition we shall discuss some of the fundamental properties of elliptic operators, first in the case of the Laplacian, and then in general. These results, especially the "splitting theorems" are of considerable use in proving certain subsets of the function spaces are actually submanifolds. This will find application in hydrodynamics and general relativity. Finally, we shall consider some elementary properties of the space of maps of one manifold to another.

We begin then with a discussion of the Sobolev spaces.

Sobolev spaces.

Let $\Omega \subset R^n$ be an open bounded set with C^∞ boundary. Let $\bar{\Omega}$ be the closure of Ω. Define $C^\infty(\Omega, R^n)$ to be the set of functions from Ω into R^n that can be extended* to a C^∞ function on some open set in R^n containing $\bar{\Omega}$. Let $C_0^\infty(\Omega, R^m) =$ $\{f \in C^\infty(\Omega, R^m) |$ the support of f is contained in a compact subset of $\Omega\}$.

To describe the Sobolev spaces in an elementary fashion, we temporarily introduce some more notation. An n multi-index is

---

* This definition is the same as saying that the functions are C^∞ on the closed set $\bar{\Omega}$ (with difference quotients taken within $\bar{\Omega}$) by virtue of the Whitney extension theorem. See the appendix of Abraham-Robbin [1]. The same technique can be applied to Sobolev spaces; cf. the Calderon extension theorem below and Marsden [8].

an ordered set of n non-negative integers. If $k = (k_1, \ldots, k_n)$ is an n multi-index, then put $|k| = k_1 + k_2 + \ldots + k_n$. If $u \in C^\infty(\Omega, R^m)$, define $D^k u$ by the formula

$$D^k u = (\partial^{|k|} u / \partial x_1^{k_1} \ldots \partial x_n^{k_n})$$

and $D^0(u) = u$. For $u \in C^\infty(\Omega, R^m)$ (or $C_0^\infty(\Omega, R^n)$) , define

$$\|u\|_s^2 = \int_\Omega \sum_{0 \le |k| \le s} |D^k u(x)|^2 dx .$$

Now $H^s(\Omega, R^m)$ (resp. $H_0^s(\Omega, R^m)$) is defined to be the completion of $C^\infty(\Omega, R^m)$ (resp. $C_0^\infty(\Omega, R^m)$) under the $\| \ \|_s$ norm. These H^s spaces are called the <u>Sobolev spaces</u>. Note that $H_0^0(\Omega, R^m) = H^0(\Omega, R^m) = L_2(\Omega, R^m) \supset H^s(\Omega, R^m)$; but for $s \geq 1$, $H_0^s(\Omega, R^m) \neq H^s(\Omega, R^m)$ as we shall see below.

There is another equivalent, and perhaps better, definition of the Sobolev norm. Let $d^k u$ be the kth total derivative of u so that $d^k u : \Omega \to L^k(R^n, R^m)$ where $L^k(R^n, R^m)$ denotes the k-linear maps on $\underbrace{R^n \times R^n \times \ldots \times R^n}_{k \text{ times}} \to R^m$ with the standard norm. Then if we set

$$|u|_s^2 = \int_\Omega \sum_{0 \le k \le s} \|d^k u(x)\|^2 dx$$

the $| \ |_s$ and $\| \ \|_s$ norms are equivalent. This is a simple exercise.

Also note that $H^s(\Omega, R^n)$ and $H_0^s(\Omega, R^n)$ are Hilbert spaces with the inner product

$$\langle u, v \rangle = \int_\Omega \sum_{0 \le |k| \le s} D^k u(x) \cdot D^k v(x)\, dx \ .$$

Sobolev Theorem.

(a) Let $s > (n/2) + k$. Then $H^s(\Omega, R^m) \subset C^k(\Omega, R^m)$ and the inclusion map is continuous (in fact is compact) when $C^k(\Omega, R^m)$ has the standard C^k topology, (the sup of the derivatives of order $\le k$).

(b) If $s > (n/2)$ then $H^s(\Omega, R^m)$ is a ring under pointwise multiplication of components. (This is often called the Schauder ring.)

(c) If $s > \frac{1}{2}$ and $f \in H^s(\Omega, R^m)$ then $f|\partial\Omega \in H^{s-\frac{1}{2}}$.

