

4. The Motion of an Incompressible Fluid.

This lecture is concerned with some fundamental properties of perfect fluids. We shall begin with some motivation and an intuitive outline of the results. Then we shall fill in a number of the gaps. The results of this section lean on work of Arnold [1] and Ebin-Marsden [1]. They are primarily concerned with interpreting the equations as a Hamiltonian system and with the associated existence theory. These go hand in hand, and as a bonus, when one regards the equations from the Hamiltonian point of view the existence theory unexpectedly becomes easier. The difference is essentially that between "Eulerian" and "Lagrangian (following the fluid)" coordinates, as we hope to explain. A similar bit of analysis can be made for elasticity.*

Basic Ideas in Hydrodynamics.

Throughout, let M be a fixed compact, oriented, Riemannian, n -manifold, possibly with a C^∞ boundary. Intuitively, M is the space in which the fluid moves. For example, M might be the unit ball in R^3 . As an aside, for the general theory there seems to be no particular advantage of assuming M is open in R^n . This is because the spaces of mappings of M to M that we will shortly discuss are still very nonlinear.

A diffeomorphism on M is a C^∞ bijective map $\eta : M \rightarrow M$ such that η^{-1} is also C^∞ .

*This remark is based on some recent joint work with T. Hughes.

We let $\mathcal{D} = \{\text{orientation preserving diffeomorphisms on } M\}$.

If the Riemannian structure is given locally by $g_{ij} : M \rightarrow R$, then the volume element μ on M is the n -form which, in a (positively oriented) coordinate chart, is given by

$$\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n$$

or, intrinsically, $\mu(v_1, \dots, v_n) = \sqrt{\det\langle v_i, v_j \rangle}$ for v_1, \dots, v_n oriented tangent vectors. We say a diffeomorphism η is volume preserving if $\eta^*\mu = \mu$. The condition $\eta^*\mu = \mu$ means that the Jacobian of η is one.

By the change of variables formula, it follows that a diffeomorphism η is volume preserving if and only if for every measurable set $A \subset M$, $\mu(A) = \mu(\eta(A))$. Here we also use μ to stand for the measure defined by μ (cf. Abraham [2], §12).

Set $\mathcal{D}_\mu = \{\eta \in \mathcal{D} \mid \eta \text{ is volume preserving}\}$.

For technical reasons it will be convenient to enlarge \mathcal{D} and \mathcal{D}_μ to slightly larger spaces. Namely let \mathcal{D}^S (resp. \mathcal{D}_μ^S) be the completion of \mathcal{D} , (resp. \mathcal{D}_μ) under the Sobolev H^S topology; this will be discussed in detail later.

At least in the beginning, we will be discussing perfect fluids; i.e., fluids which are nonviscous, homogeneous and incompressible. We also ignore external forces for simplicity.

Consider, then, our manifold M whose points are supposed to represent the fluid particles at $t = 0$. Let us look at the fluid moving in M . As t increases, call $\eta_t(m)$ the curve followed by the fluid particle which is initially at $m \in M$. For fixed t , each η_t will be a diffeomorphism of M . In fact, since the fluid is incompressible, we have $\eta_t \in \mathcal{D}_\mu$. The function $t \mapsto \eta_t$ is thus a curve in \mathcal{D}_μ (they are easily seen to be orientation preserving since they are connected to η_0 , the identity function on M). Note that if M has a fixed boundary the flow will be parallel to ∂M .

The motions of a perfect fluid are governed by the Euler equations which are as follows

$$\text{(Euler equations)} \quad \left\{ \begin{array}{l} \frac{\partial v_t}{\partial t} + \nabla_{v_t} v_t = -\text{grad } p_t \\ \text{div } v_t = 0 \\ v_t \text{ is tangent to } \partial M. \end{array} \right.$$

In these equations, $\nabla_{v_t} v_t$ is the covariant derivation and its i th component is given in a coordinate chart by

$$(\nabla_{v_t} v_t)^i = \sum_j v_t^j \frac{\partial v_t^i}{\partial x^j} + \sum_{j,k} \Gamma_{jk}^i v_t^j v_t^k$$

and $p_t = p(t)$ is some (unknown) real valued function on M called the pressure.

In the case of Euclidean space, each $\Gamma_{jk}^i = 0$ and then we get, using vector analysis notation

$$\nabla_{\mathbf{v}} \mathbf{v} = (\mathbf{v} \cdot \nabla) \mathbf{v} .$$

Note. We shall always use a subscripted variable to denote that the variable is held fixed, as in v_t . It will never denote differentiation.

The physical derivation of these equations is quite simple in R^n . We use Newton's Law $F = ma$. We can ignore the mass because of homogeneity (i.e., constant mass density) and we are assuming there are no external forces, so the only forces result from the internal pressure. We wish to deal with conservative force fields and therefore one assumes these internal forces arise as the gradient of a real valued function, the pressure. So we have

$$\text{acceleration} = -\text{grad } p_t .$$

Clearly the acceleration is given by

$$\begin{aligned} a &= \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t, x(t + \Delta t)) - v(t, x(t))}{\Delta t} \\ &= \frac{\partial v}{\partial t} + \sum_i \frac{\partial v}{\partial x^i} \frac{\partial x^i}{\partial t} = \frac{\partial v}{\partial t} + \sum_i \frac{\partial v}{\partial x^i} v^i . \end{aligned}$$

Here we have just used the chain rule. This gives us the correct equation for $\partial v / \partial t$. Now $\text{div } v = 0$ is the same as assuming η_t is

volume preserving, and v parallel to ∂M just corresponds to particles not moving across ∂M .

As the Euler equations stand, they are not Hamiltonian. In fact the way they are written, their form as an evolution equation is not manifest. To rectify the latter problem, we use:

Theorem. Let X be an (H^s) vector field on M . Then X can be uniquely written

$$X = Y + \text{grad } p$$

for Y an (H^s) divergence free vector field parallel to ∂M and p an (H^{s+1}) function.* Here $s \geq 0$. We let $P(X) = Y$ and call P the projection onto the divergence free part.

This follows directly from the Hodge decomposition applied to the corresponding one forms discussed in lecture 3.

Let E denote the space of all divergence free vector fields on M which are parallel to ∂M . Define $T : E \rightarrow E$ by $T(v) = -P(\nabla_v v)$. Note that

$$-P(\nabla_v v) = -(\nabla_v v - \text{grad } p) = -\nabla_v v + \text{grad } p$$

and therefore we can rewrite the Euler equation (modulo a trivial sign convention on p) as a differential equation on the linear space E :

$$\frac{\partial v_t}{\partial t} = T(v_t) v_0 \text{ is given.}$$

* In case M were non-compact, e.g. $M = \mathbb{R}^3$, p would only be locally H^{s+1} .

Notice that T maps H^s to H^{s-1} and so, as discussed in lecture one, the usual existence and uniqueness theorem doesn't apply. It is possible to use other methods however, such as the Nash-Moser technique and Galerkin methods.

Observe that the equation is non-local. This is due to the non-local operator P ; i.e., given X in a neighborhood of x_0 , it is not possible to compute $P(X)$ at x_0 . Rather P is an integral operator. Indeed, from

$$\begin{aligned} X &= Y + \text{grad } p, \\ \delta X &= \Delta p \\ \text{so } p &= \Delta^{-1} \delta X \end{aligned}$$

and Δ^{-1} is an integral operator involving convolution with a suitable green's function.

Summary of the Main Results.

Before getting down to some technical details we would like to present the punch line. To do this we need to state a few facts proven below.

As above, let \mathcal{D} denote all C^∞ diffeomorphisms of M , $\eta : M \rightarrow M$. One can show that \mathcal{D} is (in a certain sense) a smooth manifold modelled on a Fréchet space and it is a "Lie group" in that the group operations of composition and inversion are smooth.

The tangent space to \mathcal{D} at the identity, $T_e \mathcal{D}$ consists of

all vector fields on M . This is as in lecture 3: Indeed, if $\eta(t) \in \mathcal{D}$ is a curve, $\left. \frac{d}{dt} \eta(t)(m) \right|_{t=0}$ represents a vector field on M if $\eta(0)(m) = m$. Generally, $T_{\eta} \mathcal{D} = \{X : M \rightarrow TM \mid \pi \circ X = \eta\}$ where $\pi : TM \rightarrow M$ is the projection.

Also as above, we let $\mathcal{D}_{\mu} = \{\eta \in \mathcal{D} \mid \eta \text{ is volume preserving}\}$. Then \mathcal{D}_{μ} is also a "Lie group" and

$$T_e \mathcal{D}_{\mu} = \{X \in T_e \mathcal{D} \mid \text{divergence of } X = \text{div } x = 0\} .$$

If M has boundary we must always add the condition that X is parallel to the boundary.

Now put a metric on \mathcal{D} and hence \mathcal{D}_{μ} by

$$(X, Y) = \int_M \langle X(m), Y(m) \rangle \eta(m) d\mu(m)$$

for $X, Y \in T_{\eta} \mathcal{D}$. It is easy to see that $(,)$ is right invariant on \mathcal{D}_{μ} . This metric corresponds exactly to the total kinetic energy of the fluid:

$$\text{Energy} = \frac{1}{2} \int_M \|v\|^2 d\mu$$

where v is the velocity field of the fluid:

$$v_t(\eta_t(m)) = \left. \frac{d}{ds} \eta_s(m) \right|_{s=t}$$

Given a time dependent vector field $v(t, x)$ satisfying the

Euler equations, we can construct its flow η_t ; it is the solution to

$$\begin{cases} \frac{d}{dt} \eta_t(m) = v_t(\eta_t(m)) \\ \eta_0(m) = m \end{cases}$$

and of course conversely, given η_t we can obtain v_t .

