6. Symmetry Groups in Mechanics.

In this lecture we shall discuss the conservation laws resulting when one has a Hamiltonian system with symmetry. Intuitively one should think of linear and angular momentum which arise from translational and rotational invariance respectively. However one can have more sophisticated conservation laws too such as those dealing with spin, with the rigid body etc. Following these topics, we shall explain how one can shrink down the phase space in order to eliminate the variables which were obtained from the conservation laws. Some of these are subtle yet very fundamental, viz Jacobi's "elimination of the node" in celestial mechanics. Finally we shall discuss a completeness theorem in geometry and how various conservation laws can be used to prove it. Other completeness theorems are proved in lecture 8. This lecture is based on Souriau [1] and Marsden-Weinstein [1].

Before beginning the actual mechanics, we shall need a little notation and a few facts concerning Lie groups.

Preliminaries on Lie Groups and Group Actions.

Let G be a Lie group; i.e. a C^∞ manifold which is also a group and the group operations are C^∞. Let $\mathfrak{g}$ denote the Lie algebra of G; we can think of $\mathfrak{g}$ either as the vector space T_eG or as the space of all left invariant vector fields on G. The latter gives us a bracket $[\xi, \eta]$ on $\mathfrak{g}$ making it into a Lie algebra; i.e. $[[\xi,\eta],\zeta]\ [\zeta, \xi], \eta] + [\eta, \zeta], \xi] = 0$ holds.

Example 1. $SO(3)$ the group of all 3×3 orthogonal matrices of determinant +1 is a 3 dimensional Lie group. $\mathcal{G} = T_e SO(3)$ consists of the 3×3 skew adjoint matrices with bracket equalling the commutator. This space of matrices is, in turn identifiable with R^3. Making the identification, the bracket is just the cross product.

2. See the example $\mathcal{D}$, the diffeomorphism group, discussed in lecture 4.

By an <u>action</u> (or "non-linear representation") of G on a manifold M, we mean a collection of mappings $\Phi_g : M \to M$ such that

$$\text{(i)} \quad \Phi_{gh} = \Phi_g \circ \Phi_h$$

$$\text{and} \quad \text{(ii)} \quad \Phi_e = \text{identity} \qquad e = \text{identity in } G .$$

We also require $(g, x) \mapsto \Phi_g(x)$ to be C^∞.

Notice that if $G = R$, an action is nothing more than a flow. As every flow determines a generating vector field, we are led to define the infinitesimal generators of an action. We do this in the following discussion.

Let $\xi \in \mathcal{G}$. Let $\exp \xi$ denote the exponential of ξ. (This is defined as follows; let $\tilde{\xi}$ denote the left invariant vector field which equals ξ at e; then $\exp t\xi$ is the integral curve of $\tilde{\xi}$ starting at e. For matrix groups $\exp \xi = e^A$ defined, e.g. as a power series.) Now one verifies $\exp t\xi$ is a one parameter subgroup of G; i.e., $\exp(t+s)\xi = \exp t\xi \cdot \exp s\xi$. Thus $\Phi_{\exp t\xi}$ is a flow on M. Let ξ_M denote its generator. We call the map $\xi \mapsto \xi_M$ of $\mathfrak{g}$ to vector fields on M the <u>infinitesimal generator</u> of the action. One has $[\xi_M, \eta_M] = -[\xi, \eta]_M$.

Let G act on M and let $x \in M$. The <u>isotropy group</u> of x is, by definition:

$$G_x = \{g \in G \mid \Phi_g(x) = x\} .$$

It is a subgroup of G.

G is said to act <u>freely</u> on M if each $x \in M$ has $G_x = \{e\}$.

G is said to act <u>properly</u> on M if the map $(g, x) \mapsto (x, \Phi_g(x))$ of $G \times M \to M \times M$ is proper; i.e. inverse images of compact sets are compact.

If G acts on M and $x \in M$, $\{\Phi_g(x) \mid g \in G\} = G\cdot x$ is the <u>orbit</u> of x. These are always immersed submanifolds of M. (One maps $G/G_x \to G\cdot x$ to obtain the required immersion.) Moreover, M is the disjoint union of the orbits. Thus one can consider M/G the space of all orbits.

If G acts freely and properly on M then M/G is a C^∞ manifold and $\pi : M \to M/G$ is a submersion. Let $\pi(x) = [x]$. Now

$$T_x(G\cdot x) = \{\xi_M(x) \mid \xi \in \mathfrak{g}\}$$

and

$$T_{[x]}(M/G) \cong T_xM/T_xG\cdot x.$$

(These facts are proven, for example in Bourbaki [1].) When we form quotient manifolds in the sequel we implicitly assume these hypotheses. If M consists only of one orbit, we say that we have a homogeneous space. Thus $M \cong G/G_x$. (In general M/G is not a manifold; consider S^1 acting on the plane; M/G is then a half ray.)