(d) (Calderon Extension Theorem). If $f \in H^s(\Omega, R^m)$ then f has an extension $f \in H^s(R^n, R^m)$.

Regarding (c), see Palais [1] for a discussion of continuous Sobolev chains; i.e., the definition of H^s for s not an integer; basically one can use the Fourier transform or one can interpolate. (d) means that f can be extended across $\partial\Omega$ in an H^s way.

Differentiability properties at the boundary presents some technical problems but are very important in hydrodynamics. Thus it is important to distinguish H_0^s from H^s.

The proof of the Sobolev Theorem can be found in Nirenberg

[1] and Palais [1]; see also Sobolev [1].

For most of hydrodynamics we will need $s > (n/2)+1$. One of the outstanding problems in the field is determining to what extent we can relax this condition on s. For many problems, one would like to allow corners and discontinuities in such things as the density of the fluid or the velocity field. $L_p^k = W^{k,p}$ spaces are often useful for this.

<u>H^s Spaces of Sections.</u>

Let M be a compact manifold, possibly with boundary. Also, let E be a finite dimensional vector bundle over M. For example E may be the tangent bundle, or a tensor bundle over M. Let $\pi : E \to M$ be the canonical projection. The following fact is useful and is obvious from the definition of a vector bundle (see lecture 1).

<u>Proposition</u>. <u>Suppose for each</u> $x \in M$, <u>we have</u> $\pi^{-1}(x) \cong R^m$. <u>Then there is a finite open cover</u> $\{U_i\}$ <u>of</u> M <u>such that each</u> U_i <u>is a chart of</u> M <u>and</u> $\pi^{-1}(U_i) \cong U_i \times R^m$ <u>for each</u> i.

Such a cover is called <u>trivializing</u>. Recall that a <u>section</u> of E is a map $h : M \to E$ such that $\pi \circ h = id_M$. Informally, we define, for $s \geq 0$, $H^s(E)$ to be the set of sections of E whose derivatives up to order s are in L_2.

This makes sense since in view of the proposition, a section of E can locally be thought of as a map from R^n to R^m where n

is the dimension of M. Similarly, we can put a Hilbert structure on $H^s(E)$ by using a trivializing cover. However, since this Hilbert space structure depends on the choice of charts, the norm on $H^s(E)$ is not canonical, so we call $H^s(E)$ a Hilbertible Space (ie., it is a space on which some complete inner product exists). To obtain a good norm on $H^s(E)$ one needs some additional structure such as a connection.

One has to check that the definition of $H^s(E)$ is independent of the trivialization and this can be done by virtue of compactness of M.

Of course the Sobolev theorems have analogues for $H^s(E)$. In particular if $s \geq 1$ it makes sense to restrict a section $h \in H^s(E)$ to ∂M. This is by part (c) of the Sobolev Theorem. Of course if $s > (n/2)$, h will be continuous and so this will be clear. For $s = 0$, we have $L_2(E)$ and restriction to ∂M does not make sense.

One defines $H_0^s(E)$ in a similar way. For $s > \frac{1}{2}$, when we restrict $h \in H_0^s(E)$ to ∂M, h will vanish, as will its derivatives to order $s - \frac{1}{2}$.

Much of the theory goes over for M noncompact, but we must specify a metric on M and a connection on E; further M must be complete and obey some curvature restriction such as sectional curvature bounded above; see Cantor [2].

Operations on Differential Forms.

Now, let M be a compact oriented Riemannian n-manifold without boundary.

As in Lecture 1, let Λ^k be the vector bundle over M whose fiber at $x \in M$ consists of k-linear skew-symmetric maps from T_xM, the tangent space to M at $x \in M$, to R. For each x, $\oplus_{k=0}^{n}\Lambda_x^k$ forms a graded algebra with the wedge product. Then $H^s(\Lambda^k)$ is a space of H^s differential k-forms. The exterior derivative d then is an operator:

$$d : H^{s+1}(\Lambda^k) \to H^s(\Lambda^{k+1}) .$$

It drops one degree of differentiability because d differentiates once; i.e., is a first order operator.

