5. Turbulence and Chorin's Formula.

This lecture is concerned with some aspects of the Navier Stokes equations which are connected with turbulence. We shall be beginning with a representation theorem for the solution of the Navier-Stokes equations which was discovered by A. Chorin in an attempt to find a good numerical scheme to calculate solutions. This scheme is important in that it allows good calculations at interesting Reynolds numbers. One writes the Navier-Stokes equations as

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{1}{R}\Delta v + (v\cdot\nabla)v = -\text{grad } p \\ \text{div } v = 0 \\ v = 0 \quad \text{on} \quad \partial M \end{cases}$$

and calls $R = 1/(\text{viscosity})$ the Reynolds number; if one rescales v to Vv, distances by a factor d and time by d/V we get a new solution with $R = Vd/\nu$. Most numerical schemes break down with R a few hundred, but Chorin's scheme is valid far beyond that, possibly up to $R = 50{,}000$. Our goal is to present the formula and to discuss where it comes from and its plausibility. The second part of the lecture will discuss some aspects of turbulence theory. This subject is basically concerned with qualitative features of the solutions as $R \to \infty$. The approach here follows that of Ruelle-Takens [1].

Statement of Chorin's Formula.

Let us write the Navier-Stokes equations as follows:

$$\begin{cases} \frac{\partial v}{\partial t} = \tilde{\Delta} v + Z(v) \\ v = 0 \quad \text{on} \quad \partial M \end{cases}$$

where $\tilde{\Delta} = \frac{1}{R} P \cdot \Delta$ and $Z(v) = -P((v \cdot \vec{\nabla})v)$. Here P is the projection onto the divergence free part discussed in the last lecture (Δv is divergence free, but need not be parallel to ∂M , so one still requires a P in front of Δv) .

Let H_t denote the evolution operator or semi-group defined by $\tilde{\Delta}$. It exists because it is an elementary exercise to show that $\tilde{\Delta}$ is self adjoint and ≤ 0 on the Hilbert space $L_2(M)$ with domain $H_0^2(M)$. (See the parabolic form of the Hille-Yosida theorem discussed in lecture 1). Thus H_t is defined for $t \geq 0$, and solves $\partial v/\partial t = \tilde{\Delta} v$. (This is called the "Stokes" equation.)

Let E_t denote the evolution operator for the Euler equations which was obtained in the last lecture.

Let F_t denote the full solution to the Navier-Stokes equations.

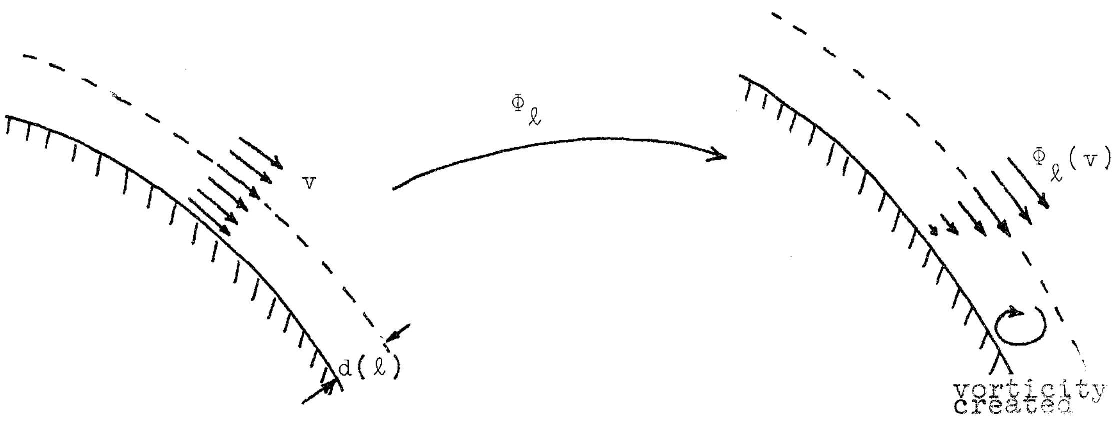
Let $\varphi(v)$ be a potential for v ; e.g.: $\varphi(v) = d\Delta^{-1}(v)$, so $v = \delta(\varphi(v))$. Here δ is the divergence operator discussed in lecture 3. (More concretely in three dimensions, $v = \vec{\nabla} \times \varphi(v)$.) Let $d(\ell)$ be a function of $\ell \in R$, $\ell \geq 0$ with $d(\ell) = \sqrt{\ell \nu}$ where $\nu = 1/R$ is the viscosity of the fluid. It will turn out that $d(\ell)$ will be a measure of the thickness of the boundary layer.