The first important fact is the following:

Theorem. (Arnold). A time dependent vector field $v(t, x)$ on M satisfies the Euler equations

$$\Leftrightarrow \text{its flow } \eta_t \text{ is a geodesic in } \mathcal{D}_\mu .$$

The second one is:

Theorem. (Ebin-Marsden). The spray governing the geodesics on \mathcal{D}_μ , $Z : T\mathcal{D}_\mu \rightarrow T^2\mathcal{D}_\mu$ is a C^∞ map in H^s : $Z : T\mathcal{D}_\mu^s \rightarrow T^2\mathcal{D}_\mu^s$. Hence the standard existence and uniqueness theorem can be used.

The first result is analogous to the way in which one can describe the motion of a rigid body either by looking at its velocity vector in Eulerian (space) coordinates or as a geodesic in the Lie group $SO(3)$ (body coordinates). In fact one can proceed in general to describe Hamiltonian systems on Lie groups in general of which hydrodynamics and the rigid body are special cases (see Arnold [1], Marsden-Abraham [1] and Iacob [1]).

The second result should be surprising in view of our

previous discussion that the standard existence and uniqueness theorem could not be used in Eulerian coordinates.

We would now like to try to give the essence of this idea. The key thing is that in Lagrangian coordinates, the equations change their character completely. Suppose then that

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\text{grad } p$$

on \mathbb{R}^3 . We let η_t be the flow of v and look at the new variables $\eta_t, X = v_t \circ \eta_t$ instead of v itself. Now

$$\begin{aligned} \frac{\partial X}{\partial t} &= \frac{\partial v}{\partial t} \circ \eta_t + \sum \frac{\partial v_t}{\partial x^i} \frac{\partial \eta_t^i}{\partial t} \\ &= \frac{\partial v}{\partial t} \circ \eta_t + \sum \left(\frac{\partial v_t}{\partial x^i} \cdot v_t^i \right) \circ \eta_t = -\text{grad } p \end{aligned}$$

since v satisfies $(\partial v / \partial t) + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\text{grad } p$.

In order for the spray Z to be smooth, the map $(\eta, X) \mapsto \left(\frac{\partial \eta}{\partial t}, \frac{\partial X}{\partial t} \right)$ has to at least map H^s to H^s . Now $\text{grad } p$ is the gradient part of $(\mathbf{v} \cdot \nabla) \mathbf{v}$, so it is not completely obvious that $\text{grad } p$ is H^s if v is H^s . However we can see it by a simple calculation: Indeed take the divergence of

$$\frac{\partial v}{\partial t} + \sum v^i \frac{\partial v}{\partial x^i} = -\text{grad } p$$

to get

$$-\Delta p = \sum_j \frac{\partial}{\partial x^j} \left(v^i \frac{\partial v^j}{\partial x^i} \right) = \sum_j \frac{\partial v^i}{\partial x^j} \frac{\partial v^j}{\partial x^i}$$

since $\operatorname{div} v = 0$. Thus if v is H^s , $\operatorname{grad} p$ will be H^s as well (regularity of the Laplace operator).

By combining the previous two theorems with the existence and uniqueness theorem, we obtain the following.

Corollary. Let $s > \frac{n}{2} + 1$ ($n = \text{dimension of } M$), and v_0 a divergence free H^s vector field parallel to ∂M . Then there is a unique $H^s v_t$ equalling v_0 at $t = 0$ which satisfies the Euler equations (that is, there is a $p(x, t)$ such that the Euler equations hold), defined for $-\epsilon < t < \epsilon$ for some $\epsilon > 0$. If v_0 is C^∞ , so is v_t .

Recently Bourguignon and Brezis [1] have obtained these results in a more classical way without using infinite dimensional manifolds. The same results are also obtained in the spaces $W^{s,p}$, $s > \frac{n}{p} + 1$.

The question naturally arises if we can infinitely extend the solutions in the corollary. Such solutions would be called global.

Theorem. (Wolibner (1933), Judovich (1964), Kato (1967)). If $\dim M = 2$, the solutions in theorem 2 can be indefinitely extended for all $t \in \mathbb{R}$ (and remain smooth).

The problem is open if $\dim M = 3$.

The problem is also open, in general, if we consider the equations with viscosity. This leads us to a hamiltonian system with a dissipative term.

$$\text{Navier-Stokes equations} \quad \left\{ \begin{array}{l} \frac{\partial v}{\partial t} - \nu \Delta v + \nabla_v v = -\text{grad } p \quad (+ \text{ forces}) \\ \text{div } v = 0 \\ v = 0 \quad \text{on } \partial M \quad (\text{note the change in boundary conditions}) \end{array} \right.$$

The term $\nu \Delta v$ is an approximation to viscous forces in the fluid which tend to slow the fluid down. Thus the chances for a global solution are increased.

For the Euler equations it is known (see Marsden-Ebin-Fischer [1]) that if the C^1 norm of v is bounded on an interval $[0, T[$, then the solution can be extended beyond T . Thus one gets global solutions if an a priori bound is known on the C^1 norm. One can do better for the Navier-Stokes equations:

Theorem. (Leray [3]). Let v_t be a solution to the Navier-Stokes equations, $\dim M = 3$. Suppose one has an a priori bound on the spatial L_p norm of v_t on finite t -intervals, where $p > 3$. Then the solution can be infinitely extended to $[0, \infty[$ as a smooth solution.

One can also show that one has global solutions if the initial data is sufficiently small (Ladyzhenskaya [2]) and for fixed

but perhaps large initial data the time of existence is of the order of ν . It is really the case of "large" initial data which is of interest and for these, Leray's theorem gives a criterion which is necessary and sufficient, but is not too easy to verify (see remarks below).

In the next lecture we shall discuss a method, using "Chorin's formula", which gives a fundamental improvement on the time of existence for general initial data.

These difficulties with global solutions bear on the nature of turbulence. (See the next lecture for further discussion.) Indeed Leray believed that it is possible for solutions to become non-smooth and non-unique after some time interval $[0, T]$, at which time they turn into weak, or Hopf solutions and this was supposed to represent turbulence.

Nowadays, the opposite point of view prevails, but it is not yet completely settled. In other words, we now believe that turbulence represents very complicated, but still smooth solutions to the equations.

But the situation is very delicate and one must be careful. For example a law of Kolmogorov,^{*} experimentally verified for turbulent flows, when translated into norms indicates that one has an a priori bound on the L_p norm of v_t for $p < 3$! This just misses the critical value of $p = 3$, but refinements of this may be able to raise the value of p above 3.

* We refer to the $-5/3$ law; see Landau-Lifschitz (1). The experimental verification is not conclusive and is also consistent with other possible laws.

There are other reasons for this view and we shall discuss them below in lecture 5. Briefly, turbulence is believed to be a result of successive losses of stability (rather than smoothness).

From Arnold's theorem we can rephrase the problem of extending solutions of the Euler equations as follows:

Problem. Let M be a (compact) 3-manifold. Then is \mathcal{D}_{μ}^S geodesically complete? (That is, do geodesics exist for all time? From Wolibner's result, the answer is yes, if $\dim M = 2$.) *

The following simple lemma bears on the problem (the lemma is standard); see Wolf [1], p. 89 and lecture 6 below.

Lemma. Let G be a finite dimensional Lie group with a right invariant riemannian metric. Then G is geodesically complete.

The lemma also holds if G is a "Hilbert group-manifold", but unfortunately, it does not apply to our problem because the topology of our metric (recall it gives the L^2 norm) does not coincide with the topology on \mathcal{D}_{μ}^S . If the requirement $\operatorname{div} v = 0$ were dropped, the result is definitely false -- this is the phenomenon of shock waves in compressible flow. (For example the solution of $(\partial u / \partial t) + u(\partial u / \partial x) = 0$ in one dimension is $u(t, x) = u_0(y)$ where $x = y + tu_0(y)$. One can see as soon as $x \mapsto y$ becomes non-invertible, that derivatives of u blow up.)

* At present the most reasonable sounding conjecture for this problem is "no" because of "vortex sheets" but "yes" for the Navier-Stokes equations for which vortex sheets are impossible by Leray's theorem.

Kelvin Circulation Theorem.

This is a standard classical theorem of hydrodynamics that is very easy to prove in our context. It says the amount of circulation about any closed loop is constant in time.

Kelvin Circulation Theorem. Let M be a manifold and $\ell \subset M$ a smooth closed loop i.e., a compact one manifold. Let u_t be a solution to the Euler Equations on M and $\ell(t)$ be the image of ℓ at time t when each particle moves under the flow η_t of u_t i.e., $\ell(t) = \eta_t(\ell)$. Then

$$\frac{d}{dt} \int_{\ell(t)} \tilde{u}_t = 0 \quad (\tilde{u}_t \text{ is the one form dual to } u_t)$$

Proof. We have the identity $L_u \tilde{u} = \nabla_u u + \frac{1}{2} d\langle u, u \rangle$, valid for any vector field u on the manifold M . We leave the verification as an exercise.

Then, identifying the differential forms with their dual vector fields, we find $P(L_u \tilde{u}) = P(\nabla_u u)$ since P annihilates exact forms. (Remember P projects onto the divergence free part).