The star operator $* : H^s(\Lambda^k) \to H^s(\Lambda^{n-k})$ is given on Λ^k at $x \in M$ by

$$*(1) = \pm dx_1 \wedge \ldots \wedge dx_n \ , \quad *(dx_1 \wedge \ldots \wedge dx_n) = \pm 1$$

and

$$*(dx_1 \wedge \ldots \wedge dx_p) = \pm dx_{p+1} \wedge \ldots \wedge dx_n$$

where the "+" is taken if the $dx_1 \wedge \ldots \wedge dx_n$ is positively oriented and "-" otherwise, $x_1, \ldots, x_n$ form a coordinate system orthogonal at x, and $*$ is extended linearly as an operator

$\Lambda^k \to \Lambda^{n-k}$. Now if $\alpha \in H^s(\Lambda^k)$ then clearly $*\alpha \in H^s(\Lambda^{n-k})$, so $*$ can be taken as an operator from $H^s(\Lambda^k)$ to $H^s(\Lambda^{n-k})$.

The space Λ^k carries, at each point $x \in M$, an inner product. It is the usual business: the metric converts covariant tensors to contravariant ones (i.e., it raises or lowers indices) and then one contracts. If α_i , β_j are one forms, we have $<\alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k> = \det[<\alpha_i, \beta_j>]$. It is not hard to check that if μ is the volume form on M then

$$<\alpha, \beta>\mu = \alpha \wedge * \beta = \beta \wedge *\alpha .$$

Note that the inner product may be defined by the above formula. See Flanders [1] for more details on these matters. Define the operator $\delta : H^{s+1}(\Lambda^k) \to H^s(\Lambda^{k-1})$ by $\delta = (-1)^{n(k+1)+1} *d*$. There is an inner product on $H^0(\Lambda^k)$ (and hence on $H^s(\Lambda)$) given by

$$(\alpha, \beta) = \int_M <\alpha, \beta>d\mu .$$

<u>Proposition</u>. <u>For</u> $\alpha \in H^s(\Lambda^k)$ <u>and</u> $\beta \in H^s(\Lambda^{k+1})$

$$(d\alpha, \beta) = (\alpha, \delta\beta) .$$

<u>Proof</u>. Note that

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^k \alpha \wedge d*\beta$$

$$= d\alpha \wedge *\beta - \alpha \wedge *\delta\beta ,$$

since $** = (-1)^{k(n-k)}$.

Since $\partial M = \emptyset$, by Stokes Theorem, we get

$$0 = \int_M d(\alpha \wedge *\beta)$$

$$= \int_M d\alpha \wedge *\beta - \int_M \alpha \wedge *\delta\beta$$

$$= (d\alpha, \beta) - (\alpha, \delta\beta) . \quad \square$$

Rephrasing, one says that d and δ are <u>adjoints</u> in the (,) inner product.

The δ operator corresponds to the classical divergence operator. This is easily seen: let X be a vector field on M . Then because of the Riemannian structure X corresponds to a 1-form $\widetilde{X}$, where $\widetilde{X}(v) = \langle X, v \rangle$.

<u>Proposition</u>. $\operatorname{div}(X) = -\delta(\widetilde{X})$.

<u>Proof and Discussion</u>. Let $L_X\mu$ be the Lie derivative of μ with respect to X . Then by definition, $\operatorname{div}(X)\mu = L_X\mu$ (see Abraham [2]). We have the general formula

$$L_X\mu = di_X(\mu) + i_X d(\mu) .$$

Now $d(\mu) = 0$ since μ is an n-form, so $L_X\mu = d(i_X\mu) = d(*\widetilde{X})$ (one easily checks that $i_X\mu = *\widetilde{X}$) . Hence

$$\operatorname{div}(X) = \operatorname{div}(X)*\mu = *(\operatorname{div}(X)\mu) = *d*\widetilde{X} = -\delta\widetilde{X} ,$$

since for $k = 1$, $(-1)^{n(k+1)+1} = -1$. $\square$

The Laplace de Rham operator is defined by $\Delta = \delta d + d\delta$. Note that $\Delta : H^s(\Lambda^k) \to H^{s-2}(\Lambda^k)$. If f is a real valued function on R^n, it is easy to check, using the above expressions for d, δ, that $\Delta(f) = -\nabla^2(f)$ where $\nabla^2 f = \mathrm{div}(\mathrm{grad}\ f)$ is the usual Laplacian. Note $\delta f = 0$ on functions.