Let G act on M and on N by actions Φ_g and Ψ_g respectively. A map $\psi : M \to N$ is called equivariant (or an intertwining map) if

$$\psi \circ \Phi_g = \Psi_g \circ \psi \quad \text{for all} \quad g \in G .$$

Consider now just a given Lie group G. Then there is an action of G on $\mathfrak{G}$ by linear transformations called the adjoint action:

$$\mathrm{Ad}_g \cdot \xi = TR_{g^{-1}} TL_g \cdot \xi .$$

Here R_g and L_g are the right and left translation maps. The infinitesimal generator of this action is $\xi \mapsto \xi_{\mathfrak{G}}$, $\xi_{\mathfrak{G}}(\zeta) = [\xi, \zeta] = \mathrm{ad}_\xi(\zeta)$.

We also get an action on the dual space $\mathfrak{G}^*$ called the coadjoint action by using $(\mathrm{Ad}_{g^{-1}})^*$.

The Moment Function of Souriau and Noethers Theorem.

We now consider a general setting for finding conservation laws. The basic results are due to Souriau [1] but were found also in Marsden [1] and Smale [4].

<u>Definition</u>. Let G be a Lie group and P a (weak) symplectic manifold. Let G act on P by symplectic diffeomorphisms. (It follows that each infinitesimal generator ξ_P satisfies $di_{\xi}\omega = 0$.)

By a <u>moment</u> for the action we mean a C^∞ map $\psi : P \to \mathfrak{G}^*$ such that if $\hat{\psi}$ denotes the dual map from $\mathfrak{G}$ to the space of smooth function on P , i.e. $\hat{\psi}(\xi)(p) = \psi(p)\cdot\xi$, we have

$$d(\hat{\psi}(\xi)) = i_{\xi_P}\omega$$

i.e., $\langle T_p\psi\cdot v, \xi\rangle = \omega_p(\xi_P(p), v)$ for $\xi \in \mathfrak{G}$, $v \in T_pP$. In other words, each infinitesimal generator ξ_P has $\hat{\psi}(\xi)$ as a Hamiltonian function. A moment, if it exists, is defined up to an arbitrary additive constant in $\mathfrak{G}^*$.

<u>Theorem</u>. <u>Let</u> $H : P \to R$ <u>be invariant under</u> Φ , <u>i.e.</u>, $H\circ\Phi_g = H$. <u>Then</u> ψ <u>is a constant of the motion for</u> X_H ; <u>i.e. if</u> F_t <u>is the flow of</u> X_H , $\psi\circ F_t = \psi$.

<u>Proof</u>. From $H\circ\Phi_g = H$ it follows that $H\circ\Phi_{\exp t\xi} = H$ and hence $L_{\xi_P} H = 0$. But this means $\{\hat{\psi}(\xi), H\} = 0$ so $\hat{\psi}(\xi)$ is a constant of the motion. □

In order to actually compute ψ we use:

<u>Theorem</u>. <u>Let</u> G <u>act symplectically on</u> P . <u>Assume</u> $\omega = -d\theta$ <u>and the action leaves</u> θ <u>invariant. Then</u> $\psi(p)\cdot\xi = (i_{\xi_P}\theta)(p)$ <u>and</u> ψ <u>is equivariant; i.e.</u> $\psi\circ\Phi_g = (Ad_{g^{-1}})^*\circ\psi$.

Proof. Since Φ_g leaves θ invariant, we have $L_{\xi_p}\theta = 0$. Hence

$$di_{\xi_p}\theta + i_{\xi_p}d\theta = 0$$

i.e.,
$$i_{\xi_p}\omega = di_{\xi_p}\theta \quad .$$

Hence we can choose $\hat{\psi}(\xi) = i_{\xi_p}\theta$ as required. We leave equivariance of this formula as an exercise. □

Let us specialize further to give an even more useful formula:

Theorem. Let G act on M . Then the action lifts to one on T^*M preserving the canonical one form (this was essentially proved in lecture 2). We have

$$\psi(\alpha)\cdot\xi = \alpha\cdot\xi_M(x) \qquad , \qquad \alpha \in T^*_xM \quad .$$

This follows in a straightforward way from the previous theorem. The quantities are sometimes written $P(X)(\alpha_x) = \alpha_x\cdot X(x)$, X a vector field on M and called the momentum functions. Equivariance can be phrased infinitesimally in terms of the commutation relations:

$$P([X, Y]) = -\{P(X), P(Y)\} \ .$$

In examples of linear or angular momentum the conserved quantity ψ reduces to the usual expressions.

We can also specialize to TM with a given metric rather than to T^*M with the canonical symplectic structure.

Theorem. Let G act on M by isometries, where M is a given Riemannian manifold. Let $V : M \to R$ be invariant and let $H(v) = \frac{1}{2}\langle v, v\rangle + V(x)$, $v \in T_xM$. Then if ξ_M is an infinitesimal generator of the action, the function

$$\hat{\psi}(\xi)(v_x) = \langle v_x, \xi_M(x)\rangle$$

is a constant of the motion for X_H .

It is also useful to present a version for general Lagrangian systems. The classical Noether theorem is a special case. Although this follows from the above, we give a separate proof here. (See lecture 2 for a general discussion of Lagrangian systems, and an explanation of the notation FL .)

In this result, observe that we do allow for the possibility that L might be degenerate. The only special assumption needed on Z is that it exist and be second order.