Let g_ℓ be a C^∞ function equal to one a distance $\geq d(\ell)$ from ∂M and $g_\ell = 0$ on a neighborhood of ∂M .

Define the operator

$$\Phi_\ell(v) = \delta(g_\ell \cdot \varphi(v)) ,$$

we call Φ_ℓ the <u>vorticity creation operator</u>. The reason for this is that $\Phi_\ell(v)$ equals v away from ∂M, but if v is only $\|\partial M$, $\Phi_\ell(v)$ will be zero on ∂M so has the effect of "chopping off" v within the boundary layer (we do not use $g_\ell \cdot v$ since that is not divergence free). Such a chopping off effectively creates vorticity. (See the figure following.)



The formula now reads as follows:

$$F_t(v) = \text{solution of Navier-Stokes equation}$$

$$= \lim_{n\to\infty} (H_{t/n} \circ \Phi_{t/n} \circ E_{t/n})^n v .$$

In this formula the power means iteration. For example:

$$(H_{t/3} \circ \Phi_{t/3} \circ E_{t/3})^3 v = H_{t/3} \circ \Phi_{t/3} \circ E_{t/3} \circ H_{t/3} \circ \Phi_{t/3} \circ E_{t/3} \circ H_{t/3} \circ \Phi_{t/3} \circ E_{t/3} \cdot v .$$

Thus one divides the time scale into n parts and then iterates the

the procedure: solve Euler's equations then create vorticity, then solve the Stokes equation then the Euler equation, etc.

This is the basic method underlying Chorin's technique. However part of the beauty of the method is the way in which he solves numerically for E_t and H_t. He uses vorticity methods for E_t and probabilistic methods for H_t. See Chorin [2] for details.

In the following figure* we reproduce one of Chorin's outputs. The 0's mark negative vorticity and *'s mark positive vorticity. This representation is for flow past a cylinder with R and t as marked and initial v corresponding to parallel flow. It is a remarkable achievement to obtain on the computer something resembling the famous "Karmen vortex street". (For a spectacular photograph, see Scientific American, January 1970, p. 40; this is reproduced on the cover of "Basic Complex Analysis", W. H. Freeman Co. (1973).) Below we shall discuss further the qualitative features of why and how such periodic phenomena can get generated.

As is well known (Nelson [3]) product formulas are closely related to Wiener integrals. Chorin has recently used this idea to improve the scheme still further, as far as computer efficiency goes, so the method is valid into the fully turbulent region.

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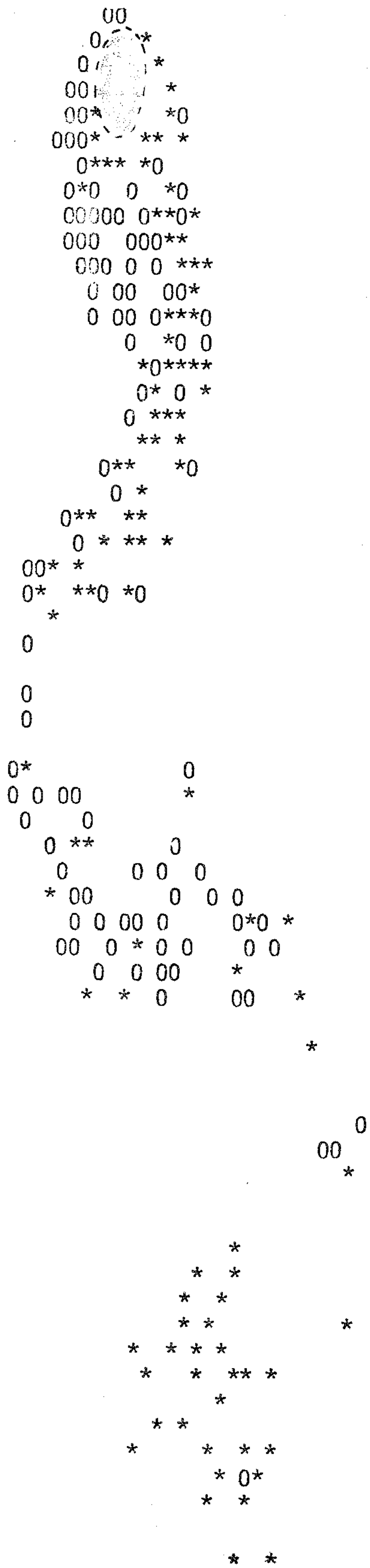
* The computer has distorted the cylinder somewhat into an ellipse.