Proposition. Let $\alpha \in H^s(\Lambda^k)$, then $\Delta\alpha = 0$ iff

$$d\alpha = 0 \quad \text{and} \quad \delta\alpha = 0 .$$

Proof. It is obvious that if $d\alpha = 0$ and $\delta\alpha = 0$ then $\Delta\alpha = 0$. To show the converse, assume $\Delta\alpha = 0$. Then $0 = (\Delta\alpha, \alpha) = ((d\delta + \delta d)\alpha, \alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha)$, so the result follows. □

A form α for which $\Delta\alpha = 0$ is called harmonic.

The Hodge decomposition theorem (for $\partial M = \emptyset$).

Theorem. Let $\omega \in H^s(\Lambda^k)$. Then there is $\alpha \in H^{s+1}(\Lambda^{k-1})$, $\beta \in H^{s+1}(\Lambda^{k+1})$ and $\gamma \in C^\infty(\Lambda^k)$ such that $\omega = d\alpha + \delta\beta + \gamma$ and $\Delta(\gamma) = 0$. Here $C^\infty(\Lambda^k)$ denotes the C^∞ sections of Λ^k. Furthermore $d\alpha$, $\delta\beta$, and γ are mutually L_2 orthogonal and so are uniquely determined.

If $\mathcal{H}^k = \{\gamma \in C^\infty(\Lambda^k) \mid \Delta_\gamma = 0\}$, then the above may be summarized by

$$H^s(\Lambda^k) = d(H^{s+1}(\Lambda^{k-1})) \oplus \delta H^{s+1}(\Lambda^{k+1}) \oplus \mathcal{H}^k .$$

The fact that the Harmonic forms $\mathcal{H}^k$ are all C^∞, follows from regularity theorems on the Laplacian. This fact is also called Weyl's lemma or, its generalization, Friedrich's theorem. We shall discuss this further below.

The Hodge theorem goes back to V. W. D. Hodge [1], in the 1930's. Substantial contributions have been made by many authors, leading up to the present theorem. See for example Weyl [1], and Morrey-Eells [1].

We can easily check that the spaces in the Hodge decomposition are orthogonal. For example

$$(d\alpha, \delta\beta) = (dd\alpha, \beta) = 0$$

since δ is the adjoint of d and $d^2 = 0$.

The basic idea behind the Hodge theorem can be abstracted as follows. We consider a linear operator T on a Hilbert space E with $T^2 = 0$. In our case $T = d$ and E is the L_2 forms. (We ignore the fact that T is only densely defined, etc.) Let T^* be the adjoint of T. Let $\mathcal{H} = \{x \in E \mid Tx = 0 \text{ and } T^*x = 0\}$. We assert

$$E = \overline{\text{Range } T} \oplus \overline{\text{Range } T^*} \oplus \mathcal{H}$$

which, apart from technical points on differentiability and so on is the essential content of the Hodge decomposition.

To see this, note, as before that the ranges of T and T^*

are orthogonal because

$$\langle Tx, T^*y \rangle = \langle T^2x, y \rangle = 0 .$$

Let C be the orthogonal complement of $\overline{\text{Range } T} \oplus \overline{\text{Range } T^*}$. Certainly $\mathcal{H} \subset C$. But if $x \in C$,

$$\langle Ty, x \rangle = 0 \quad \text{for all} \quad y \Rightarrow T^*x = 0 .$$

Similarly $Tx = 0$, so $C \subset \mathcal{H}$ and hence $C = \mathcal{H}$.

The complete proof of the theorem may be found in Morrey [1]. For more elementary expositions, also consult Flanders [1] and Warner [1].

An interesting consequence of this theorem is that $\mathcal{H}^k$ is isomorphic to the kth de Rham cohomology class (the clased k-forms mod the exact ones). This is clear since over M , each closed form ω may be written $\omega = d\alpha + \gamma$. (One can check that the $\delta\beta$ term drops out when ω is closed; indeed we get $0 = d\delta\beta$ so $(d\delta\beta, \beta) = 0$ or $(\delta\beta, \delta\beta) = 0$ so $\delta\beta = 0$.)

The Hodge theorem plays a fundamental role in incompressible hydrodynamics, as we shall see in lecture 4. It enables one to introduce the pressure for a given fluid state.