So substituting into the Euler equations, we get the following alternative form:

$$\frac{\partial \tilde{u}}{\partial t} + P(L_u \tilde{u}) = 0 \quad (*)$$

Let η_t be the flow of u_t . Then $\ell(t) = \eta_t(\ell)$, and so changing

variables,

$$\int_{\eta_t(\ell)} \tilde{u}_t = \int_{\ell} \eta_t^*(\tilde{u}_t)$$

which becomes, on carrying out the differentiation,

$$\frac{d}{dt} \int_{\eta_t(\ell)} \tilde{u}_t = \int_{\ell} \eta_t^*(L_u \tilde{u}) + \eta_t^* \frac{\partial \tilde{u}}{\partial t} .$$

Let $P(L_u \tilde{u}) = L_u \tilde{u} - \text{grad } q$. By Stokes theorem $\int_{\ell} \text{grad } q = 0$,

$$\frac{d}{dt} \int_{\ell(t)} \tilde{u}_t = \int_{\ell} \eta_t^*(L_u \tilde{u} + \frac{\partial \tilde{u}}{\partial t} - \text{grad } q) = 0 . \quad \square$$

In practical fluid mechanics, this is an important theorem. One can obtain a lot of qualitative information about specific flows by following a closed loop throughout time and using the fact the circulation is constant.

The quantity $\tilde{d}u = \omega$ is the vorticity. (In three dimensions $\vec{\omega} = \vec{\nabla} \times \vec{u}$.) From (*) we get $\frac{\partial \omega}{\partial t} + L_u \omega = 0$ and so $\omega_0 = \eta_t^* \omega_t$, showing that vorticity moves with the fluid. This is, via Stokes theorem, another way of phrasing Kelvin's theorem.

Steady Flows.

A flow is steady if its vector field satisfies $(\partial u / \partial t) = 0$, i.e., u is constant in time. This condition means that the "shape" of the fluid flow is not changing. Even if each particle is moving

under the flow, the global configuration of the fluid does not change.

Not much is really known about steady flows, their stability, or what initial conditions result in steady flows. We should mention, however, that for viscous flow quite a bit more is known. See for example Ladyzhenskaya [2] and Finn [1]. There are some elementary equivalent formulations of the Euler problem.

Proposition. Let u_t be a solution to the Euler equations on a manifold M and η_t its flow. Then the following are equivalent:

- (1) $u_0 \in T_{e_\mu}^S$ yields a steady flow (i.e., $(\partial u / \partial t) = 0$)
- (2) η_t is a one parameter subgroup of $\mathcal{D}_\mu^S(M)$
- (3) $L_{u_0} \tilde{u}_0$ is an exact form
- (4) $i_{u_0} du_0$ is an exact form.

The details are omitted.

It follows at once from (4) that if $u_0 \in T_{e_\mu}^S(M)$ is a harmonic vector field; i.e., u_0 satisfies $\delta u_0 = 0$ and $\tilde{d}u_0 = 0$ then it yields a stationary flow. Also it is known there are other steady flows for manifolds with boundary. For example, on a closed 2-disc, with polar coordinates (r, θ) , $v = f(r)(\partial/\partial\theta)$ is the velocity field of a steady flow because

$$\nabla_v v = -\nabla p, \text{ where } p(r, \theta) = \int_0^r f^2(s) s \, ds.$$

Clearly such a v need not be harmonic.

For the remainder of this chapter we shall fill in a number of details. In particular we shall prove \mathcal{D}_μ^s is a smooth manifold, will prove Arnold's theorem and outline the proof that the geodesic spray is smooth. (In Arnold [1], and Marsden-Abraham [1], the result of Arnold is proved using Lie group methods).

Groups of Diffeomorphisms.

These objects have a very interesting yet complicated structure. For this section we let M be a compact manifold without boundary. Let $\mathcal{D}^s(M) = \{f \in H^s(M, M) \mid f \text{ is one-one, orientation preserving and } f^{-1} \in H^s(M, M)\}$. The fact that $\mathcal{D}^s(M)$ is a manifold is a trivial consequence of the fact that $H^s(M, M)$ is a manifold and the following proposition;

Proposition. If $s > (n/2) + 1$, then $\mathcal{D}^s(M)$ is open in $H^s(M, M)$.

Proof. Since $s > (n/2) + 1$, we have a continuous inclusion $H^s(M, M) \subset C^1(M, M)$ (by the Sobolev Theorem). So it is sufficient to show that if a map g on M is C^1 close to a diffeomorphism, then g is a diffeomorphism. To show this, note that $G : f \mapsto \inf_{x \in M} Jf(x)$ is a continuous real valued map on $C^1(M, M)$, where $Jf(x)$ is the Jacobian of $f : f^*\omega = (Jf)_\mu$. Also, since M is compact, if $f \in \mathcal{D}^s(M)$, then $G(f) \neq 0$. By continuity of G , there is a neighborhood U of f in $C^1(M, M)$ such that if $g \in U$ then $G(g) \neq 0$. By the inverse function theorem U consists of local diffeomorphisms. It is easy to show that if $g \in U$ then g is an onto map. This is

because $g(M)$ is open in M , as g is a local diffeomorphism and since g is continuous and M is compact, then $g(M)$ is closed. Hence if M is connected $g(M) = M$. (If M is not connected, one need just remark that g maps into each component of M since f does and g is uniformly close to f .) It remains to show there is a neighborhood of f containing only 1-1 functions. (It is not true that a local diffeomorphism on a compact set is a diffeomorphism. Consider the map which wraps S^1 around itself twice.) It is an easy exercise in point set topology to show that if M is connected then any local diffeomorphism on M is a covering map; that is, is globally k to 1 for some integer k . Also, the function that assigns to a local diffeomorphism f the number of elements in $f^{-1}(x)$ for any $x \in M$ is continuous in the C^1 topology onto the integers. In particular there is a neighborhood of a diffeomorphism containing only diffeomorphisms. \square

Because of the above proposition, we will henceforth assume $s > (n/2) + 1$.

It is unknown whether, in general, the composition of two H^s maps is again H^s . In all known proofs one needs that one of the maps is a diffeomorphism or is C^∞ . The main composition properties are stated in the following.

Theorem. (a) \mathcal{D}^s is a group under composition.

(b) (α -Lemma) If $\eta \in \mathcal{D}^s$ the map $R_\eta : \mathcal{D}^s \rightarrow \mathcal{D}^s$,

$\zeta \mapsto \zeta \circ \eta$ is a C^∞ map (in fact R_η is clearly "formally linear" and continuous).

(c) (ω -Lemma-Global) If $\eta \in \mathcal{D}^S$, then $L_\eta : \zeta \mapsto \eta \circ \zeta$ is C^0 . (This map is definitely not smooth, in fact it is not even a locally Lipschitz map.*)

(c)' More generally, the map

$$\mathcal{D}^{s+l} \times \mathcal{D}^s \rightarrow \mathcal{D}^s$$

$$(\eta, \zeta) \mapsto \eta \circ \zeta$$

is C^l .

(d) \mathcal{D}^S is a topological group.

Remark. (d) follows from the other parts of the theorem because of the following lemma of Montgomery [1]:

Lemma. Let G be a group that is also a topological space. Assume further that G is a separable, metrizable, Baire space and multiplication in G is separately continuous. Then G is a topological group.

We shall not prove (a), (b), (c)', here since we have already given the basic ideas involved. The proof may be found in Ebin [1]. Another useful fact proved by Ebin is that if η is an H^S map with a C^1 inverse, then the inverse is H^S . This is analogous to what one has in the C^k inverse function theorem (Lang [2]). These results also extend to the L^k_p and $C^{k+\alpha}$ spaces; cf. Bourguinon and Brezis [1] and Ebin-Marsden [1].

* That it cannot be locally Lipschitz follows from an example given by T. Kato. See the footnote on page 118.

\mathfrak{D}^S as a "Lie group".

\mathfrak{D}^S is not precisely a Lie group, (since a left multiplication is continuous, but not smooth) but it shares some important Lie group properties. If we were to work with $\mathfrak{D} = \mathfrak{D}^\infty$, we would have Lie group, but not a Banach manifold.

In general if G is a Lie group and $e \in G$ is the unit element, then the Lie Algebra \mathcal{G} of G may be identified with $T_e G$. Hence, $T_e \mathfrak{D}^S(M) = \mathfrak{X}^S(M) = H^S(TM) = H^S$ vector fields on M (recall members of $T_e \mathfrak{D}^S(M)$ cover the identity map on M) serve as the Lie algebra for \mathfrak{D}^S . Since right multiplication is smooth, we can talk about right invariant vector fields on \mathfrak{D}^S . By the ω -lemma, if $X \in \mathfrak{X}^{S+l}$, the map $\tilde{X} : \eta \mapsto X \circ \eta$ is a C^l map from \mathfrak{D}^S to $T\mathfrak{D}^S$ ($l \geq 0$); in particular $X \circ \eta \in T_\eta \mathfrak{D}^S$ and so it is a vector field on \mathfrak{D}^S . In fact \tilde{X} is a right invariant C^l vector field on \mathfrak{D}^S (i.e., $(R_\eta)_* (\tilde{X}_\zeta) = \tilde{X}_\zeta \circ \eta$ for $\eta \in \mathfrak{D}^S$ and $\tilde{X}_\zeta = \tilde{X}(\zeta) \in T_\zeta \mathfrak{D}^S$). Conversely if \tilde{X} is a right invariant C^l vector field, then $\tilde{X}(e) \in \mathfrak{X}^{S+l}$. In fact the right invariant C^l vector fields are isomorphic to \mathfrak{X}^{S+l} by evaluation at e , and in particular $T_e \mathfrak{D}^S$ is isomorphic to the C^0 right invariant vector fields.