Proposition. Let Z be a Lagrangian vector field for $L : TM \to R$ and suppose Z is a second order equation.

Let Φ_t be a one parameter group of diffeomorphisms of M generated by the vector field $Y : M \to TM$. Suppose that for each real number t , $L \circ T\Phi_t = L$. Then the function $P(Y) : TM \to R$, $P(Y)(v) = FL(v)\cdot Y$ is constant along integral curves of Z .

Proof. Let $v(t)$ be an integral curve for Z . Then we shall show that $(d/dt)\{P(Y)(v(t))\} = 0$. Indeed, in a coordinate chart, if $(u(t), v(t))$ is the integral curve,

$$\frac{d}{dt}\{FL(v(t))\cdot Y\} = \frac{d}{dt}\{D_2L(u(t), v(t))\cdot Y(u(t))\}$$

$$= D_1D_2L(u(t), v(t))\cdot Y(u(t))\cdot u(t) + D_2D_2L(u(t), v(t))$$

$$\cdot\ Y(u(t))\cdot \dot{v}(t) + D_2L(u(t), v(t))\cdot DY(u(t))\cdot \dot{u}(t) .$$

Now the condition that Z be the Lagrangian vector field of L means exactly that the first two terms equal $D_1L(u(t), v(t))\cdot Y(u(t))$ (see the results given in lecture 2.). However if we differentiate $L\circ T\Phi_t$ with respect to t we obtain for any point (u, v) ,

$$0 = \frac{d}{dt}L(\Phi_t(u), D\Phi_t(u)\cdot v)\Big|_{t=0}$$

$$= D_1L(u, v)\cdot Y(u) + D_2L(u, v)\cdot DY(u)\cdot v .$$

Comparing this with the above gives $(d/dt)\{FL(v)\cdot Y\} = 0$ and proves the assertion. □

The Reduced Phase Space.

As mentioned in the introduction, when one has a group of symmetries, it is a classical procedure to eliminate a number of variables in order to get rid of the symmetries. We present now, following Marsden-Weinstein [1] a unified, as well as simplified, scheme for carrying out such a program.

For simplicity we shall always assume the moment ψ is equivalent with respect to the $Ad_{g^{-1}}^*$ action. (This is not really necessary, for Souriau has shown that one can suitably modify the action.)

Let μ be a regular value of ψ ; i.e., ψ is a submersion on $\psi^{-1}(\mu)$, so $\psi^{-1}(\mu)$ is a submanifold.

Let G_μ be the isotropy group of μ for the G action on $\mathcal{G}^*$. By equivarience, $\psi^{-1}(\mu)$ is invariant under G_μ so the orbit space $\psi^{-1}(\mu)/G_\mu$ is defined. Note also that by equivariance if $p \in \psi^{-1}(\mu)$ and $\Phi_g(p) \in \psi^{-1}(\mu)$ then $g \in G_\mu$. We let

$$P_\mu = \psi^{-1}(\mu)/G_\mu$$

and call P_μ the <u>reduced phase space</u>.

The main result is as follows.

<u>Theorem</u>. <u>Let</u> G <u>be a Lie group acting symplectically on the symplectic manifold</u> P , ω . <u>Let</u> ψ <u>be a moment for the action. Let</u> $\mu \in \mathcal{G}^*$ <u>be a regular value of</u> ψ . <u>Suppose</u> G_μ <u>acts freely and properly on the manifold</u> $\psi^{-1}(\mu)$. <u>Then if</u> $i_\mu : \psi^{-1}(\mu) \to P$ <u>is inclusion</u>, <u>there is a unique symplectic structure</u> ω_μ <u>on the reduced phase space</u> P_μ <u>such that</u> $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, <u>where</u> π_μ <u>is the projection of</u> $\psi^{-1}(\mu)$ <u>onto</u> P_μ .

To prove this we shall make use of the following:

<u>Lemma</u>. <u>For</u> $p \in \psi^{-1}(\mu)$ <u>we have</u>

(i) $T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_p(\psi^{-1}(\mu))$

and (ii) $T_p(\psi^{-1}(\mu))$ is the ω-orthogonal complement of $T_p(G \cdot p)$.

Proof. (i) Let $\xi \in \mathcal{G}$, so $\xi_P(p) \in T_p(G \cdot p)$. We must show $\xi_P(p) \in T_p(\psi^{-1}(\mu))$ iff $\xi \in \mathcal{G}_\mu$ the Lie algebra of G_μ . Equivariance gives $T_p\psi \cdot \xi_P(p) = \xi_{\mathcal{G}^*}(\mu)$, so $\xi \in \mathcal{G}_\mu$ iff $\xi_{\mathcal{G}^*}(\mu) = 0$ iff $\xi_P(p) \in \ker T_p\psi = T_p(\psi^{-1}(\mu))$.