R = 1000

t = 24

The interesting feature of the above formula is that the error for large n is $0(1/n)$ independent of R. Furthermore using the formula as an existence theorem we find that smooth solutions to the Navier-Stokes equations exist for a time interval $T > 0$ independent of R as $R \to \infty$ and converge in L_p to solutions of the Euler equations.

This is an important result, for it guarantees as positive time of existence for given initial data, no matter how small the viscosity. This is strong evidence for the existence of smooth turbulent solutions. (See below.)

In case $\partial M = \emptyset$ (for example using periodic boundary conditions) the formula reads

$$F_t v = \lim_{n \to \infty} (H_{t/n} \circ E_{t/n})^n v \ .$$

This formula was proven in Ebin-Marsden [1] and Marsden [5]. It enabled us to show that as $\nu \to 0$ (or $R \to \infty$) the solutions converge in H^s to solutions of the Euler equations. (See also Swann [1], Kato [2].) Basically this means that turbulence cannot occur if no boundaries are present. Such convergence will not occur if $\partial M \neq \emptyset$ in topologies stronger than L_p because the boundary conditions and the vorticity carried into the mainstream flow will not allow it.

The complete proofs of these results are too technical for us to go into here. Rather we shall confine ourselves, in the next section, to an elementary exposition of where these formulas come from. We shall also include some additional intuition below.

The Lie-Trotter Formula.

Let X and Y be vector fields with flows H_t and E_t. Then the flow F_t of $X + Y$ is given by

$$F_t = \lim_{n \to \infty} (H_{t/n} \circ E_{t/n})^n .$$

Theorem. This is valid if X, Y are C^r vector fields for those t for which F_t is defined.

Let us give the idea (for details, see e.g., Nelson [1]). We first show F_t defined by the limit is a flow. One shows $F_{t+s} = F_t \circ F_s$ first if s, t are rationally related and takes limits. Consider, e.g.: $t = s$.

$$\begin{aligned} F_{2t} &= \lim_{n \to \infty} (H_{2t/n} \circ E_{2t/n})^n \\ &= \lim_{n \to \infty} (H_{2t/2n} \circ E_{2t/2n})^{2n} \\ &= \lim_{n \to \infty} (H_{t/n} \circ E_{t/n})^{2n} \\ &= \lim_{n \to \infty} (H_{t/n} \circ E_{t/n})^n \circ (H_{t/n} \circ E_{t/n})^n \\ &= F_t \circ F_t . \end{aligned}$$

Next one shows $\frac{d}{dt} F_t(x) \big|_{t=0} = X(x) + Y(x)$. Indeed, formally,

$$\frac{d}{dt} F_t(x) \Big|_{t=0} = \lim_{n \to \infty} \frac{d}{dt} (H_{t/n} \circ E_{t/n})^n x \Big|_{t=0}$$

$$= \lim_{n\to\infty} \frac{d}{dt}(H_{t/n}\circ E_{t/n}\circ H_{t/n}\circ\cdots\circ H_{t/n}\circ E_{t/n})x\Big|_{t=0}$$

$$= \lim_{n\to\infty} [\frac{1}{n}(X(x)+Y(x))+\ldots+\frac{1}{n}(X(x)+Y(x))]$$

$$= X(x) + Y(x) .$$

It follows now that F_t is the flow of $X+Y$ since

$$\frac{d}{dt}F_t(x) = \frac{d}{ds}F_{s+t}(x)\Big|_{s=0}$$

$$= \frac{d}{ds}F_s(F_t(x))\Big|_{s=0}$$

$$= X(F_t(x)) + Y(F_t(x)) .$$

The above formula arose historically in Lie group theory. It tells us how to exponentiate the sum of two elements in the Lie algebra. In the case of matrix groups it is the classical formula:

$$e^{t(A+B)} = \lim_{n\to\infty} (e^{tA/n}e^{tB/n})^n .$$

Of course if $[X, Y] = 0$ the formula reads $F_t = H_t\circ E_t$, but it really is the case in which X, Y do not commute that is of interest.

The above formula has been generalized to linear evolution equations, as in the Hille-Yosida theorem by Trotter [1], and to certain non-linear semi-groups by Brezis-Pazy [1] and Marsden [5]. These results can be used to establish the claims made about the Navier Stokes equation if $\partial M = \emptyset$. Indeed one takes $X = \tilde{\Delta}$ and $Y = Z$.