For $l \geq 1$, there is a natural Lie bracket operation on the C^l right invariant vector fields on \mathfrak{D}^S . This defines the bracket operation on the corresponding members of $T_e \mathfrak{D}^S(M)$. We now establish that the Lie algebra structure of \mathfrak{D}^S is the usual Lie algebra structure on the vector fields.

Theorem. Let $l \geq 1$ and for $X, Y \in H^{s+l}(TM)$, let \tilde{X} and \tilde{Y} be the corresponding right invariant vector fields on \mathcal{D}^s . Then $[\tilde{X}, \tilde{Y}]_e = [X, Y]$, the usual Lie bracket of vector fields on M .

Proof. Recall that locally $[X, Y] = DX \cdot Y - DY \cdot X$ (where DX is the derivative of X ; cf. lecture 1. However, as shown above, for $\eta \in \mathcal{D}^s$, $\tilde{X}(\eta) = X \circ \eta$ and $\tilde{Y}(\eta) = Y \circ \eta$, so in particular since $\tilde{Y} \cdot X = TY \cdot X$ we get $[\tilde{X}, \tilde{Y}]_e = (D\tilde{X} \cdot \tilde{Y} - D\tilde{Y} \cdot \tilde{X})_e = D\tilde{X}(e) \cdot \tilde{Y}(e) - D\tilde{Y}(e) \cdot \tilde{X}(e) = DX \cdot Y - DY \cdot X$. \square

Note since $DX \cdot Y \in H^{s+l-1}(TM)$, we really cannot put this bracket on $T_e \mathcal{D}^s = \mathfrak{X}^s$ and none of the $\mathcal{D}^{s+l}(TM)$ are Lie algebras since they are not closed under the bracket operation; one would have to pass to $\mathcal{D} = \mathcal{D}^\infty$.

For any Lie group G , there is a standard \exp map from \mathcal{G} onto a neighborhood of the identity e in G . If $X \in \mathcal{G}$, there is a unique one parameter smooth subgroup c in G (i.e., $c(t+s) = c(t) \cdot c(s)$ and $c(0) = e$) such that $c'(0) = X$. In this case X is the infinitesimal generator of c ; c is the solution of $c'(t) = \tilde{X}(c(t))$ where \tilde{X} is the right invariant vector field equaling X at e . Define $\exp(X) = c(1)$.

If G has a Riemannian structure, then there is another map $\overline{\exp} : \mathcal{G} \rightarrow G$ defined (as above) by following geodesics instead of subgroups. If the metric is bi-invariant (i.e., if $g = (g_{ij})$ is the Riemannian metric, then for $a \in G$, $(R_a)^*(g) = (L_a)^*(g) = g$)

then it is easy to show the two exp maps coincide.

In the case of \mathfrak{D}^S , we will construct a metric that is right invariant, but not left invariant, and so the two exp maps will in general be different.

Actually \mathfrak{D}^S (and \mathfrak{D}^S_μ) have no bi-invariant metrics. (Indeed, as in Sternberg [1], a group G has a bi-invariant metric iff the image of G under the adjoint map is relatively compact.)

Let $X \in T_e \mathfrak{D}^S$. Then $X \in H^S(TM) = \mathfrak{X}^S$ and therefore has a flow F_t ($F_t(m)$ is the integral curve of X starting at m). This is a one parameter group since $F_{s+t} = F_s \circ F_t$. Since M is compact, F_t is defined on all of M for all $t \in \mathbb{R}$. (Flows of C^r vector fields on compact manifolds are always complete.)

Let us argue that we should have

$$\exp X = F_1$$

where \exp is the (right) exponential map on \mathfrak{D} . Indeed we need to show F_t is an integral curve of \tilde{X} defined above. But

$$\frac{d}{dt} F_t(m) = X(F_t(m))$$

i.e.

$$\frac{d}{dt} F_t = X \circ F_t = \tilde{X}(F_t)$$

Hence F_t is an integral curve in \mathfrak{D}^S of \tilde{X} . This justifies us in

saying that $\exp X = F_1$.

Actually it is not obvious that $F_t \in \mathcal{D}^S$; i.e., the flow of an H^S vector field is H^S . (This, of course, is well-known in the C^k case -- see lecture 1.) However the H^S version is also true. See Ebin-Marsden [1], Bourguignon and Brezis [1] and Fischer-Marsden [2] for proofs.

So via this theorem and the remark that $F_0 = \text{id}$, we have a sort of Lie group exponential map from $T_e \mathcal{D}^S(M)$ into a neighborhood of identity, $X \mapsto F_1$. It is natural to ask why not use this exp map to directly define charts on $\mathcal{D}^S(M)$. We cannot do this because it is a fact that exp does not map onto any neighborhood of the identity in $\mathcal{D}^S(M)$. This is equivalent to saying that there are diffeomorphisms near e not embeddable in a flow. In other words for any neighborhood U of e in \mathcal{D}^S , there is $\eta \in U$ such that there is no flow F_t with $F_1 = \eta$. In fact η will not, in general, have a square root. Explicit examples have been given by several people such as Eells and Smale. One is written down in Omori [1] and in Friefeld [1].

A consequence of this is that the exp map on $T_e \mathcal{D}^S$ is not C^1 , for if it were, it would be locally onto by the inverse function theorem.

Volume Preserving Diffeomorphisms.

For now let M be a compact Riemannian manifold without boundary. (The boundary case is done below.) Let μ be the volume

form given by the metric on M . Recall from the introduction that

$\mathcal{D}_\mu^s = \{f \in \mathcal{D}^s \mid f^*(\mu) = \mu\}$. We shall show that \mathcal{D}_μ^s is a smooth submanifold of \mathcal{D}^s .

Recall that if $f : P \rightarrow Q$ is a smooth map between manifolds, f is a submersion on a set $A \subset P$ if $T_x f : T_x P \rightarrow T_{f(x)} Q$ is a surjection, for each $x \in A$ and the kernel splits. We showed in lecture one that if P, Q are Hilbert manifolds and $f : P \rightarrow Q$ is a C^∞ map, then for $g \in Q$, $f^{-1}(g)$ is a C^∞ submanifold of P , if f is a submersion on $f^{-1}(g)$.

We shall need the following:

Lemma. Let λ be an n -form on M such that $\int_M \lambda = 0$. Then λ is exact; $\lambda = d\alpha$ for an $n-1$ form α .

This is a special case of de Rham's theorem, stating that a closed form is exact if all its periods vanish. For the proof, see for example Warner [1]. A discussion is also found in Flanders [1].

Theorem. Let $s > (n/2) + 1$. Then \mathcal{D}_μ^s is a closed C^∞ submanifold of \mathcal{D}^s .

Proof. Let μ be the volume form on M . By the Hodge theorem, $[\mu]_s = \mu + d(H^{s+1}(\Lambda^{n-1}))$ is a closed affine subspace of $H^s(\Lambda^n)$, being the translate of the closed subspace $d(H^{s+1}(\Lambda^{n-1}))$ by μ . Define the map

$$\psi : \mathcal{D}^{s+1}(M) \rightarrow [\mu]_s$$

$$\eta \mapsto \eta^*\mu .$$

Now $\eta^*\mu \in [\mu]_s$ since

$$\int_M (\mu - \eta^*\mu) = \int_M \mu - \int_M \eta^*\mu = 0 .$$

Hence $\mu - \eta^*\mu = d\alpha$ by the lemma. By the ω -Lemma, one can easily see that ψ is a C^∞ map. Now $\mathcal{D}_\mu^{s+1}(M) = \psi^{-1}(\mu)$, so if ψ is a submersion then $\mathcal{D}_\mu^{s+1}(M)$ is a C^∞ submanifold of $\mathcal{D}^{s+1}(M)$.

We shall show this at $e \in \mathcal{D}^s(M)$ (e is the identity map).

It turns out that $T_e\psi(X) = L_X\mu$ where $X \in T_e\mathcal{D}^{s+1}(M)$. Indeed let

η_t be a curve tangent to X , such as its flow. Then

$T_e\psi(X) = (d/dt)\eta_t^*\mu|_{t=0}$ which is indeed the Lie derivative. Using the "magic" formula $L_X\mu = di_X\mu + i_Xd\mu$ for the Lie derivative and the fact that $d\mu = 0$, we get

$$T_e\psi(X) = L_X\mu = di_X\mu .$$

Hence to show $T_e\psi$ is a surjection, we only need show that

$$\{i_X\mu : X \in T_e\mathcal{D}^{s+1}(M)\} = H^{s+1}(\Lambda^{n-1}) .$$

But $i_X\mu = *\tilde{X}$ and $*$ is a bijection between $n-1$ forms and 1-forms.

Hence $T_e\psi$ is onto. Similarly $T_\eta\psi$ is onto. \square

However this last step in the proof only holds if μ is a

(nowhere zero) n -form or a closed nondegenerate 2-form. This remark allows us to show that the diffeomorphisms that preserve a symplectic form form a submanifold of the diffeomorphism group using the same sort of argument.

It follows from the basic connection between Lie derivatives and flows given in Lecture one that a vector field generates volume preserving diffeomorphisms if and only if it is divergent free. In our context this means $T_{e, \mu}^{\mathcal{D}^S(M)} = \{X \in \mathcal{D}^S(M) \mid \delta X = 0\}$. This is clearly a subspace of $T_e^{\mathcal{D}^S(M)}$ and in fact $T_{e, \mu}^{\mathcal{D}^S(M)}$ is closed under the bracket operation, in the same sense as $T_e^{\mathcal{D}^S(M)}$ (see page 92 above).