Below we shall generalize the Hodge theorem to yield some decomposition theorems for general elliptic operators (rather than the special case of the Laplacian). However, we first pause to discuss what happens if a boundary is present.

Hodge theory for manifolds with boundary.

This theory was worked out by Kodaira [1], Duff-Spencer [1], and Morrey [1]. (See Morrey [2], Chapter 7.) Differentiability across the boundary is very delicate, but important. The best possible results in this regard were worked out by Morrey.

Also note that d and δ may not be adjoints in this case, because boundary terms arise when we integrate by parts. Hence we must impose certain boundary conditions.

Let $\alpha \in H^s(\Lambda^k)$. Then α is parallel or tangent to ∂M if the normal part, $n\alpha = i^*(*\alpha) = 0$ where $i : \partial M \to M$ is the inclusion map. Analogously α is perpendicular to ∂M if $t\alpha = i^*(\alpha) = 0$.

Let X be a vector field on M. Using the metric, we know when X is tangent or perpendicular to ∂M. X corresponds to the one-form $\tilde{X}$ and also to the $n - 1$ form $i_X\mu$ (μ is, as usual, the volume form). Then X is tangent to ∂M if and only if $\tilde{X}$ is tangent to ∂M iff $i_X\mu$ is normal to ∂M. Similarly X is normal to ∂M iff $i_X\mu$ is tangent to ∂M. Set

$$H_t^s(\Lambda^k) = \{\alpha \in H^s(\Lambda^k) \mid \alpha \text{ is tangent to } \partial M\}$$

$$H_n^s(\Lambda^k) = \{\alpha \in H^s(\Lambda^k) \mid \alpha \text{ is perpendicular to } \partial M\}$$

and

$$\mathcal{H}^s(\Lambda^k) = \{\alpha \in H^s(\Lambda^k) \mid d\alpha = 0,\ \delta\alpha = 0\} .$$

The condition that $d\alpha = 0$ and $\delta\alpha = 0$ is now stronger than $\Delta\alpha = 0$. Following Kodaira [1], one calls elements of $\mathcal{H}^s$, <u>harmonic fields</u>.

<u>The Hodge Theorem</u>.

$$H^s(\Lambda^k) = d(H_t^{s+1}(\Lambda^{k-1})) \oplus \delta(H_n^{s+1}(\Lambda^{k+1})) \oplus \mathcal{H}^s(\Lambda^k) .$$

One can easily check from the formula

$$(d\alpha, \beta) = (\alpha, \delta\beta) + \int_{\partial M} \alpha \wedge *\beta$$

that the summands in this decomposition are orthogonal.

There are two other closely related decompositions that are of interest.

<u>Theorem</u>.

$$\text{(a)} \qquad H^s(\Lambda^k) = d(H^{s+1}(\Lambda^{k-1})) \oplus D_t^s$$

<u>where</u>

$$D_t^s = \{\alpha \in H_t^s(\Lambda^k) \mid \delta\alpha = 0\}$$

<u>and dually</u>

$$\text{(b)} \qquad H^s(\Lambda^k) = \delta(H^{s+1}(\Lambda^{k+1})) + C_n^s$$

<u>where</u> C_n^s <u>are the closed forms normal to</u> ∂M .

<u>Differential Operators and Their Symbols</u>.

Let E and F be vector bundles over M and let

$C^\infty(E)$, $H^s(E)$ denote the C^∞ and H^s sections of E as above. Assume M is Riemannian and the fibers of E and F have inner products.

A kth order differential operator is a linear map $D : C^\infty(E) \to C^\infty(F)$ such that if $f \in C^\infty(E)$ and f vanishes to kth order at $x \in M$, then $D(f)(x) = 0$. (Vanishing to kth order makes intrinsic sense independent of charts.)

Then in local charts D has the form

$$D(f) = \sum_{0\leq|j|\leq k} a_j \frac{\partial^{|j|} f}{\partial x^{j_i} \cdots \partial x^{j_s}}$$

where $j = (j_1, \ldots, j_s)$ is a multi-index and a_j is a C^∞ function mapping E to F .