For $\partial M \neq \emptyset$ the composition $H_t \circ E_t$ doesn't even make sense (except perhaps in $L_2(M)$, but that is not too useful) because $E_t(v)$, even if $v = 0$ on ∂M , will not be 0 on ∂M , but will only be parallel to ∂M . The purpose of the vorticity creation operator is to correct for this failure of the boundary conditions.

Some additional intuition on Chorin's Formula.

Consider again the formula

$$F_t(v) = \lim_{n \to \infty} (H_{t/n} \circ \Phi_{t/n} \circ E_{t/n})^n v \; .$$

The term $E_{t/n} v$ gives the main overall features of the flow past the boundary. Let us call it the downstream drift. Consider the effect: $E_{t/n}$ drifts us downstream, then $\Phi_{t/n}$ creates vorticity near ∂M , then $H_{t/n}$ has the effect of diffusing this vorticity away from ∂M then $E_{t/n}$ tends to sweep this vorticity downstream etc. The net effect is a lot of vorticity swept downstream. This is exactly what happens in examples such as the von Karmen vortex street.

The proof of Chorin's formula is based on a generalization of the Lie Trotter product formula due to Chernoff [1] in the linear case and Brezis-Pazy [1] and Marsden [5] in the non-linear case. We discuss this formula next.

Chernoff's Formula.

Suppose $K(t)$ is a family of operators, $t \geq 0$ (satisfying suitable hypotheses). Let $X = K'(0)$. Then the flow of X is

$$F_t(x) = \lim_{n \to \infty} [K(t/n)]^n (x).$$

This is Chernoff's generalization of the Lie-Trotter formula. We obtain the previous formula for $X+Y$ using $K(t) = H_t \circ E_t$.

For details on the hypotheses, see the aformentioned references and Chernoff-Marsden [1] and Nelson [1].

In applications to hydrodynamics it is important to use Lagrangian coordinates, for as we have stressed in the previous lecture, the Euler equations then become a C^∞ vector field. This is a great advantage in dealing with these product formulas (in the linear case it corresponds to adding a bounded operator to an unbounded one -- a relatively easy procedure).

For example one can give an almost trivial proof of the formula

$$E_t = \lim_{n \to \infty} (P\mathring{E}_{t/n})^n$$

where $\mathring{E}_t$ is the evolution operator for $\frac{\partial u}{\partial t} + (u \cdot \nabla)u = 0$ whose solution is known explicitly. A similar theorem proved using Euler coordinates and with more effort was done by Chorin [1].

To obtain Chorin's formula as previously described, one chooses $K(t) = H_t \circ \Phi_t \circ E_t$.

Calculation of the Generator.

Probably the most crucial thing in Chorin's formula is the formal reason why $K'(0) = \tilde{\Delta} + Z$. Indeed we claim that Φ_t contributes nothing to $K'(0)$. This is, of course, crucial if our resulting flow is to be associated with the Navier-Stokes equations. In the following we attempt to show why $K'(0) = \tilde{\Delta} + Z$ with $K(t)$ as above.

In order to see this, write

$$\frac{1}{t}\{H_t\Phi_t E_t v - v\} = \frac{1}{t}\{[H_t\Phi_t E_t v - H_t\Phi_t v] + [H_t\Phi_t v - H_t v] + [H_t v - v]\} .$$

The first and last terms converge, respectively to $Z(v)$ and $\tilde{\Delta} v$ (one needs to know $H_t\Phi_t$ is t-continuous for this). Thus the validity is assured by the following key lemma: if v is suitably smooth, $v = 0$ on ∂M, then in L_p,

$$\lim_{t \to 0} \frac{1}{t} [H_t\Phi_t v - H_t v] = 0 .$$

Indeed, if $K(t, x, y)$ is a Green's function for $\tilde{\Delta}$ on M then

$$\frac{1}{t}(H_t\Phi_t v - H_t v)(x) = \frac{1}{t} \int_M K(t, x, y)[(\Phi_t v)(y) - v(y)]dy$$

$$= \int_M \frac{1}{t} dK(t, x, y)[g_t\varphi(v)(y) - \varphi(v)(y)]dy$$

$$= \int_{B_t} \frac{1}{t} dK(t, x, y)[g_t\varphi(v)y - \varphi(v)(y)]dy$$

where $B_t = \{x \in M | d(x, \partial M) \leq d(t)\}$. Taking into account the nature of the singularity of K and the choice of $d(t)$ it is easy to see that in L_p norm, the above is majorized by

$$C_t \times \sup_M [g_t\varphi(v) - \varphi(v)]$$

where C_t , the L_p norm of $\frac{1}{t}\int_{B_t} dK(t, x, y)dy$ goes to zero as $t \to 0$ on account of the rapidity with which the volume of B_t goes to zero as $t \to 0$. This gives the formula.