Manifolds with Boundary.

Suppose M is a compact, oriented, Riemannian Manifold with smooth boundary. Let \tilde{M} be the double of M , i.e., \tilde{M} is two copies of M with the boundaries identified, with the obvious differential structure. Now \tilde{M} is a compact, oriented, Riemannian manifold without boundary and M has a natural imbedding in \tilde{M} . We have the manifold structure of $H^S(M, \tilde{M})$ by our above work. Clearly $\mathcal{D}^S(M) \subset H^S(M, \tilde{M})$ and in fact:

Theorem. $\mathcal{D}^S(M)$ is a C^∞ submanifold of $H^S(M, \tilde{M})$.

Sketch of Proof. Briefly, we put a metric on \tilde{M} such that $\partial M \subset \tilde{M}$ is totally geodesic. Then let $E : TH^S(M, \tilde{M}) \rightarrow H^S(M, \tilde{M})$ be the exponential map associated with this metric.

Let $\eta \in \mathcal{D}^S(M) \subset H^S(M, \tilde{M})$ and choose an exponential chart

$E : U \subset T_\eta H^S(M, \tilde{M}) \rightarrow H^S(M, \tilde{M})$ about η . Also we should have

$$T_\eta \mathcal{D}^S(M) = \{X \in H^S(M, \tilde{TM}) \mid X \text{ covers } \eta \text{ and} \\ X(x) \in T_{\eta(x)} \partial M \text{ for all } x \in \partial M\}$$

which is a closed subspace of

$$H^S(M, \tilde{TM}) = TH^S(M, \tilde{M}) .$$

Since ∂M is totally geodesic, E takes $U \cap T_\eta \mathcal{D}^S(M)$ onto a neighborhood of η in $\mathcal{D}^S(M)$. See Ebin-Marsden [1] for details. \square

By inspecting the above argument we see $T_e \mathcal{D}^S(M) = \{H^S \text{ vector fields on } M \text{ that are tangent to } \partial M\}$. Formally, this is a Lie algebra in the same sense as we had when M had no boundary.

Theorem. If μ is the volume on M and $\mathcal{D}_\mu^S(M)$ is the set of volume preserving diffeomorphisms, then $\mathcal{D}_\mu^S(M) \subset \mathcal{D}^S(M)$ is a smooth submanifold.

This is proven as in the case that M has no boundary. This proof works here because we have the Hodge theorems for manifolds with boundary. The rest of the material from the no boundary case (such as the α and ω -lemmas) carries over to the case when M has a boundary. For the non-compact case, see Cantor [1,2].

If M has boundary, then $H^S(M, M)$ will not be a smooth manifold, but will have "corners". Thus it is interesting that nevertheless, $\mathcal{D}^S(M)$ is a smooth manifold.

Topology of the Diffeomorphism Group.

For topological theorems we can work in $\mathcal{D}(M) = \mathcal{D}^\infty(M)$.

Indeed it follows from very general results of Cerf [1] and Palais [3] that the topology of \mathcal{D}^S and \mathcal{D} are the same; one uses the fact that the injection of \mathcal{D} into \mathcal{D}^S is dense. The first theorem in this field was proven by Smale [1] in 1959. He showed that $\mathcal{D}(S^2)$ is contractable to $SO(3)$; here S^2 is the 2-sphere, and $SO(3)$ is the special orthogonal group on R^3 , which we can regard as the (identity component of the) isometry group of S^2 . This theorem was extended to all compact 2-manifolds by Earle and Eells [1] and to the boundary case by Earle and Schatz [1].

It is fairly simple to show that $\mathcal{D}(S^1)$ is contractable to $SO(2)$. The following argument is based on a suggestion of J. Eells.

First fix $s \in S^1$. Let $\theta : [0, 1] \rightarrow S^1$ be a parameterization of S^1 such that $\theta(0) = \theta(1) = s$. Now let f be a diffeomorphism that leaves s fixed. Then the map

$$h_f(t, x) : [0, 1] \times S^1 \rightarrow S^1$$

$$(t, x) \mapsto \begin{cases} \theta(t\theta^{-1}(x) + (1-t)\theta^{-1}(f(x))) & x \neq s \\ s & x = s \end{cases}$$

is an homotopy from f to id_{S^1} .

Suppose $g : S^1 \rightarrow S^1$ maps s to $g(s) \neq s$; then there is a rotation $r : S^1 \rightarrow S^1$ that carries $g(s)$ to s and therefore

$r \circ g(s) = s$. Hence, by the above argument $r \circ g$ is homotopic to the identity. Therefore g is homotopic to r^{-1} , which is, naturally, also a rotation. \square

For dimension 3 the situation is much more complicated and little is known. The work of Cerf [2] seems indicative of the complexity. Antoneli et al. [1] have shown that if M has high dimension $\mathcal{D}(M)$ will not have the homotopy type of a finite cell complex. Various people have also been working towards showing $\mathcal{D}(M)$ is a simple group; cf. Herman [1], Epstein [1] and Herman-Sergeraert [1]. This result was actually known to von Neumann for the case of homeomorphisms. It has recently been announced for $\mathcal{D}(M)$ by W. Thurston.

Another important result in this field is that of Omori [1]. He proved that for any compact Riemannian manifold without boundary $\mathcal{D}(M)$ is contractible to $\mathcal{D}_\mu(M)$, the set of volume preserving diffeomorphisms. In fact if $V = \{v \in C^\infty(\Lambda^n) \mid v \text{ is nondegenerate, positively oriented and } \int_M v = \int_M \mu\}$ ($C^\infty(\Lambda^n)$ are the C^∞ n -forms) then $\mathcal{D}(M)$ is diffeomorphic to $\mathcal{D}_\mu \times V$. This implies $\mathcal{D}(M)$ is contractible to $\mathcal{D}_\mu(M)$ since V is contractible to μ . (In fact V is convex.) The proof that $\mathcal{D}(M) \approx \mathcal{D}_\mu(M) \times V$ uses an important result of Moser [1].

Theorem. [Moser]. If on a compact manifold M , there are 2 volume elements μ and ν such that $\int_M \nu = \int_M \mu$, then there is map $f \in \mathcal{D}(M)$ such that $f^*(\nu) = \mu$.

We formulate the results following Ebin-Marsden [1].

Theorem. Let M be compact without boundary with a smooth volume
element μ . Let

$$V = \{v \in C^\infty(\Lambda^n) \mid v > 0, \int_M v = \int_M \mu\} .$$

Then \mathcal{D} is diffeomorphic to $\mathcal{D}_\mu \times V$. In particular (since V is
convex), \mathcal{D}_μ is a deformation retract of \mathcal{D} .

For the proof, we begin by proving Moser's result.

Lemma. There is a map $\chi : V^s \rightarrow \mathcal{D}^s$, $s > (n/2) + 1$ such that
 $\psi_\mu : \mathcal{D}^s \rightarrow V^{s-1}$, $\psi_\mu(\eta) = \eta^*(\mu)$ satisfies $\psi_\mu \circ \chi = \text{identity}$. Further,
 $\chi : V \rightarrow \mathcal{D}$ is a C^∞ map.

Proof. For $v \in V_s$, let $v_t = tv + (1-t)\mu$, so that $v_t \in V^s$.
Since $\int \mu = \int v$, we can write, as before, $\mu - v = d\alpha$. Define X_t
by $i_{X_t} v_t = \alpha$ so that $X_t \in H^s(TM)$. Let η_t be the flow of X_t ,
so $\eta_t \in \mathcal{D}^s$. Define $\chi(v) = \eta_1^{-1}$. We want to show that $\eta_t^*(v_t) = \mu$
by showing $d/dt(\eta_t^*(v_t)) = 0$. Indeed, we have, from the basic fact
about Lie derivatives

$$\begin{aligned} \frac{d}{dt}(\eta_t^*(v_t)) &= \eta_t^*(L_{X_t} v_t + \frac{d}{dt}v_t) \\ &= \eta_t^*(di_{X_t} v_t - (\mu - v)) \\ &= \eta_t^*(d\alpha - \mu + v) = 0 . \quad \square \end{aligned}$$

Note that χ is canonically defined, given the Riemannian metric on M .

Proof. Define $\Phi : \mathcal{D}_\mu \times V \rightarrow \mathcal{D}$ by $\Phi(\zeta, v) = \zeta \circ \chi(v)$. Then $\Phi^{-1}(\eta) = (\eta \circ (\chi \eta^*)(\mu))^{-1}$, $\eta^*(\mu)$ as is easily checked. \square

This can be generalized to the boundary case as well.

The basic technique used here is essentially the same as that used in the proof of Darboux's theorem in lecture 2.