Now D has an adjoint operator D^* given in charts (with the standard Euclidean inner product on fibers) by

$$D^*(h) = \sum_{0\leq|j|\leq k} (-1)^{|j|} \frac{1}{\rho} \frac{\partial^{|j|}}{\partial x^{j_i} \cdots \partial x^{j_s}} (\rho a_j^* h)$$

where $\rho\, dx^1 \wedge \ldots \wedge dx^n$ is the volume element and a_j^* is the transpose of a_j . The crucial property of D^* is

$$(g, D^*h) = (Dg, h)$$

where $(\,,\,)$ denotes the L_2 inner product, $g \in C_0^\infty(E)$, and $h \in C_0^\infty(F)$.

A kth order operator induces naturally a map

$$D : H^s(E) \to H^{s-k}(F) .$$

For example we have the operators

$$d : H^s(\Lambda^k) \to H^{s-1}(\Lambda^{k+1})$$

$$\delta : H^s(\Lambda^k) \to H^{s-1}(\Lambda^{k-1})$$

and $\quad \Delta : H^s(\Lambda^k) \to H^{s-2}(\Lambda^k) .$

The <u>symbol</u> of D assigns to each $\xi \in T^*_xM$, a linear map

$$\sigma_\xi : E_x \to F_x .$$

It is defined by

$$\sigma_\xi(e) = D(\frac{1}{k!}(g - g(x))^k f)(x)$$

where $g \in C^\infty(M, R)$, $dg(x) = \xi$ and $f \in C^\infty(E)$, $f(x) = e$. If there is danger of confusion we write $\sigma_\xi(D)$ to denote the dependence on D. By writing this out in coordinates one sees that σ_ξ is a polynomial expression in ξ of degree k obtained by substituting each ξ_{j_i} in place of a $\partial/\partial x^{j_i}$ in the highest order term. For example, if

$$D(f) = \Sigma\, g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + (\text{lower order terms})$$

then

$$\sigma_\xi = \Sigma\, g^{ij}\xi_i\xi_j$$

(g^{ij} is for each ij a map of E_x to F_x). For real valued functions, the classical definition of an elliptic operator is that the above quadratic form be definite. This can be generalized as follows:

D is called <u>elliptic</u> if σ_ξ is an isomorphism for each $\xi \neq 0$.

We have now seen all three classical types of partial differential equations:

elliptic: typified by $\Delta\alpha = \beta$

parabolic: typified by $\dfrac{\partial u}{\partial t} = \Delta u$

hyperbolic: typified by $\dfrac{\partial^2 u}{\partial t^2} = \Delta u$.

To see that $\Delta : H^s(\Lambda^k) \to H^{s-2}(\Lambda^k)$ is elliptic one uses the facts that

(1) the symbol of d is $\sigma_\xi = \xi\wedge$

(2) the symbol of δ is $\sigma_\xi = i_\xi$

and (3) the symbol is multiplicative: $\sigma_\xi(D_1 \circ D_2) = \sigma_\xi(D_1) \circ \sigma_\xi(D_2)$.

<u>The Regularity Theorem and Splitting Theorems</u>.

<u>Theorem</u>. <u>Let</u> M <u>be compact without boundary. Let</u> D <u>be elliptic of order</u> k . <u>Let</u> $f \in L_2(E)$ <u>and suppose</u> $D(f) \in H^s(F)$. <u>Then</u> $f \in H^{s+k}(E)$.

One can allow boundaries if the appropriate boundary conditions are used. See Nirenberg [1]. As a special case of this theorem we get Weyl's lemma: $\Delta f = 0 \Rightarrow f$ is C^∞.

The proof of the theorem is too intricate to go into here; see Palais [1] or Yosida [1]. It is important to note that this sort of result is certainly false if we use C^k spaces, although Holder spaces $C^{k+\alpha}$, $0 < \alpha < 1$ would be suitable.

Theorem. (Fredholm Alternative) *Let* D *be as above. Then*

$$H^s(F) = D(H^{s+k}(E)) \oplus \ker D^*$$

$(D^* : H^s(F) \to H^{s-k}(E))$. *Indeed this holds true if we merely assume that either* D *or* D^* *has injective symbol.*

The proof of this leans heavily on the regularity theorem. The main technical point is to show that $D(H^{s+k})$ is closed. (One uses the fact that $\|f\|_{s+k} \leq \text{const}(\|f\|_s + \|Df\|_s)$, for D elliptic.) Then one shows that the L_2 orthogonal complement of $D(H^{s+k})$ is in $\ker D^*$, just as in the Hodge argument. This yields an L_2 splitting and we get an H^s splitting via regularity. The splitting in case D has injective symbol relies on the fact that D^*D is, in this case, elliptic. One could use, e.g.: $D = d$ to get the Hodge theorem. For details on this, see Berger-Ebin [1].