The Hopf Bifurcation.

We now turn our attention to the qualitative nature of turbulence. Actually the literature is very confusing -- a few representative works are listed in the bibliography. However we wish to describe a theory due to Ruelle-Takens [1] which has several very attractive features.

Basically we want to study the Navier-Stokes equations and let $R \to \infty$. Thus we are interested in studying dynamical systems depending on a parameter. One of the most basic results in this regard is a theorem of Hopf from 1942 (Hopf [1]).

In order to understand Hopf's theorem, let us review some standard material in ordinary differential equations. For a complete discussion of this material, see Coddington-Levinson [1] and Abraham-Robbin [1]. Let $X : R^n \to R^n$ be a linear map. Then regarding X as a vector field on R^n , its flow is given by $F_t(a) = e^{tX}(a)$, where

$a \in R^n$ and $e^{tX} = \sum_{n=0}^{\infty} (t^n X^n / n!)$; in this expression $X^0 = I$ and multiplication is as matrices. Let $\lambda_1, \ldots, \lambda_k$ be the (possibly complex) eigenvalues of X . Since X has only real entries when considered as a matrix, the λ_i appear in conjugate pairs. Clearly $e^{t\lambda_1}, \ldots, e^{t\lambda_k}$ are the eigenvalues of F_t .

Now suppose that for all i , we have $\mathrm{Re}(\lambda_i) < 0$. Then as t increases $|e^{\lambda_i t}|$ is decreasing and hence the orbit of a point $a \in R^n$ i.e., the curve $t \mapsto F_t(a)$, is approaching zero. (This is clear if X is diagonalizable; for the general case one uses the Jordan canonical form.) Since F_t is linear, for each t we have $F_t(0) = 0$. In this situation, we say 0 is an <u>attracting</u> or <u>stable fixed point</u>.

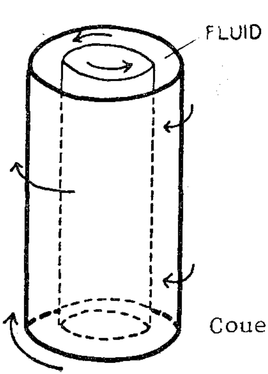
Now if all $\mathrm{Re}(\lambda_i) > 0$, it is clear that each $|e^{t\lambda_i}|$ is increasing with t , and so the orbit of a point under the flow is away from 0 . Here, we say 0 is a <u>repelling</u> or <u>unstable fixed point</u>.

For the nonlinear case, we linearize and apply the above results as follows. Let X be a vector field on some manifold M . Suppose there is a point $m_0 \in M$ such that $X(m_0) = 0$. Then F_t , the flow of X leaves m_0 fixed; $F_t(m_0) = m_0$. It makes sense to consider $DX(m_0) : T_{m_0}M \to T_{m_0}M$. If $y_1, \ldots, y_n$ is a coordinate system for M at m_0 , the coordinate matrix expression for $DX(m_0)$ is just $DX(m_0) = (\partial X^i / \partial y^j)(m_0)$. Now, $DX(m_0)$ can be treated as a linear map on R^n and the same analysis as above applies. Hence m_0 is an attracting or repelling fixed point (or neither) for the flow of X depending on the sign of the real part of the eigenvalues of

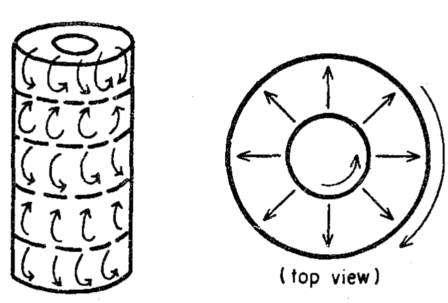
$(\partial X^i/\partial y^j)(m_0)$. However if m_0 is attracting (when the real parts of the eigenvalues are < 0) , it is only nearby points which $\to m_0$ as $t \to \infty$.