It is also possible to study other groups of diffeomorphisms. For example, let M be a compact manifold and let G be a compact group. Let $\Phi : G \times M \rightarrow M$ be a group action, and let $\Phi_g(m) = \Phi(g, m)$. Set

$$\mathcal{D}_\Phi^S(M) = \{\eta \in \mathcal{D}^S(M) \mid \eta \circ \Phi_g = \Phi_g \circ \eta\}.$$

This is a subgroup of $\mathcal{D}^S(M)$, a C^∞ submanifold and has "Lie algebra"

$$T_e \mathcal{D}_\Phi^S(M) = \{V \in T_e \mathcal{D}^S(M) \mid V \text{ commutes with all infinitesimal generators of } \Phi\}.$$

Of course, we can also take $\mathcal{D}_{\mu, \Phi}^S(M) = \mathcal{D}_\mu^S(M) \cap \mathcal{D}_\Phi^S(M)$. Since this intersection is not in general transversal, it is not obvious that $\mathcal{D}_{\mu, \Phi}^S(M)$ is a submanifold. It is true, but requires some argument (Marsden [7]). The group $\mathcal{D}_{\mu, \Phi}^S(M)$ is important in the study of flows with various symmetries (e.g., a flow in R^3 that is symmetric with respect to a given axis). Also, in general we find that $\dim(\mathcal{D}_\Phi^S(M))$ and $\text{codim}(\mathcal{D}_\Phi^S(M))$ are both infinite so Frobenius methods do not work

(Leslie [2] and Omori [1, 3] have shown that if \mathcal{J} is a Lie sub-algebra of $T_e \mathcal{D}$ with finite dimension or codimension, then \mathcal{J} comes from a smooth subgroup of \mathcal{D}).

The metric on \mathcal{D}_μ^S .

It follows from the results we established above that the tangent space to $\mathcal{D}_\mu^S(M)$ at a point $\eta \in \mathcal{D}_\mu^S$ is given by $T_{\eta} \mathcal{D}_\mu^S(M) = \{X \in H^S(M, TM) \mid X \text{ covers } \eta, \delta(X \circ \eta^{-1}) = 0, \text{ and } X \text{ is parallel to } \partial M\}$. Note that if $X \in T_{\eta} \mathcal{D}_\mu^S(M)$ then $X \circ \eta^{-1}$ is a vector field on M . If we are working on \mathcal{D}^S then the divergence condition $\delta(X \circ \eta^{-1}) = 0$ is dropped, so $T_{\eta} \mathcal{D}^S$ consists of H^S sections parallel to ∂M which cover η .

Let M be a compact Riemannian manifold $m \in M$ and let \langle, \rangle_m be the inner product on $T_m M$. Now we put a metric on $\mathcal{D}^S(M)$ as follows: Let $\eta \in \mathcal{D}^S(M)$ and $X, Y \in T_{\eta} \mathcal{D}^S(M)$. Then $X(m)$ and $Y(m)$ are in $T_{\eta(m)} M$. Now define:

$$(X, Y)_{\eta} = \int_M \langle X(m), Y(m) \rangle_{\eta(m)} d\mu(m).$$

This is a symmetric bilinear form on each tangent space $T_{\eta} \mathcal{D}^S$ of $\mathcal{D}^S(M)$. By restriction it also defines a symmetric bilinear form on each tangent space of \mathcal{D}_μ^S .

The norm induced by this inner product is clearly an L^2 norm and hence the topology it induces is weaker than the H^S topology on each $T_{\eta} \mathcal{D}^S(M)$. Thus, in the terminology of lecture 2, $(,)$ is a

weak metric. It is important to allow weak metrics although most definitions of Riemannian manifolds exclude this (as in Lang [1]). Also, recall that this is the physically appropriate metric for hydrodynamics, since for $X \in T_{\eta}^{\mathcal{D}}(M)$, $\frac{1}{2}(X, X)$ represents the total kinetic energy of a fluid in state η and velocity field $v = X \circ \eta^{-1}$. So finding geodesics is formally the same as finding a flow satisfying a least energy condition. (This is the connection with variational principles or least action principles in fluid mechanics.)

This metric $(\cdot, \cdot)_{\eta}$ just constructed is smooth in this sense: If $B(T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}}, T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}})$ is the vector bundle of bilinear maps over the tangent spaces of $\mathcal{D}^{\mathcal{S}}(M)$ (i.e., if $g_{\eta} \in B(T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}}, T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}})$ then $g_{\eta} : T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}}(M) \times T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}}(M) \rightarrow \mathbb{R}$ is bilinear), then the map $\eta \mapsto (\cdot, \cdot)_{\eta}$ is a section of this bundle, and to say the metric is smooth is to say this section is smooth. (Here each fiber of $B(T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}}, T_{\eta}^{\mathcal{D}}_{\mu}^{\mathcal{S}})$ has the standard topology put on bilinear maps on Banach spaces, and one constructs the bundle as in Lang [1], Ch. III, §4.)

Note. It is not always true that a weak metric yields geodesics. For example, suppose $\partial M \neq \emptyset$. Then on $\mathcal{D}^{\mathcal{S}}(M)$, this weak metric would yield geodesics which would try to cross the boundary of M . We shall see this in more detail below.

The Spray on $\mathcal{D}_{\mu}^{\mathcal{S}}$.

We now wish to construct the spray on $\mathcal{D}_{\mu}^{\mathcal{S}}$ corresponding to the metric (\cdot, \cdot) . Recall from lecture 2 that this means finding the Hamiltonian vector field on $T_{\mu}^{\mathcal{D}}^{\mathcal{S}}$ corresponding to the energy

$K(X) = \frac{1}{2}(X, X)$. Assume

Theorem. Let Z be the spray of the metric on M . Then the spray of (\cdot, \cdot) on $\mathcal{D}^S(M)$ is given by

$$\bar{Z} : T\mathcal{D}^S \rightarrow T^2(\mathcal{D}^S) ; X \mapsto Z \circ X .$$

We shall just make the result plausible, leaving details to the reader. See also Ebin-Marsden [1] and Eliasson [1].

Note. As with $T\mathcal{D}^S$, it is not hard to see that $T_X(T\mathcal{D}^S)$ consists of H^S maps $Y : M \rightarrow T^2M$ which cover X ; i.e., such that $\pi_1 \circ Y = X$, where $\pi_1 : T^2M \rightarrow TM$ is the projection. The spray Z satisfies $\pi_1 \circ Z = \text{identity}$, since Z is a vector field. Thus $Z(X) \in T_X T\mathcal{D}^S$ so Z is indeed a vector field on $T\mathcal{D}^S$.

The idea behind the proof is to realize that we can explicitly write down what should be the geodesics on $\mathcal{D}^S(M)$. From the construction of charts on $\mathcal{D}^S(M)$, there is the map $\overline{\text{exp}} : T\mathcal{D}^S(M) \rightarrow \mathcal{D}^S(M)$ where $\overline{\text{exp}}(X) = \text{exp} \circ X$ and $\text{exp} : TM \rightarrow M$ is the Riemannian exponential map on M . First we assert that for $X \in T_e \mathcal{D}^S(M)$, the geodesic on $\mathcal{D}^S(M)$ through e in the direction X is given by $t \mapsto \text{exp}(tX)$. What this geodesic looks like is seen by considering any $m \in M$. Then $t \mapsto \overline{\text{exp}}(tX)(m) = \text{exp}(tX_m)$ is the geodesic starting at m in the direction X_m . So $\text{exp}(tX)$ represents all of the geodesics on M in the direction of the vector field X evaluated at $m \in M$. Now in general , as t increases it is likely that some pair of geodesics will intersect. Say this happens at $t = t_0$. Then $\text{exp}(t_0 X)$ is not

a diffeomorphism. Hence even if M is a simple manifold (like the flat 2-torus), $\mathcal{D}^S(M)$ is not geodesically complete.

If we can show $t \mapsto \exp tX$ is a geodesic on \mathcal{D}^S , then the formula for Z follows at once, since for each $m \in M$, $v(t) = (d/dt)\exp(tX(m))$ satisfies $(d/dt)v(t) = Z(v(t))$, and $v(0) = X(m)$. Hence it suffices to establish our assertion concerning the geodesics on \mathcal{D}^S .

Of course a fundamental property of geodesics is that they locally minimize length. Suppose we have a family of geodesic curves $t \mapsto \eta(t)(m)$, starting at $m \in M$, where for $t_0 \in \mathbb{R}$, t_0 near 0, the map $m \mapsto \eta(t_0)(m)$ is a diffeomorphism so that $t \mapsto \eta_t$ is a curve in \mathcal{D}^S . Then since the length of a curve in $\mathcal{D}^S(M)$ given by our weak metric is the integral over M of the lengths of each curve, $t \mapsto \eta_t(m)$, this integrated length is also minimized. Hence it is reasonable that $t \mapsto \eta(t)$ should be a geodesic on $\mathcal{D}^S(M)$. The curves $t \mapsto \overline{\exp(tX)}(m)$ have all the above properties so $t \mapsto \exp(tX)$ should be a geodesic on $\mathcal{D}^S(M)$. This concludes our justification

Corollary. \bar{Z} is a C^∞ vector field on $T\mathcal{D}^S$.

This is a consequence of the omega lemma since Z is a C^∞ map.

Let us consider a simple example. Let \mathcal{T}^2 be the flat 2-torus. Then $T(\mathcal{T}^2) \cong \mathcal{T}^2 \times \mathbb{R}^2$ is also a flat 4-manifold and $T(T\mathcal{T}^2) \cong (\mathcal{T}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2)$. In this case the spray for the flat metric is given by

$$Z : T(\mathcal{T}^2) \rightarrow T(T(\mathcal{T}^2)) : (x, v) \mapsto ((x, v), (v, 0)) .$$

The x in the first coordinate is just the base point of the tangent vector in $T\mathcal{T}^2$. The v in the third coordinate is an important formal property of sprays reflecting the fact that the geodesic equations are "second order" (see Lang [1]) and the 0 in the last coordinate reflects the fact that the metric is flat, hence each $\Gamma_{jk}^i = 0$. In this case the geodesics are of the form $\eta(t)(m) = m + tX(m)$ (where $X \in T_e \mathcal{D}^S(\mathcal{T}^2)$ and using the obvious identification). These are straight lines and hence $\mathcal{D}^S(\mathcal{T}^2)$ is "flat". In general, in coordinates $x = (x^1, \dots, x^n)$ on a manifold M , we have $Z(x, v) = ((x, v), (v, -\Gamma_{jk}^i v^j v^k))$.