In later applications (see lectures 4 and 10) we will use this result in the following way. Certain sets in which we are interested will be defined by constraints $f(x) = 0$. The relation $v \mapsto T_x f \cdot v$

will be a differential operator. To show it is surjective (and hence $f^{-1}(0)$ is a submanifold) we can show $(T_x f)^*$ is injective with injective symbol. For then $\ker(T_x f)^* = 0$, so $T_x f$ itself will be onto.

Manifolds of Maps.

History.

The basic idea was first laid down by Eells [1] in 1958. He constructed a smooth manifold out of the continuous maps between two manifolds. In 1961, Smale and Abraham worked out the more general case of C^k mappings. Their notes are pretty much unavailable, but the 1966 survey article by Eells [2] is a good reference. The H^s case is found in a 1967 article by Elliasson [1]. This is also found in Palais [4] where it is done in the more general context of fiber bundles.

Making the manifold out of the C^k diffeomorphism group on a compact manifold without boundary was done independently by Abraham (see Eells [2]) and Leslie [1] around 1966. The H^s case is found in a paper by Ebin [1] and one by Omori [1] around 1968. Ebin also showed that the volume preserving diffeomorphisms form a manifold. Finally Ebin-Marsden [1] worked out the manifold structure for the H^s diffeomorphisms, the symplectic and volume preserving diffeomorphisms for a compact manifold with smooth boundary.

Other papers on manifolds of maps include those of Saber [1], Leslie [2, 3], Omori [2], Gordon [1], Penot [2, 3], and Graff [1]. Some further references are given below.

Local Structure.

Let M and N be compact manifolds and assume N is without boundary. Let n be the dimension of M, and ℓ the dimension of N. Say $f \in H^s(M, N)$ if for any $m \in M$ and any chart (U, φ) containing m and any chart (V, ψ) at $f(m)$ in N, the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to R^\ell$ is in $H^s(\varphi(U), R^\ell)$. This can be shown to be a well defined notion, independent of charts for $s > (n/2)$. The basic fact one needs is that by the Sobolev Theorem we have $H^s(M, N) \subset C^0(M, N)$. Things are not as nice, however, for $s < (n/2)$. It is possible for a map to have a (derivative) singularity which is L_2 integrable in one coordinate system on N and not be integrable in another. So for $s < (n/2)$, $H^s(M, N)$ cannot be defined invariantly. Hence, from now on we assume $s > (n/2)$.

In order to find charts in $H^s(M, N)$ we first need to determine the appropriate modeling space. Let $f \in H^s(M, N)$. The modeling space, should it exist, must be isomorphic to $T_f H^s(M, N)$, whatever that is. So a way to begin is to find a plausible candidate for $T_f H^s(M, N)$. If P is any manifold and $p \in P$ then $T_p P$ can be constructed by considering any smooth curve c in P such that $c(0) = p$; then $c'(0) \in T_p P$ (see lecture 1).

With this in mind, let us consider a curve $c_f :]-1, 1[\to H^s(M, N$

such that $c_f(0) = f$. Now if $m \in M$, then the function $t \mapsto c_f(t)(m)$ is a curve in N (i.e., for each $t \in]-1, 1[$, $c_f(t) \in H^s(M, N)$ and therefore $c_f(t) : M \to N$.) Now $c_f(0)(m) = f(m)$, so the derivative of this curve at 0 , $(d/dt)c_f(t)(m)|_{t=0}$ is an element of $T_{f(m)}N$. So the map $m \mapsto (d/dt)c_f(t)(m)|_{t=0}$ maps M to TN and covers f , i.e., if $\pi_N : TN \to N$ is the canonical projection, this diagram commutes:

$$\begin{tikzcd} & TN \arrow[d, "\pi_N"] \\ M \arrow[ur, "c_f'(0)"] \arrow[r, "f"'] & N \end{tikzcd} \qquad c_f'(0) = \frac{d}{dt}c_f(t)\Big|_{t=0}$$

where

$$c_f'(0)(m) = \frac{d}{dt}c_f(t)(m)\Big|_{t=0} .$$

Making the identification

$$\left(\frac{d}{dt}c_f(t)\Big|_{t=0}\right)(m) = \frac{d}{dt}c_f(t)(m)\Big|_{t=0} ,$$

$c_f'(0)$ is a good candidate for the tangent to c_f at f .