(ii) For $\xi \in \mathcal{G}$, $v \in T_pP$ we have $\omega(\xi_P(p), v) = \langle T_p\psi \cdot v, \xi \rangle$ since ψ is a moment. Thus $v \in \ker T_p\psi$ iff $\omega(\xi_P(p), v) = 0$ for all $\xi \in \mathcal{G}$. □

In the following proof we use the fact that if $F \subset E$ is a subspace of a symplectic space E , then $(F^\perp)^\perp = F$ where $\perp$ is the ω-orthogonal complement. In finite dimensions this follows by dimension counting. It is also true in infinite dimensions for weak symplectic forms if E is reflexive. We now prove our theorem.

Proof. For $v \in T_p(\psi^{-1}(\mu))$, let $[v] \in T_{\pi_\mu(p)}P_\mu$ denote the corresponding equivalence class in $T_p\psi^{-1}(\mu)/T_p(G_\mu \cdot p)$, so $[v] = T\pi_\mu \cdot v$. The assertion $\pi_\mu^*\omega_\mu = i_\mu^*\omega$ becomes

$$\omega_\mu([v], [w]) = \omega(v, w) \ , \text{ for all } v, w \in T_p\psi^{-1}(\mu) \ .$$

Thus ω_μ is unique. Moreover, ω_μ is well-defined because of the lemma. Also ω_μ is smooth because quantities on a quotient M/G are smooth when they have smooth pull-backs to M . Thus ω_μ is a well-defined smooth two-form on P_μ .

To show ω_μ is symplectic we first show ω_μ is non-degenerate; $\omega_\mu([v], [w]) = 0$ for all $w \in T_p\psi^{-1}(\mu)$ implies $v \in T_p(G_\mu \cdot p)$ by the lemma, or $[v] = 0$. It remains to show ω_μ is closed. But from $\pi^*_\mu\omega_\mu = i^*_\mu\omega$ and $d\omega = 0$, we conclude that $\pi^*_\mu(d\omega_\mu) = 0$, so $d\omega_\mu = 0$ since $T\pi_\mu$ is surjective. □

Remarks. Even if $\omega = -d\theta$ and the action leaves θ invariant, ω_μ need not be exact. For $\mu \neq 0$, θ does not project to a one-form on P_μ because $\theta(\xi_P)(p) = \psi(p)\xi \neq 0$.

As a consequence, observe that (in the finite dimensional case) P_μ is even-dimensional. If ψ is a submersion, then $\dim P_\mu = \dim P - \dim G - \dim G_\mu$.

If μ is a regular value of ψ , the action is always locally free near $\psi^{-1}(\mu)$.

Examples. 1. Let us begin by recalling the cotangent bundle case. Namely, if G acts on a manifold M , we obtain a symplectic action on T^*M which preserves the canonical one-form θ on T^*M . A moment for this action is given by $\psi : T^*M \to \mathfrak{G}^*$:

$$\langle\psi(\alpha), \xi\rangle = \langle\alpha, \xi_M(m)\rangle , \quad \alpha \in T_mM .$$

By an earlier general theorem, this moment is Ad*-equivarient.

We conclude that if G_μ acts freely and properly on

$\psi^{-1}(\mu) = \{\alpha \in T^*M | \langle\alpha, \xi_M(m)\rangle = \langle\mu, \xi\rangle$ for all $\xi \in \mathfrak{G}\}$, then $\psi^{-1}(\mu)/G_\mu$ is a symplectic manifold. If the $\xi_M(m)$ span a space of dimension = dim $\mathfrak{G}$ at m , then it is easy to see that each point of $T^*_m M$ is regular.

2. If we specialize example 1, taking $M = G$ with G acting on itself by left multiplication, then the moment $\psi : T^*G \to \mathfrak{G}^*$ is given by

$$\psi(\alpha) = (TR_g)^* \cdot \alpha \in T^*_e G = \mathfrak{G}^* , \quad \alpha \in T_g G$$

where R_g denotes right translation (cf. Arnold [1], Marsden-Abraham [1]). Thus each $\mu \in \mathfrak{G}^*$ is regular and $\psi^{-1}(\mu)$ is the graph of the right invariant one-form ω_μ whose value at e is μ . Now $G_\mu = \{g \in G | L^*_g \omega_\mu = \omega_\mu\}$, so the action of G_μ on $\psi^{-1}(\mu)$ is left translation on the base point. Thus $\psi^{-1}(\mu)/G_\mu \approx G/G_\mu \approx G\cdot\mu \subset \mathfrak{G}^*$. Thus the reduced phase space is just the orbit of μ in $\mathfrak{G}^*$. That this is a symplectic manifold then follows from the above theorem. The rather special construction in this case is due to Kirillov-Kostant; see Kostant [1]. If one traces through the definitions one finds for $\beta \in G\cdot\mu$, $\gamma_1 = (\mathrm{ad}\, u_1)^*\beta$ and $\gamma_2 = (\mathrm{ad}\, u_2)^*\beta$, that

$$\omega_\mu(\gamma_1, \gamma_2) = \beta([u_2, u_1]) .$$

When viewed directly, the symplectic structure on $G\cdot\mu \subset \mathfrak{G}^*$ seems rather special. However, it becomes natural when viewed in the context of reduced phase spaces. Moreover, the proof becomes more transparent. This example is studied further below.