To begin our study of the Hopf theorem, let us consider a physical example of the general phenomenon of bifurcation. The idea in each case is that the system depends on some real parameter, and the system undergoes a sudden qualitative change as the parameter crosses some critical point. (For research in a slightly different direction and for more examples, consult the papers in Antman-Keller [1] and Zarantonello [1].)

Example __. (Couette Flow). Suppose we have a viscous fluid between two concentric cylinders (see the following figure). Suppose further we forcibly rotate the cylinders in opposite directions at some constant angular velocity ρ which is our parameter. For ρ near 0 , we get a steady horizontal laminar flow in the fluid. However as ρ reaches some critical point, the fluid breaks up into what are called Taylor cells and the fluid moves radially in cells from the inner cylinder to the outer one and vice versa. Note, that the directions of flow are such that flow is continuous.



Couette Flow



Taylor Cells

In the above example, we have a situation described by differential equations and at some critical point of the parameter, the given solution becomes unstable and the system shifts to a "stable" solution. This sharp division of solutions is the sort of bifurcation we shall encounter in Hopf's theorem.

For simplicity, let us consider the case where the underlying space is simply R^2. Let X_μ be a vector field on R^2 depending smoothly on some real parameter μ. Actually it is convenient to put X_μ in R^3 by considering the map $\tilde{X} : (x, y, \mu) \mapsto (X_\mu(x, y), 0)$. This way we can graph the flow F_t^μ of X_μ and keep track of the parameter μ. The flow G_t of $\tilde{X}$ is $G_t(x, y, \mu) = (F_t^\mu(x, y), \mu)$. Similarly, we consider X_μ acting on the plane $\mu = \text{const.}$

Now suppose $X_\mu(0, 0) = (0, 0)$ for each μ; more generally one could consider a curve (x_μ, y_μ) of critical points of X_μ. We can apply the analysis we developed for vector fields, i.e., for each μ, we look at the eigenvalues of $DX_\mu(0, 0)$ say $\lambda(\mu)$ and $\overline{\lambda(\mu)}$. (They are complex conjugate.) Note that the eigenvalues depend on μ and by our earlier analysis of flows, we know the qualitative behaviour of the flow depends on the sign of $\text{Re}(\lambda(\mu))$ and $\text{Re}(\overline{\lambda(\mu)})$ (which are equal in case $\lambda(\mu)$ itself is not real). So if we know how $\lambda(\mu)$ depends on μ then we can hope to extract some information about the flow near $(0, 0)$ as μ increases. We make these hypotheses:

<u>Suppose</u> $\text{Re}(\lambda(\mu)) < 0$ <u>for</u> $\mu < 0$ <u>and</u> $\text{Re}(\lambda(0)) = 0$ <u>and</u> $\text{Re}(\lambda(\mu))$ <u>is increasing as</u> μ <u>increases across</u> 0. <u>Also assume that</u>

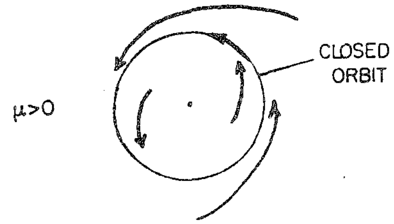
$\lambda(\mu)$ is not real and that for $\mu = 0$, (0, 0) is an attracting fixed point for X

(perhaps with a weaker or slower attraction than when $Re(\lambda(\mu)) < 0$).

Now for $\mu < 0$, we know from the above that the flow is "stable," i.e., points near (0, 0) are carried towards (0, 0) by the flow, as is the case for $\mu = 0$ (only slower) by assumption. The surprising case is the behavior for $\mu > 0$.

Theorem.* (E. Hopf). In the situation described above, there is a stable periodic orbit for X_μ when $0 < \mu < \epsilon$ for some $\epsilon > 0$. (Stable here means points near the periodic orbit will remain near and eventually be carried closer to the orbit by the flow.)

So as in the example we get a qualitative change in the stable solutions as μ crosses 0 , from an attracting fixed point at (0, 0) to a periodic solution away from (0, 0) .



This theorem does generalize to R^n where we can get tori forming as the stable solutions (instead of closed orbits) as further bifurcations take place; see Ruelle-Takens [1] for details.

The proof of the theorem occurs in many places besides Hopf [1]. See, for instance Andronov and Chaikin [1], or Bruslinskaya

* See Ruelle [4] for a version suitable for systems with symmetry, such as Couette flow.

[3], or Ruelle-Takens [1].