With the above motivation, let us define

$$T_fH^s(M, N) = \{X \in H^s(M, TN) | \pi_N \circ X = f\} .$$

Note this is a linear space, for if V_f and X_f are in $T_fH^s(M, N)$, we can define $aV_f + X_f$ $(a \in R)$ as the map $m \mapsto aV_f(m) + X_f(m)$ where

$V_f(m)$ and $X_f(m)$ are in $T_{f(m)}N$. It is this space which we use as a model for $H^s(M, N)$ near f.

To show this we need the map $\exp_N : T_pN \to N$ for $p \in N$. Recall that if $v_p \in T_pN$ there is a unique geodesic σ_{v_p} through p whose tangent vector at p is v_p. Then $\exp_p(v_p) = \sigma_{v_p}(1)$. In general $\exp_v$ is a diffeomorphism from some neighborhood of 0 in T_pN onto a neighborhood p in N. However, since N is compact and without boundary, it is geodesicly complete and hence $\exp_v$ is defined on all of T_pN. This map can be extended to a map $\exp : TN \to N$ such that if $v_p \in TN$ then $\exp(v_p) = \exp_p(v_p)$. With this map we define the map $\overline{\exp}_f : T_fH^s(M, N) \to H^s(M, N)$

$$X \mapsto \exp \circ X .$$

We assert that $\overline{\exp}_f$ maps the linear space $T_fH^s(M, N)$ <u>onto</u> a neighborhood of f in $H^s(M, N)$ taking 0 to f and hence is a candidate for a chart in $H^s(M, N)$. It should be remarked that in spite of the use of the map $\exp$, the structure is independent of the metric on N. The assertion is easy to check in case things are C^∞ or C^s, by using standard properties of $\exp$; Milnor [1].

For the H^s case and to show that the change of charts is well defined (i.e., maps into the right spaces) and is smooth, one needs the following lemma.

<u>Local ω-Lemma. (Left Composition of Maps).</u> <u>Let</u> U <u>be a bounded</u>

<u>open set in</u> R^p , <u>and</u> $h : R^n \to R^m$ <u>be</u> C^∞ . <u>Then</u> $\omega_h : H^s(U, R^n) \to H^s(U, R^m)$ <u>defined by</u> $\omega_h(f) = h \circ f$ <u>is a</u> C^∞ <u>map</u>.

This conclusion is not true if h is merely an H^s or C^s map. The problem can be seen in this way. If M and N are manifolds and $g : M \to N$ is C^1 then for $p \in M$ and $v_p \in T_pM$, we have $T_pg : T_pM \to T_{g(p)}N$, which is determined in the usual way: let $c :]-1, 1[\in M$ be a curve such that $c(0) = p$ and $c'(0) = v_p$. Then $T_pg(v_p) = (d/dt)g(c(t))|_{t=0}$. Applying this procedure to ω_h , and using the chain rule, we find for $X \in T_fH^s(U, R^n)$ that the tangent of ω_h is the map $T_f\omega_h : X \mapsto Th \circ X$. But since Th is only H^{s-1} , $Th \circ X$ is, at best, in $H^{s-1}(U, R^m)$ and $T\omega_h$ does <u>not</u> map into the tangent space of $H^s(U, R^m)$ at $\omega_h(f)$.

This necessity of differentiating h is a crucial difference between composition on the left and composition on the right.

The exact proof of the ω-lemma may be found in Ebin [1] and the other references above. In fact, the result essentially goes back to Sobolev [1] p. 223. See also Marcus-Mizel [1], and Bourguinon - Brezis [1].

Using the ω-lemma, it is now routine to check that $\exp_f$ yields smooth charts on $H^s(M, N)$. For other methods of obtaining charts, see Palais [4], Penot [3] and Krikorian [1].