3. If G acts on M and leaves a given closed two-form F on M invariant, then we get a symplectic action on T^*M with the symplectic form $\omega_F = \omega + \pi^*F$ where ω is the canonical form and $\pi : T^*M \to M$ the projection. Such a situation arises when one has a particle moving in the "electromagnetic field" F (see Souriau [1] and Sniatycki-Tulczyjew [2]). Now suppose $F = dA$ is exact and A is invariant. Then the moment is given by

$$\langle \psi(\alpha), \xi \rangle = \langle \alpha - A, \xi_M(m) \rangle$$

(this corresponds to the classical prescription of replacing p by $p - \frac{e}{c} A$ in an electromagnetic potential A). The verification is the same as in example 1. Thus again, if μ is a regular value and G_μ acts freely and properly on $\psi^{-1}(\mu)$, we can form the reduced phase space P_μ.

4. Let $G = SO(3)$ and P a symplectic manifold. Here $\mathcal{G} \approx R^3$ and the adjoint action is the usual one. For $\mu \in R^3$, $\mu \neq 0$, $G_\mu = S^1$ corresponding to rotations about the axis μ. (Since G is semi-simple, a symplectic action of G on P has an Ad*-equivariant moment ψ by Souriau [1]). One refers to ψ as "angular momentum" in this case. The reduction of P to $\psi^{-1}(\mu)/S^1$ is a generalization of the procedure called "elimination of the nodes" (cf. Smale [4] and Whittaker [1, p. 344]).

5. Suppose we have the situation of the above theorem, and in addition G is abelian. Ad*-equivariance means that the generating

functions $\hat{\psi}(\xi)$ are all in involution on P. Furthermore, $G_\mu = G$ for each $\mu \in \mathcal{G}^*$. If the action is free and μ is a regular value, we can form $P_\mu = \psi^{-1}(\mu)/G$. In this case $\dim P_\mu = \dim P - 2 \dim G$. The reduction to P_μ represents the classical reduction of a Hamiltonian system by integrals in involution.

As a special case, let X_H be a Hamiltonian vector field on P, so that the flow of X_H yields an action of R on P. The moment is just H itself so we get a symplectic structure on $H^{-1}(e)/R$ which is just the space of orbits on each energy surface (we assume e is a regular value of H).

6. Let $\mathcal{D}$ denote the group of C^∞-diffeomorphisms of a finite dimensional Riemannian manifold M. Suppose M is compact, or restrict to diffeomorphisms which are "asymptotic to the identity". Now as we saw in lecture 4, $T_e\mathcal{D} = \mathfrak{X}(M)$ = the vector fields on M and we put on $\mathcal{D}$ the L_2 metric which is obtained from $\mathfrak{X}(M)$ by right invariance. Thus $\mathcal{D}$ acting on $T\mathcal{D}$ on the right is a symplectic action. As in example 2, we conclude that for each $X \in \mathfrak{X}(M)$, the set $\{\eta^*X \mid \eta \in \mathcal{D}\} \subset \mathfrak{X}(M)$ is a weak symplectic manifold. The symplectic structure is

$$\omega_X(\eta^*L_{Y_1}X, \eta^*L_{Y_2}X) = \int_M \langle X, [Y_2, Y_1]\rangle dx .$$

One may similarly restrict to volume preserving diffeomorphisms and divergence free vector fields. This symplectic manifold is left invariant by the Euler equations on $\mathfrak{X}(M)$ and they define a Hamiltonian system so restricted. (See the following theorem and corollary).

7. Let M and $\mathcal{D}$ be as in example 6. Let $\mathfrak{M}$ denote the space of all Riemannian metrics on M. Define the DeWitt metric on $\mathfrak{M}$ by

$$\mathcal{G}_g(h, k) = \int_M [\langle h, k\rangle - (\operatorname{tr} h)(\operatorname{tr} k)]\, d\mu_g$$

Where h, $k \in T_g\mathfrak{M}$ = the symmetric 2-tensors on M, $\langle h, k\rangle$ is the inner product of h, k using the metric g, tr denotes the trace, and μ_g is the volume element associated with g. $\mathcal{G}_g$ is a weak metric and gives a (weak) symplectic structure on $T\mathfrak{M}$.

The space $T\mathfrak{M}$ is a basic (weak) symplectic manifold used in general relativity. We will now describe its reduced phase space in the presence of the symmetry group $\mathcal{D}$. (See lecture 9 for the connections of these ideas with general relativity.) $\mathcal{D}$ acts symplectically on $T\mathfrak{M}$ by pull-back. The moment for this action is not difficult to compute. It is:

$$\psi(g, k)\cdot X = 2\int_M \langle X, \delta\pi\rangle d\mu_g$$

where $\pi = k - \frac{1}{2}(\operatorname{tr} k)g$ and δ is the divergence taken with respect to g. Of particular interest is the case $\psi^{-1}(0) = \{(g, k) \in T\mathfrak{M} \mid \delta\pi = 0\}$ (referred to as the divergence constraint in general relativity).