We now consider the metric for $\mathcal{D}_\mu^S(M) \subset \mathcal{D}^S(M)$. Even if $\mathcal{D}^S(M)$ is geometrically relatively simple, as above for \mathcal{T}^2 , $\mathcal{D}_\mu^S(M)$ may be geometrically very complicated. Consider the above example. It should be clear that the diffeomorphism specified by having each point moving along straight lines is generally not volume preserving. So requiring each point on a geodesic in \mathcal{D}^S to be volume preserving must introduce some curvature. In fact the curvature of the space \mathcal{D}_μ^S is rather complicated. For $M = \mathcal{T}^2$ it is worked out in Arnold [1].

Suppose S is a submanifold of a Riemannian manifold Q such that we have an orthogonal projection of $T_p Q$ onto $T_p S$ for each $p \in S$. This gives us a bundle map $P : TQ \uparrow S \rightarrow TS$ (where $TQ \uparrow S = \{v_p \in T_p Q \mid p \in S\}$). This is of course the situation we have

for $\mathcal{D}_\mu^S(M)$ as a submanifold of $\mathcal{D}^S(M)$ where the projection is given by the Hodge theorem (i.e., we project onto the divergent free part of X for $X \in T_e \mathcal{D}^S(M)$). In this situation, the following tells us how to put the spray on the submanifold.

Lemma. If Z is the spray on Q then $TP \circ Z$ is the spray on S .

This is a standard result in Riemannian geometry, see e.g. Hermann [1]. A proof using Hamiltonian theory may be found in Ebin-Marsden [1].

Now Z is a vector field on TQ as is $TP \circ Z$ on TS . However their difference, say h , can be identified (technically by means of the vertical lift -- see below) with a map of TS into $TQ \uparrow S$, which turns out to be (the quadratic part of) the second fundamental form of S as a submanifold. Specifically for $v \in TS$, $h(v)$ is the normal component of $\nabla_v v$; see Hermann [1] or Chernoff-Marsden [1] for details. Thus this difference h in the sprays tells us how curved S is in Q . (More exactly the curvatures on Q and on S are related through this second fundamental form by the Gauss-Codazzi equations; cf. Yano [1], p. 94 and lecture 9.)

Define

$$P_e : T_e \mathcal{D}^S(M) \rightarrow T_e \mathcal{D}_\mu^S(M)$$

by carrying a vector field to its divergent free part. As we mentioned above, this is an L^2 orthogonal projection as it is orthogonal for the

weak inner product on $T_e \mathcal{D}^S(M)$. We define for $X \in T_\eta \mathcal{D}^S(M)$;
 $\eta \in \mathcal{D}_\mu^S(M)$

$$P_\eta(X) = (P_e(X \circ \eta^{-1})) \circ \eta .$$

This makes P right invariant and is correct since the metric on $\mathcal{D}_\mu^S(M)$ is right invariant as we now show

Proposition.

(i) Let $\eta \in \mathcal{D}^S(M)$; then $(R_\eta)_* X_\zeta = X_{\zeta \circ \eta}$ (where $\zeta \in \mathcal{D}^S(M)$,
 $(R_\eta)_* : T_\zeta \mathcal{D}^S(M) \rightarrow T_{\zeta \circ \eta} \mathcal{D}^S(M)$) .

(ii) If $\eta \in \mathcal{D}_\mu^S(M)$ then $((R_\eta)_* X , (R_\eta)_* Y)_{\zeta \circ \eta} = (X , Y)_\zeta$,
where $X , Y \in T_\zeta \mathcal{D}^S(M)$.

Proof. Part (i) has been used before and is easily seen. We will show the second part. Let $\eta \in \mathcal{D}_\mu^S(M)$; then:

$$\begin{aligned} ((R_\eta)_* X , (R_\eta)_* Y)_{\zeta \circ \eta} &= (X \circ \eta , Y \circ \eta)_{\zeta \circ \eta} \\ &= \int_M \langle X \circ \eta(m) , Y \circ \eta(m) \rangle_{\zeta \circ \eta(m)} d\mu \\ &= \int_{\eta^{-1}(M)} \langle X(m) , Y(m) \rangle_{\zeta(m)} (\eta^{-1})^*(d\mu) . \end{aligned}$$

But, since η^{-1} is volume preserving, $(\eta^{-1})^*(d\mu) = d\mu$ and $\eta^{-1}(M) = M$.

Hence

$$\begin{aligned} ((R_\eta)_* X , (R_\eta)_* Y) &= \int_M \langle X(m) , Y(m) \rangle_{\zeta(m)} d\mu \\ &= (X , Y)_\zeta . \quad \square \end{aligned}$$

Note that the metric on \mathcal{D}^S is not right invariant.

Putting all this together we can write down the spray S on $\mathcal{D}_\mu^S(M)$. Namely, for $X \in T\mathcal{D}_\mu^S(M)$ we have $S(X) = TP(Z(X)) = TP(Z \circ X)$. There is a major assumption in writing down this formula. When we write TP , we assume P is a C^∞ map. This is not at all obvious since if $X \in T_{\eta\mu}^S(M)$, we compose X with η^{-1} , project, and then compose with η . As we have seen, composition of H^S maps is not smooth but is at most continuous. However, we have

Theorem. P is a C^∞ bundle map. That is $P : T\mathcal{D}^S(M) \uparrow \mathcal{D}_\mu^S(M) \rightarrow T\mathcal{D}_\mu^S(M)$ is C^∞ . Hence the spray S on \mathcal{D}_μ^S , $S(X) = TP(Z \circ X)$, is also a C^∞ vector field on $T\mathcal{D}_\mu^S$.

For a proof see Ebin-Marsden [1]. There is an alternative and perhaps simpler proof to the one in the aforementioned paper. In this proof one defines another metric on $T\mathcal{D}^S(M)$; namely for $X, Y \in T_e\mathcal{D}^S(M)$, set

$$(X, Y)_S = (X, Y) + (\Delta^{S/2}X, \Delta^{S/2}Y)$$

where $(,)$ is the L^2 metric on $T_e\mathcal{D}^S(M)$, and Δ is the Laplacian. Then extend $(,)_S$ to make it right invariant.

It turns out that this metric is smooth and by regularity properties of Δ is equivalent to the H^S metric. Smoothness facts like this again are not obvious but are proven in Ebin [1]. These facts are also useful for other purposes. The Hodge decomposition is

then easily seen to be orthogonal in this strong metric $(\cdot, \cdot)_S$ and hence it follows automatically that the projection P is smooth.

This result is important for we are going to apply the Picard theorem from ordinary differential equations to the equation:

$$\frac{dX_t}{dt} = S(X_t) = TP(Z \circ X_t)$$

and this requires that S is at least a Lipshitz map.

In case M has boundary, we do not get a spray on \mathcal{D}_μ^S , but we do get one on \mathcal{D}_μ^S . This is basically because P projects from vector fields sticking out of M , onto vector fields parallel to ∂M . We shall just accept as plausible that this extension can be made.

As mentioned earlier, it is unknown whether $\mathcal{D}_\mu^S(M)$ is geodesically complete. (By Arnold's theorem, this is the same thing as saying solutions to the Euler equations go for all time, and remain in H^S). Note that this is not equivalent to saying the induced distance metric is complete since the metric is only weak. In fact $\mathcal{D}_\mu^S(M)$ is not complete in this distance sense since the completion of $\mathcal{D}_\mu^S(M)$ under an L^2 topology is much larger than $\mathcal{D}_\mu^S(M)$. (Presumably it consists of a class of measure preserving maps from M to M .)

Derivation of the Euler Equations.

To show geodesics in $\mathcal{D}_\mu^S(M)$ satisfy the Euler equations, we

need to know a bit more about T^2M . Let $\pi : TM \rightarrow M$ be the projection so that $T\pi : T^2M \rightarrow TM$. An element $w \in T^2M$ is called vertical if $T\pi(w) = 0$ (in coordinates this means the third component is 0). Now let $v, w \in T_m M$; define the vertical lift of w with respect to v to be

$$(w)_v^\ell = \frac{d}{dt}(v + tw) \Big|_{t=0} \in T_v^2M = T_v(TM).$$

In coordinates this is simply

$$(w)_v^\ell = (m, v, 0, w).$$

The proof that geodesics in \mathcal{D}_μ^S yield solutions to the Euler equations essentially is calculations. The idea is to show that if a curve $X_t \in T\mathcal{D}_\mu^S$ satisfies the spray equation

$$\frac{dX_t}{dt} = S(X_t), \quad X_t \in T\mathcal{D}_\mu^S(M),$$

then X_t gives rise to a solution to the Euler equations in a sense explained below. For alternative proofs, see Arnold [1], Marsden-Abraham [1], or Chernoff-Marsden [1]; see also Hermann [1].

Lemma. $Z(X) = Z \circ X = TX \circ X - (\nabla_X X)_X^\ell$ for $X \in T_e \mathcal{D}^S$.