Hopf's theorem is closely related to a linear model used in physics known as the "Turing model." As D. Ruelle, S. Smale, N. Kopell and H. Hartman have remarked, these sort of phenomena may be basic for understanding a large variety of qualitative changes which occur in nature, including biological and chemical systems. See for instance Turing [1], Selkov [1]. We have examined here only one of many types of possible bifurcations. There are many others which occur in Thom's theory of morphogenesis (see articles in Chillingworth [1] and Abraham [4] for more details and bibliography). Meyer [1] and Abraham [5] are representative of the Hamiltonian case.

For applications to fluid mechanics one wishes the vector field X_μ to be the Navier-Stokes equations and μ to be the Reynolds number. One is hampered by the fact that X_μ in this case is not a C^r vector field (even in Lagrangian coordinates). However this difficulty can be overcome and indeed the Hopf theorem is valid. For details see Marsden [3], Joseph-Sattinger [1], Iooss [1, 2], Judovich [3, 4], Bruslinskaya [1] etc.

Moreover, an important feature is that one can show that when a bifurcation does occur <u>one retains global existence of smooth solutions near the closed orbit</u>. This is in fact good evidence in the direction of verifying that the Navier-Stokes equations do <u>not</u> break down when turbulence develops.

Stability and Turbulence.

Shortly we shall explain more fully the Ruelle-Takens theory of turbulence. For now we just wish to stress the point that turbulence appears to be some complicated flow which sets in after successive bifurcations have occurred. In this process, stable solutions become unstable, as the Reynolds number is increased. Hence turbulence is supposed to be a necessary consequence of the equations and in fact of the "generic case" and just represents a complicated solution. For example in Couette flow as one increases the angular velocity Ω_1 of the inner cylinder one finds a shift from laminar flow to Taylor cells or related patterns at some bifurcation value of Ω_1. Eventually turbulence sets in. In this scheme, as has been realized for a long time, one first looks for a stability theorem and for when stability fails (Hopf [4], Chandresekar [1], Lin [1] etc.). For example, if one stayed closed enough to laminar flow, one would expect the flow to remain approximately laminar. Serrin [2] has a theorem of this sort which we present as an illustration:

Stability Theorem. *Let $D \subset R^3$ be a bounded domain and suppose the flow v_t^ν is prescribed on ∂D (this corresponds to having a moving boundary, as in Couette flow). Let* $V = \max_{\substack{x \in D \\ t \geq 0}} \|v_t^\nu(x)\|$, $d =$ *diameter of D and ν equal the viscosity. Then if the Reynolds number $R = (Vd/\nu) \leq 5.71$, v_t^ν is universally L^2 stable.*

Universally L^2 stable means that if $\bar{v}_t^\nu$ is *any* other solution to the equations and with the same boundary conditions, then the L^2 norm (or energy) of $\bar{v}_t^\nu - v_t^\nu$ goes to zero as $t \to 0$.

The proof is really very simple and we recommend reading Serrin [2] for the argument.

Chandresekar [1], Serrin [2], and Velte [1] have analyzed criteria of this sort in some detail for Couette flow.

As a special case, we recover something that we expect. Namely if $v_t^\nu = 0$ on ∂M is any solution for $\nu > 0$ then $v_t^\nu \to 0$ as $t \to \infty$ in L^2 norm, since the zero solution is universally stable.

Couette flow is not the only situation where this Taylor cell type of phenomenon occurs and where the above analysis is possible. For example, in the Bénard Problem one has a vessel of water heated from below. At a critical value of the temperature gradient, one observes convection currents, which behave like Taylor cells; cf. Rabinowitz [1].

This transition from laminar to periodic motion (the Hopf bifurcation) occurs in many other physical situations such as flow behind an obstable.

<u>A Definition of Turbulence</u>.

A traditional definition (as in Hopf [2], Landau-Lifschitz [1]) says that turbulence develops when the vector field v_t can be described as $v_t(w_1, \ldots, w_n) = f(tw_1, \ldots, tw_n)$ where f is a quasi-periodic function, i.e., f is periodic in each coordinate, but the periods are not rationally related. For example, if the orbits of the v_t on the tori given by the Hopf theorem can be described by spirals with irrationally related angles, then v_t would such a flow.