The isotropy group is all of $\mathcal{D}$, so the reduced phase space is $\psi^{-1}(0)/\mathcal{D}$. If we work near a metric with no isometries (asymptotically the identity if M is not compact), then $\psi^{-1}(0)/\mathcal{D}$ is a manifold by using methods explained in lecture 10. We conclude that $\psi^{-1}(0)/\mathcal{D}$ is a (weak) symplectic manifold.* This is the basic space one uses for a dynamical formulation of general relativity. It is related to "superspace" $\mathfrak{M}/\mathcal{D}$ in that all "geometrically equivalent" objects have been identified. See Marsden-Fischer [1] for further results along these lines.

8. Let $\mathcal{H}$ be complex Hilbert space with $\omega = \mathrm{Im}\langle , \rangle$ and $G = S^1$. Then G acts symplectically on $\mathcal{H}$ by $\Phi_z(\varphi) = z \cdot \varphi$, $|z| = 1$, $\varphi \in \mathcal{H}$. A moment is easily seen to be

$$\psi(\varphi) \cdot z = \tfrac{1}{2} \langle \varphi, \varphi \rangle \cdot z \; .$$

Thus $\psi^{-1}(1)$ is the unit sphere, so $\psi^{-1}(1)/G$ is projective Hilbert space. We recover the well-known fact that projective Hilbert space is a symplectic manifold (in fact it has a Kahler structure). This result will be useful for the next lecture.

Hamiltonian Systems on the Reduced Phase Space.

Theorem. Let the conditions of the above theorem hold. Let K be another group acting symplectically on P with a moment φ. Let the actions of K and G commute and φ be invariant under G. Then

(i) K leaves ψ invariant

(ii) the induced action of K on P_μ is symplectic and has a moment which is naturally induced from the moment φ.

* It is a conjecture of D. Ebin that this is true globally.

Proof. (i) This follows as in the proof that ψ is conserved by any G invariant Hamiltonian system on P (see above).

To prove (ii), let Ψ_k denote the action of $k \in K$ on P. By (i), $\psi^{-1}(\mu)$ is invariant under this action, and since the action commutes with that of G, we get a well-defined action on P_μ. Also, if $\tilde{\Psi}_k$ is the induced action on P_μ,

$$\pi_\mu^* \tilde{\Psi}_k^* \omega_\mu = \Psi_k^* \pi_\mu^* \omega_\mu = \Psi_k^* i_\mu^* \omega = i_\mu^* \Psi_k \omega = i_\mu^* \omega .$$

Hence $\tilde{\Psi}_k^* \omega_\mu = \omega_\mu$. Similarly, from the definition of moment we see that the induced moment is a moment for the induced action: namely, the induced moment $\tilde{\varphi}$ satisfies $\hat{\varphi} \circ \pi_\mu = \varphi$, so for $[v] = T\pi_\mu \cdot v \in TP_\mu$, $\xi \in \mathfrak{G}_K$, we have

$$\langle T\hat{\varphi} \cdot [v], \xi \rangle = \langle T\varphi \cdot v, \xi \rangle = \omega(\xi_P, v) = \omega_\mu(\xi_{P_\mu}, [v])$$

since, as is easy to see, the generators ξ_P, ξ_{P_μ} on $\psi^{-1}(\mu)$ and P_μ are related by the projection π_μ. □

For example, if we consider example 2 and let $G = K$ acting on T^*G by lifting the right action, we can conclude that the natural action of G on the orbit $G\cdot\mu \subset \mathfrak{G}^*$ is a symplectic action. The induced moment is easily seen to be just the identity map: $\tilde{\varphi}(Ad_g^* \mu) = Ad_g^* \mu \in \mathfrak{G}^*$.

The fact that G acts symplectically on the orbit $G\cdot\mu$, so that $G\cdot\mu$ is a "homogeneous Hamiltonian G-space", is a known and useful result. See Kostant [1] and Souriau [1, p. 116].

Taking $K = R$, we are led to:

Corollary. Let the conditions of the theorem preceeding the above hold and let X_H be a Hamiltonian vector field on P with H invariant under the action of G. Then the flow of X_H induces a Hamiltonian flow on P_μ whose energy $\tilde{H}$ is that induced from H; i.e., $\tilde{H} \circ \pi_\mu = H \circ i_\mu$.

For example if $\langle \, , \rangle$ is a left invariant metric on a group G, the Hamiltonian $H(v) = \frac{1}{2}\langle v, v\rangle$, which yields geodesics on G, induces a Hamiltonian system on the orbits in $\mathfrak{G}^* \approx \mathfrak{G}$. Note that the original Hamiltonian system on P is completely determined by the induced systems on the reduced spaces P_μ.

Similarly, in each of the other examples above, if we start with a given Hamiltonian system on P, invariant under G, then we can, with no essential loss of information, pass to the Hamiltonian system on the reduced phase space.

Relative Equilibria and Relative Periodic Points.

Definition. In the situation of the above corollary, a point $p \in P$ such that $\pi_\mu(p) \in P_\mu$ is a critical point [resp. periodic point] for the induced Hamiltonian system on P_μ is called a relative equilibrium [resp. relative periodic point] of the original system.

Poincaré [1] considered relative periodic points in the n-body problem on an equal footing with ordinary periodic points. Indeed, in general, the only "true" dynamics is that taking place in the reduced phase space P_μ .