Proof. In coordinates

$$(\nabla_X X)^i = \sum_j X^j \frac{\partial X^i}{\partial x^j} + \sum_{j,k} \Gamma_{jk}^i X^j X^k.$$

Now

$$(TX \circ X)^i = \sum_j X^j \frac{\partial X^i}{\partial x^j} \quad \text{so} \quad (TX \circ X - (\nabla_X X))^i = - \sum_{jk} \Gamma_{jk}^i X^j X^k .$$

This then puts the right expressions in the fourth component. \square

Note that both $TX \circ X$ and $(\nabla_X X)_X^\ell$ are elements of $T_{X \mu}^S$. The latter is by construction of the vertical lift. To see this for $TX \circ X$, let $\pi_1 : T^2M \rightarrow TM$ be the projection; then since $\pi_1 \circ TX = X \circ \pi$ we have

$$\pi_1 \circ TX \circ X = X \circ \pi \circ X = X$$

since $\pi \circ X$ is the identity.

As we have observed, the map $X \mapsto Z \circ X$ (for $X \in T_{\mu}^S(M)$) is C^∞ . Hence even though $TX \circ X$ and $\nabla_X X$ are only H^{s-1} , their difference must be H^s .

Lemma. Let σ and X be in $T_{\mu}^S(M)$ then $TP[(\sigma)_X^\ell] = (P(\sigma))_X^\ell$.

Proof. Since P is linear on each fiber and $P(X) = X$, we get

$$\begin{aligned} (P(\sigma))_{P(X)}^\ell &= \frac{d}{dt}(P(X) + tP(\sigma)) \Big|_{t=0} \\ &= \frac{d}{dt}P(X + t\sigma) \Big|_{t=0} \\ &= TP\left(\frac{d}{dt}(X + t\sigma)\right) \Big|_{t=0} \quad (\text{chain rule}) \\ &= TP[(\sigma)_X^\ell] . \quad \square \end{aligned}$$

Lemma. Let $\eta \in \mathcal{D}_\mu^S$ and $X \in T_{\eta} \mathcal{D}^S(M)$; then $TP(T(X \circ \eta^{-1}) \circ X) = \{T(P_e[X \circ \eta^{-1}])\} \circ X$.

Proof. $X \circ \eta^{-1}$ is an H^S vector field on M . Let F_t be its flow (or any curve tangent to X) . Let $G_t = (X \circ \eta^{-1}) \circ F_t$. Then $G_0 = X \circ \eta^{-1}$ and $(dG_t/dt) = T(X \circ \eta^{-1}) \circ (X \circ \eta^{-1})$. Thus we get

$$\begin{aligned} TP(T(X \circ \eta^{-1}) \circ (X \circ \eta^{-1})) &= \frac{d}{dt} P(G_t) \Big|_{t=0} \quad (\text{chain rule}) \\ &= \frac{d}{dt} (P_e(X \circ \eta^{-1}) \circ F_t) \Big|_{t=0} \\ &= T(P_e(X \circ \eta^{-1})) \circ (X \circ \eta^{-1}) . \end{aligned}$$

But by right invariance $TP(T(X \circ \eta^{-1}) \circ (X \circ \eta^{-1})) = TP(T(X \circ \eta^{-1}) \circ X) \circ \eta^{-1}$. \square

Proposition. The spray on $T\mathcal{D}_\mu^S$ is given by

$$S(X) = T(X \circ \eta^{-1}) \circ X - (P_e \nabla_{X \circ \eta^{-1}} X \circ \eta^{-1}) \Big|_{X \circ \eta^{-1}} \circ \eta \quad \text{where } X \in T_{\eta} \mathcal{D}^S(M) .$$

Proof. This follows directly from the above lemmas. \square

So now that we have an explicit formula for the spray, let us inspect the Euler equations. Recall that these describe the time evolution of the velocity vector field on M . The equations are written out in Eulerian coordinates and are equations involving elements of $T_e \mathcal{D}^S(M)$. The spray on the other hand is a map on all of $T\mathcal{D}^S(M)$. The integral curves of the spray are the velocities written in Lagrangian coordinates. So if $X_t \in T_{\eta(t)} \mathcal{D}^S(M)$ is an integral curve of the spray, we wish to show that the pullback of X_t , i.e., $X_t \circ \eta_t^{-1} \in T_e \mathcal{D}^S(M)$, is

a solution of the Euler equations. Let us recall that the vector field $v(t) = X_t \circ \eta_t^{-1}$ is justified as follows. We want η_t to be the flow of v , so this means that

$$\frac{d}{dt} \eta_t(m) = v_t(\eta_t(m)) .$$

Since we are dealing with geodesics and hence $(d\eta/dt) = X$, we get the desired relation $v_t = X_t \circ \eta_t^{-1}$.

It turns out, as we shall see momentarily, that the derivative loss of the Euler equations occurs in this pullback operation (or "coordinate change").

We are interested in computing (dv/dt) , and so we need this lemma.

Lemma. We have:

$$\frac{dv(t)}{dt} = \frac{d}{dt}(X_t \circ \eta_t^{-1}) = \frac{dX_t}{dt} \circ \eta_t^{-1} - TX_t \circ T\eta_t^{-1} \circ X_t \circ \eta_t^{-1} .$$

Proof. This follows by differentiating both places t occurs, using the chain rule and the formula

$$\frac{d}{dt}(\eta_t^{-1}) = -T\eta_t^{-1} \circ \frac{d\eta}{dt} \circ \eta_t^{-1} .$$

The last formula follows from the chain rule applied to $\eta_t \circ \eta_t^{-1} = \text{id}$. \square

So, putting this together, we get:

$$\frac{dv_t}{dt} = S(X_t) \circ \eta_t^{-1} - T(X_t \circ \eta_t^{-1}) \circ X_t \circ \eta_t^{-1} .$$

Now using the previous formula for $S(X)$, this becomes
 $= -(P_e \nabla_v v)_v^{\ell}$. Note especially the cancellation of the $Tv \circ v$ terms which has occurred. But as we recall $P_e(\nabla_{v_t} v_t) = \nabla_{v_t} v_t - \nabla p_t$ where p is a smooth function. We can identify $P_e(\nabla_{v_t} v_t)_v^{\ell}$ with $P_e(\nabla_{v_t} v_t)$ (since dv/dt really stands for its vertical lift) and hence get the Euler equations

$$\frac{dv_t}{dt} = -\nabla_{v_t} v_t + \nabla p_t$$

or

$$\frac{dv_t}{dt} + \nabla_{v_t} v_t = \nabla p_t .$$

(The minus sign on the pressure can be recovered by using $-p_t$.) Thus we have proved:

Theorem. If X_t is an integral curve of the spray on \mathcal{D}_μ^S , its pullback $v_t = X_t \circ \eta_t^{-1}$ does satisfy the Euler equations. In other words, η_t is a geodesic on \mathcal{D}_μ^S iff its velocity field satisfies the Euler equations.

By inspecting the above calculation it becomes clear where the derivative loss occurs. If X is an H^S vector field on M , we know $S(X)$ is an H^S vector field on TM . However it is the sum of two H^{S-1} vector fields on TM . The top derivatives cancel, but when this is pulled back to Eulerian coordinates one of these terms

disappears, namely $TX_t \circ X$ and so what we are left with is one of the H^{s-1} summands.

All of the above goes through for manifolds with boundary since the Hodge theorem projects vector fields at the boundary onto those which are tangent to the boundary as mentioned before.

As a consequence of these calculations we have this theorem, mentioned earlier.

Theorem. Given $v_0 \in T_{e_\mu} \mathcal{D}_\mu^S$ there is an $\epsilon > 0$ and a unique vector field $v(t) \in T_{e_\mu} \mathcal{D}_\mu^S$ for $-\epsilon < t < \epsilon$ which satisfies the Euler equation. Moreover, these solutions v_t depend continuously on the initial data v_0 .

Proof. For the existence part of the theorem it is sufficient to find short-time solutions to the geodesic spray on $\mathcal{D}_\mu^S(M)$. But since $\mathcal{D}_\mu^S(M)$ is a Hilbert manifold and the spray is smooth the existence follows immediately from the existence theorem for C^r vector fields on Banach manifolds (see lecture 1).

The continuous dependence on initial conditions follows from the fact that the pullback $v_t = X_t \circ \eta_t^{-1}$ involves left composition so it is continuous (but not smooth). The initial condition for the spray on $\mathcal{D}_\mu^S(M)$ is an element of $T_{e_\mu} \mathcal{D}_\mu^S(M)$ since we are interested in flows in $\mathcal{D}_\mu^S(M)$ starting at the identity. \square

This existence theorem has been proved in weaker forms by

Lichtenstein [1] and Guynter [1]. The general case of manifolds with boundary is due to Ebin-Marsden [1].

The flow in Lagrangian coordinates is C^∞ . In Euler coordinates, let $E_t(v_0) = v_t$ be the solution flow. Then for fixed t , E_t is a continuous map, but is probably not differentiable.* Various smoothness properties of the Euler and Navier-Stokes equations are important in developments discussed in the next lecture (see Marsden [7]).

The proof of the above Theorem is based on the existence of integral curves for the spray S . This in turn follows from the fundamental existence theorem for ordinary differential equations. Recall that this theorem is proven by showing an iteration (called Picard iteration) always yields solutions. So, by inspecting the above proof it should be possible to find an approximation procedure which converges to solutions.

This in fact, points out an essential difference between working with the whole spray and working with its pullback $P_e(\nabla_v v)$. The Picard method will not in general converge for the Euler equation as it stands. Indeed in practical numerical computations, one often uses Lagrangian coordinates. (See also the next lecture.)

* Indeed, Kato [5] has shown that the evolution operator $U_t : H^s \rightarrow H^s$ for $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$ on \mathbb{R} is continuous, but is not Holder continuous for any exponent α , $0 < \alpha \leq 1$.