Considering the above example a bit further, it should be clear there are many orbits that the v_t could follow which are qualitatively like the quasi-periodic ones but which fail themselves to be quasi-periodic. In fact a small neighborhood of a quasi-periodic function may fail to contain many other such functions. One might desire the functions describing turbulence to contain most functions and not only a sparse subset. More precisely, say a subset U of a topological space S is <u>generic</u> if it is a Baire set (i.e., the countable intersection of open dense subsets). It seems reasonable to expect that the functions describing turbulence should be generic, since turbulence is a common phenomena and the equations of flow are never exact. Thus we would want a theory of turbulence that would not be destroyed by adding on small perturbations to the equations of motion.

The above sort of reasoning lead Ruelle-Takens [1] to point out that since quasi-periodic functions are not generic, it is unlikely they "really" describe turbulence.* In its place, they propose the use of "strange attractors." (See Smale [2] and Williams [1].) These exhibit much of the qualitative behavior one would expect from "turbulent" solutions to the Navier-Stokes equations and they are stable under perturbations.

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* See also Joseph-Sattinger [1].

Here is an example of a strange attractor. Let $U \subset R^n$ be open and $\sigma_t : U \to U$ some flow; suppose further for $x \in U$, there is an $s \in R$ such that $\sigma_{s+t}(x) = \sigma_t(x)$, i.e., x belongs to a periodic orbit of the flow. Let $(d/dt)\sigma_t(x)|_{t=0} = Y_x$ and let V be the affine hypersurface in U orthogonal to Y_x. For a small neighborhood S of x in V, there is a map $P : S \to V$ called the Poincaré map, defined as follows: For $w \in S$, it is easy to show there is a smallest $P_w \in R$ such that $\sigma_{P_w}(w) \in V$. Call $P(w) = \sigma_{P_w}(w)$. Now of course one can do this for each point of the periodic orbit. By doing this one gets a map on a small "tubular" neighborhood of the periodic orbit in U. (Here one must check that there is a neighborhood N os the orbit such that if $x \in N$ then x belongs to a unique hypersurface orthogonal to the orbit.) Also one can drop the condition that P be defined about a closed orbit by requiring that the vector field be almost parallel and everywhere transversal to a hypersurface V. In this case one can define a Poincaré map P over the entire space U by letting $P(x)$ be the first intersection of the integral curve through x with V.

In particular consider V to be a solid torus in three space and suppose we have a flow σ_t on U such that its Poincaré map wraps the torus around twice. Then the attracting set of the flow (i.e., $\{x \in U | x = \lim_{t\to\infty} \sigma_t(y)$ for some $y \in U\}$ is locally a Cantor set cross a 2-manifold (see Smale [2]). This is certainly a strange attractor! Ruelle-Takens [1] have shown if we define a strange attractor to be one which is neither a closed orbit or a point, then there are stable strange attractors on T^4 in the sense that a whole neighborhood of vector fields has a strange attractor as well.

If the attracting set of the flow, in the space of vector fields, which is generated by Navier-Stokes equations is strange, then a solution attracted to this set will clearly behave in a complicated, turbulent manner and since strange attractors are "generic", this sort of behavior should not be uncommon. Thus we have the following reasonable definition of turbulence as proposed by Ruelle-Takens:

"... the motion of a fluid system is turbulent when this motion is described by an integral curve of a vector field X_μ which tends to a set A, and A is neither empty nor a fixed point nor a closed orbit."

This turbulent motion is supposed to occur on one of the tori T^k that occurs in the Hopf bifurcation. This takes place after a finite number of successive bifurcations have occurred. However as S. Smale and C. Simon pointed out to us, there may be an infinite number of other qualitative changes which occur during this onset of turbulence (such as stable and unstable manifolds intersecting in various ways etc).

Since this sort of phenomena is supposed to be "generic," one would expect it to occur in other similar phenomena such as the Benard problem. (As the temperature gradient becomes very large, the flow becomes "turbulent.")

Recently Ruelle [1] (and unpublished work) has shown how the usual statistical mechanics of ergodic systems can be used to study the case of strange attractors, following work of Bowen [1] and Sinai [1]. It remains to connect this up with observed statistical properties of fluids, like the time average of the pressure in turbulent flow.

For the analytical nature of turbulent solutions, the work of Bass [1, 2] seems to be important.

In summary then, this view of turbulence may be phrased as follows. Our solutions for small μ (= Reynolds number in many fluid problems) are stable and as μ increases, these solutions become <u>unstable</u> at certain critical values of μ and the solution falls to a more complicated <u>stable</u> solution; eventually, after a certain finite number of such bifurcations, the solution falls to a strange attractor (in the space of all time dependent solutions to the problem). Such a solution, which is wandering close to a strange attractor, is called <u>turbulent</u>.