The following shows that our definition coincides with the standard ones (Smale [4], Robbin [4]).

Theorem. (i) $p \in P$ is a relative equilibrium iff there is a one-parameter subgroup $g(t) \in G$ such that for all $t \in R$, $F_t(p) = \Phi_{g(t)}(p)$ where F_t is the flow of X_H and Φ is the action of G .

(ii) $p \in P$ is a relative periodic point iff there is a $g \in G$, and $\tau > 0$ such that for all $t \in R$, $F_{t+\tau}(p) = \Phi_g(F_t(p))$.

Proof. (i) p is a relative equilibrium iff $\pi_\mu(p)$ is a fixed point for the induced flow on P_μ iff $\pi_\mu(F_t(p)) = \pi_\mu(p)$. If this holds there is a unique curve $g(t) \in G_\mu$ such that $F_t(p) = \Phi_{g(t)}(p)$ since the action of G_μ on $\psi^{-1}(\mu)$ is free. The flow property $F_{t+s}(p) = F_t \circ F_s(p)$ immediately gives $g(t+s) = g(t)g(s)$, so $g(t)$ is a one-parameter subgroup of G_μ . Conversely, if $F_t(p) = \Phi_{g(t)}(p)$ where $g(t)$ is a one-parameter subgroup of G , we must show $g(t) \in G_\mu$. But this follows from invariance of $\psi^{-1}(\mu)$ under F_t and equivariance (see §2 above).

One proves (ii) in a similar way. □

As a result of our definition we have the following theorem of Smale, whose proof has also been simplified by Robbin [4] and Souriau. We present yet another proof.

Theorem. Let μ be a regular value of ψ. Then $p \in \psi^{-1}(\mu)$ is a relative equilibrium iff p is a critical point of $\psi \times H : P \to \mathcal{G}^* \times R$.

Proof. By our definition and the non degeneracy of the symplectic form on P_μ, P is a relative equilibrium iff p is a critical point of $\tilde{H}$, the reduced Hamiltonian. Since we have invariance under G, this is equivalent to p being a critical point of $H|\psi^{-1}(\mu)$ i.e. of $\psi \times H$ (Lagrange multiplier theorem). □

Thus the advantage of passing to P_μ is that relative equilibria really become equilibria and, moreover, we have a Hamiltonian system on P_μ with a (non-degenerate) symplectic form.

In the above theorem, it is necessary that μ be a regular value. For example, in the n-body problem (where $G = SO(3)$), if all the bodies are lined up with velocities headed towards the center of mass, we have a critical point of $\psi \times H$ but the bodies do not travel in circles (theorem 4(i) fails).

There are a number of equivalent ways to rephrase the above result if $P = TM$ and $H = K + V$. (In particular see Smale [4]; some interesting conditions have also been given by O. Lanford.)

Using these ideas, Smale is able to estimate the number of relative equilibria by using Morse theory to count the critical points. The results yield quite interesting information for the n-body problem (see Smale [4], Iacob [2]).

Stability of Relative Equilibria.

Let us recall the classical definition of Liapunov stability (see also lectures 5, 8). Let x be a critical point of a flow F_t ; i.e., $F_t(x) = x$. Then x is stable if for every neighborhood U of x there is a neighborhood V of x such that $y \in V$ implies $F_t y \in U$ for all t .

Now we can define stability of relative equilibria as follows:

Definition. Let $p \in P$ be a relative equilibrium of the Hamiltonian vector field X_H . We call p relatively stable if the point p is (Liapunov) stable for the induced flow on the quotient space P/G , (on P/G , p is a fixed point).

Theorem. Let the conditions guaranteeing the symplectic structure on P_μ and the above corollary hold and let $p \in P$ be a relative equilibrium. Let $\tilde{H}$ be the induced Hamiltonian on P_μ . If $d^2\tilde{H}$ is definite at $\pi_\mu(p)$, then p is relatively stable.

Proof. The condition tells us that $\pi_\mu(p)$ is a stable fixed point on P_μ , by conservation of energy. Thus we conclude that within each $\psi^{-1}(\mu)/G_\mu$, p is stable. But by openness of the conditions, the same is true of nearby reduced phase spaces $P_{\mu'}$, μ' near μ . Thus p is actually relatively stable. □

If G is a Lie group with a left invariant metric, a relative equilibrium represents a fixed point v in the Lie algebra, or a one-parameter subgroup of G. We can use theorem 6 to test for its stability. If we do so, we recover a result of V. Arnold [1] (who proved it directly by an apparently more complicated procedure) as follows. The quadratic form $d^2\tilde{H}$ at $v \in \mathcal{G}$ is, in this case, worked out to be -- after a short straightforward computation:

$$Q_v(w_1, w_2) = \langle B(v, w_1), B(v, w_2)\rangle + \langle [w_1, v], B(v, w_2)\rangle$$

where $\langle B(u, v), w\rangle = \langle [u, w], v\rangle$. Thus the condition requires Q_v to be definite. In case of a rigid body $(G = SO(3))$ this yields the classical result that a rigid body spins stably about its longest and shortest principal axes, but unstably about the middle one. For fluids $(G = \mathcal{D}_\mu$ = group of volume-preserving diffeomorphisms) the situation is complicated by the fact that the metric is only weak so the criterion is not directly applicable. In celestial mechanics stability of the relative equilibria often depends on stability criteria much deeper than that above, such as Moser's "twist stability theorem"; (see Abraham [2]).

<u>Completeness of Homogeneous Spaces</u>.

Recall that a homogeneous space is a manifold together with a transitive group action Φ on it. The following is a classical and useful result.

<u>Theorem</u>. <u>Let</u> M <u>be a riemannian manifold and suppose either</u>

(a) M <u>is compact</u>

<u>or</u> (b) M <u>is a homogeneous space, the transitive action consisting</u>

of isometries.

Then M *is geodesically complete*.

Here, geodesically complete means that the geodesic flow on TM is complete; i.e. geodesics can be indefinitely extended (without running off M) . In the finite dimensional case it is equivalent to M being complete as a metric space and to closed balls being compact.

To prove (a) one uses the fact that if an integral curve stays in a compact set then it can be indefinitely extended (this follows from the local existence theory). But TM is a union of compact invariant sets, namely the sets $S^c = \{v \in TM \mid \|v\| = c\}$, $c \in R$, $c \geq 0$. Hence (a) holds.

One proves (b) by using the homogeneity to keep translating vectors to a fixed point say x_0 , to estimate the time of existence. This time does not shrink because of conservation of energy. Hence one can keep on extending a geodesic by a definite ε time interval, independent of the base point. Hence a geodesic can be indefinitely extended.

For pseudo-riemannian manifolds (i.e. the metric need not be positive definite) this argument does not work. However we have the following (see Wolf [1], p. 95, Marsden [9]).

Theorem. *Let* M *be a compact pseudo-riemannian manifold. Let* G *be a Lie group which acts transitively on* M *by isometries. Then* M *is* geodesically complete.

This result was proved by Hermann [3] in the special case of a semi-simple compact Lie group carrying a left invariant pseudo-riemannian metric. It should be noted that in the statement of the theorem neither the homogeneity nor the compactness may be dropped. For example it has become well-known to relativists that there are incomplete Lorentz metrics on the two torus. These were constructed by Y. Clifton and W. Pohl. (See Markus [1], p. 189.) An incomplete metric on the noncompact group SO(2, 1) is constructed in Hermann [3] although this is a special case of a whole class of incomplete pseudo-riemannian manifolds constructed by J. A. Wolf. (See Wolf [1,2]).

Proof. We shall show that the tangent bundle TM of M is the union of compact subsets S_α parametrized by elements α of the dual $\mathcal{G}^*$ of the Lie algebra of G, with S_α invariant under the geodesic flow. Since a vector field whose integral curves remain in a compact set has a complete flow, this is clearly enough to prove the theorem, as above.

Let $P : TM \to \mathcal{G}^*$, $P(v)\cdot\xi = \langle v, \xi_M(x)\rangle$ be the moment and for $\alpha \in \mathcal{G}^*$, set $S_\alpha = P^{-1}(\alpha)$. By the conservation theorems, S_α is invariant under the flow. Obviously TM is the union of the S_α. Therefore, it remains only to prove the following lemma. In this lemma we use the fact that $T_xM = \{\xi_M(x) \mid \xi \in \mathcal{G}\}$ which follows from the fact that M is homogeneous, i.e. there is only one orbit.

Lemma. Each of the sets S_α is a compact subset of TM .

Proof. Certainly S_α is closed. Furthermore, the restriction of the canonical projection $\pi : TM \to M$ to S_α is one-to-one because from the fact that the $\xi_M(m)$ span T_mM , we see that S_α intersects each fiber in at most one point.

We claim first of all that $\pi(S_\alpha)$ is closed and hence compact. Indeed $x \notin \pi(S_\alpha)$ means that α is not in the range of the linear map obtained by restricting P to T_xM . Thus α is not in the range of $P|T_yM$ for y in a whole neighborhood of x . Hence $\pi(S_\alpha)$ is closed.

Now let v_x , $v_y \in S_\alpha$, so $\langle v_x, \xi_M(x)\rangle = \langle v_y, \xi_M(y)\rangle = \alpha(\xi)$ for all $\xi \in \mathfrak{G}$. From the fact that $\xi_M(m)$ span T_mM and non-degeneracy of $\langle\ ,\ \rangle$, we may conclude that v_x is close to v_y if x is close to y . Hence the inverse $\pi^{-1} : \pi(S_\alpha) \to S_\alpha$ is continuous. Thus S_α is compact. □

Remarks. 1. If dim G = dim M , then S_α is actually a submanifold because $P : TM \to \mathfrak{G}^*$ is a submersion in that case (the derivative of P along the fibers is one-to-one and hence surjective).

2. Of course we have actually proved more. We only require that the infinitesimal generators span at each point, and that we have an invariant Hamiltonian system. Clearly conservation of energy, which is the basis of the proof for the Riemannian case (see above), plays no role here.